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► **To cite this version:**

Eric Goles, Diego Maldonado, Pedro Montealegre, Nicolas Ollinger. On the complexity of the stability problem of binary freezing totalistic cellular automata. *Information and Computation*, 2020, 274, pp.104535 -. 10.1016/j.ic.2020.104535 . hal-03491633

**HAL Id: hal-03491633**

**<https://hal.science/hal-03491633v1>**

Submitted on 21 Sep 2022

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# On the Complexity of the Stability Problem of Binary Freezing Totalistic Cellular Automata

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## Abstract

In this paper we study the family of two-state Totalistic Freezing Cellular Automata (TFCA) defined over the triangular and square grids with von Neumann neighborhoods. We say that a Cellular Automaton is Freezing and Totalistic if the active cells remain unchanged, and the new value of an inactive cell depends only of the sum of its active neighbors.

We classify all the Cellular Automata in the class of TFCA, grouping them in five different classes: the Trivial rules, Turing Universal rules, Algebraic rules, Topological rules and Fractal Growing rules. At the same time, we study in this family the STABILITY problem, consisting in decide whether an inactive cell becomes active, given an initial configuration. We exploit the properties of the automata in each group to show that:

- For Algebraic and Topological Rules the STABILITY problem is in **NC**.
- For Turing Universal rules the STABILITY problem is **P**-Complete.

*Keywords:* Cellular Automata, Computational Complexity, Freezing Cellular Automata, Totalistic Cellular Automata, Fast parallel algorithms, P-Complete

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*Preprint submitted to Elsevier*

*March 25, 2018*

## 1. Introduction

Consider a one or two dimensional *cellular automaton* (CA) on a finite torus. This consists of a regular toroidal grid of cells, each in one of a finite number of states, such as *active* and *inactive* or 1 and 0. For each cell, a set of cells called its *neighborhood* is defined relative to the specified cell. Each cell changes synchronously by the action of a *local function* depending of the states of its neighborhood. We call *configuration* to a coloration of the grid, and the CA change the configuration in each iteration.

An important problem in complexity of CA consists in prediction of the future state of a cell, given an initial configuration. This decision problem is called Prediction [1]. The complexity question how fast we could determine the solution of Prediction, and in particular if we may answer faster than simulating of the automaton.

That leads us to consider in the context of the *Computational Complexity Theory*, the classes,  $\mathbf{P}$  of polynomial problems (i.e., problems solved by a polynomial algorithm) an  $\mathbf{NC}$  of problems that can be solved in poly logarithmic time in a PRAM. Clearly  $\mathbf{NC} \subseteq \mathbf{P}$  although the equality is a very hard open problem. Usually, the simple simulation of an automaton leads to a polynomial-time solution of the Prediction problem.

We classify CA according whether its associated Predictionproblem. Our goal is to show that an automaton belongs to  $\mathbf{NC}$  if it is  $\mathbf{P}$ -Complete, i.e., every problem in the class  $\mathbf{P}$  can be reduced to it. So, unless  $\mathbf{P} = \mathbf{NC}$ , our goal is to classify a CA in two groups, those in  $\mathbf{NC}$  (where we say that Prediction is easy to decide) and those they are  $\mathbf{P}$ -Complete (where the problem is *hard*)

To our knowledge, the first study related with the computational complexity of Cellular Automata was done by E. Banks. In his PhD thesis he studied the possibility for simple Cellular Automata in two dimensional grids, to simulate logical gates. If such simulation is possible, the automaton is capable of universal Turing computation [2]. Directly in the context of prediction problems C. Moore et al [1] studied the *Majority Automata* (next state of a site will be the most represented in the neighborhood). They proved that for a given  $T \geq 0$ , Predictionis  $\mathbf{P}$ -Complete in three and more dimensions. The complexity remains open in two dimensions.

Recently, in [3], is study a particular family of CA, the *freezing CA* (FCA). This are CA where the state of a cell can only change to another bigger state, e.g. if the states are *active* and *inactive* the active cell remains active forever.

It is direct that, every initial configuration converges in at most  $N$  steps to a fixed point (where  $N$  the size of the torus), thus if we could decide Prediction for a sufficiently large time, then we could compute if a cell change or remains *stable* (always in the same state). We call this problem STABILITY. In [3] it is shown that the *stability* problem is in **NC**, for every one-dimensional freezing cellular automata.

In order to find a FCA with higher complexity, the result of [3] shows that it is necessary to study FCA in more than one dimensions. In this context, we should mention that D. Griffeath and C. Moore studied the *Life without Death* Automaton (i.e. the game of life such that active sites remains unchanged), showing that the STABILITY problem for this rule is **P**-Complete [4]. We remark that the Life without Death is a two-dimensional cellular automaton with Moore neighborhood.

On other hand, in [5], it was studied the *freezing majority cellular automaton*, also known as *bootstrap percolation model*, in arbitrary undirected graph. In this case, a inactive cell becomes active if and only if the active cells are the most represented in its neighborhood. It was proved that STABILITY is **P**-Complete over graphs such that its maximum degree (number of neighbors)  $\geq 5$ . Otherwise (graphs with maximum degree  $\leq 4$ ), the problem is in **NC**. This clearly includes the two dimensional case, with von Neuman neighborhood.

In this work we study the two simplest ways to tessellate the bi-dimensional grid: tessellation with triangles (where each cell has three neighbors) and with squares (where each cell has four neighbors, i.e. the two dimensional CA with von Neumann neighborhood). In each one of this grids we study the family of *freezing totalistic cellular automata* (FTCA). The name totalistic means that the new value of a cell only depends of the sum of its neighbors. We show that this family of CAs exhibit a broad and rich range of behaviors.

More precisely, we classify FTCA's in four groups:

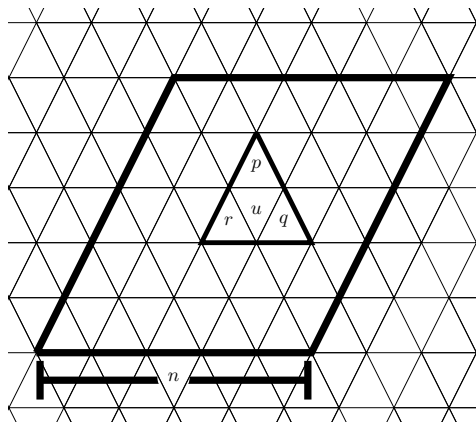
- Simple rules: Rules that exhibit very simple dynamics, which reach fixed points in a constant number of steps.
- Topological rules: Rules where the stability of a cell depends on some topological property given by the initial configuration.
- Algebraic rule: Rules where the dynamics can be accelerated, exploiting some algebraic properties given by the rule.

- Turing Universal rules: Rules capable of simulating Boolean Circuits and capable to simulate Turing Computation.
- Fractal growing rules: Rules that produce patterns with which grow forming fractal shapes.

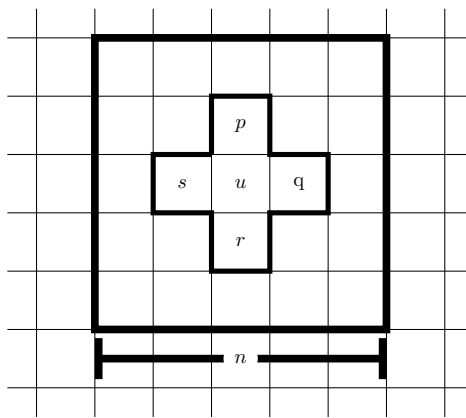
The paper is structured as follows: first in Section 2 definitions and notations are introduced. In Section 3, the FTCA for the triangular grid are studied. In Section 4, we study the FTCA on the square grid. Finally, in Section 5 we give some conclusions.

## 2. Preliminaries

Consider the plane tessellated by triangles, as depicted in Figure 1a, or tessellated by squares, as depicted in Figure 1b. We call such tessellations the *triangular grid* and *squared grid*, respectively. In each case, a triangle or square is called a *cell*. In the triangular grid each cell (triangle) has three adjacent cells, and in the square grid each cell (square) has four adjacent cells. A cell that is adjacent to a cell  $u$  is called a *neighbor* of  $u$ . The set of neighbors of  $u$  is denoted by  $N(u)$ . In the triangular and squared grid, this definition of neighbors is called the *von Neumann Neighborhood*, and it is denoted  $N(0,0)$ .



(a) Triangular grid, with its neighbors  $p$ ,  $q$  and  $r$  of a cell  $u$ .



(b) Square grid. Neighbors  $p$ ,  $q$ ,  $r$  and  $s$  of a cell  $u$ .

Figure 1: Triangular (a) and square (b) grids with the von Neumann neighborhood of a cell  $u$ .

Each cell in a grid has two possible *states*, which are denoted 0 and 1. We say that a site in state 1 is *active* and a site in state 0 is *inactive*. A configuration of a grid (triangular or squared) is a function that assigns a state to every cell. In a squared grid, a *finite configuration*  $x$  of dimension  $n \times n$  is a function that assigns values in  $\{0, 1\}$  to squared shaped area of  $n^2$  cells. Analogously, in a triangular grid, a finite configuration  $x$  of dimension  $n \times n$  is a function that assigns values in  $\{0, 1\}$  to a rhomboid shaped area of  $2n^2$  cells. The value of the cell  $u$  in the configuration  $x$  is denoted  $x_u$  (See figure 1). We remark that a finite configuration  $x$  of dimension  $n \times n$  has  $2n^2$  cells in a triangular grid and has  $n^2$  cells in a square grid. In both cases the number of covered cells is  $\mathcal{O}(n^2)$ .

Given a finite configuration  $x$  of dimension  $n \times n$ , the periodic configuration  $c = c(x)$  is an infinite configuration over the grid, obtained by repetitions of  $x$  in all directions. The configuration  $c(x)$  is a spatially periodic, and will be interpreted as a torus, where each cell in the boundary of  $x$  has a neighbor placed in the opposite boundary of  $x$ .

We call  $\mathcal{C}$  is the set of all possible configurations over a (triangular or squared) grid. A *cellular automaton* (CA) with set of states  $\{0, 1\}$  is a function  $F : \mathcal{C} \rightarrow \mathcal{C}$ , defined by a *local function*  $f : \{0, 1\}^{\mathcal{N}(u)} \rightarrow \{0, 1\}$  as  $F(c)_u = f(c_{N(u)})$ . Computing  $F$  is equivalent to compute in synchronously in each site of the grid, the application of the local function  $f$  cell by cell. A cellular automaton is called *freezing* [3] (FCA) if the local rule  $f$  satisfies that the active cells always remain active. A cellular automaton is called *totalistic* [6] (TCA) if the local rule  $f$  satisfies  $f(c_{N(u)}) = f(c_u, \sum_{v \in N(u)} c_v)$ , i.e. it depends only in the sum of the states in the neighborhood of a cell.

We call FTCA the family of two-state freezing totalistic cellular automata, over the square and triangular grids, with von Neumann neighborhood. In this family, the active cells remain active, because the rule is freezing, and the inactive cells become active depending only in the sum of their neighbors. Notice that this sum of the states of the neighbors of a site is at most the size of the neighborhood, that we call  $|N(0,0)|$ , and equals 3 in the case of the triangular grid, and 4 in the case of the squared grid.

Let  $F$  be a FTCA. We can identify  $F$  with a set  $\mathcal{I}_F \subseteq \{1, \dots, |N(0,0)|\}$  such that, for every configuration  $c$  and site  $u$ :

$$f(c_{N(u)}) = \begin{cases} 1 & \text{if } (c_u = 1) \vee (\sum_{v \in N(u)} c_v \in \mathcal{I}_F), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that  $\mathcal{I}_F \subseteq \{0, 1, 2, 3\}$  in the triangular grid and  $\mathcal{I}_f \subseteq \{0, 1, 2, 3, 4\}$

in the squared grid. We will name the FTCA according to the elements contained in  $\mathcal{I}_F$ , as the concatenation of the elements of  $\mathcal{I}_F$  in increasing order (except when  $\mathcal{I}_F = \emptyset$ , that we call  $\phi$ ). For example, let *Maj* be the freezing majority vote CA, where an inactive cell becomes active if the majority of its neighbors is active. Note that  $\mathcal{I}_{Maj} = \{2, 3\}$  in the triangular grid and  $\mathcal{I}_{Maj} = \{2, 3, 4\}$  in the square grid. We call then *Maj* the rule 23 in the first case and 234 in the later.

We deduce that there are  $2^{|\mathcal{N}(0,0)|}$  different FTCA, each one of them represented by the corresponding set  $\mathcal{I}_F$ . Notice that the number of different FTCA is 16 in the triangular grid and 32 different in the squared grid. We will focus our analysis in the FTCA where the inactive state is a quiescent state, which means that the inactive sites where the sum of their neighborhoods is 0 remain inactive. Therefore, we will consider 8 different FTCA in the triangular grid, and 16 in the squared grid.

Recall that in an FTCA the active cells remain always active. We will be interested in the inactive cells that always remain inactive.

**Definition 2.1.** Given a configuration  $c \in \{0, 1\}^{\mathbb{Z}^2}$ , we say that a site  $v$  is *stable* if and only if  $c_v = 0$  and it remains inactive after any iterated application of the rule, i.e.,  $F^t(c)_v = 0$  for all  $t \geq 0$ .

From the previous definition, we consider the problem STABILITY, which consists in deciding if a cell on a periodic configuration  $c$  is stable. More formally, if  $F$  is a cellular automaton, then:

STABILITY

**Input:** A finite configuration  $x$  of dimensions  $n \times n$  and a site  $u \in [n] \times [n]$  such that  $x_u = 0$ .

**Question:** Is  $u$  stable for configuration  $c = c(x)$ ?

In other words, the answer of STABILITY is *no* if there exists  $T > 0$  such that  $F^T(c(x))_u = 1$ . Our goal is to understand the difficulty of STABILITY in terms of its computational complexity, for every FTCA defined over a triangular or squared grid. We consider two classes of complexity of problems: **P** and **NC**.

The class **P** is the class of problems that can be solved by a deterministic Turing machine in time  $n^{\mathcal{O}(1)}$ , where  $n$  is the size of the input. Let  $F$  be a freezing cellular automaton (FCA) and  $x$  be a finite configuration of dimensions  $n \times n$  cells. Notice that the dynamics of  $F$  over  $c(x)$  reach a *fixed point*

(a configuration  $c'$  such that  $F(c') = c'$ ) in  $\mathcal{O}(n^2)$  steps. Indeed, after each application of  $F$  before reaching the fixed point, at least one inactive site become active in each copy of  $x$ . The application of one step of any FCA can be simulated in polynomial time, simply computing the local function of every cell. Therefore, for every FCA (and then for every FTCA)  $F$  problem  $\text{STABILITY}(F)$  is in  $\mathbf{P}$ .

The class  $\mathbf{NC}$  is a subclass of  $\mathbf{P}$ , consisting of all problems solvable by a *fast-parallel algorithm*. A fast-parallel algorithm is one that runs in a parallel random access machine (PRAM) in poly-logarithmic time (i.e. in time  $(\log n)^{\mathcal{O}(1)}$ ) using  $n^{\mathcal{O}(1)}$  processors. It is direct that  $\mathbf{NC} \subseteq \mathbf{P}$ , and it is a wide-believed conjecture that the inclusion is proper [7]. Indeed,  $\mathbf{NC} = \mathbf{P}$  would imply that for any problem solvable in polynomial time, there is a parallel algorithm solving that problem *exponentially faster*. Back in our context, the fact that for some FTCA the problem  $\text{STABILITY}$  belongs to  $\mathbf{NC}$  will imply that one can solve the problem significantly faster than simply simulating the steps of the automaton.

The problems in  $\mathbf{P}$  that are the most likely to not belong to  $\mathbf{NC}$  are the  $\mathbf{P}$ -Complete problems. A problem  $p$  is  $\mathbf{P}$ -Complete if it is contained in  $\mathbf{P}$  and every other problem in  $\mathbf{P}$  can be reduced to  $p$  via a function computable in logarithmic-space. For further details we refer to the book of [7].

### 2.1. Some graph terminology

For a set of cells  $S \subseteq \mathbb{Z}^2$ , we call  $G[S] = (S, E)$  the graph defined with vertex set  $S$ , where two vertices are adjacent if the corresponding sites are neighbors for the von Neumann neighborhood.

For a graph  $G = (V, E)$ , a sequence of vertices  $P = v_1, \dots, v_k$  is called a  $v_1, v_k$ -path if  $\{v_i, v_{i+1}\}$  is an edge of  $G$ , for each  $i \in [k - 1]$ . Two  $u, v$ -paths  $P_1, P_2$  are called *disjoint* if  $P_1 \cap P_2 = \{u, v\}$ . A  $u, v$ -path where  $u$  and  $v$  are adjacent is called a *cycle*.

**Definition 2.2.** A graph  $G$  is called  $k$ -connected if for every pair of vertices  $u, v \in V(G)$ ,  $G$  contains  $k$  disjoint  $u, v$ -paths. A 1-connected graph is simply called *connected*, a 2-connected graph is called *bi-connected* and a 3-connected graph is called *tri-connected*

A maximal set of vertices of a graph  $G$  that induces a  $k$ -connected subgraph is called a  *$k$ -connected component* of  $G$ .



## 2.2. Parallel subroutines

In this subsection, we will give some **NC** algorithms that we will use as subroutines of our fast-parallel algorithm solving STABILITY.

### 2.2.1. Prefix-sum

First, we will study a general way to compute in **NC** called *prefix sum algorithm* [8]. Given an associative binary operation  $*$  defined on a group  $G$ , and an array  $A = (a_1, \dots, a_n)$  of  $n$  elements of  $G$ , the prefix sum of  $A$  is the vector  $B$  of dimension  $n$  such that  $B_i = a_1 * \dots * a_i$ . Computing the prefix sum of a vector is very useful. For example, it can be used to compute the parity of a Boolean array, the presence of a nonzero coordinate in an array, etc.

**Proposition 2.1** ([8]). *There is an algorithm that computes the prefix-sum of an array of  $n$  elements in time  $\mathcal{O}(\log n)$  with  $\mathcal{O}(n)$  processors.*

### 2.2.2. Connected components

The following propositions state that the connected, bi-connected and tri-connected components of an input graph  $G$  can be computed by fast-parallel algorithms.

**Proposition 2.2** ([8]). *There is an algorithm that computes the connected components of a graph with  $n$  vertices in time  $\mathcal{O}(\log^2 n)$  with  $\mathcal{O}(n^2)$  processors.*

**Proposition 2.3** ([9]). *There is an algorithm that computes the bi-connected components of a graph with  $n$  vertices in time  $\mathcal{O}(\log^2 n)$  with  $\mathcal{O}(n^3/\log n)$  processors.*

**Proposition 2.4** ([9]). *There is an algorithm that computes the tri-connected components of a graph in time  $\mathcal{O}(\log^2 n)$  with  $\mathcal{O}(n^4)$  processors.*

### 2.2.3. Vertex level algorithm

Given a rooted tree we are interested in to compute the level  $level(v)$  of each vertex  $v$ , which is the distance (number of edges) between  $v$  and the root  $r$ . The following proposition shows that there is a fast-parallel algorithm that computes the level of every vertex of the graph.

**Proposition 2.5** ([8]). *There is an algorithm that computes, on an input rooted tree  $(T, r)$  the level  $level(v)$  of every vertex  $v \in V(T)$  in time  $\mathcal{O}(\log n)$  and using  $\mathcal{O}(n)$  processors, where  $n$  is the size of  $T$ .*

#### 2.2.4. All pairs shortest paths

Given a graph  $G$  of size  $n$ . Name  $v_1, \dots, v_n$  the set of vertices of  $G$ . A matrix  $B$  is called an *All Pairs Shortest Paths matrix* if  $B_{i,j}$  corresponds to the length of a shortest path from vertex  $v_i$  to vertex  $v_j$ . The following proposition states that there is a fast-parallel algorithms computing an All Pairs Shortest Path matrix of an input graph  $G$ .

**Proposition 2.6** ([8]). *There is an algorithm that computes all Pairs Shortest Paths matrix of a graph with  $n$  vertices in time  $\mathcal{O}(\log^2 n)$  with  $\mathcal{O}(n^3 \log n)$  processors.*

### 3. Triangular Grid

We will start our study over the regular grid where each cell has three neighbors, as Figure 1a. In this topology, the sixteen FTCA are reduced to eight non-equivalent, considering the inactive state as a quiescent state. According to our classifications, the eight FTCAs in the triangular grid are grouped as follows:

- Simple rules:  $\phi$ , 123 and 3.
- Topological rules: 2 and 23.
- Algebraic rule: 12.
- Fractal growing rules: 1 and 13.

Is easy to check that Simple rules are in **NC**. For rule  $\phi$ , we note that every configuration is a fixed point (then **STABILITY** for this rule is trivial). For rule 123, no site is stable unless the configuration consists in every cell inactive. We can check in time  $\mathcal{O}(\log n)$  and  $\mathcal{O}(n^2)$  processors whether a configuration contains an active cell using a prefix-sum algorithm (sum the states of all cells, and then decide if the result is different than 0). Finally, for rule 3 we notice that all dynamics reach a fixed point after one step. Therefore, we check if the initial neighborhood of site  $u$  makes it active in the first step (this can be decided in  $\mathcal{O}(\log n)$  time in a sequential machine).

We continue our study with the Topological FTCA.

### 3.1. Topological Rules

We say that rules 2 and 23 are *topological* because, as we will see, we can characterize the stable sites according to some topological properties of the initial configurations.

As we mentioned before, rule 23 is a particular case of the freezing majority vote CA (that we called *Maj*). In [5] the authors show that STABILITY for *Maj* is in **NC** over any graph with degree at most 4. This result is based on a characterization of the set of stable cells, that can be verified by a fast-parallel algorithm. Thus we can apply this result to solve STABILITY for rule 23, considering the triangular grid as a graph of degree 3. Then we have the next theorem:

**Theorem 3.1** ([5]). *There is a fast-parallel algorithm that solves STABILITY for 23 in time  $\mathcal{O}(\log^2 n)$  and  $\mathcal{O}(n^4)$  processors. Then STABILITY for 23 is in **NC**.*

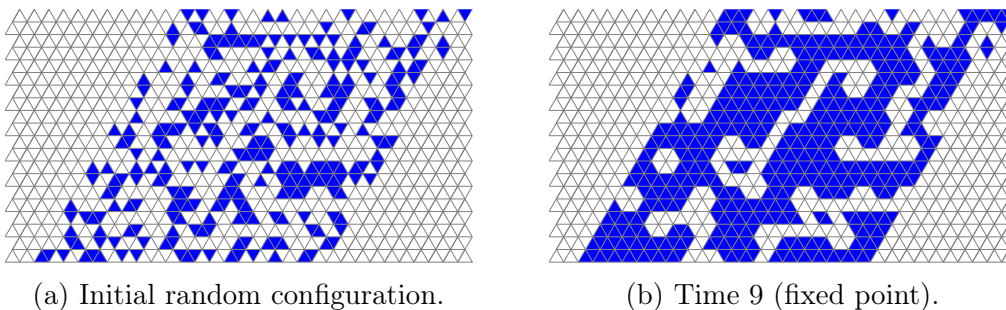


Figure 2: Example of fixed point for the rule 23. The cells in state 0 in the fixed point are stable cells.

For sake of completeness, we give the main ideas used to prove Theorem 3.1. The main idea is a characterization of the set of stable sites.

**Proposition 3.2** ([5]). *Let *Maj* be the freezing majority vote CA defined over a graph  $G$  of degree at most 4. Let  $c$  be a configuration of  $G$ , and let  $G[0]$  be the subgraph of  $G$  induced by the vertices (cells) which are inactive according to  $c$ .*

*An inactive vertex  $u$  is stable if and only if,*

- (i)  $u$  belongs to a cycle in  $G[0]$ , or*

(ii)  $u$  belongs to a path  $P$  in  $G[0]$  where both endpoints of  $P$  are contained in cycles in  $G[0]$ .

Moreover, there is a fast-parallel algorithm that checks conditions (i) and (ii) in time  $\mathcal{O}(\log^2 n)$  using  $\mathcal{O}(4^2)$  processors.

Therefore, the proof of Theorem 3.1 consists in (1) notice that a finite configuration on the triangular grid, seen as a torus, is a graph of degree 3 (then in particular is a graph of degree at most 4); (2) use the algorithm given in Proposition 3.2 to check whether the given site  $u$  is stable.

We will use the previous result to solve the stability problem for rule 2.

**Theorem 3.3.** STABILITY is in *NC* for the freezing CA 2.

*Proof.* When we compare rule 2 and rule 23, we noticed that they exhibit quite similar dynamics. Indeed, a cell  $u$  which is stable for rule 23 is also stable for rule 2. Therefore, to solve STABILITY for 2 on input configuration  $x$  and cell  $u$ , we can first solve STABILITY for 23 on those inputs using the algorithm given by Theorem 3.1. When the answer of STABILITY for 23 is *Accept*, we know that STABILITY for 2 will have the same answer. In the following, we focus in the case where the answer of STABILITY for 23 is *Reject*, i.e.  $u$  is not stable on configuration  $x$  in the dynamics of rule 23.

Suppose that  $u$  is stable for rule 2, but is not stable for rule 23. Let  $t$  be the first time-step where  $u$  becomes active in the dynamics of rule 23. Note that, since  $u$  is stable for rule 2, necessary in step  $t - 1$  the three neighbors of  $u$  are active. Moreover, at least two of them simultaneously became active in time  $t - 1$ .

Let now  $G$  be the graph representing the cells of the triangular grid covered by configuration  $x$ . Let  $G[0]$  be the subgraph of  $G$  induced by the initially inactive cells, and let  $G[0, u]$  be the connected component of  $G[0]$  containing cell  $u$ . We claim that  $G[0, u]$ , in the dynamics of rule 23, every vertex (cell) in  $G[0, u]$  must become active before  $u$ , i.e. in a time-step strictly smaller than  $t$ .

**Claim 1:** Every vertex of  $G[0, u]$  is active after  $t$  applications of rule 23.

Indeed, suppose that there exists a vertex (cell)  $w$  in  $G[0, u]$  that becomes active in a time-step greater than  $t$ . Call  $P$  a shortest path in  $G[0, u]$  that connects  $u$  and  $w$ , and let  $u^*$  be the neighbor of  $u$  contained in  $P$ . Note that

except the endpoints, all the vertices (cells) in  $P$  have at least two neighbors in  $P$ , which are inactive. Moreover, both endpoints of  $P$  are inactive at time  $t$ . Therefore, all the vertices in  $P$  will be inactive in time  $t$ . This contradicts the fact that the three neighbors of  $u$  become active before  $u$ .

**Claim 2:**  $G[0, u]$  is a tree.

Indeed,  $G[0, v]$  is connected, since it is defined as a connected component of  $G[0]$  containing  $u$ . On the other hand, suppose that  $G[0, u]$  contains a cycle  $C$ . From Proposition 3.2, we know that all the cells in  $C$  are stable, which contradicts Claim 1.

Call  $T_u$  the tree  $G[0, u]$  rooted on  $u$ . Let  $d$  be the depth of  $T_u$ , i.e. longest path between  $u$  and a leaf of  $T_u$ .

**Claim 3:** Every vertex of  $G[0, u]$ , except  $u$ , is active after  $d$  applications of rule 2.

Notice that necessarily a leaf of  $T_u$  has two active neighbors (because they are outside  $G[0, u]$ ) and one inactive neighbor (its parent in  $T_u$ ). Therefore, in one application of rule 2, all the leaves will become active. We will reason by induction on  $d$ . Suppose that  $d = 1$ . Then all vertices  $w$  of  $T_u$  except  $u$  are leaves, so the claim is true. Suppose now that the claim is true for all trees of depth smaller or equal than  $d$ , but  $T_u$  is a tree of depth  $d + 1$ . We notice in one step the leaves are the only vertices of  $T_u$  that become active (every other vertex has two inactive neighbors). Then, after one step, the inactive sites of  $T_u$  induce a tree  $T'_u$  of depth  $d$ . By induction hypothesis, all the cells in  $T'_u$ , except  $u$ , become active after  $d$  applications of rule 2. We deduce the claim.

Let  $u_1, u_2, u_3$  be the three neighbors of  $u$ . For  $i \in \{1, 2, 3\}$ , call  $T_{u_i}$  the subtree of  $T_u$  rooted at  $u_i$ , obtained taking all the descendants of  $u_i$  in  $T_u$ . Call  $d_i$  the depth of  $T_{u_i}$ . Without loss of generality,  $d_1 \geq d_2 \geq d_3$ .

**Claim 4:**  $u$  is stable for the dynamics of rule 2 but not for the dynamics of rule 23, if and only if  $d_1 = d_2 \geq d_3$ .

Recall that  $u$  is stable for the dynamics of rule 2 but not for the dynamics of rule 23 if and only if  $u$  has three active neighbors at time-step  $t$ , and at

least two of them become active at time  $t - 1$ . The claim follows from the application of Claim 3 to trees  $T_{u_1}, T_{u_2}$  and  $T_{u_3}$ .

We deduce the following fast-parallel algorithm solving STABILITY for 2: Let  $x$  be the input configuration and  $u$  the cell that we want to decide stability. First, use the fast-parallel algorithm given by Theorem 3.1 to decide if  $u$  is stable for the dynamics of rule 23 on configuration  $x$ . If the answer is affirmative, then we decide that  $(x, u)$  is a *Accept*-instance of STABILITY for 2. If the answer is negative, the algorithm looks for cycles in  $G[0, u]$ . If there is a cycle, then the algorithm *Rejects*, because Claim 2 implies that  $u$  cannot be stable for rule 2. If  $G[0, u]$  is a tree, then the algorithm computes in parallel the depth  $d_v$  of the subtrees  $T_v$ , for each  $v \in N(u)$ . Finally, the algorithm accepts if the conditions of Claim 4 are satisfied, and otherwise rejects.

The steps of the algorithm are represented in Algorithm 1.

---

**Algorithm 1** Solving STABILITY 2

---

**Input:**  $x$  a finite configuration of dimensions  $n \times n$  and  $u$  a cell.

```

1: if the answer of STABILITY for rule 23 is Accept on input  $(x, u)$  then
2:   return Accept
3: else
4:   Compute  $G[0, u]$ 
5:   Compute  $C$  the set of cycles of  $G[0, u]$ 
6:   if  $C \neq \emptyset$  then
7:     return Reject
8:   else
9:     for all  $v \in N(u)$  do in parallel
10:      Compute  $d_v$  the depth of  $T_v$ 
11:     end for
12:     if  $\exists a, b, c \in (N(u)) : d_a = d_b \geq d_c$  then
13:       return Accept
14:     else
15:       return Reject
16:     end if
17:   end if
18: end if

```

---

Let  $N = n^2$  the size of the input. Algorithm 1 runs in time  $\mathcal{O}(\log^2 N)$  using  $\mathcal{O}(N^3/\log N)$  processors. Indeed, the condition of line 1 can be checked

in time  $\mathcal{O}(\log^2 N)$  using  $\mathcal{O}(N^2)$  processors according the algorithm of Theorem 3.1. Step 4 can be done in time  $\mathcal{O}(\log^2 N)$  using  $\mathcal{O}(N^2)$  processors using a connected components algorithm given in [8]. Step 5 can be done in time  $\mathcal{O}(\log^2 N)$  using  $\mathcal{O}(N^3/\log N)$  processors using a bi-connected components algorithm given in [8]. Step 10 can be solved in time  $\mathcal{O}(\log N)$  using  $\mathcal{O}(N)$  processors using a vertex level algorithm given in [8]. Finally, Step 12 can be done in  $\mathcal{O}(\log N)$  time in a sequential machine. □

### 3.2. Algebraic Rule

We now continue with the study of rule 12. We say that this rule is *algebraic* because, as we will see, we can speed-up the dynamics using some algebraic properties of this rule. This speed-up we will provide an algorithm that decides the stability of a cell much faster than the simple simulation of the automaton. In other words, we will show that STABILITY for rule 12 is in NC.

Let  $x$  be a finite configuration on the triangular grid,  $u$  a cell. Let  $v$  be a neighbor of  $u$ . We define a *semi-plane*  $S_v$  as a partition of the triangular grid in two parts, cut by the edge of the triangle that share cell  $u$  and  $v$ , as shown in Figure 3.

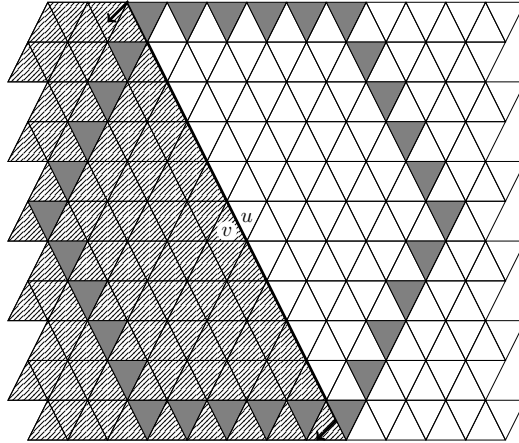


Figure 3: Triangular grid divided in semi-planes according to  $u$  and  $v$ . The hatch pattern represent the semi-plane  $S_v$  Gray cells are at the same distance from  $u$ .

We say that two cells  $v_1, v_2$  are at *distance*  $d$  if a shortest path connecting  $v_1$  and  $v_2$  is of length  $d$ . In the following, we call  $D_d$  the set of cells at distance

$d$  from  $u$ . We can be more explicit and use the property above to give a way to speed-up the dynamics of rule 12.

**Lemma 3.4.** *Let  $d \geq 2$  be the distance from  $u$  to the nearest active cell. Then the distance to the nearest cell to  $u$  in  $F(c)$  is  $d - 1$*

*Proof.* Let  $w$  be an active cell at distance  $d$  of  $u$  in configuration  $x$ , and call  $P$  a shortest  $u, w$ -path. Call  $w_1$  the neighbor of  $w$  contained in  $P$ , and let  $w_2$  be the neighbor of  $w_1$  in  $P$  different than  $w$  (this cells exists since  $d \geq 2$ ). Note that  $w_2$  might be equal to  $u$ . Since  $P$  is a shortest path,  $w_2$  is at distance  $d - 2$  from  $u$ . Then all the neighbors of  $w_2$  are inactive, so  $w_2$  it is necessarily inactive in  $F(c)$ . Moreover,  $w_1$  has more than one active neighbor, and less than three active neighbors, so  $w_1$  is active in  $F(c)$ . Then the distance from  $u$  to the nearest active cell in  $F(c)$  is  $d - 1$ .  $\square$

**Lemma 3.5.** *Let  $d \geq 2$  be the distance from  $u$  to the nearest active cell, and let  $v \in N(u)$ . Then  $v$  is active after  $d - 1$  applications of rule 12 (i.e.  $F^{d-1}(c)_v = 1$ ) if and only there exists an active cell in  $S_v \cap D_d$*

*Proof.* We reason by induction on  $d$ . In the base case,  $d = 2$ , suppose that  $S_v$  does not contain an active site at distance 2. Then every neighbor of  $v$  is inactive in the initial configuration, so  $v$  is inactive after one application of rule 12 (i.e.  $F(c)_v = 0$ ). Conversely, if  $F(c)_v = 0$ , then every neighbor of  $v$  is initially inactive, in particular all the sites in  $S_v$  at distance 2 from  $u$ .

Suppose now that the statement of the lemma is true on configurations where the distance is  $d$ , and let  $c$  be a configuration where the distance from  $u$  to nearest active cell is  $d + 1$ . Let  $c'$  be the configuration obtained after one application on  $c$  of rule 12 (i.e.  $c' = F(c)$ ).

**Claim 1:**  $F^{d-1}(c')_v = 1$  if and only if in  $c'$  there exists an active cell in  $S_v \cap D_d$ .

From Lemma 3.4, the distance from  $u$  to the nearest active cell in  $c'$  is  $d$ . The claim follows from the induction hypothesis.

**Claim 2:** Suppose that  $F^{d-1}(c')_v = 0$ . Then in  $c$ , all the cells in  $S_v \cap D_{d+1}$  are inactive.

Notice that, from Claim 1, the fact that  $F^{d-1}(c')_v = 0$  implies that in  $c'$  all the cells in  $D_v \cap S_v$  must be inactive. Suppose, by contradiction, that



there is a cell  $w$  in  $S_v \cap D_{d+1}$  that is active in  $c$ . Let  $w'$  be a neighbor of  $w$  contained in  $S_v \cap D_d$ , and let  $w''$  be a neighbor of  $w'$  not contained in  $D_{d+1}$  (then  $w'$  belongs to  $D_d \cup D_{d-1}$ ). Note that  $w'$  has an active neighbor in  $c$ , but must be inactive in  $c'$ . The only option is that all the neighbors of  $w'$  are active in  $c$ , in particular  $w''$  is active in  $c$ . This contradicts the fact the nearest active cell is at distance  $d + 1$  in  $c$ .

**Claim 3:** Suppose that  $F^{d-1}(c')_v = 1$ . Then there is a cell in  $S_v \cap D_{d+1}$  that is active in  $c$ .

From Claim 1, the fact that  $F^{d-1}(c')_v = 0$  implies that there is a cell  $w \in S_v \cap D_d$  that is active in  $c'$ . Suppose by contradiction that all the cells in  $S_v \cap D_{d+1}$  are inactive in  $c$ . Since  $w$  is active in  $c'$ , necessarily  $w$  has at least one neighbor  $w'$  that is active in  $c$ . Since  $w'$  is not contained in  $S_v \cap D_{d+1}$  (because we are supposing that all those cells are inactive in  $c$ ), we deduce that  $w'$  belongs to  $D_d \cup D_{d-1}$ . This contradicts the fact the nearest active cell is at distance  $d + 1$  in  $c$ .

We deduce that  $F^{d-1}(c')_v = 1$  if and only if there is a cell in  $S_v \cap D_{d+1}$  that is active in  $c$ . Since  $c' = F(c)$ , we obtain that  $F^d(c)_v = 1$  if and only if there is a cell in  $S_v \cap D_{d+1}$  that is active in  $c$ .  $\square$

**Theorem 3.6.** STABILITY is in NC for the freezing CA 12.

*Proof.* In our algorithm solving STABILITY for 12, we first compute the distance  $d$  to the nearest active cell from  $u$  (if every cell is inactive, our algorithm trivially accepts). Then, for each  $v \in N(u)$ , the algorithm computes the set of cells  $S_v \cap D_d$ , and checks if that set contains an active cell. If it does, we mark  $v$  as *active*, and otherwise we mark  $v$  as *inactive*. Finally, the algorithm rejects if the three neighbors of  $u$  are active, and accepts otherwise. The steps of this algorithm are described in Algorithm 2

From Lemma 3.5, we know that  $v$  becomes active at time  $d - 1$  if and only if  $S_v \cap D_d$  contains an active cell in the initial configuration. Since the nearest active cell from  $u$  is at distance  $d$ , necessarily after  $d - 1$  steps at least one of the three neighbors of  $u$  will become active. If the three neighbors of  $u$  satisfy the condition of Lemma 3.5, then the three of them will become active in time  $d - 1$ , so  $u$  will remain inactive forever. Otherwise,  $u$  will have more than one and less than three active neighbors at time-step  $d - 1$ , so it will become active at time  $d$ .

---

**Algorithm 2** Solving STABILITY 12

---

**Input:**  $x$  a finite configuration of dimensions  $n \times n$  and  $u$  a cell.

```
1: if For all cell  $w$ ,  $x_w = 0$  then
2:   return accept
3: else
4:   Compute a matrix  $M = (m_{ij})$  of dimensions  $2n^2 \times 2n^2$  such that
       $m_{ij}$  is the distance from cell  $i$  to cell  $j$ .
5:   Compute the distance  $d$  to the nearest active cell from  $u$ .
6:   for all  $v \in N(u)$  do in parallel
7:     Compute the set of cells  $S_v \cap D_d$ 
8:     if there exists  $w \in S_v \cap D_d$  such that  $x_w = 1$  then
9:       Mark  $v$  as active
10:    else
11:      Mark  $v$  as inactive
12:    end if
13:  end for
14:  if there exists  $v$  in  $N(u)$  that is marked inactive then
15:    return Reject
16:  else
17:    return Accept
18:  end if
19: end if
```

---

Let  $N = n^2$  the size of the input. This algorithm runs in time  $\mathcal{O}(\log N)$  using  $\mathcal{O}(N)$  processors. Indeed, the verifications on lines 1-3 and 8-10 can be done in time  $\mathcal{O}(\log N)$  using  $\mathcal{O}(N)$  processors using a prefix-sum algorithm. Finally, step 7 can be done in time  $\mathcal{O}(\log N)$  using  $\mathcal{O}(N)$  processors, assigning one processor per cell and solving three inequations of kind  $ax + by < c$ .  $\square$

#### 4. Square Grid

We now continue our study, considering the square grid. As we said in the preliminaries section, we can define 32 different FTCAs over this topology. Again, considering the inactive state as a quiescent state, the set of non-equivalent FTCAs is reduced to 16. According to our classifications, this list of FTCAs is grouped as follows:

- Simple rules:  $\phi$ , 1234 and 4.
- Topological rules: 234, 3 and 34.
- Algebraic rules: 12, 123, and 124.
- Turing Universal rules: 2, 24.
- Fractal growing rules: 1, 13, 14 and 134.

In complete analogy to the triangular topology, we verify that the STABILITY problem in Simple rules is **NC**. We will directly continue then with the Topological Rules.

#### 4.1. Topological Rules

We study this rules characterizing their fixed points and building a faster algorithms for to find them. This characterization is based the structure of the set of stable cells, called *stable sets*. Naturally, the structure of stable-sets depends on the rule.

##### 4.1.1. Rules 34 and 3.

First, notice that the rule 34 corresponds to freezing version of the majority automaton (Maj) over the squared grid. We remark that a finite configuration over the squared grid, seen as a torus, is a regular graph of degree 4. Therefore, we can use the Algorithm given in Proposition 3.2 to check whether a given site is stable for rule 34. We deduce the following theorem (also given in [5])

**Theorem 4.1** ([5]). STABILITY is in **NC** for rule 34.

Likewise, in analogy of the behavior of rule 2 with respect to rule 23 in the triangular grid, we can use the algorithm solving STABILITY for the rule 34 to solve STABILITY for the rule 3. Let  $(x, u)$  be an instance of problem STABILITY. Clearly, if  $u$  is stable for rule 34 we have that  $u$  is stable for rule 3. Suppose now that  $u$  is not stable for rule 34 but it is stable for rule 3. Let  $G[0, u]$  be connected component of  $G[0]$  containing  $u$ . Using the exact same proof used for rule 2 on the triangular grid, we can deduce that  $G[0, u]$  is a tree, and we call  $T_u$  this tree rooted in  $u$ . Moreover, let  $u_1, u_2, u_3, u_4$  be the four neighbors of  $u$ , and let  $T_{u_i}$  be the subtree of  $T_u$  obtained taking all the descendants of  $u_i$ ,  $i \in \{1, 2, 3, 4\}$ . Call  $d_i$  the depth of  $T_{u_i}$ , which without loss

of generality we assume that  $d_1 \geq d_2 \geq d_3 \geq d_4$ . We have that  $u$  is stable for rule 3 but not for rule 34 if and only if  $d_1 = d_2 \geq d_3 \geq d_4$ . We deduce that, with very slight modifications, Algorithm 1 solves STABILITY for rule 3. We deduce the following theorem.

**Theorem 4.2.** *STABILITY is in NC for rule 3.*

*4.1.2. Rule 234.*

Notice that rule 234 is the freezing version of the non-strict majority automaton, the CA where the cells take the state of the majority of its neighbors, and in tie case they decide to become active. In the following, we will show that the stability problem for this rule is also in NC, characterizing the set of stable sets. This time, the topological conditions of the stable sets will be the property of being tri-connected.

**Theorem 4.3.** *STABILITY is in NC for the freezing CA 234.*

**Lemma 4.4.** *Let  $x \in \{0, 1\}^{[n] \times [n]}$  be a finite configuration and  $u \in [n] \times [n]$  a site. Then,  $u$  is stable for  $c = c(x)$  if and only if there exist a set  $S \subseteq [n] \times [n]$  such that:*

- $u \in S$ ,
- $c_u = 0$  for every  $u \in S$ , and
- $G[S]$  is a graph of minimum degree 3.

*Proof.* Suppose that  $u$  is stable and let  $S$  be the subset of  $[n] \times [n]$  containing all the sites that are stable for  $c$ . We claim that  $S$  satisfy the desired properties. Indeed, since  $S$  contains all the sites stable for  $c$ , then  $u$  is contained in  $S$ . On the other hand, since the automaton is freezing, all the sites in  $S$  must be inactive on the configuration  $c$ . Finally, if  $G[S]$  contains a vertex  $v$  of degree less than 3, it means that necessarily the corresponding site  $v$  has two non-stable neighbors that become 1 in the fixed point reached from  $c$ , contradicting the fact that  $v$  is stable.

On the other direction suppose that  $S$  contains a site that is not stable and let  $t > 0$  be the minimum step such that a site  $v$  in  $S$  changes to state 1, i.e.,  $v \in S$  and  $t$  are such  $F^{t-1}(c)_w = 0$  for every  $w \in S$ , and  $F^t(c)_v = 1$ . This implies that  $v$  has at least two active neighbors in the configuration  $F^{t-1}(c)$ . This contradicts the fact that  $v$  has three neighbors in  $S$ . We conclude that all the sites contained in  $S$  are stable, in particular  $u$ .  $\square$

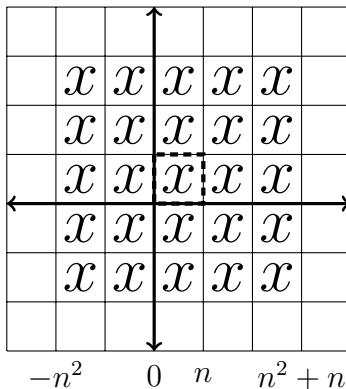


Figure 4: Construction of the finite configuration  $D(x)$  obtained from a finite configuration  $x$  of dimension  $n \times n = 2 \times 2$ . Note that  $D(x)$  is of dimensions  $7 \times 7$ .

For a finite configuration  $x \in \{0, 1\}^{[n] \times [n]}$ , let  $D(x) \in \{0, 1\}^{\{-n^2-n, \dots, n^2+2n\}^2}$  be the finite configuration of dimensions  $m \times m$ , where  $m = 2n^2 + 3n$ , constructed with repetitions of configuration  $x$  in a rectangular shape, as is depicted in Figure 4, and inactive sites elsewhere. We also call  $D(c)$  the periodic configuration  $c(D(x))$ .

**Lemma 4.5.** *Let  $x \in \{0, 1\}^{[n]^2}$  be a finite configuration, and let  $u$  be a site in  $[n] \times [n]$  such that  $x_u = 0$ . Then  $u$  is stable for  $c = c(x)$  if and only if it is stable for  $D(c)$ .*

*Proof.* Suppose first that  $u$  is stable for  $c$ , i.e. in the fixed point  $c'$  reached from  $c$ ,  $c'_u = 0$ . Call  $c''$  the fixed point reached from  $D(c)$ . Note that  $D(c) \leq c$  (where  $\leq$  represent the inequalities coordinate by coordinate). Since the 234 automata is monotonic, we have that  $c'' \leq c'$ , so  $c''_u = 0$ . Then  $u$  is stable for  $D(c)$ .

Conversely, suppose that  $u \in [n] \times [n]$  is not stable for  $c$ , and let  $S$  be the set of all sites at distance at most  $n^2$  from  $u$ . We know that in each step on the dynamics of  $c$ , at least one site in the periodic configuration changes its state, then in at most  $n^2$  steps the site  $u$  will be activated. In other words, the state of  $u$  depends only on the states of the sites at distance at most  $n^2$  from  $u$ . Note that for every  $v \in S$ ,  $c_v = D(c)_v$ . Therefore,  $u$  is not stable in  $D(c)$ .  $\square$

Note that the perimeter of width  $n$  of  $D(x)$  contain only inactive sites. We call this perimeter the *border* of  $D(x)$ , and  $D(x) - B$  the *interior* of

$D(x)$ . Note that  $B$  is tri-connected and forms a set of sites stable for  $D(c)$  thanks to Lemma 4.4. We call  $Z$  the set of sites  $w$  in  $[m] \times [m]$  such that  $D(x)_w = 0$ .

**Lemma 4.6.** *Let  $u$  be a site in  $[n] \times [n]$  stable for  $D(c)$ . Then, there exist three disjoint paths on  $G[Z]$  connecting  $u$  with sites of the border  $B$ . Moreover, the paths contain only sites that are stable for  $B(c)$ .*

*Proof.* Suppose that  $u$  is stable. From Lemma 4.4 this implies that  $u$  has three stable neighbors. Let  $0 \leq i, j \leq n$  be such that  $u = (i, j)$ . We divide the interior of  $D(c)$  in four quadrants:

- The first quadrant contain all the sites in  $D(x)$  with coordinates at the north-east of  $u$ , i.e., all the sites  $v = (k, l)$  such that  $k \geq i$  and  $l \geq j$ .
- The second quadrant contain all the sites in  $D(x)$  with coordinates at the north-west of  $u$ , i.e., all the sites  $v = (k, l)$  such that  $k \leq i$  and  $l \geq j$ .
- The third quadrant contain all the sites in  $D(x)$  with coordinates at the south-west of  $u$ , i.e., all the sites  $v = (k, l)$  such that  $k \leq i$  and  $l \leq j$ .
- The fourth quadrant contain all the sites in  $D(x)$  with coordinates at the south-east of  $u$ , i.e., all the sites  $v = (k, l)$  such that  $k \geq i$  and  $l \leq j$ .

We will construct three disjoint paths in  $G[Z]$  connecting  $u$  with the border, each one passing through a different quadrant. The idea is to first choose three quadrants, and then extend three paths starting from  $u$  iteratively picking different stable sites in the chosen quadrants, until the paths reach the border.

Suppose without loss of generality that we choose the first, second and third quadrants, and let  $u_1, u_2$  and  $u_3$  be three stable neighbors of  $u$ , named according to Figure 5.

Starting from  $u, u_1$ , we extend the path  $P_1$  through the endpoint different than  $u$ , picking iteratively a stable site at the east, or at the north if the site in the north is not stable. Such sites will always exist since by construction the current endpoint of the path will be a stable site, and stable sites must have three stable neighbors (so either one neighbor at east or one neighbor

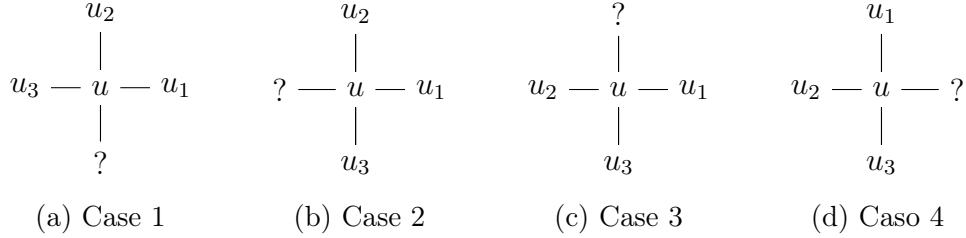


Figure 5: Four possible cases for  $u_1, u_2$  and  $u_3$ . Note that one of these four cases must exist, since  $u$  has at least three stable neighbors. From  $u_1$  we will extend a path through the first quadrant, from  $u_2$  a path through the second quadrant, and from  $u_3$  a path through the third quadrant.

at north). The iterative process finishes when  $P_1$  reaches the border. Note that necessarily  $P_1$  is contained in the first quadrant. Analogously, we define paths  $P_2$  and  $P_3$ , starting from  $u_2$  and  $u_3$ , respectively, and extending the corresponding paths picking neighbors at the north-west or south-west, respectively. We obtain that  $P_2$  and  $P_3$  belong to the second and third quadrants, and are disjoint from  $P_1$  and from each other.

This argument is analogous for any choice of three quadrants. We conclude there exist three disjoint paths of stable sites from  $u$  to the border  $B$ .  $\square$

**Lemma 4.7.** *Let  $u, v$  be two sites in  $[n] \times [n]$  stable for  $D(c)$ . Then, there exist three disjoint  $u, v$ -paths in  $G[Z]$  consisting only of sites that are stable for  $D(c)$ .*

*Proof.* Let  $u, v$  be stable vertices. Without loss of generality, we can suppose that  $u = (i, j)$ ,  $v = (k, l)$  with  $i \leq k$  and  $j \leq l$  (otherwise we can rotate  $x$  to obtain this property). In this case  $u$  and  $v$  divide the interior of  $D(x)$  into nine regions (see Figure 6). Let  $P_{u,2}, P_{u,3}, P_{u,4}$  be three disjoint paths that connect  $u$  with the border through the second, third and fourth quadrants of  $u$ . These paths exist according to the proof of Lemma 4.6. Similarly, define  $P_{v,1}, P_{v,2}, P_{v,3}$  three disjoint paths that connect  $v$  to the border through the first, second and third quadrants of  $v$ .

Observe first that  $P_{u,3}$  touch regions that are disjoint from the ones touched by  $P_{v,1}, P_{v,2}$  and  $P_{v,3}$ . The same is true for  $P_{v,1}$  with respect to  $P_{u,2}, P_{u,3}, P_{u,4}$ . The first observation implies that paths  $P_{u,3}$  and  $P_{v,1}$  reach the border without intersecting any other path. Let  $w_1$  and  $w_2$  be respectively the intersections of  $P_{u,3}$  and  $P_{v,1}$  with the border. Let now  $P_{w_1, w_2}$  be any path in  $G_B$

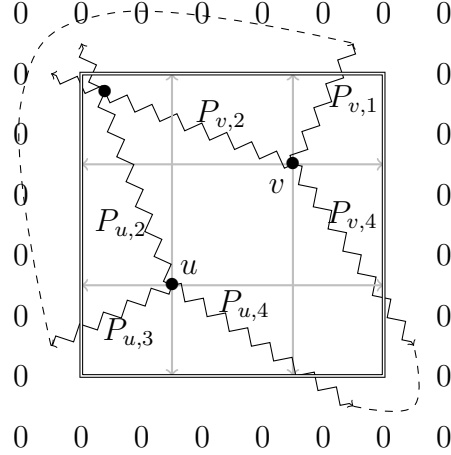


Figure 6: Vertices  $u$  and  $v$  divides the interior of  $D(x)$  into four regions each one. Together they split the space into nine regions. According to Lemma 4.6, we can choose three disjoint paths connecting  $u$  and  $v$ , in such a way that each of the nine regions intersect at most one path. We use the border of  $D(x)$  to connect the paths that do not intersect in the interior of  $D(x)$ .

connecting  $w_1$  and  $w_2$ . We call  $P_{1,3}$  the path induced by  $P_{u,3} \cup P_{w_1,w_2} \cup P_{v,1}$ .

Observe now that  $P_{u,2}$  and  $P_{v,4}$  must be disjoint, as well as  $P_{u,4}$  and  $P_{v,2}$ . This observation implies that  $P_{u,2}$  either intersects  $P_{v,2}$  or it do not intersect any other path, and the same is true for  $P_{u,4}$  and  $P_{v,4}$ . If  $P_{u,2}$  does not intersect  $P_{v,2}$ , then we define a path  $P_{2,2}$  in a similar way than  $P_{1,3}$ , i.e., we connect the endpoints of  $P_{u,2}$  and  $P_{v,2}$  through a path in the border (we can choose this path disjoint from  $P_{1,3}$  since the border is tri-connected). Suppose now that  $P_{u,2}$  intersects  $P_{v,2}$ . Let  $w$  the first site where  $P_{u,2}$  and  $P_{v,2}$  intersect, let  $P_{u,w}$  be the  $u, w$ -path contained in  $P_{u,2}$ , and let  $P_{w,v}$  be the  $w, v$ -path contained in  $P_{v,2}$ . We call in this case  $P_{2,2}$  the path  $P_{u,w} \cup P_{w,v}$ . Note that also in this case  $P_{2,2}$  is disjoint from  $P_{1,3}$ . Finally, we define  $P_{4,4}$  in a similar way using paths  $P_{u,4}$  and  $P_{v,4}$ . We conclude that  $P_{1,3}$ ,  $P_{2,2}$ , and  $P_{4,4}$  are three disjoint paths of stable sites connecting  $u$  and  $v$  in  $G[Z]$ .  $\square$

We are now ready to show our characterization of stable set of vertices.

**Lemma 4.8.** *Let  $x \in \{0, 1\}^{[n] \times [n]}$  be a finite configuration, and let  $u$  be a site in  $[n] \times [n]$ . Then,  $u$  is stable for  $c = c(x)$  if and only if  $u$  is contained in a tri-connected component of  $G[Z]$ .*



*Proof.* From Lemma 4.5, we know that  $u$  is stable for  $c$  if and only if it is stable  $D(c)$ . Let  $S$  be the set of sites stable for  $D(c)$ . We claim that  $S$  is a tri-connected component of  $G[Z]$ . From Lemma 4.7, we know that for every pair of sites in  $S$  there exist three disjoint paths in  $G[S]$  connecting them, so the set  $S$  must be contained in some tri-connected component  $T$  of  $G[Z]$ . Since  $G[T]$  is a graph of degree at least three, and the sites in  $T$  are contained in  $Z$ , then Lemma 4.4 implies that  $T$  must form a stable set of vertices, then  $T$  equals  $S$ .

On the other direction, Lemma 4.4 implies that any tri-connected component of  $G[Z]$  must form a stable set of vertices for  $D(c)$ , so  $u$  is stable for  $c$ .  $\square$

We are now ready to study the complexity of STABILITY for this rule.

*Proof of Theorem 4.3 .* Let  $(x, u)$  be an input of STABILITY, i.e.  $x$  is a finite configuration of dimensions  $n \times n$ , and  $u$  is a site in  $[n] \times [n]$ . Our algorithm for STABILITY first computes from  $x$  the finite configuration  $D(x)$ . Then, the algorithm uses the algorithm of Proposition 2.4 to compute the tri-connected components of  $G[Z]$ , where  $Z$  is the set of sites  $w$  such that  $D(x)_w = 0$ . Finally, the algorithm answers *no* if  $u$  belongs to some tri-connected component of  $G[Z]$ , and answer *yes* otherwise.

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**Algorithm 3** Solving STABILITY 234

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**Input:**  $x$  a finite configuration of dimensions  $n \times n$  and  $u \in [n] \times [n]$  such that  $x_u = 0$ .

- 1: Compute the finite configuration  $D(x)$  of dimensions  $m \times m$  with  $m = 2n^2 + 3n$
  - 2: Compute the set  $Z = \{w \in [m] \times [m] : D(x)_w = 0\}$ .
  - 3: Compute the graph  $G[Z]$ .
  - 4: Compute the set  $\mathcal{T}$  of tri-connected components of  $G[Z]$ .
  - 5: **for all**  $T \in \mathcal{T}$  **do in parallel**
  - 6:     **if**  $u \in T$  **then**
  - 7:         **return** *Accept*
  - 8:     **end if**
  - 9: **end for**
  - 10: **return** *Reject*
- 

The correctness of Algorithm 3 is given by lemma 4.8. Indeed, the algorithm answers *Reject* on input  $(x, u)$  only when  $u$  does not belong to a

tri-connected component of  $G[Z]$ . From lemma 4.8, it means that  $u$  is not stable, so there exists  $t > 0$  such that  $F^t(c(x))_u = 1$ .

Let  $N = n^2$  the size of the input. Step **1** can be done in  $\mathcal{O}(\log N)$  time with  $m^2 = \mathcal{O}(N^2)$  processors: one processor for each site of  $B(x)$  computes from  $x$  the value of the corresponding site in  $B(x)$ . Step **2** can be done in time in  $\mathcal{O}(\log N)$  with  $\mathcal{O}(N^2)$  processors, representing  $Z$  as a vector in  $\{0, 1\}^{m^2}$ , each coordinate is computed by a processor. Step **3** can be done in time  $\mathcal{O}(\log N)$  and  $\mathcal{O}(N^2)$  processors: we give one processor to each site in  $Z$ , which fill the corresponding four coordinates of the adjacency matrix of  $G[Z]$ . Step **4** can be done in time  $\mathcal{O}(\log^2 N)$  with  $\mathcal{O}((N^2)^4)$  processors using the algorithm of Proposition 2.4. Finally, steps **5** to **10** can be done in time  $\mathcal{O}(\log N)$  with  $\mathcal{O}(N^2)$  processors: the algorithm checks in parallel if  $u$  is contained in each tri-connected components. All together the algorithm runs in time  $\mathcal{O}(\log^2 N)$  with  $\mathcal{O}(N^8)$  processors.  $\square$

#### 4.2. Algebraic Rules

We will now study the family of FTCA where the cells become active with one or two neighbors. We consider there the rules 12, 123, 124. Of course, rule 1234 will fit in our analysis, but we already know that this rule is trivial. As we already mentioned, these rules are *algebraic* in the sense that, in order answer the STABILITY problem, we will *accelerate* the dynamics using algebraic properties of these rules.

In the following, we assume that the cells are placed in the Cartesian coordinate system, where each cell is placed in a coordinate in  $\mathbb{N} \times \mathbb{N}$ . Moreover, without loss of generality, our decision cell is  $u = (0, 0)$  and the configuration  $c$  has at least one active cell. Let  $\tau > 1$  be the distance from  $u$  to the first active cell. Like for rule 12 in the triangular grid, we called  $D_\tau$  the set of cells at distance  $\tau$  from  $u$ . Due to its shape in the squared grid, the set  $D_\tau$  is called in this context the *diamond at distance  $\tau$  from  $u$* . Note that  $D_\tau = \{(i, j) \in \mathbb{N}^2 : |i - j| \leq \tau\}$ . We also call  $d_I(\tau)$  the *diagonal at distance  $\tau$  of  $u$  in the first quadrant*, defined as follows:

$$d_I(\tau) := \{(i, j) \in \mathbb{N}^2 : |i - j| = \tau \text{ and } i, j > 0\}.$$

Then, we place ourselves in the case where all the cells in  $D_{\tau-1}$  are inactive.

Let  $c'$  be the configuration obtained after one step, i.e.  $c' = F(c)$ , where  $F$  is one of the rules in  $\{12, 123, 124\}$ . Notice that all the cells in  $D_{\tau-1}$  will remain inactive in  $c'$ . Moreover, the states of cells in  $d_I(\tau - 1)$  can be

computed as follows (see figure 7a):

$$\forall (i, j) \in d_I(\tau - 1), c'_{i,j} = c_{i+1,j} \vee c_{i,j+1}.$$

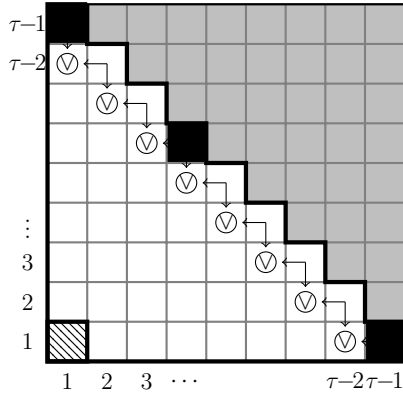
Where  $\vee$  is the OR operator (i.e.  $c'_{i,j} = 1$  if  $c_{i+1,j} = 1$  or  $c_{i,j+1} = 1$ ). If we inductively apply this formula, we deduce:

$$F^{\tau-2}(c)_{1,1} = \bigvee_{(i,j) \in d_I(\tau)} c_{i,j}.$$

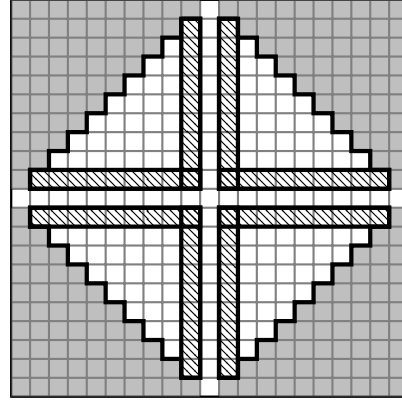
Note that if the cell  $(1, 1)$  is inactive at time  $\tau - 1$ , then necessarily all the cells in  $d_I(\tau)$  are inactive at time 0. Moreover, by we can apply the same ideas to every cell  $(i, j) \in \bigcup_{1 \leq k \leq \tau} d_I(k)$  such that  $i, j \geq 1$ , obtaining:

$$F^{\tau-1-i}(c)_{i,1} = \bigvee_{\substack{(k,j) \in D(\tau) \\ k \geq i}} c_{k,j} \quad \text{and} \quad F^{\tau-1-j}(c)_{1,j} = \bigvee_{\substack{(i,k) \in D(\tau) \\ k \geq j}} c_{i,k}. \quad (1)$$

Analogously, we can define  $d_{II}(\tau)$  (resp.  $d_{III}(\tau)$ ,  $d_{IV}(\tau)$ ) the *diagonals at distance  $\tau$  of  $u$  in the second (resp. third, fourth) quadrant*, and deduce similar formulas in the other three quadrants. Concretely we can compute the states the states of cells  $(\pm i, \pm 1)$ ,  $i = 1, \dots, \tau - 1$  in time  $\tau - i$  and the states of cells  $(\pm 1, \pm j)$ ,  $j = 1, \dots, \tau - 1$  in time  $\tau - j$ . This cells are represented as the hatch patterns in Figure 7b. This way of computing cells we call it *OR technique*.



(a) Computation of  $d_I(\tau-1)$  after one application of rule  $F$ .



(b) Cells  $(\pm i, \pm 1)$  and  $(\pm 1, \pm j)$ , where  $i, j, \dots, \tau - 1$ , represented with a hatch pattern.

Figure 7: Computation in a diamond of size  $\tau$ , where only the cells at distance  $\tau$  from  $(0, 0)$  can be initially active.

We define the following sets of cells.

- The **north-east triangle** is set of cells in the first quadrant between the cells in the hatch pattern (including them) and the gray zone, i.e. is the set  $D_{\tau,I} = \{(i, j) \in \mathbb{N}^2 : |i - j| \leq \tau \text{ and } i, j \geq 1\}$ . Analogously we define north-west, south-west and south-east triangles, and denote them  $D_{\tau,II}$ ,  $D_{\tau,III}$  and  $D_{\tau,IV}$ , respectively.
- The **north corridor** is the set of cells in the positive x-axis contained in the diamond, i.e. is the set  $\{(0, i) \in \mathbb{N}^2 : 1 \leq i \leq \tau\}$ . Analogously we define west, south and east corridors.

Consider now a smaller diamond at distance 2 of  $u = (0, 0)$  depicted in Figure 8. As we explained, we can compute the states of cells  $b, d, f$  and  $h$  in time  $\tau - 1$  using the OR technique. The use of this information in order to solve STABILITY, will depend on the which rule we are considering. In the following, we will show how to use this information to solve stability for rule 123, then for rule 12 and finally for rule 124.

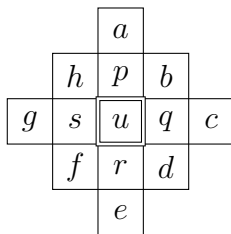


Figure 8: Notation for the cells in the small diamond. Cells  $p, q, r$  and  $s$  correspond to the neighbors of cell  $u$ .

#### 4.2.1. Solving STABILITY for rule 123

Rule 123 is the simplest of the Algebraic rules. Its simplicity, follows mainly from the following claim. Remember that  $\tau$  is the distance from  $u$  to the nearest active cell.

**Claim 1:** Either  $u$  becomes active at time  $\tau$  or  $u$  is stable.

We know that at least one of the cells in  $\{b, d, f, h\}$  will be active at time  $\tau - 2$ . Indeed the states of those cells depend only on the logical disjunction of the sites in the border of the diamond  $D_\tau$ , and we are assuming that there is at least one active site in  $D_\tau \setminus D_{\tau-1}$ . Therefore, in time  $\tau - 1$ , necessary at least one neighbor  $w \in N(u)$  will become active, since it will have more than one active neighbor, and less than four (because  $u$  is inactive at time  $\tau - 1$ ). Suppose now that  $u$  does not become active in time  $\tau$ . Since  $u$  has one active neighbor in time  $\tau - 1$ , the only possibility is that the four neighbors of  $u$  are active  $\tau - 1$ . Since the rule is freezing,  $u$  will remain stable in inactive state.

At this point, we know how to compute the states of  $b, d, f$  and  $h$  in time  $\tau - 2$ , and we know that the only possibility for  $u$  to become active is on time  $\tau$ . Therefore, in order to decide STABILITY for rule 123 we need to compute the states of cells  $p, q, r$  and  $s$  in time  $\tau - 1$ . In the following, we show how to compute the state of site  $p$  in time  $\tau - 1$ . The arguments for computing cells  $q, r$  and  $s$  will be deduced by analogy (considering the same arguments in another quadrant).

Call  $x_{(i,j)}^t$  the state of cell  $(i, j)$  in time  $t$ , with the convention of  $x_{(i,j)}^0$  is the input state of  $(i, j)$ . First, note that, for all  $i \in \{0, \dots, \tau - 2\}$ , the state of  $(0, i)$  in time 1 will be inactive. Moreover, the state of cell  $(0, \tau - 1)$  will be active if and only if at least one of its three neighbors  $(-1, \tau - 1), (1, \tau - 1)$  or

$(0, \tau)$  is active at time 0. Then, we deduce the following formula for  $x_{(0, \tau-1)}^1$ :

$$x_{(0, \tau-1)}^1 = x_{(-1, \tau-1)}^0 \vee x_{(0, \tau)}^0 \vee x_{(1, \tau-1)}^0$$

For the same reasons, we notice that at time  $j > 0$  the nearest neighbor from  $u$  is in the border of  $D_{\tau-j}$ . Therefore,

$$x_{(0, \tau-j)}^j = x_{(-1, \tau-j)}^{j-1} \vee x_{(0, \tau)}^{j-1} \vee x_{(1, \tau-j)}^{j-1}$$

In particular

$$x_p^{\tau-1} = x_{(0,1)}^{\tau-1} = x_{(-1,1)}^{\tau-2} \vee x_{(0,1)}^{\tau-2} \vee x_{(1,1)}^{\tau-2}.$$

Remember that we know how to compute  $x_{(-1,1)}^{\tau-2}$  and  $x_{(1,1)}^{j-1}$  according to Equation 1. We deduce that we can compute  $x_p^{\tau-1}$  as follows:

$$x_p^{\tau-1} = \bigvee_{k=1}^{\tau} x_{(-k, \tau-k)}^0 \vee x_{(0, \tau)}^0 \vee \bigvee_{k=1}^{\tau} x_{(k, \tau-k)}^0 \quad (2)$$

In words, the state of  $p$  at time  $\tau - 1$  can be computed as the OR of all the cells to the north of the  $u$  contained in  $D_{\tau} \setminus D_{\tau-1}$ . Analogously we can compute  $x_q^{\tau-1}$ ,  $x_r^{\tau-1}$  and  $x_s^{\tau-1}$ .

#### 4.2.2. Solving stability for rule 12

For rule 12 the computation of  $a$ ,  $c$ ,  $e$  and  $g$  is not so simple as the previous case. First of all, there is one case when cell  $u$  remain active, though we can also assume **Claim 1** for this rule.

Indeed, remember that we know that at least one of cells in  $\{b, d, f, h\}$  will be active at time  $\tau - 2$ . Suppose that  $u$  remains inactive at time  $\tau$ . There are two three possibilities: (1) none of the neighbors of  $u$  will become active at time  $\tau - 1$ , (2) three neighbors of  $u$  become active at time  $\tau - 1$ ; and (3) the four neighbors of  $u$  become active at time  $\tau - 1$ . See figure 9.

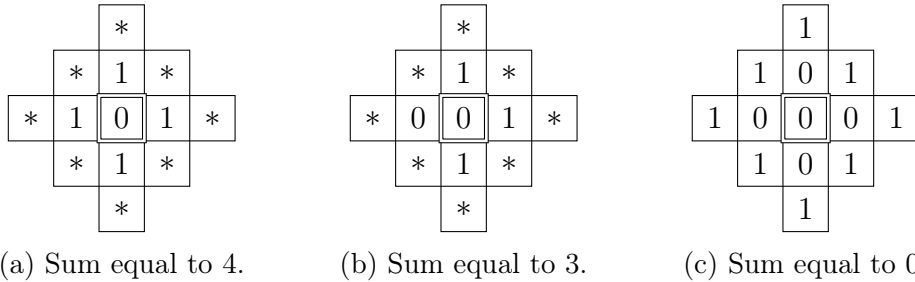


Figure 9: Possibles cases of rule 12 at time  $\tau$  such that  $u$  remain inactive.

Note that in cases when the sum in its neighborhood is 3 or 4 (Figure 9a and 9b respectively), we directly obtain that  $u$  is stable, because the rule is freezing.

The case when the sum in its neighborhood is 0 is slightly more complicated. As we said, we know that at least one of  $\{b, d, f, h\}$  becomes active at time  $\tau - 2$ . Suppose, without loss of generality, that  $b$  satisfies this condition. On the other hand, we are assuming that  $p$  and  $q$  remain inactive at time  $\tau - 1$ . Since these cells have one active neighbor at time  $\tau - 2$ , the sole possibility is that cells  $h, a, c$  and  $d$  are active at time  $\tau - 1$ . Applying the same arguments to cells  $r$  and  $s$ , we deduce that all cells  $a, b, c, d, e, f, g$  and  $h$  will be active at time  $\tau - 2$  (as depicted in Figure 9c). Since the rule is freezing, we deduce that cells  $p, q, r$  and  $s$  are stable, obtaining that also  $u$  is stable.

From Equation 1, we know how to compute the states of cells  $b, d, f$  and  $h$  in time  $\tau - 2$ . To decide the stability of  $u$ , we need to compute the states of  $p, q, r$  and  $s$  in time  $\tau - 1$ . In this case, however, the dynamics in the corridors is more complicated. In the following, we will show how to compute the east corridor (in order to compute  $q$ ), depicted in Figure 10. We will study only this case, since the other three corridors are analogous.

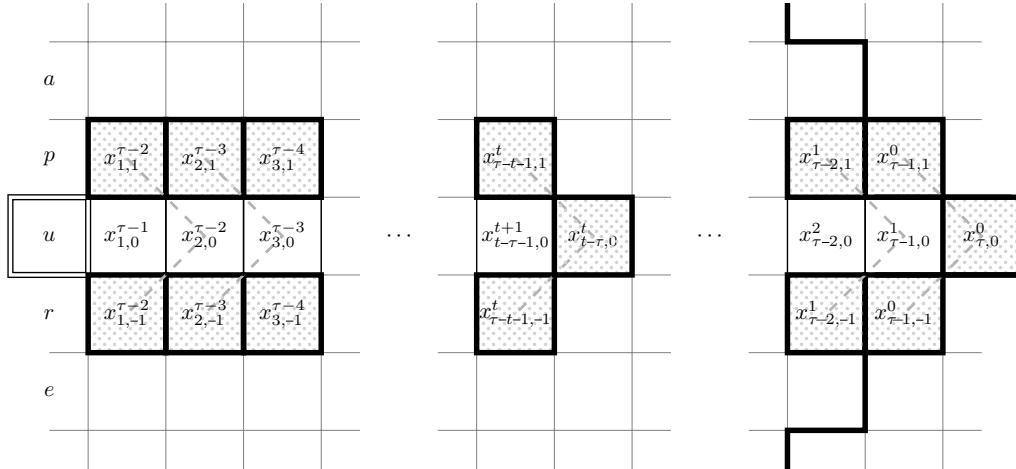


Figure 10: Computation of east corridor. Recall that  $x_{(i,j)}^t$  is the state of cell  $(i, j)$  in time-step  $t$ . The gray cells were previously computed using the OR technique. Dashed lines connect cells which potentially change states at the same time.

Remember that, using Equation 1 we can compute the values of  $x_{(i,1)}^{\tau-1-i}$  and  $x_{(-i,1)}^{\tau-1-i}$ , for every  $i \in \{1, \dots, \tau - 2\}$ . Notice first that, if  $x_{(i,1)}^{\tau-1-i} \neq x_{(-i,1)}^{\tau-1-i}$ ,

then necessarily  $x_{(i,0)}^{\tau-i} = 1$ . Indeed, we know that  $x_{(i-1,0)}^{\tau-i-1} = 0$  (otherwise we contradict the definition of  $\tau$ ). Then,  $x_{(i,1)}^{\tau-1-i} \neq x_{(-i,1)}^{\tau-1-i}$  implies that in time  $\tau - 1 - i$  the cell  $(i, 0)$  will have more than one and less than three active neighbors, so it will become active in time  $\tau - i$ .

Let  $i^*$  be the minimum value of  $i \in \{1, \dots, \tau\}$  such that  $x_{(i,1)}^{\tau-1-i} \neq x_{(-i,1)}^{\tau-1-i}$ . If no such  $i$  exists, then fix  $i^* = \tau$ . Call  $I^*$  the set  $\{1, \dots, i^* - 1\}$ . In other words, we know that  $x_{(i,1)}^{\tau-1-i} = x_{(-i,1)}^{\tau-1-i}$  for every  $i \in I^*$ . Moreover, we also know that  $x_{(i^*,0)}^{\tau-i^*} = 1$ .

We now identify two situations, concerning the values of  $x_{(i,1)}^{\tau-1-i}$ , for  $i \in I^*$ .

**If  $x_{(i,1)}^{\tau-1-i} = x_{(-i,1)}^{\tau-1-i} = 0$  then,** the value of  $x_{(i,0)}^{\tau-i}$  will equal the value of  $x_{(i+1,0)}^{\tau-1-i}$ . Indeed, in time  $\tau - 1 - i$ , the cell  $(i, 0)$  will have three inactive neighbors  $((-i, 1), (i, 1), (i - 1, 0))$ . Then it will take the same state than cell  $(i + 1, 0)$  at time  $\tau - i - 1$ .

**If  $x_{(i,1)}^{\tau-1-i} = x_{(-i,1)}^{\tau-1-i} = 1$  then,** the value of  $x_{(i,0)}^{\tau-i}$  will be the opposite than value of  $x_{(i+1,0)}^{\tau-1-i}$ . Indeed, in time  $\tau - 1 - i$ , the cell  $(i, 0)$  will have two active neighbors  $((-i, 1), (i, 1))$  and one inactive neighbor  $((i - 1, 0))$ . Then cell  $(i, 0)$  is active at time  $\tau - i$  if and only if cell  $(i + 1, 0)$  is inactive at time  $\tau - 1 - i$ .

We imagine that a signal drive along the corridor. The signal starts at  $(i^*, 0)$  with value  $x_{(i^*,1)}^{\tau-i^*}$ . The movement of the signal satisfies that, each time it encounters an  $i \in I^*$  such that  $x_{(i,1)}^{\tau-1-i} = x_{(-i,1)}^{\tau-1-i} = 1$ , the state switches to the opposite value. Let  $z = |\{i \in I^* : x_{(i,1)}^{\tau-1-i} = x_{(-i,1)}^{\tau-1-i} = 1\}|$  (i.e.  $z$  is the number of switches). From the two situations explained above, we deduce the following lemma.

**Lemma 4.9.**  $x_{(1,0)}^{\tau-1}$  equals  $x_{(i^*,1)}^{\tau-i^*}$  if  $z$  is even, and  $x_{(1,0)}^{\tau-1}$  is different than  $x_{(i^*,1)}^{\tau-i^*}$  when  $z$  is odd.

Therefore, to solve STABILITY for rule 12, we compute the values of  $x_p^{\tau-1}$ ,  $x_q^{\tau-1}$ ,  $x_r^{\tau-1}$ ,  $x_s^{\tau-1}$  according to Lemma 4.9.

#### 4.2.3. Solving STABILITY for rule 124

The analysis for the rule 124 is more complicated than the one we did for rule 12 and 123. In fact, one great difference is that **Claim 1** is no longer



true for this rule. In other words,  $u$  might not be stable but change after time-step  $\tau$ .

For this rule, the cases when the cell  $u$  remain inactive are the cases when  $u$  has zero or three active neighbors. The possible cases when the cell  $u$  remain inactive at time  $\tau$  are given in Figure 11.

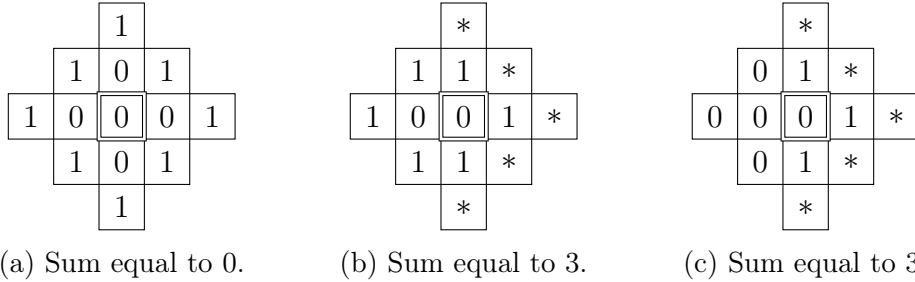
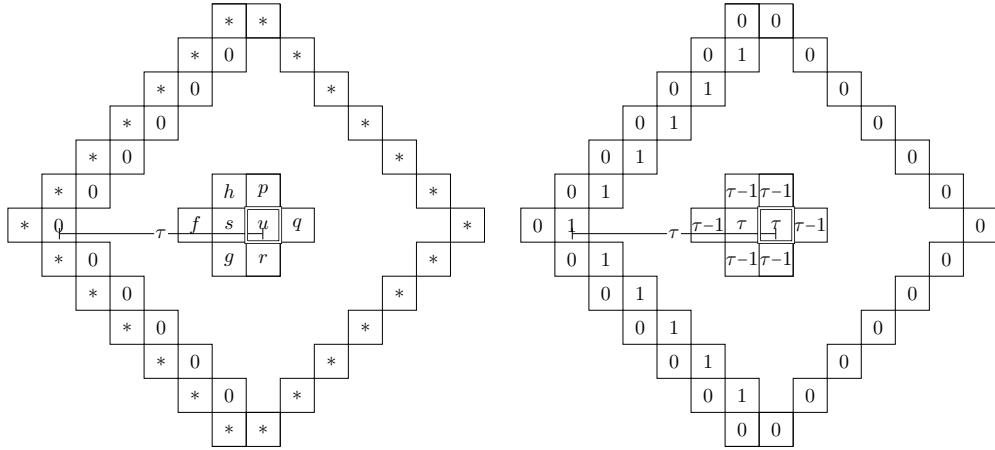


Figure 11: Possible cases of rule 124 at time  $\tau$  such that  $u$  remain inactive.

The case when  $u$  has four inactive neighbors at time  $\tau - 1$  is exactly the same that we explained for rule 12 (see Figure 11a). Suppose that  $u$  has three active neighbors, and without loss of generality assume that  $p, q, r$  are active and  $s$  is inactive. Then there are two possibilities, either  $s$  has three active neighbors  $(h, g, f)$ , in which case  $u$  and  $s$  remain inactive (see Figure 11b) The difference with the rule 12 is that we can not decide immediately if the cell  $u$  remains inactive when the sum at time  $\tau$  is 3. Indeed, in the case depicted in Figure 11b, it is possible that  $s$  becomes active in a time-step later than  $\tau - 1$ .

Thus we need study only the case when the sum at time  $\tau - 1$  of the states of neighbors of  $s$  is 0 or equivalently  $x_f^{\tau-2} = x_g^{\tau-2} = x_h^{\tau-2} = 0$ . Note that, by the OR technique, the fact that  $x_f^{\tau-2} = x_g^{\tau-2} = x_h^{\tau-2} = 0$  means that all the cells in the left side border of  $D_\tau$  are initially inactive, as shown in Figure 12.

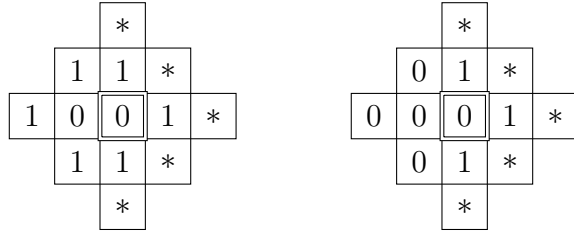


(a) Configuration at time 0. (b) Diagram with the times when a cell can be activated.

Figure 12: Schedule for to compute  $x_u^\tau$  if  $x_f^{\tau-2} = x_g^{\tau-2} = x_h^{\tau-2} = 0$ .

Knowing that  $x_f^{\tau-2} = x_g^{\tau-2} = x_h^{\tau-2} = 0$ , we can compute their states at time-step  $\tau - 1$ , considering the OR-technique in the diamond  $D_{\tau+1}$ . Using this information, we can compute the state of  $s$  in time  $\tau$ , i.e. compute  $x_s^\tau$ .

Remember that we are in the case where  $x_p^{\tau-1} = x_q^{\tau-1} = x_r^{\tau-1} = 1$  and  $x_s^{\tau-1} = 0$ . If the cell  $s$  becomes activate at time  $\tau$ , then  $u$  will have four active neighbors at time  $\tau$ , and it will become active. Now we suppose that  $s$  also remain inactive at time  $\tau$ . Again, we have two possible cases:



(a) Sum equal to 3. (b) Sum equal to 3.

Figure 13: Possible cases of rule 124 at time  $\tau + 1$  such that  $u$  remain at state 0.

In the case in Figure 13a (i.e., when  $s$  remains inactive at time  $\tau$  because  $f, g$  and  $h$  were active at time  $\tau - 1$ ) the cell  $u$  is stable.

For the case shown in Figure 13b (i.e.  $s$  remains inactive at time  $\tau$  because  $e, f, g$  and  $h$  were inactive at time  $\tau - 1$ ) we must repeat the previous analysis.



We deduce that either  $u$  becomes active at time  $\tau, \tau + 1$  or  $\tau + 2$ , or  $u$  is stable.

**Lemma 4.10.** *Let  $F$  be rule 124. Given a finite configuration  $x$ , a cell  $u$  and  $\tau$  the distance from  $u$  to the nearest active cell. Then  $u$  becomes active at time  $\tau, \tau + 1$  or  $\tau + 2$ , or  $u$  is stable.*

Now we give an algorithm for to decide the STABILITY inn **NC** for the rule 124. The algorithms for rules 12 and 123 can be deduced from this algorithm.

**Theorem 4.11.** *STABILITY is in **NC** for the freezing CA 12, 123 and 124.*

*Proof.* Let  $(x, u)$  be an input of STABILITY,  $x$  is a finite configuration of dimensions  $n \times n$  and  $u$  is a site in  $[n] \times [n]$ . The following parallel algorithm is able to decide STABILITY using the fast computation of the first neighbors of  $u$  by the OR technique. Let  $N^2(u)$  be the set of cells at distance at most 2 from  $u$ . For  $t \geq 0$ , we call  $x_{N(u)}^t$  the set of states at time  $t$  of all cells in  $N(u)$ .

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**Algorithm 4** Solving STABILITY 124

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**Input:**  $x$  a finite configuration of dimensions  $n \times n$  and  $u \in [n] \times [n]$ .

- 1: Compute  $\tau$  the distance form  $u$  to the nearest active cell in  $x$ .
  - 2: Compute  $x_{N^2(u)}^{\tau-2}$  using the OR technique and the corridors.
  - 3: **if**  $x_u^\tau = 1$  **then**
  - 4:     **return** *Reject*
  - 5: **end if**
  - 6: **if**  $x_u^\tau = 0$  and  $x_{N^2(u)}^{\tau-2}$  is as 11b **then**
  - 7:     **return** *Accept*
  - 8: **end if**
  - 9: Compute  $s$  the neighbor of  $u$  such that  $x_s^{\tau-1} = 0$ .
  - 10: Compute  $x_{N^2(s)}^{\tau-1}$  using the OR technique and the corridors.
  - 11: **if**  $x_u^{\tau+1} = 1$  **then**
  - 12:     **return** *Reject*
  - 13: **end if**
  - 14: **if**  $x_u^{\tau+1} = 0$  and  $x_{N^2(u)}^{\tau-1}$  is as 11b **then**
  - 15:     **return** *Accept*
  - 16: **end if**
  - 17: **return** *Reject*
-

Let  $N = n^2$  the size of the input. Step **1** can be done in  $\mathcal{O}(\log N)$  time with  $\mathcal{O}(N)$  processors: one processor for each cell for to choose the actives cells and to compute its distances with  $u$ , then in  $\mathcal{O}(\log N)$  compute the nears cell to  $u$ . Steps **2** and **10** can be done in  $\mathcal{O}(\log N)$  time with  $\mathcal{O}(N)$  processors: the OR technique and the corridors can be computed with prefix sum algorithm (see proposition 2.1) for the computation of consecutive  $\vee$  and parity of  $z$  in the corridors. The others steps can be computed in  $\mathcal{O}(\log n)$  time in using a sequential algorithm.  $\square$

#### 4.3. Turing Universal Rules

For the rules 2 and 24 the problem STABILITY is **P**-Complete by reducing a restricted version of the Circuit Value Problem [10] to this problem. Instances of circuit value problem are encoded into a configuration of the CA 2 and 24 using the idea in the proof of the **P**-completeness of Planar Circuit Value Problem (PCV) [11]. Moreover, we use an aproach given in [12], were the authors show that a two-dimensional automaton capable of simulating *wires*, *OR gates*, *AND gates* and *crossing gadgets* is **P**-Complete.

In Figure 15 we will give the gadgets that simulate this structures for rules 2 and 24. We remark that both rules have the same structures, because the patterns with four active neighbors never appear in the gadgets.

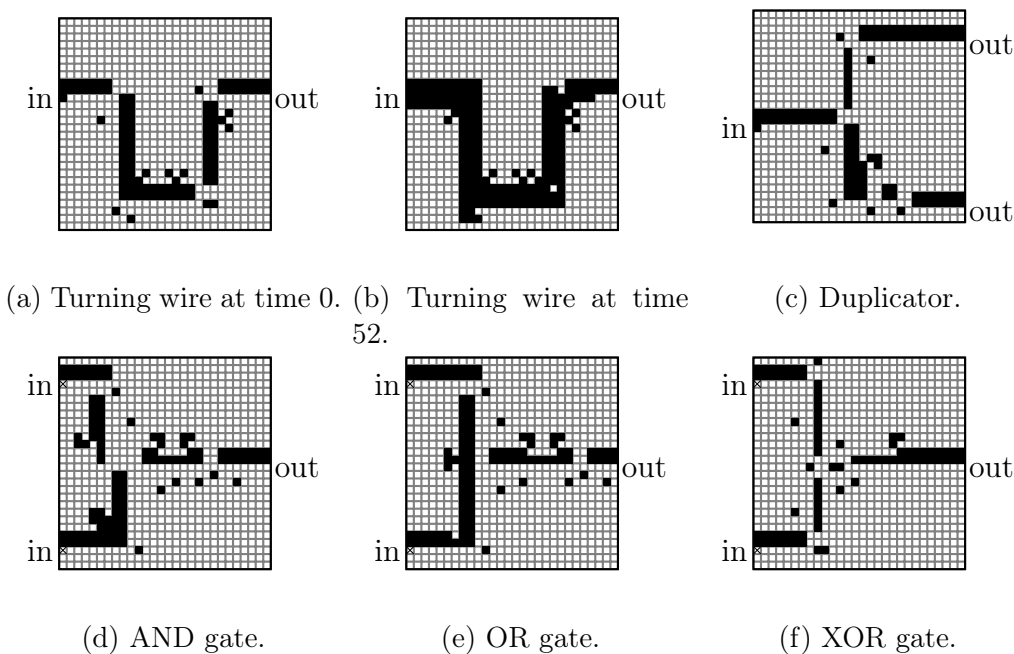


Figure 15: Gadgets for the implementation of logic circuits for the rules 2 and 24.

We represent the information flowing through wires, which are based, roughly, on a line of active sites. Then, all the sites over (under) this line will have one active neighbor. If a cell over the line becomes active, then in the next step a neighbor of this cell will become active, so the information flows over the wire.

These constructions of the gates are quite standard. Maybe with one exception is the XOR gate. The crucial observation is that we manage to simulate the XOR using the synchronicity of information. An XOR gate consists roughly in two confluent wires. If a signal arrives from one of the two wires, the signal simply passes. If two signals arrive at the same time, the next cell in the wire will have more than two neighbors, so it will remain active.

Using the XOR gate (Figure 15f), one can build a planar crossing gadget, concluding the  $\mathbf{P}$ -completeness constructions.

**Theorem 4.12.** *STABILITY is  $\mathbf{P}$ -complete for the freezing CA 2 and 24.*

Remark: In our construction we use strongly neighborhoods composed only of inactive cells, so these constructions can not be used for rules 02 and

024, where zero is not a quiet state. So in these cases stability could have less complexity.

## 5. Concluding Remarks

### 5.1. Summary of our results

In this paper we have studied the complexity of the STABILITY problem for the Freezing Totalistic Cellular Automata (FTCA) on the triangular and square grid with von Neumann neighborhood and two states. We find different complexities for this FTCA, including a **P**-complete case on square grid. For the rules where STABILITY is in **NC** we have considered two approaches: a topological approach (theorems 3.3, 3.1, 4.1, 4.2, and 4.3) and an algebraic approach (theorems 3.6 and 4.11).

Rule	STABILITY	<i>Theorem</i>
$\phi$	$\mathcal{O}(1)$	Trivial
3	<b>NC</b>	Trivial
2	<b>NC</b>	Thm 3.3
23	<b>NC</b>	Thm 3.1
12	<b>NC</b>	Thm 3.6
123	<b>NC</b>	Trivial

Table 1: Summary of rules and their complexity of STABILITY on triangular grid.

Rule	STABILITY	<i>Theorem</i>	Rule	STABILITY	<i>Theorem</i>
4	<b>NC</b>	Trivial	234	<b>NC</b>	Thm 4.3
3	<b>NC</b>	Thm 4.2	12	<b>NC</b>	Thm 4.11
34	<b>NC</b>	Thm 4.1	124	<b>NC</b>	Thm 4.11
2	<b>P-Complete</b>	Thm 4.12	123	<b>NC</b>	Thm 4.11
24	<b>P-Complete</b>	Thm 4.12	1234	$\mathcal{O}(1)$	Trivial

Table 2: Summary of rules and their complexity of STABILITY on square grid.

### 5.2. About Fractal-Growing Rules

In this paper we have not included a study of fractal growing rules. In fact, the complexity of STABILITY remains open for these rules, even for fractal rules defined over a triangular grid.

To have an intuition about the dynamical complexity of those rules, see Figures 16 and 17, where starting with only the center active we obtain a fractal behavior.

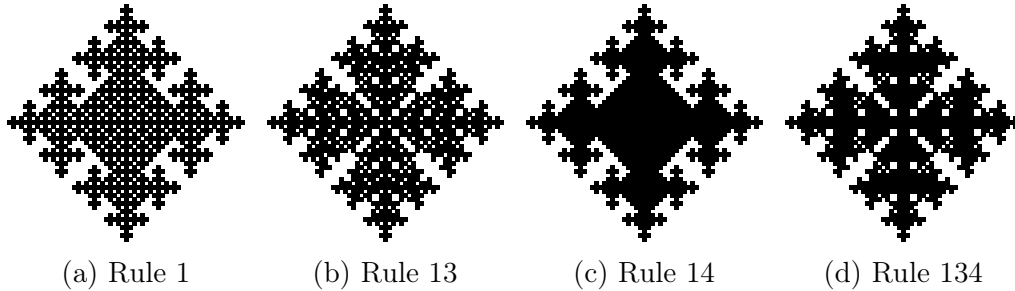


Figure 16: Examples of different rules with the similar fractal dynamics starting with a single active cell on square grid.

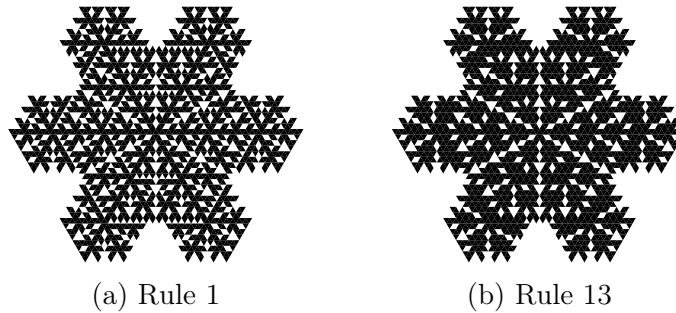


Figure 17: Examples of different rules with the similar fractal dynamics starting with a single active cell on triangular grid.

It is interesting to remark that non-freezing version of rule 13 is the usual XOR between the four neighbors, which is a linear cellular automaton. Using a prefix-sum algorithm, we can compute any step of a linear cellular automaton, so the *non-freezing* rule 13 is in **NC**. Although we might imagine that adding *freezing* property simplifies the dynamics of a rule, the non-linearity of rule 13 increases enough the difficulty to prevent us to characterize its complexity.

### 5.3. On P-Completeness on the triangular grid

It is important to point out that for triangular graph (despite rule 1 or 13 might be candidates), we do not exhibit a rule such that STABILITY is



**P**-complete. The reader might think that, like in the one-dimensional case, every freezing rule defined in a triangular grid is **NC**. This is not the case. Moreover, three states (that we call 0, 1, 2) suffice to define a **P**-complete FTCA. The general freezing property means that states may only grow (so, in this case state 2 is stable). In this context, for a triangular grid, consider the following the local function.

$$f\left(x_u, \sum_{z \in N^+u} x_z\right) = \begin{cases} 1, & \text{if } x_u = 0 \wedge (\sum_{z \in N^+u} x_z = 2 \vee \sum_{z \in N} x_z = 12) \\ 1, & \text{if } x_u = 10 \wedge \sum_{z \in N^+u} x_z = 11 \\ x_u, & \text{otherwise} \end{cases}$$

The proof of **P**-Completeness follows similar arguments than the ones we used for rules 2 and 24 in the squared grid (Theorem 4.12), i.e. reducing the Circuit Value Problem (CVP) to STABILITY on this rule. Instances of CVP are encoded into a configuration of this FTCA using the idea in the construction of the logical gates. In figures 15d and 18d we exhibit the gadgets.

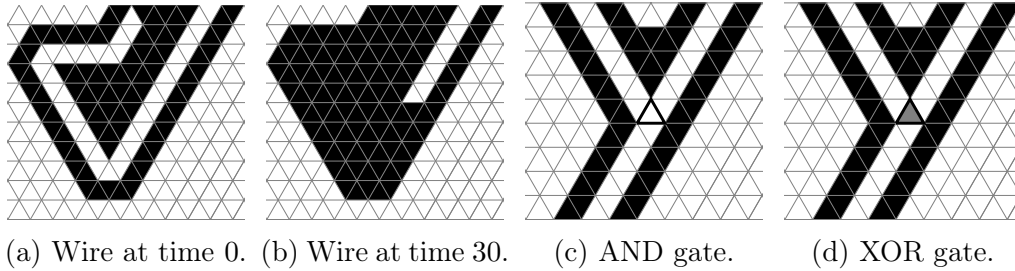


Figure 18: Gadgets for the implementation of logic circuits. The thick line marks the cell that makes the calculation from signals. The color code is:  $\blacktriangle$  : 1,  $\blacktriangle$  : 10 and  $\triangle$  : 0

#### 5.4. About non-quiescent rules

Finally, it is convenient to say a word about rules where cells become active with zero active neighbors, i.e., rules where state 0 is not quiescent. Clearly, after one step for those rules, every cell will have at least one active neighbor. Then, their complexity is at most the complexity of the same rule, not considering the case of zero active neighbors as an activating state. For example, consider rule 034 in the squared grid. After one step of rule 034, the dynamics are exactly the same that the one of rule 34. Therefore, rule

034 is in **NC**. Although, there are some interesting cases. First, notice that rule 01 is trivial (in the triangular or squared grids), because after only one step the rule reaches a fixed point. This contrasts with rule 1, which is a Fractal-Growth rule. Second, consider rule 02 or 024 in the squared grid. We know that rule 2 and 24 are **P**-Complete. However, the reader can verify that the gadgets used to reduce CVP to STABILITY do not work for rule 02 and 024. This fact opens the possibility that rules 02 and 024 belong to **NC**.

## 6. Acknowledgment

Research funded by CONICYT+PAI+CONVOCATORIA NACIONAL SUBVENCION A INSTALACION EN LA ACADEMIA CONVOCATORIA AÑO 2017 + PAI77170068 (P.M.)

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