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Tykhonov Well-posedness of a Rate-type Viscoplastic Constitutive Law[1]

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Abstract

We consider a rate-type constitutitve law given by an implicit nonlinear differential equation in the space of second order symmetric tensors on $\mathbb{R}^d$, in which the unknowns are the stress and the linearized strain fields. We list the assumptions on the constitutive functions then we state and prove its well-posedness with respect to two different Tykhonov triples. We use these well-posedness properties in order to deduce two convergence results. Finally, we provide the mechanical interpretation of these results as well as some concluding remarks.

Keywords: rate-type viscoplastic constitutive law, Tykhonov well-posedness, approximating sequence, convergence.

1. Introduction

Rate-type viscoelastic or viscoplastic constitutive laws have been used in the literature in order to model the properties of metals, rubbers, polymers, rocks and soils, among others. Usually, they are expressed in terms of differential equations in which the unknowns are the stress and the strain field. References in the field include the books [1, 3, 4, 6, 7, 10]. A relevant example is given by the constitutive law

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\dot{\boldsymbol{\varepsilon}} + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}), \qquad (1)$$

where $\boldsymbol{\sigma}$ denotes the stress tensor, $\boldsymbol{\varepsilon}$ represents the linearized strain tensor, $\mathcal{E}$ is a fourth order elasticity tensor and $\mathcal{G}$ is a viscoplastic constitutive function, respectively. In (1) and everywhere in this paper the dot above a variable represents the derivative of that variable with respect to the time.

Constitutive laws of the form (1) have been introduced by Cristescu in [1] and then used by many authors. Various examples and mechanical interpretations can be found in [2, 3, 10]. A concrete example is the Perzyna constitutive law

$$\dot{\boldsymbol{\varepsilon}} = \mathcal{E}^{-1}\dot{\boldsymbol{\sigma}} + \frac{1}{\mu}(\boldsymbol{\sigma} - \mathcal{P}_K\boldsymbol{\sigma}). \qquad (2)$$

in which $\mathcal{E}$ is a fourth order invertible tensor, $\mathcal{E}^{-1}$ denotes its inverse, $\mu > 0$ is a viscosity constant, K is a nonempty closed convex subset of the space of symmetric second order tensors and $\mathcal{P}_K$ represents the projection operator on K. Note that in this case the function $\mathcal{G}$ does not depend on $\boldsymbol{\varepsilon}$ and is given by

$$\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = -\frac{1}{\mu}\mathcal{E}(\boldsymbol{\sigma} - \mathcal{P}_K\boldsymbol{\sigma}). \qquad (3)$$

Since $\boldsymbol{\sigma} = \mathcal{P}_K\boldsymbol{\sigma}$ iff $\boldsymbol{\sigma} \in K$, from (2) we see that viscoplastic deformations could occur only for the stress tensors $\boldsymbol{\sigma}$ which do not belong to K. A relatively simple one-dimensional example of constitutive law of the form (1) in which a full coupling of the stress and the strain is involved in the function $\mathcal{G}$ is given by

$$\dot{\sigma} = E\dot{\varepsilon} + G(\sigma, \varepsilon), \qquad (4)$$

where $E > 0$ is the Young modulus and $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the function

$$G(\sigma, \varepsilon) = \begin{cases} -k_1 F_1(\sigma - f(\varepsilon)) & \text{if} \quad \sigma > f(\varepsilon), \\ 0 & \text{if} \quad g(\varepsilon) \le \sigma \le f(\varepsilon), \\ k_2 F_2(g(\varepsilon) - \sigma) & \text{if} \quad \sigma < g(\varepsilon). \end{cases} \qquad (5)$$

Here $k_1, k_2 > 0$ are viscosity constants, $f, g : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions such that $g(\varepsilon) \le f(\varepsilon)$ for all $\varepsilon \in \mathbb{R}$, and $F_1, F_2 : \mathbb{R}_+ \to \mathbb{R}$ are increasing functions which satisfy $F_1(0) = F_2(0) = 0$. More details on the constitutive law (4), (5) can be found in [3, p. 35].

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Note that the domain of elastic behavior is characterized by the inequalities $g(\varepsilon) \le \sigma \le f(\varepsilon)$. Assume now that $g(\varepsilon) < 0 < f(\varepsilon)$ for all $\varepsilon \in \mathbb{R}$. In this case viscoplastic deformations occur only for $\sigma > f(\varepsilon)$, in traction, and for $\sigma < g(\varepsilon)$, in compression. Therefore, since the yield limit (in traction and in compression) depends on the deformation, we conclude that the viscoplastic constitutive law (4), (5) describes a hardening property of the material.

The variational analysis of mathematical models which describe the contact of materials with a constitutive laws of the form (1) was carried out in [7, 14] and, more recently, in [15]. There, existence, uniqueness and convergence results have been obtained, by using various functional methods. The numerical analysis of the corresponding contact models, including error estimates and numerical modelling, can be found in [7, 8] and the references therein.

The concept of Tykhonov well-posedness (well-posedness, for short) was introduced in [18] for a minimization problem and then it has been generalized for different mathematical problems, including optimization, fixed point and various inequality problems. It is based on two main ingredients: the existence and uniqueness of solution and the convergence to it of any approximating sequence. References in the field include [5, 9, 11, 12, 13, 19]. A general framework which unifies the view on well-posedness for abstract problems in metric spaces was recently considered in [16]. There, the well-posedness concept has been introduced by using approximating sequences which are defined by a family of subsets $\{\Omega(\omega)\}_\omega$ indexed upon a positive parameter $\omega > 0$. The results in [16] have been extended in [20], where a more general concept of well-posedness was introduced, based on the notion of Tykhonov triple $\mathcal{T} = (I, \Omega, C)$. Here I is set of parameters, Ω represents a family approximating sets and C is a set which defines a criterion of convergence.

As mentionned above, the concept of well-posedness was used in the literature in the study of many problems. Nevertheless, at the best of our knowledge, it has not been used in the study of constitutive laws for deformable solids. Our aim in this paper is to fill this gap. Thus, we study here the well-posedness of the rate-type constitutive laws (1) by using the mathematical tools provided in [20], based on the properties of Tykhonov triples. Proving that, under appropriate assumptions on $\mathcal{E}$ and $\mathcal{G}$, the rate-type constitutive law (1) is well-posed in the sense of Tykhonov represents the main trait of originalty of this work. It allows us to obtain existence, uniqueness and convergence results for which we provide the corresponding mechanical interpretation.

The rest of the manuscript is structured as follows. In Section 2 we list the assumptions on the data, then we prove an existence and uniqueness result, Theorem 1. In Section 3 we introduce the concept of well-posedness for the constitutive law (1), then we state and prove our main results, Theorems 6 and 8. In Section 4 we use these theorems in in order to deduce two convergence results and, finally, in Section 5 we present some concluding remarks.

We end this Introduction with some notation and preliminaries. Everywhere in this paper $d \in \{1, 2, 3\}$, the indices i, j, k, l run form 1 to d and the convention summation upon a repreated index is used. We denote by $\mathbb{N}$ the set of positive integers, i.e., $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{R}_+$ will represent the set of nonnegative real numbers, i.e., $\mathbb{R}_+ = [0, +\infty)$. We use $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$, endowed with the canonical inner product and the Euclidean norm given by

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \sigma_{ij}\tau_{ij}, \quad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2} \qquad (6)$$
$$\forall\, \boldsymbol{\sigma} = (\sigma_{ij}), \quad \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.$$

We also use $C(\mathbb{R}_+; \mathbb{S}^d)$ and $C^1(\mathbb{R}_+; \mathbb{S}^d)$ for the space of continuous and continuously differentable functions on $\mathbb{R}_+$ with values in $\mathbb{S}^d$, respectively. The convergence of a sequence $\{\boldsymbol{\tau}_n\}$ in the space $C(\mathbb{R}_+; \mathbb{S}^d)$ is described as follows:

$$\begin{cases} \boldsymbol{\tau}_n \to \boldsymbol{\tau} \ \text{ in } \ C(\mathbb{R}_+; \mathbb{S}^d) \ \text{ as } \ n \to \infty \\ \text{if and only if} \\ \max\limits_{t\in[0,m]} \|\boldsymbol{\tau}_n(t) - \boldsymbol{\tau}(t)\| \to 0 \text{ as } n \to \infty, \ \forall\, m \in \mathbb{N}. \end{cases} \qquad (7)$$

Moreover, the convergence of a sequence $\{\boldsymbol{\tau}_n\}$ in the space $C^1(\mathbb{R}_+; \mathbb{S}^d)$ is described in the following way:

$$\begin{cases} \boldsymbol{\tau}_n \to \boldsymbol{\tau} \ \text{ in } \ C^1(\mathbb{R}_+; \mathbb{S}^d) \ \text{ as } \ n \to \infty \\ \text{if and only if} \\ \max\limits_{t\in[0,m]} \|\boldsymbol{\tau}_n(t) - \boldsymbol{\tau}(t)\| + \max\limits_{t\in[0,m]} \|\dot{\boldsymbol{\tau}}_n(t) - \dot{\boldsymbol{\tau}}(t)\| \to 0 \\ \text{ as } \ n \to \infty, \ \forall\, m \in \mathbb{N}. \end{cases} \qquad (8)$$

It follows from (7) and (8) that $\boldsymbol{\tau}_n \to \boldsymbol{\tau}$ in $C^1(\mathbb{R}_+; \mathbb{S}^d)$ if and only if $\boldsymbol{\tau}_n \to \boldsymbol{\tau}$ in $C(\mathbb{R}_+; \mathbb{S}^d)$ and $\dot{\boldsymbol{\tau}}_n \to \dot{\boldsymbol{\tau}}$ in $C(\mathbb{R}_+; \mathbb{S}^d)$.

2. An existence and uniqueness result

In the study of the constitutive law (1) we assume that the elasticity tensor $\mathcal{E}$ is symmetric and positively defined and the viscoplastic function $\mathcal{G}$ is Lipschitz continuous, i.e.,

$$\left\{\begin{array}{l}\mathcal{E}=(\mathcal{E}_{ijkl})\colon \mathbb{S}^d\to\mathbb{S}^d \text{ is such that}\\ \text{(a) } \mathcal{E}_{ijkl}=\mathcal{E}_{klij}=\mathcal{E}_{jikl}\ ,\ 1\le i,j,k,l\le d,\\ \text{(b) there exists } m_{\mathcal{E}}>0 \text{ such that}\\ \qquad (\mathcal{E}\boldsymbol{\tau},\boldsymbol{\tau})\ge m_{\mathcal{E}}\|\boldsymbol{\tau}\|^2 \text{ for all } \boldsymbol{\tau}\in\mathbb{S}^d.\end{array}\right. \tag{9}$$

$$\left\{\begin{array}{l}\mathcal{G}\colon \mathbb{S}^d\times\mathbb{S}^d\to\mathbb{S}^d \text{ and there exists } L_{\mathcal{G}}>0\\ \text{such that}\\ \quad \|\mathcal{G}(\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1)-\mathcal{G}(\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)\|\\ \qquad \le L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|)\\ \qquad\quad \text{for all } \boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2\in\mathbb{S}^d.\end{array}\right. \tag{10}$$

Note that these assumptions guarantee that the stress function $\boldsymbol{\sigma}$ and the strain function $\boldsymbol{\varepsilon}$ play a symmetric role, since (1) is equivalent with the rate-type constitutive law

$$\dot{\boldsymbol{\varepsilon}}=\widetilde{\mathcal{E}}\dot{\boldsymbol{\sigma}}+\widetilde{\mathcal{G}}(\boldsymbol{\sigma},\boldsymbol{\varepsilon}). \tag{11}$$

where $\widetilde{\mathcal{E}}=\mathcal{E}^{-1}$ represents the inverse of the tensor $\mathcal{E}$ and $\widetilde{\mathcal{G}}=-\mathcal{E}^{-1}\mathcal{G}$. Moreover, note that if (9) and (10) hold, then $\widetilde{\mathcal{E}}$ satisfies condition (9) and $\widetilde{\mathcal{G}}$ satisfies condition (10), too. Finally, note that assumption (9) implies that there exists $L_{\mathcal{E}}>0$ and $L_{\mathcal{E}^{-1}}>0$ such that

$$\|\mathcal{E}\boldsymbol{\varepsilon}_1-\mathcal{E}\boldsymbol{\varepsilon}_2\|\le L_{\mathcal{E}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|, \tag{12}$$

$$\|\mathcal{E}^{-1}\boldsymbol{\varepsilon}_1-\mathcal{E}^{-1}\boldsymbol{\varepsilon}_2\|\le L_{\mathcal{E}^{-1}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|, \tag{13}$$

$$(\mathcal{E}\boldsymbol{\varepsilon}_1-\mathcal{E}\boldsymbol{\varepsilon}_2,\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2)\ge m_{\mathcal{E}}\|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2\|^2 \tag{14}$$

for all $\boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2\in\mathbb{S}^d$.

Next, we consider a stress function $\boldsymbol{\sigma}$ and an initial data $\boldsymbol{\varepsilon}_0$ such that

$$\boldsymbol{\sigma}\in C^1(\mathbb{R}_+;\mathbb{S}^d), \tag{15}$$

$$\boldsymbol{\varepsilon}_0\in\mathbb{S}^d. \tag{16}$$

Under these assumptions, we consider the following problem.

Problem $\mathcal{P}$. *Find a strain function $\boldsymbol{\varepsilon}\in C^1(\mathbb{R}_+;\mathbb{S}^d)$ such that*

$$\dot{\boldsymbol{\sigma}}(t)=\mathcal{E}\dot{\boldsymbol{\varepsilon}}(t)+\mathcal{G}(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(t))\quad\forall\, t\in\mathbb{R}_+, \tag{17}$$

$$\boldsymbol{\varepsilon}(0)=\boldsymbol{\varepsilon}_0. \tag{18}$$

Our main result in this section is the following.

Theorem 1. *Assume that* (9), (10), (15) *and* (16) *hold. Then Problem $\mathcal{P}$ has a unique solution.*

Proof. We use a fixed point argument. To this end, we consider the operator $\Lambda: C(\mathbb{R}_+;\mathbb{S}^d)\to C(\mathbb{R}_+;\mathbb{S}^d)$ defined by

$$\Lambda\boldsymbol{\eta}(t)=\boldsymbol{\varepsilon}_0-\mathcal{E}^{-1}\boldsymbol{\sigma}(0) \tag{19}$$
$$-\int_0^t\mathcal{E}^{-1}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\eta}(s)+\mathcal{E}^{-1}\boldsymbol{\sigma}(s))\,ds$$

for each $t\in\mathbb{R}_+$ and $\boldsymbol{\eta}\in C(\mathbb{R}_+;\mathbb{S}^d)$. Then, using assumption (10) and inequality (13) it follows that there exists a constant $L>0$ such that

$$\|\Lambda\boldsymbol{\eta}_1(t)-\Lambda\boldsymbol{\eta}_2(t)\|_V\le L\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|\,ds$$
$$\forall\,\boldsymbol{\eta}_1,\ \boldsymbol{\eta}_2\in C(\mathbb{R}_+;\mathbb{S}^d),\ t\in\mathbb{R}_+.$$

This inequality shows that Λ is a so-called history history-dependent operator, and, therefore, using Theorem 26 in [15] we deduce that there exists a unique element $\boldsymbol{\eta}^*\in C(\mathbb{R}_+;\mathbb{S}^d)$ such that $\boldsymbol{\eta}^*=\Lambda\boldsymbol{\eta}^*$. This equality combined with (19) shows that $\boldsymbol{\eta}\in C^1(\mathbb{R}_+;\mathbb{S}^d)$. Denote by $\boldsymbol{\varepsilon}$ the function

$$\boldsymbol{\varepsilon}=\boldsymbol{\eta}^*+\mathcal{E}^{-1}\boldsymbol{\sigma} \tag{20}$$

and note that, obviously, $\boldsymbol{\varepsilon}\in C^1(\mathbb{R}_+;\mathbb{S}^d)$. Moreover, (20), equality $\boldsymbol{\eta}^*=\Lambda\boldsymbol{\eta}^*$ and (19) imply that

$$\boldsymbol{\varepsilon}(t)=\mathcal{E}^{-1}\boldsymbol{\sigma}(t)-\int_0^t\mathcal{E}^{-1}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(s))\,ds \tag{21}$$
$$+\boldsymbol{\varepsilon}_0-\mathcal{E}^{-1}\boldsymbol{\sigma}(0)\quad\forall\, t\in\mathbb{R}_+$$

or, equivalently,

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}(t)+\int_0^t\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(s))\,ds \tag{22}$$
$$+\boldsymbol{\sigma}(0)-\mathcal{E}\boldsymbol{\varepsilon}_0\quad\forall\, t\in\mathbb{R}_+.$$

Equalities (22) and (21) show that (17) and (18) hold and, therefore, $\boldsymbol{\varepsilon}$ is a solution of Problem $\mathcal{P}$. This proves the existence part in Theorem 1. The uniqueness is a consequence of the uniqueness of the fixed point of the operator Λ, guaranteed by Theorem 26 in [15]. □

3. Tykhonov well-posedness

Everywhere in this section we assume that (9), (10), (15) and (16) hold, even if we do not mention it explicitly. As already mentioned in the Introduction, the concept of well-posedness for Problem $\mathcal{P}$ is associated to a so-called Tykhonov triple which is defined as follows.

Definition 2. *A Tykhonov triple is a mathematical object of the form $\mathcal{T} = (I, \Omega, C)$ where I is a given nonempty set, $\Omega : I \to 2^X - \{\emptyset\}$ and $C \subset \mathcal{S}(I)$ is a nonempty set, where $X = C^1(\mathbb{R}_+; \mathbb{S}^d)$, 2^X is the power set of X and $\mathcal{S}(I)$ denotes the set of sequences whose elements belongs to I.*

Below in this paper, for any $\omega \in I$, we refer to the sets $\Omega(\omega) \subset C^1(\mathbb{R}_+; \mathbb{S}^d)$ as the approximating sets. Next, following our work [20], we consider the following definitions.

Definition 3. *Given a Tykhonov triple $\mathcal{T} = (I, \Omega, C)$, a sequence $\{\boldsymbol{\varepsilon}_n\}_n \subset \mathbb{S}^d$ is called a $\mathcal{T}$-approximating sequence if there exists a sequence $\{\omega_n\}_n \in C$, such that $\boldsymbol{\varepsilon}_n \in \Omega(\omega_n)$ for each $n \in \mathbb{N}$.*

Note that approximating sequences always exist, since, by assumption, $C \neq \emptyset$ and, moreover, for any sequence $\{\omega_n\} \in C$ and any $n \in \mathbb{N}$, the set $\Omega(\omega_n)$ is not empty.

Definition 4. *Problem $\mathcal{P}$ is said to be well-posed with respect to the Tykhonov triple $\mathcal{T} = (I, \Omega, C)$ if it has a unique solution and every $\mathcal{T}$-approximating sequence for Problem $\mathcal{P}$ converges in $C^1(\mathbb{R}_+, \mathbb{S}^d)$ to the solution.*

In other words, Problem $\mathcal{P}$ is well-posed with respect to $\mathcal{T}$ if there exists a unique function $\boldsymbol{\varepsilon} \in C^1(\mathbb{R}_+; \mathbb{S}^d)$ which satisfies (17) and (18) and, moreover, for any $\mathcal{T}$-approximating sequence $\{\boldsymbol{\varepsilon}_n\}_n$, we have

$$\boldsymbol{\varepsilon}_n \to \boldsymbol{\varepsilon} \quad \text{in} \quad C^1(\mathbb{R}_+; \mathbb{S}^d) \quad \text{as} \quad n \to \infty. \tag{23}$$

Note that the concept of well-posedness defined above depends on the Tykhonov triple $\mathcal{T}$. A relevant example of such triple is the following.

Example 5. *Keep the assumption in Theorem 1 and take $\mathcal{T} = (I, \Omega, C)$ where*

$$I = \{\, \omega = (\{\theta^m\}_m, \widetilde{\theta}) \ : \ \theta^m > 0 \ \ \forall\, m \in \mathbb{N}, \ \ \widetilde{\theta} > 0 \,\}, \tag{24}$$

$$C = \Big\{ \{\omega_n\} \ : \ \omega_n = (\{\theta_n^m\}_m, \widetilde{\theta}_n) \in I \ \ \forall\, n \in \mathbb{N}, \tag{25}$$
$$\theta_n^m \to 0 \ \text{ as } \ n \to \infty \ \ \forall\, m \in \mathbb{N},$$
$$\widetilde{\theta}_n \to 0 \ \text{ as } \ n \to \infty \Big\}$$

and, for each $\omega = (\{\theta^m\}_m, \widetilde{\theta}) \in I$, the set $\Omega(\omega)$ is defined as follows:

$$\Omega(\omega) = \Big\{ \boldsymbol{\varepsilon} \in C^1(\mathbb{R}_+; \mathbb{S}^d) \ : \tag{26}$$

$$\|\dot{\boldsymbol{\sigma}}(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}(t) - \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(t))\| \leq \theta^m$$
$$\forall\, t \in [0, m], \ m \in \mathbb{N}, \tag{27}$$

$$\|\boldsymbol{\varepsilon}(0) - \boldsymbol{\varepsilon}_0\| \leq \widetilde{\theta} \Big\}.$$

Note that for each $\omega \in I$ the solution $\boldsymbol{\varepsilon}$ obtained in Theorem 1 belongs to $\Omega(\omega)$ and, therefore, $\Omega(\omega) \neq \emptyset$, which shows that $\mathcal{T}$ satisfies Definition 2.

Our main result in this section is the following.

Theorem 6. *Assume* (9), (10), (15) *and* (16). *Then Problem $\mathcal{P}$ is well-posed with respect to the Tykhonov triple in Example* 5.

Proof. We start by recalling that the existence of a unique solution $\boldsymbol{\varepsilon} \in C^1(\mathbb{R}; \mathbb{S}^d)$ to Problem $\mathcal{P}$ was provided in Theorem 1. To proceed, we consider a $\mathcal{T}$-approximating sequence for the Problem $\mathcal{P}$, denoted by $\{\boldsymbol{\varepsilon}_n\}_n$. Then, according to Definition 3, there exists a sequence $\{\omega\}_n \subset C$ with $\omega_n = (\{\theta_n^m\}_m, \widetilde{\theta_n}) \in I$ such that

$$\theta_n^m \to 0 \quad \text{as} \quad n \to \infty \quad \forall\, m \in \mathbb{N}, \tag{28}$$
$$\widetilde{\theta}_n \to 0 \quad \text{as} \quad n \to \infty \tag{29}$$

and, moreover, for each $n \in \mathbb{N}$, the following properties hold:

$$\boldsymbol{\varepsilon}_n \in C^1(\mathbb{R}_+; \mathbb{S}^d), \tag{30}$$

$$\|\dot{\boldsymbol{\sigma}}(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) - \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t))\| \leq \theta_n^m \tag{31}$$
$$\forall\, t \in [0, m], \ m \in \mathbb{N},$$

$$\|\boldsymbol{\varepsilon}_n(0) - \boldsymbol{\varepsilon}_0\| \leq \widetilde{\theta}_n. \tag{32}$$

Let n, $m \in \mathbb{N}$ be fixed and let $t \in [0, m]$. We use (14) and (17) to write

$$\begin{aligned}
m_{\mathcal{E}}\|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\|^2 &\leq (\mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t), \dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)) \\
&\leq \|\mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}(t)\|\|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\| \\
&\leq \|\mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\sigma}}(t) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(t))\|\|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\| \\
&\leq \|\mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t)) - \dot{\boldsymbol{\sigma}}(t)\|\|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\| \\
&\quad + \|\mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(t)) - \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t))\|\|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\|.
\end{aligned}$$

Then, using (31) and assumption (10) it follows that

$$m_{\mathcal{E}}\|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\| \leq \theta_n^m + L_{\mathcal{G}}\|\boldsymbol{\varepsilon}_n(t) - \boldsymbol{\varepsilon}(t)\|. \tag{33}$$

On the other hand, the initial condition (18) implies that

$$\boldsymbol{\varepsilon}_n(t) - \boldsymbol{\varepsilon}(t) = \int_0^t (\dot{\boldsymbol{\varepsilon}}_n(s) - \dot{\boldsymbol{\varepsilon}}(s))\, ds + \boldsymbol{\varepsilon}_n(0) - \boldsymbol{\varepsilon}_0$$

and, using (32) yields

$$\|\boldsymbol{\varepsilon}_n(t) - \boldsymbol{\varepsilon}(t)\| \leq \int_0^t \|\dot{\boldsymbol{\varepsilon}}_n(s) - \dot{\boldsymbol{\varepsilon}}(s)\|\, ds + \widetilde{\theta}_n. \tag{34}$$

We now combine inequalities (33) and (34) to deduce that

$$\|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\| \le \frac{\theta_n^m + L_{\mathcal{G}}\widetilde{\theta}_n}{m_{\mathcal{E}}} + \frac{L_{\mathcal{G}}}{m_{\mathcal{E}}}\int_0^t \|\dot{\boldsymbol{\varepsilon}}_n(s) - \dot{\boldsymbol{\varepsilon}}(s)\|\, ds$$

and, after using the Gronwall argument we find that

$$\|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\| \le \frac{\theta_n^m + L_{\mathcal{G}}\widetilde{\theta}_n}{m_{\mathcal{E}}}\, e^{\frac{L_{\mathcal{G}}}{m_{\mathcal{E}}} t}.$$

We now use the convergences (28) and (29) to obtain that

$$\max_{t\in[0,m]} \|\dot{\boldsymbol{\varepsilon}}_n(t) - \dot{\boldsymbol{\varepsilon}}(t)\| \to 0 \qquad \text{as} \quad n \to \infty. \tag{35}$$

Next, inequality (34) combined with convergences (29) and (35) guarantees that

$$\max_{t\in[0,m]} \|\boldsymbol{\varepsilon}_n(t) - \boldsymbol{\varepsilon}(t)\| \to 0 \qquad \text{as} \quad n \to \infty. \tag{36}$$

Finally, it follows from (35), (36) and (8) that the convergence (23) holds, which concludes the proof. □

We now consider a second example of Tykhonov triple with whom Problem $\mathcal{P}$ is also well-posed.

Example 7. *Keep the assumption in Theorem* 1 *and take* $\mathcal{T} = (I, \Omega, C)$ *where*

$$I = \{\, \omega = (\theta, \widetilde{\theta}) \ :\ \theta > 0,\ \widetilde{\theta} > 0 \,\}, \tag{37}$$

$$C = \Big\{ \{\omega_n\} \ :\ \omega_n = (\theta_n, \widetilde{\theta}_n) \in I \ \ \forall\, n \in \mathbb{N}, \quad \theta_n \to 0,\ \widetilde{\theta}_n \to 0 \ \text{as}\ n \to \infty \Big\} \tag{38}$$

and, for each $\omega = (\theta, \widetilde{\theta}) \in I$*, the set* $\Omega(\omega)$ *is defined as follows:*

$$\Omega(\omega) = \Big\{ \boldsymbol{\varepsilon} \in C^1(\mathbb{R}_+; \mathbb{S}^d) \ : \ \|\dot{\boldsymbol{\sigma}}(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}(t) - \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(t))\| \le \theta \quad \forall\, t \in \mathbb{R}_+, \quad \|\boldsymbol{\varepsilon}(0) - \boldsymbol{\varepsilon}_0\| \le \widetilde{\theta} \Big\}. \tag{39}$$

Note that, again, using Theorem 1 *it follows that* $\Omega(\omega) \neq \emptyset$*, for each* $\omega \in I$*.*

We have the following well-posednes result.

Theorem 8. *Assume* (9)*,* (10)*,* (15) *and* (16)*. Then Problem* $\mathcal{P}$ *is well-posed with respect to the Tykhonov triple* $\mathcal{T}$ *in Example* 7*.*

The proof of this theorem is similar to that of Theorem 6 and, therefore, we skip it. Note that, in contrast with the proof of Theorem 6, in the proof of Theorem 6 some estimates are simpler since they do not depend on $m \in \mathbb{N}$.

4. Continuous dependence results

The solution of Problem $\mathcal{P}$ depends on the data $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}_0$. Its continuous dependence with respect these data is provided by the following convergence result.

Theorem 9. *Assume* (9)*,* (10)*,* (15)*,* (16)*, denote by* $\boldsymbol{\varepsilon}$ *the solution of Problem* $\mathcal{P}$ *and, for each* $n \in \mathbb{N}$*, denote by* $\boldsymbol{\varepsilon}_n$ *the solution of Problem* $\mathcal{P}$ *for the data* $\boldsymbol{\sigma}_n$*,* $\boldsymbol{\varepsilon}_{0n}$ *which satisfy* $\boldsymbol{\sigma}_n \in C^1(\mathbb{R}_+; \mathbb{S}^d)$*,* $\boldsymbol{\varepsilon}_{0n} \in \mathbb{S}^d$*. Moreover, assume that*

$$\boldsymbol{\sigma}_n \to \boldsymbol{\sigma} \ \text{ in } \ C^1(\mathbb{R}_+; \mathbb{S}^d), \tag{40}$$

$$\boldsymbol{\varepsilon}_{0n} \to \boldsymbol{\varepsilon}_0 \ \text{ in } \ \mathbb{S}^d \tag{41}$$

as $n \to \infty$*. Then,*

$$\boldsymbol{\varepsilon}_n \to \boldsymbol{\varepsilon} \ \text{ in } \ C^1(\mathbb{R}_+; \mathbb{S}^d) \quad \text{as}\ n \to \infty. \tag{42}$$

Proof. Let n, $m \in \mathbb{N}$ and $t \in [0, m]$. We have

$$\dot{\boldsymbol{\sigma}}_n(t) = \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) + \mathcal{G}(\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}_n(t)), \tag{43}$$

$$\boldsymbol{\varepsilon}_n(0) = \boldsymbol{\varepsilon}_{0n}. \tag{44}$$

We now use (43), (17) and assumption (10) to write

$$\begin{aligned}
&\|\dot{\boldsymbol{\sigma}}(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) - \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t))\| \\
&\ = \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_n(t) + \dot{\boldsymbol{\sigma}}_n(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) - \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t))\| \\
&\ = \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_n(t) + \mathcal{G}(\boldsymbol{\sigma}_n(t), \boldsymbol{\varepsilon}_n(t)) - \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t))\| \\
&\ \le \|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_n(t)\| + L_{\mathcal{G}}\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}_n(t)\|
\end{aligned}$$

and, using notation

$$\theta_n^m = \max_{t\in[0,m]} \|\dot{\boldsymbol{\sigma}}_n(t) - \dot{\boldsymbol{\sigma}}(t)\| + L_{\mathcal{G}} \max_{t\in[0,m]} \|\boldsymbol{\sigma}_n(t) - \boldsymbol{\sigma}(t)\|, \tag{45}$$

we find that

$$\|\dot{\boldsymbol{\sigma}}(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) + \mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t))\| \le \theta_n^m. \tag{46}$$

Let

$$\widetilde{\theta}_n = \|\boldsymbol{\varepsilon}_{0n} - \boldsymbol{\varepsilon}_0\| \tag{47}$$

and note that the initial conditions (44), (18) yield

$$\|\boldsymbol{\varepsilon}_n(0) - \boldsymbol{\varepsilon}(0)\| \le \widetilde{\theta}_n. \tag{48}$$

Consider now the sequence $\{\omega_n\}_n$ with $\omega_n = (\{\theta_n^m\}_m, \widetilde{\theta}_n)$, defined by (45), (47). Then, (46) and (48) show that the inequalities (31) and (32) hold, which implies that $\boldsymbol{\varepsilon}_n \in \Omega(\omega_n)$ where the multivalued function Ω is defined by (26). On the other hand, assumptions (40) and (41) show that $\theta_n^m \to 0$ for each $m \in \mathbb{N}$ and $\widetilde{\theta}_n \to 0$ as $n \to \infty$, respectively. We conclude from Definition 3 that $\{\boldsymbol{\varepsilon}_n\}_n$ is an approximating sequence for Problem $\mathcal{P}$, with respect to the Tykhnonov triple in Example 5.

Therefore, Theorem 6 and Definition 4 guarantee that the convergences (42) hold, which ends the proof. □

In additional to the mathematical interest in the convergence (42) it is important from mechanical point of view since it provides a continuous dependence result for the rate-type constitutive equation (1). Indeed, it shows that small perturbations on the stress function together with small perturbations on the initial strain give rise to small perturbations of the solution to Problem $\mathcal{P}$.

We now remark that Theorem 9 cannot be proved by using Theorem 8 instead of Theorem 6. A counterexample which proves this statement is the following.

Example 10. *Assume that* (9), (10), (15) *and* (16) *hold and denote by* $\boldsymbol{\varepsilon}$ *the solution of Problem* $\mathcal{P}$ *obtained in Theorem* 1. *Moreover, for each* $n \in \mathbb{N}$, *consider the function* $\boldsymbol{\sigma}_n \in C^1(\mathbb{R};\mathbb{S}^d)$ *and the element* $\boldsymbol{\varepsilon}_{0n}$ *defined by*

$$\boldsymbol{\sigma}_n(t) = \boldsymbol{\sigma}(t) + \frac{t^2}{n} I_d \quad \forall\, t \in \mathbb{R}_+, \tag{49}$$

$$\boldsymbol{\varepsilon}_{0n} = \boldsymbol{\varepsilon}_0 + \frac{1}{n} I_d \tag{50}$$

where $I_d \in \mathbb{S}^d$ *represents the identity tensor. Note that,* (49), (50) *imply that conditions* (40) *and* (41) *hold, respectively. Denote by* $\boldsymbol{\varepsilon}_n$ *the solution of Problem* $\mathcal{P}$ *for the data* $\boldsymbol{\sigma}_n$, $\boldsymbol{\varepsilon}_{0n}$. *Therefore, it follows from Theorem* 9 *that the convergences* (42) *hold. Nevertheless, we claim that the sequence* $\{\boldsymbol{\varepsilon}_n\}_n$ *is not a* $\mathcal{T}$*- approximating sequence for Problem P where, here and below in this example,* $\mathcal{T}$ *represents the Tykhonov triple* (37)–(39).

To prove this claim we assume in what follows that the function $\mathcal{G}$ *does not depend on* $\boldsymbol{\sigma}$. *Arguing by contradiction, assume that* $\{\boldsymbol{\varepsilon}_n\}_n$ *is a* $\mathcal{T}$*-approximating sequence. Then, using* (39) *we deduce that there exists a sequence* $\{\theta_n\}_n$ *such that* $\theta_n \to 0$ *and, for each* $n \in \mathbb{N}$, *and* $t \in \mathbb{R}_+$, *the following inequality holds:*

$$\|\dot{\boldsymbol{\sigma}}(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) - \mathcal{G}(\boldsymbol{\varepsilon}_n(t))\| \le \theta_n. \tag{51}$$

On the other hand, $\mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) + \mathcal{G}(\boldsymbol{\varepsilon}_n(t)) = \dot{\boldsymbol{\sigma}}_n(t)$ *and, therefore,* (51) *yields*

$$\|\dot{\boldsymbol{\sigma}}(t) - \dot{\boldsymbol{\sigma}}_n(t)\| \le \theta_n. \tag{52}$$

We now combine (49) *and* (52) *to deduce that*

$$\frac{2t}{n}\|I_d\| \le \theta_n. \tag{53}$$

Recall that this inequality holds for each $n \in \mathbb{N}$ *and* $t \in \mathbb{R}_+$. *Thus, taking* $t = n$ *we deduce that* $2d \le \theta_n$ *for each* $n \in \mathbb{N}$ *which contradicts the convergence* $\theta_n \to 0$. *We conclude from above that the sequence* $\{\boldsymbol{\varepsilon}_n\}_n$ *is not a* $\mathcal{T}$*-approximating sequence for Problem P and, therefore, Theorem* 8 *cannot be used to obtain the convergence* (42), *as claimed.*

We end this section with a second convergence result. It concerns a versiom of the constitutive law (1) given by

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\dot{\boldsymbol{\varepsilon}} + \mu\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}), \tag{54}$$

where μ is a given viscosity coefficient. Our aim is to show that for $\mu \in \mathbb{R}$ small enough (54) can be approached by the elastic constitutive law $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}$. To this end let $\{\mu_n\}_n \subset \mathbb{R}$ and consider the following additional assumptions.

$$\left\{\begin{array}{l} \mathcal{G}\colon \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d \ \text{ and there exists } M_{\mathcal{G}} > 0 \\ \text{such that } \ \|\mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon})\| \le M_{\mathcal{G}} \quad \forall\, \boldsymbol{\sigma},\, \boldsymbol{\varepsilon} \in \mathbb{S}^d. \end{array}\right. \tag{55}$$

$$\mu_n \to 0 \quad \text{as} \quad n \to \infty. \tag{56}$$

Theorem 11. *Assume* (9), (10), (15), (16), (55) *and denote by* $\boldsymbol{\varepsilon}$ *the function defined by* $\boldsymbol{\varepsilon} = \mathcal{E}^{-1}\boldsymbol{\sigma}$. *Then, for each* $n \in \mathbb{N}$ *there exists a unique function* $\boldsymbol{\varepsilon}_n \in C^1(\mathbb{R}_+;\mathbb{S}^d)$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t) + \mu_n\mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t)) \quad \forall\, t \in \mathbb{R}_+, \tag{57}$$

$$\boldsymbol{\varepsilon}_n(0) = \boldsymbol{\varepsilon}_0. \tag{58}$$

Moreover, if $\boldsymbol{\sigma}(0) = \mathcal{E}\boldsymbol{\varepsilon}_0$ *and* (56) *hold, then the convergence* (42) *holds, too.*

Proof. The existence of a unique solution $\boldsymbol{\varepsilon}_n \in C^1(\mathbb{R};\mathbb{S}^d)$ to problem (57), (58) follows from Theorem 1. Assume now that $\boldsymbol{\sigma}(0) = \mathcal{E}\boldsymbol{\varepsilon}_0$ and (56) holds. Let $n \in \mathbb{N}$ and $t \in \mathbb{R}_+$. Then, we use (57) and assumption (55) to write

$$\|\dot{\boldsymbol{\sigma}}(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t)\| = \|\mu_n\mathcal{G}(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}_n(t))\| \le M_{\mathcal{G}}|\mu_n|$$

and, using notation $\theta_n = M_{\mathcal{G}}|\mu_n|$, we find that

$$\|\dot{\boldsymbol{\sigma}}(t) - \mathcal{E}\dot{\boldsymbol{\varepsilon}}_n(t)\| \le \theta_n. \tag{59}$$

Let $\widetilde{\theta}_n = \frac{1}{n}$ and note that the initial conditions (58), and equalities $\boldsymbol{\sigma}(0) = \mathcal{E}\boldsymbol{\varepsilon}(0)$, $\boldsymbol{\sigma}(0) = \mathcal{E}\boldsymbol{\varepsilon}_0$ show that $\boldsymbol{\varepsilon}_n(0) = \boldsymbol{\varepsilon}(0) = \boldsymbol{\varepsilon}_0$ and, therefore,

$$\|\boldsymbol{\varepsilon}_n(0) - \boldsymbol{\varepsilon}(0)\| \le \widetilde{\theta}_n. \tag{60}$$

Consider now the sequence $\{\omega_n\}_n$ defined by $\omega_n = (\theta_n, \widetilde{\theta}_n)$. Then, (59) and (60) show that $\boldsymbol{\varepsilon}_n \in \Omega(\omega_n)$ where the multivalued function Ω is defined by (39) with $\mathcal{G} \equiv 0$. On the other hand assumption (56) show that $\theta_n \to 0$ as $n \to \infty$ and, obviously, $\widetilde{\theta}_n \to 0$ as $n \to \infty$. We conclude from Definition 3 that $\{\boldsymbol{\varepsilon}_n\}_n$ is an approximating sequence for Problem $\mathcal{P}$, with respect to the Tykhonov triple in Example 7. Therefore, Theorem 8 and Definition 4 guarantee that the convergence (42) holds, which concludes the proof. □

In additional to the mathematical interest in the convergence in Theorem 11, it is important from mechanical point of view since it shows that the viscoplastic constitutiver law (54) can be approached by the elastic constitutive law $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}$ for a small coefficient of viscosity.

5. Conclusion

In this paper we proved the well-posedness of the rate-type constitutive law (1) with respect to two Tykhonov triples. Then, we used the well-posedness in order to prove two convergence results. We also showed that, on this matter, the choice of an appropriate Tykhonov triples plays a crucial role. The material presented in this paper leads to a better knowledge of rate-type consitutive law of the form (1) since, besides the mathematical interest in our results, they lead to interesting mechanical interpretations. Moreover, their analysis reveals some subjects for future research, which could represent a continuation of this paper.

The first subject would be to extend the results presented in this paper to various clases of viscoplastic constitutive laws. One example would be the viscoplastic laws of the form

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\dot{\boldsymbol{\varepsilon}} + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}), \tag{61}$$

in which κ is a hardening parameter or an internal state variable, assumed to satisfy a differential equation of the form

$$\dot{\boldsymbol{\kappa}} = \Phi(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\kappa}). \tag{62}$$

Results similar to those in Theorms 6, 8 and 9 can be obtained in the study of (61), (62), under appropriate assumption of the functions Φ and $\mathcal{G}$.

The second subject would be to study the well-posedness of boundary value problems with rate-type constitutive laws of the form (1) and, in particular, the study of frictional or frictionless contact problems. A first step in this direction was made in [17]. Proving the Tykhnonov well-posedness of contact problems would be a powerfull mathematical tool which could be used to obtain various convergence results that describe the behaviour of the solution with respect the data and parameters. It also allows us to establish the link between models of contact constructed by using different constitutive laws and different interface bounday conditions.

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