



HAL
open science

Periodic in time flow in thin structure: Equation on the graph

Grigory Panasenko, Konstantin Pileckas

► To cite this version:

Grigory Panasenko, Konstantin Pileckas. Periodic in time flow in thin structure: Equation on the graph. *Journal of Mathematical Analysis and Applications*, 2020, 490, pp.124335 -. <10.1016/j.jmaa.2020.124335>. <hal-03491588>

HAL Id: hal-03491588

<https://hal.science/hal-03491588v1>

Submitted on 15 Jul 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY-NC 4.0 - Attribution - Non-commercial use - International License

Periodic in time flow in thin structure: equation on the graph *

GRIGORY PANASENKO[†] and Konstantin Pileckas[‡]

Abstract

The dimension reduction for the non-stationary time periodic Navier-Stokes equations in thin structures leads to a non-local in time Reynolds type equation on the graph with Kirchhoff type junction conditions in the vertices. The existence and uniqueness of a solution to this problem is proved.

1 Introduction. Main definitions

The Newtonian flows in tube structures were considered in [6], [2], [7], [8], [9], [10], [11], [3]. These domains are finite connected unions of thin cylinders (in the two-dimensional case respectively thin rectangles). Such tube structure is represented by its graph: if the diameter of cross-sections of cylinders tends to zero then the tubes degenerate and "tend" to the edges of the graph. In the case of stationary Navier-Stokes equations set in a thin structure we can derive for the pressure a second order ordinary differential equations at the edges of the graph and some Kirchhoff-type junction conditions at the nodes (see

* *AMS Mathematical Subject classification:* 35Q35; 76D07; 65N55; *Keywords:* Stokes and Navier-Stokes equations, thin structures, equation on the graph, existence and uniqueness of a solution

[†]Univ Lyon, UJM, Institute Camille Jordan UMR CNRS 5208 and SFR MODMAD FED 4169, F-42023, Saint-Etienne, France; grigory.panasenko@univ-st-etienne.fr and Institute of Applied Mathematics, Vilnius University, Naugarduko Str., 24, LT-03225 Vilnius, Lithuania

[‡]Institute of Applied Mathematics, Vilnius University, Naugarduko Str., 24, LT-03225 Vilnius, Lithuania, konstantinas.pileckas@mif.vu.lt

[6],[2], [7],[9]). In the non-stationary case for the initial-boundary value problem for the Navier-Stokes equations the equation on the graph becomes non-stationary and non-local in time (see [11]). It is explained by a non-local in time relations between the pressure slope and the flow rate in an infinite cylinder in this case (see [1], [4] for the time-periodic flow). In the present paper we introduce the periodic in time one-dimensional model on the graph for a pipe-wise network. This model was derived from the non-stationary time periodic Navier-Stokes equation in a tube structure when the time derivative has large coefficient (fast periodic oscillations) [5]. Below we study the existence and uniqueness of a solution of the time-periodic problem on the graph.

Let us remind the definition of the graph of a tube structure. Consider N different points in $\mathbb{R}^n, n = 2, 3, \dots$: O_1, O_2, \dots, O_N , let e_1, e_2, \dots, e_M be M closed segments called edges such that every segment connects two points of the list: $e_j = \overline{O_{i_j} O_{k_j}}$, where $i_j, k_j \in \{1, \dots, N\}, i_j \neq k_j$. Each point O_i of the list is an end point of some edge.

A point O_i is called node if it is the common end of at least two edges and O_i is called vertex if it is the end of the only one edge. Any two edges e_j and e_i can intersect only at the common node. Assume that there exists at least one vertex.

The graph is a connected union of all edges: $\mathcal{B} = \bigcup_{j=1}^N e_j$. Let $x \in \mathbb{R}^n$. Denote $x' = (x_1, \dots, x_{n-1})$.

The union of all edges having the same end point in O_l is called the bundle $\mathcal{B}^{(l)}$.

Consider an edge $e = \overline{O_i O_j}$. Let us introduce two local Cartesian coordinate systems in \mathbb{R}^n . The first one has the origin in O_i and the axis $O_i x_n^{(e)}$ has the direction of the ray $[O_i O_j)$ while the second one has the origin in O_j and the opposite direction.

Denote the local variable in both cases as $x^{(e)}$.

Let us introduce an operator expressing the relation between the pressure slope and the time periodic flux in an infinite cylinder.

Let T be a positive number. The notation $V_{per}(0, T)$ means that elements of this space are T -periodic functions, i.e., $u(\cdot, t) = u(\cdot, t + T) \forall t \in \mathbb{R}$. Consider two spaces of periodic functions: $L^2_{per}(0, T)$ and $H^1_{per}(0, T)$. The spaces $L^2_{per}(0, T)$ and $H^1_{per}(0, T)$ are supplied with the inner product of $L^2(0, T)$ and $H^1(0, T)$, respectively.

Let $\tau^{(e)} \subset \mathbb{R}^{n-1}$ be a bounded domain with Lipschitz boundary $\partial\tau^{(e)}$, associated to some edge e . Denote by $\mathcal{L}^{(e)}$ the operator relating the pressure slope \mathcal{S} and the time-periodic flux \mathcal{H} in an infinite cylindrical pipe with section $\tau^{(e)}$. Namely, consider the following periodic in time heat equation in the domain $\tau^{(e)}$: for given function $\mathcal{S} \in L^2_{per}(0, T)$ find $V \in L^2_{per}(0, T; H^1_0(\tau^{(e)}))$ with $\frac{\partial V}{\partial t} \in L^2_{per}(0, T; L^2(\tau^{(e)}))$ such that

$$\begin{aligned} \frac{\partial V}{\partial t}(y^{(e)'}, t) - \nu \Delta'_{y^{(e)'}} V(y^{(e)'}, t) &= \mathcal{S}(t), \quad y^{(e)'}, t > 0, \\ V(y^{(e)'}, t)|_{\partial\tau^{(e)}} &= 0. \end{aligned} \tag{1}$$

Denote

$$\mathcal{L}^{(e)} \mathcal{S}(t) = \int_{\tau^{(e)}} V(y^{(e)'}, t) dy^{(e)'}$$

Obviously, $\mathcal{L}^{(e)}$ is bounded linear operator acting from $L^2_{per}(0, T)$ to $H^1_{per}(0, T)$. It is also proved in [1], [4] that $\mathcal{L}^{(e)}$ has the bounded inverse $(\mathcal{L}^{(e)})^{-1} : L^2_{per}(0, T) \mapsto H^1_{per}(0, T)$.

Theorem 1. *Let $\mathcal{H} \in H^1_{per}(0, T)$ be given. There exists a unique couple (V, \mathcal{S}) satisfying in the sense of distributions (1) and the flux condition*

$$\int_{\tau^{(e)}} V(y^{(e)'}, t) dy^{(e)'} = \mathcal{H}(t).$$

Moreover, $V \in L^2_{per}(0, T; H^1_0(\tau^{(e)}))$, $\frac{\partial V}{\partial t} \in L^2_{per}(0, T; L^2(\tau^{(e)}))$, $\mathcal{S} \in L^2_{per}(0, T)$, and the following estimates

$$\|V\|_{L^2(0, T; H^1_0(\tau^{(e)}))} + \left\| \frac{\partial V}{\partial t} \right\|_{L^2(0, T; L^2(\tau^{(e)}))} + \|\mathcal{S}\|_{L^2(0, T)} \leq c \|\mathcal{H}\|_{H^1(0, T)}, \tag{2}$$

$$C_e^{-1} \|\mathcal{Q}\|_{L^2(0, T)} \leq \|\mathcal{L}^{(e)} \mathcal{Q}\|_{H^1(0, T)} \leq C_e \|\mathcal{Q}\|_{L^2(0, T)}, \forall \mathcal{Q} \in L^2_{per}(0, T), \tag{3}$$

hold. Here C_e is a positive constant depending on $\tau^{(e)}$ and T .

Introduce the notation: $\langle \cdot \rangle = \frac{1}{T} \int_0^T \cdot dt$.

Theorem 2. *The following formula*

$$\langle \mathcal{L}^{(e)} S \rangle = \kappa^{(e)} \langle S \rangle,$$

is valid, where $\kappa^{(e)} = \int_{\tau^{(e)}} U^{(e)}(y^{(e)'}) dy^{(e)'} > 0$ and U is the solution (unique) to the problem

$$-\nu \Delta'_{y^{(e)'}} U^{(e)}(y^{(e)'}) = 1, \quad y^{(e)'} \in \tau^{(e)},$$

$$U^{(e)}(y^{(e)'})|_{\partial\tau^{(e)}} = 0.$$

Proof. Integrating (1) by t over the interval $[0, T]$ and using the time periodicity of the solution, we get

$$-\nu \Delta'_{y^{(e)'}} \langle V \rangle (y^{(e)'}) = \langle S \rangle, \quad y^{(e)'} \in \tau^{(e)},$$

$$\langle V \rangle (y^{(e)'})|_{\partial\tau^{(e)}} = 0,$$

and, hence $\langle V \rangle = \langle S \rangle U^{(e)}$. Then

$$\langle \mathcal{L}^{(e)} S \rangle = \int_{\tau^{(e)}} \langle V \rangle (y^{(e)'}) dy^{(e)'} = \langle S \rangle \int_{\tau^{(e)}} U^{(e)}(y^{(e)'}) dy^{(e)'} = \kappa^{(e)} \langle S \rangle.$$

□

Corollary 1. $\langle \mathcal{L}^{(e)} S \rangle = 0$ if and only if $\langle S \rangle = 0$.

Corollary 2. For any constant C there holds the equality: $\mathcal{L}^{(e)}(C) = \kappa^{(e)} C$.

Proof. It is straightforward to check that

$$\mathcal{L}^{(e)}(1) = \int_{\tau^{(e)}} U^{(e)}(y^{(e)'}) dy^{(e)'} = \kappa^{(e)},$$

and by linearity of the operator $\mathcal{L}^{(e)}$ we get the assertion of the corollary. □

2 Formulation of the time periodic problem for macroscopic pressure on the graph

Let us introduce functions $\Psi_l \in H_{per}^1(0, T)$, $l = 1, \dots, N$, and $\mathcal{F}^{(e_i)} \in H_{per}^1(0, T; L^2(\mathcal{B}))$, $i = 1, \dots, M$. Consider the following problem: find a function $p \in L_{per}^2(0, T; H^2(\mathcal{B}))$

satisfying the following equations on the edges and junction conditions at the nodes and vertices:

$$\begin{aligned}
-\frac{\partial}{\partial x_n^{(e)}} \left(\mathcal{L}^{(e)} \frac{\partial p}{\partial x_n^{(e)}}(x_n^{(e)}, t) \right) &= \mathcal{F}^{(e)}(x_n^{(e)}, t), \quad x_n^{(e)} \in (0, |e|), \forall e = e_j, j = 1, \dots, M, \\
-\sum_{e: O_l \in e} \left(\mathcal{L}^{(e)} \frac{\partial p}{\partial x_n^{(e)}} \right)(0, t) &= \Psi_l(t), \quad l = 1, \dots, N_1, \\
-\left(\mathcal{L}^{(e)} \frac{\partial p^{(e)}}{\partial x_n^{(e)}} \right)(0, t) &= \Psi_l(t), \quad l = N_1 + 1, \dots, N.
\end{aligned} \tag{4}$$

Here equation (4)₁ holds on each edge e of \mathcal{B} , condition (4)₂ holds at each node O_l with the sum taken over all edges having O_l as an end point (i.e. over all edges of $\mathcal{B}^{(l)}$), condition (4)₃ is a Neumann type boundary condition which holds at each vertex O_l . In conditions (4)₂ and (4)₃ the local coordinate system has the origin O_l . This problem is a one-dimensional model for the macroscopic pressure defined on the graph \mathcal{B} ; the right-hand sides stand for the sources. The pressure is supposed to be a continuous function on the graph because it belongs to $H^1(\mathcal{B})$, however at the nodes one can consider some given jumps of the pressure. This generalized setting can be reduced to (4) by a change of the unknown function and the right-hand sides.

3 Existence and uniqueness of the solution

Let us prove that problem (4) admits a unique solution vanishing at the vertex O_N , if the following compatibility condition

$$\sum_{i=1}^M \int_0^{|e_i|} \mathcal{F}^{(e_i)} dx_n^{(e_i)} + \sum_{l=1}^N \Psi_l(t) = 0, \tag{5}$$

holds for almost all $t \in [0, T]$.

Differentiating relations (4) with respect to time t , multiplying by $\psi \in L_{per}^2(0, T; H^1(\mathcal{B}))$ and integrating by parts in $\mathcal{B} \times (0, T)$, we obtain the following variational formulation: to find a function $p \in L_{per}^2(0, T; H^1(\mathcal{B}))$ satisfying the integral identity

$$\alpha_T(p, \psi) = \beta_T(\psi), \quad \forall \psi \in L_{per}^2(0, T; H^1(\mathcal{B})), \tag{6}$$

where

$$\alpha_T(p, \psi) = \sum_{i=1}^M \int_0^T \int_0^{|e_i|} \frac{\partial(\mathcal{L}^{(e_i)}p)_t}{\partial x_n^{(e_i)}} \frac{\partial \psi}{\partial x_n^{(e_i)}} dx_n^{(e_i)} dt$$

and

$$\beta_T(\psi) = \sum_{i=1}^M \int_0^T \int_0^{|e_i|} (\mathcal{F}^{(e_i)})_t \psi dx_n^{(e_i)} dt + \sum_{l=1}^N \int_0^T (\Psi_l)_t(t) \psi(O_l, t) dt.$$

Here $g_t = \frac{\partial g}{\partial t}$. Let us study this auxiliary problem.

Lemma 1. *Suppose that the compatibility condition (5) is valid. Then for any given $\Psi_l \in H_{per}^1(0, T)$, $l = 1, \dots, N$, and $\mathcal{F}^{(e_i)} \in H_{per}^1(0, T; L^2(\mathcal{B}))$, $i = 1, \dots, M$, there exists a unique function $p_0 \in L_{per}^2(0, T; H^1(\mathcal{B}))$ satisfying the integral identity (6), vanishing at the vertex O_N and having mean value zero in time: $p_0(O_N, t) = 0$, $\langle p_0 \rangle = 0$. Any solution p to problem (6) has the form: $p(x, t) = p_0(x, t) + \phi(x) + \psi(t)$, where $\phi \in H^1(\mathcal{B})$, $\psi \in L_{per}^2(0, T)$.*

Proof. We will use the Lax-Milgram lemma argument and inequality (3). Consider the subspace $H_{0,N}^1(\mathcal{B})$ of functions belonging to $H^1(\mathcal{B})$ and vanishing at the vertex O_N , and the subspace $\widehat{L}_{per}^2(0, T; H_{0,N}^1(\mathcal{B}))$ of the space $L_{per}^2(0, T; H_{0,N}^1(\mathcal{B}))$ with elements ψ having vanishing mean value in time $\langle \psi \rangle = 0$ (i.e., ψ satisfies relation $\int_0^T \mathcal{L}^{(e)} \psi dt = 0$, see Corollary 1).

Let us check, using (3), that α_T is a bilinear form continuous with respect to the norm of $\widehat{L}_{per}^2(0, T; H_{0,N}^1(\mathcal{B}))$ and that β_T is a bounded linear functional:

$$\begin{aligned} |\alpha_T(p, \psi)| &\leq \sqrt{\sum_{i=1}^M \int_0^T \int_0^{|e_i|} \left| \frac{\partial(\mathcal{L}^{(e_i)}p)_t}{\partial x_n^{(e_i)}} \right|^2 dx_n^{(e_i)} dt} \sqrt{\sum_{i=1}^M \int_0^T \int_0^{|e_i|} \left| \frac{\partial \psi}{\partial x_n^{(e_i)}} \right|^2 dx_n^{(e_i)} dt} \\ &\leq \max_{1 \leq j \leq M} C_{e_j} \sqrt{\sum_{i=1}^M \int_0^T \int_0^{|e_i|} \left| \frac{\partial p_t}{\partial x_n^{(e_i)}} \right|^2 dx_n^{(e_i)} dt} \sqrt{\sum_{i=1}^M \int_0^T \int_0^{|e_i|} \left| \frac{\partial \psi}{\partial x_n^{(e_i)}} \right|^2 dx_n^{(e_i)} dt}. \end{aligned}$$

The boundedness of β_T follows from the Poincaré-Friedrichs inequality in the space $H_{0,N}^1(\mathcal{B})$ and the inclusion of the space $H_{0,N}^1(\mathcal{B})$ into $C(\mathcal{B})$.

The coerciveness of α_T is a consequence of the following estimates, which hold for any edge e and any time periodic solution $(V(y^{(e)'}, t), \mathcal{S})$ of (1) :

$$\begin{aligned}
\int_0^T (\mathcal{L}^{(e)} \mathcal{S})_t \mathcal{S} dt &= \int_0^T \int_{\tau^{(e)}} (V(y^{(e)'}, t))_t \mathcal{S} dy^{(e)'} dt \\
&= \int_0^T \int_{\tau^{(e)}} \left\{ |(V(y^{(e)'}, t))_t|^2 + \frac{\nu}{2} \frac{\partial}{\partial t} |\nabla_{y^{(e)'}} V(y^{(e)'}, t)|^2 \right\} dy^{(e)'} dt \\
&= \int_0^T \int_{\tau^{(e)}} |(V(y^{(e)'}, t))_t|^2 dy^{(e)'} dt \\
&\geq \frac{1}{|\tau^{(e)}|} \int_0^T \left| \int_{\tau^{(e)}} ((V(y^{(e)'}, t))_t dy^{(e)'}) \right|^2 dt = \frac{1}{|\tau^{(e)}|} \int_0^T |(\mathcal{L}^{(e)} \mathcal{S})_t|^2 dt.
\end{aligned}$$

Note that $\langle \mathcal{L}^{(e)} \mathcal{S} \rangle = 0$. Thus, by the Poincaré inequality and (3),

$$\begin{aligned}
\frac{1}{|\tau^{(e)}|} \int_0^T |(\mathcal{L}^{(e)} \mathcal{S})_t|^2 dt &\geq \frac{1}{2|\tau^{(e)}|} \int_0^T |(\mathcal{L}^{(e)} \mathcal{S})_t|^2 dt + \frac{1}{T^2 |\tau^{(e)}|} \int_0^T |\mathcal{L}^{(e)} \mathcal{S}|^2 dt \\
&\geq \min\left(\frac{1}{2}, \frac{1}{T^2}\right) \frac{C_e^{-2}}{|\tau^{(e)}|} \int_0^T |\mathcal{S}(t)|^2 dt.
\end{aligned}$$

Finally, we get

$$\alpha_T(p, p) \geq \min\left(\frac{1}{2}, \frac{1}{T^2}\right) \frac{\min_{i=1, \dots, M} C_{e_i}^{-2}}{\max_{i=1, \dots, M} |\tau^{(e_i)}|} (p, p)_T, \quad (7)$$

where

$$(p, \psi)_T = \sum_{i=1}^M \int_0^T \int_0^{|\mathbf{e}_i|} \frac{\partial p}{\partial x_n^{(e_i)}} \frac{\partial \psi}{\partial x_n^{(e_i)}} dx_n^{(e_i)} dt.$$

Thus, the existence and uniqueness of the solution p_0 of the projection of problem (6) onto the subspace $\widehat{L}_{per}^2(0, T; H_{0,N}^1(\mathcal{B}))$ follows from the Lax-Milgram lemma.

Let us check that (6) still remains to hold for any test function from $\widehat{L}_{per}^2(0, T; H^1(\mathcal{B}))$ (i.e., without assuming test functions to be zero at the vertex O_N). Indeed, if a test function $\psi \in \widehat{L}_{per}^2(0, T; H^1(\mathcal{B}))$, then (6) holds for the function $\psi(x, t) - \psi(O_N, t)$. But, obviously, $\alpha_T(p, \psi(O_N, t)) = 0$ and, due to the compatibility condition (5), $\beta_T(\psi(O_N, t)) =$

0. If a test function belongs to $L^2_{per}(0, T; H^1(\mathcal{B}))$ (i.e., has nonzero mean value), we can take in (6) the function $\psi(x_n^{(e)}, t) - \langle \psi \rangle(x_n^{(e)})$. Due to periodicity in time, for any function $\eta(x_n^{(e)})$ independent of t we have $\alpha_T(p, \eta) = 0$ and $\beta_T(\eta) = 0$ (this follows by integrating by parts in the variable t). Thus, (6) is valid for every $\psi \in L^2_{per}(0, T; H^1(\mathcal{B}))$.

Consider two solutions p_1 and p_2 of problem (6). Then the difference $p_1 - p_2$ can be presented in the form

$$p_1(x, t) - p_2(x, t) = P(x, t) + \langle p_1 - p_2 \rangle(x) + (p_1(O_N, t) - p_2(O_N, t) - \langle p_1(O_N, t) - p_2(O_N, t) \rangle),$$

where P is a solution to the homogeneous problem (6) vanishing at the vertex O_N and having vanishing mean value in time. Such solution is unique, so $P = 0$. Therefore, any solution of problem (6) has a form $p(x, t) = p_0(x, t) + \phi(x) + \varphi(t)$, where $p_0 \in \widehat{L}^2(0, T; H_0^1(\mathcal{B}))$, $\phi \in H^1(\mathcal{B})$, $\varphi \in L^2_{per}(0, T)$. \square

The proved lemma gives the existence of a solution to the variational formulation (6) of problem (4) differentiated with respect to t . Now we have to return to the original (before the differentiating in time) formulation of problem (4) and to prove the existence of its solution and its regularity with respect to x .

Theorem 3. *Let the compatibility condition (5) holds. Then problem (4) admits a solution $p \in L^2(0, T; H_0^1(\mathcal{B}))$. The solution is unique up to an additive function φ from the space $L^2_{per}(0, T)$ depending only on the time variable t .*

Proof. 1. Consider an arbitrary edge e_i and take a test function having a compact support belonging to this edge. Then identity (6) reads as

$$\int_0^T \int_0^{|e_i|} \frac{\partial(\mathcal{L}^{(e_i)} p)_t}{\partial x_n^{(e_i)}} \frac{\partial \psi}{\partial x_n^{(e_i)}} dx_n^{(e_i)} dt = \int_0^T \int_0^{|e_i|} (\mathcal{F}^{(e_i)})_t \psi dx_n^{(e_i)} dt.$$

This yields that for almost all t and for any smooth test function ψ depending on x only and having the support belonging to e_i , holds the identity

$$\int_0^{|e_i|} \frac{\partial(\mathcal{L}^{(e_i)} p)_t}{\partial x_n^{(e_i)}} \frac{\partial \psi}{\partial x_n^{(e_i)}} dx_n^{(e_i)} dt = \int_0^{|e_i|} (\mathcal{F}^{(e_i)})_t \psi dx_n^{(e_i)} dt.$$

This last identity corresponds to the definition of a weak derivative for functions from $H^1(e_i)$ and means that $-(\mathcal{F}^{(e_i)})_t$ is a weak derivative of $\frac{\partial(\mathcal{L}^{(e_i)}p)_t}{\partial x_n^{(e_i)}}$, so that equation (4)₁ differentiated with respect to the time variable t holds for almost all x and t . In a similar way taking test functions having support in a ball $B(O_i, \varepsilon)$ of the radius ε and the center in O_i and letting $\varepsilon \rightarrow 0$ we get boundary conditions (4)₂ and (4)₃ differentiated with respect to the time variable for almost all t .

2. Integrating these three relations in time we see that the function p_0 from Lemma 1 satisfies problem (4) with the right hand sides

$$\int_0^t (\mathcal{F}^{(e)}(x_n^{(e)}, \theta))_\theta d\theta - \frac{\partial}{\partial x_n^{(e)}} \left(\mathcal{L}^{(e)} \frac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)}, 0) \right)$$

in the equation (4)₁ and with the right hand sides

$$\int_0^t (\Psi_l(\theta))_\theta d\theta - \sum_{e: O_l \in e} \left(\mathcal{L}^{(e)} \frac{\partial p}{\partial x_n^{(e)}} \right)(0, 0)$$

in the Kirchhoff type boundary conditions (4)₂ and (4)₃, respectively, i.e. with

$$\mathcal{F}^{(e)}(x_n^{(e)}, t) - \mathcal{F}^{(e)}(x_n^{(e)}, 0) - \frac{\partial}{\partial x_n^{(e)}} \left(\mathcal{L}^{(e)} \frac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)}, 0) \right)$$

and

$$\Psi_l(t) - \Psi_l(0) - \sum_{e: O_l \in e} \left(\mathcal{L}^{(e)} \frac{\partial p}{\partial x_n^{(e)}} \right)(0, 0).$$

3. On the other hand, problem (4) with the right hand sides

$$\mathcal{F}^{(e)}(x_n^{(e)}, 0) + \frac{\partial}{\partial x_n^{(e)}} \left(\mathcal{L}^{(e)} \frac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)}, 0) \right)$$

and

$$\Psi_l(0) + \sum_{e: O_l \in e} \left(\mathcal{L}^{(e)} \frac{\partial p}{\partial x_n^{(e)}} \right)(0, 0)$$

is equivalent to

$$\begin{aligned} -\frac{\partial}{\partial x_n^{(e)}} \left(\kappa^{(e)} \frac{\partial p}{\partial x_n^{(e)}}(x_n^{(e)}, t) \right) &= \mathcal{F}^{(e)}(x_n^{(e)}, 0) + \frac{\partial}{\partial x_n^{(e)}} \left(\mathcal{L}^{(e)} \frac{\partial p_0}{\partial x_n^{(e)}}(x_n^{(e)}, 0) \right), \quad x_n^{(e)} \in (0, |e|), \\ -\sum_{e: O_l \in e} \left(\kappa^{(e)} \frac{\partial p}{\partial x_n^{(e)}} \right)(0, t) &= \Psi_l(0) + \sum_{e: O_l \in e} \left(\mathcal{L}^{(e)} \frac{\partial p}{\partial x_n^{(e)}} \right)(0, 0), \quad l = 1, \dots, N_1, \\ -\left(\kappa^{(e)} \frac{\partial p}{\partial x_n^{(e)}} \right)(0, t) &= \Psi_l(0) + \left(\mathcal{L}^{(e)} \frac{\partial p}{\partial x_n^{(e)}} \right)(0, 0), \quad l = N_1 + 1, \dots, N. \end{aligned} \quad (8)$$

which admits a unique (up to an additive function of t) solution (see [9]). So, the solution to problem (4) is a sum of the solution to problem (6) and of the solution to problem (8). Thus, we proved the existence of solution to (4).

4. The uniqueness up to an additive function of time can be proved as follows. Consider solutions p_1 and p_2 to problem (8). Their difference $p_1 - p_2$ satisfies the homogeneous problem (6), and can be represented as the sum: $(p_1 - p_2)(x, t) = \phi(x) + \psi(t)$, where $\phi \in H^1(\mathcal{B})$, $\psi \in L^2_{per}(0, T)$, ϕ is a stationary solution to problem (6). According to Corollary 2, ϕ is a solution to the homogeneous problem (8). From [9] it follows that $\phi = const$, so $\phi + \psi$ is a function of $L^2_{per}(0, T)$. \square

Remark 1. Note that the condition (5) is a necessary and sufficient condition of existence of T -periodic in time solution of problem (4). Indeed, integrating the equation (4)₁ over \mathcal{B} we obtain condition (5).

4 Acknowledgments

This project has received funding from European Social Fund (project No 09.3.3-LMT-K-712-01-0012) under grant agreement with the Research Council of Lithuania (LMTLT).

References

- [1] H. BEIRAO DA VEIGA, On time periodic solutions of the Navier-Stokes equations in an unbounded cylindrical domains. Leray's problem for periodic flows, *Arch. Rat. Mech. Anal.*, **178**, No. 3, 2005, 301-325.
- [2] F. BLANC, O. GIPOULOUX, G. PANASENKO, A.M. ZINE, Asymptotic analysis and partial asymptotic decomposition of the domain for Stokes equation in tube structure, *Mathematical Models and Methods in Applied Sciences*, **9**, No. 9, 1999, 1351-1378.
- [3] R. BUNOIU, A. GAUDIELLO, A. LEOPARDI, Asymptotic analysis of a Bingham fluid in a thinT-like shaped structure, *J. Math. Pures Appl.*, 123, 2019, 148-166.

- [4] G.P. GALDI, A.M. ROBERTSON, The relation between flow rate and axial pressure gradient for time periodic Poiseuille flow in a pipe, *J. Math. Fluid Mech.*, **7**, supplement No. 2, 2005, 215-223.
- [5] R.JUODAGALVYTÉ, G.PANASENKO, K.PILECKAS, Time periodic Navier-Stokes equations in a thin tube structure, *Boundary Value Problems*, 2020:28, 2020, 35 pp. <https://doi.org/10.1186/s13661-020-01334-3>
- [6] G. PANASENKO, Asymptotic expansion of the solution of Navier-Stokes equation in a tube structure, *C.R. Acad. Sci. Paris*, **326**, Série IIb, 1998, 867-872.
- [7] G. PANASENKO, *Multi-scale Modeling for Structures and Composites*, Springer, Dordrecht, 2005.
- [8] G. PANASENKO, K. PILECKAS, Asymptotic analysis of the nonsteady viscous flow with a given flow rate in a thin pipe, *Applicable Analysis*, **91**, No. 3, 2012, 559-574.
- [9] G. PANASENKO, K. PILECKAS, Flows in a tube structure: equation on the graph, *Journal of Mathematical Physics*, **55**, 081505 (2014); <http://dx.doi.org/10.1063/1.4891249>
- [10] G. PANASENKO, K. PILECKAS, Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure.I. The case without boundary layer-in-time. *Nonlinear Analysis, Series A, Theory, Methods and Applications*, **122**, 2015, 125-168, <http://dx.doi.org/10.1016/j.na.2015.03.008>
- [11] G. PANASENKO, K. PILECKAS, Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure.II. General case. *Nonlinear Analysis, Series A, Theory, Methods and Applications*, **125**, 2015, 582-607, <http://dx.doi.org/10.1016/j.na.2015.05.018>