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Mohammad M. GHABRIES[a,b,*], Hassane ABBAS[c], Bassam MOURAD[c], Abdallah ASSI[a]

[a] *LAREMA, Faculte des Sciences-Departement de mathematiques, Angers, France*
[b] *KALMA, Department of Mathematics, Faculty of Science, Lebanese University, Beirut, Lebanon*
[c] *Department of Mathematics, Faculty of Science, Lebanese University, Beirut, Lebanon*

**Abstract**

The main goal of this paper is to prove the following determinantal inequality:

$$\det(A^k + |B^{\frac{s}{2}} A^{\frac{s}{2}}|^{\frac{2t}{s}}) \leq \det(A^k + A^t B^t) \leq \det(A^k + |A^{\frac{s}{2}} B^{\frac{s}{2}}|^{\frac{2t}{s}})$$

for any positive semi-definite matrices $A$ and $B$, and for all $0 \leq t \leq s \leq k$. It generalizes several known determinantal inequalities, and one main consequence of it confirms Lin's conjecture which states that for positive semi-definite matrices $A$ and $B$,

$$\det(A^2 + A^t B^t) \leq \det(A^2 + |AB|^t) \quad \text{for} \quad 0 \leq t \leq 2.$$

We conclude with another related determinantal inequality.

*Keywords:* Determinantal inequalities; Hermitian matrix; Positive semi-definite matrix; Log-majorization; Eigenvalues
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## 1. Introduction

Let $A$ and $B$ be two $n \times n$ positive semi-definite matrices with $n \geq 1$. Audenaert [3] proved the following determinantal inequality:

$$\det(A^2 + |BA|) \leq \det(A^2 + AB) \tag{1}$$

that answers a question arising in the study of interpolation methods for image processing in diffusion tensor imaging when comparing geodesics induced by different metrics. Recently, Lin [7] generalized Audenaert's result by proving

$$\det(A^2 + |BA|^t) \leq \det(A^2 + A^t B^t), \qquad 0 \leq t \leq 2. \tag{2}$$

In the same paper, the author introduced the following conjecture which is a complement of (2).

**Conjecture 1.1.** *Let $A$ and $B$ be $n \times n$ positive semi-definite matrices. Then*

$$\det(A^2 + A^t B^t) \leq \det(A^2 + |AB|^t), \qquad 0 \leq t \leq 2. \tag{3}$$

*Corresponding author
*URL:* mahdi.ghabries@gmail.com (Mohammad M. GHABRIES)

It is worthy to note that this conjecture has been proved in several special cases but remains open for the general case. In particular, for the special cases $t = 1$ and $t = 2$, Inequality (3) was proved in Lin [7]. Moreover, for $0 \le t \le \dfrac{4}{3}$ the authors confirmed this conjecture in the very recent paper [5]. Some related results can also be found in [2].

Our objective of this note is to prove the following determinantal inequality which obviously generalizes Inequalities (1) and (2) as well as Inequality (3) and hence it gives an affirmative answer to Conjecture 1.1.

**Theorem 1.1.** *Let $A$ and $B$ be two positive semi-definite matrices. Then, for all $0 \le t \le s \le k$,*

$$\det(A^k + |B^{\frac{s}{2}} A^{\frac{s}{2}}|^{\frac{2t}{s}}) \le \det(A^k + A^t B^t) \le \det(A^k + |A^{\frac{s}{2}} B^{\frac{s}{2}}|^{\frac{2t}{s}}).$$

Clearly, it suffices to take the case $k = s = 2$ in order to see that the preceding theorem gives a positive answer to Conjecture 1.1.

To proceed, we first fix some notation. Let $M_n$ be the space of $n \times n$ complex matrices where its identity matrix is denoted by $I_n$. The modulus of a complex matrix $X$ is defined as $|X| = (X^*X)^{1/2}$. For Hermitian matrices $X, Y \in M_n$, we write $X \ge Y$ if $X - Y$ is positive semi-definite matrix. The spectrum of a matrix $X$ is the multiset of the eigenvalues of $X$ denoted by $Sp(X)$. If the eigenvalues $\lambda_1(X), \lambda_2(X), \dots, \lambda_n(X)$ of a matrix $X$ are real, then we will always assume that they are arranged in decreasing order, that is

$$\lambda_1(X) \ge \lambda_2(X) \ge \cdots \ge \lambda_n(X).$$

For a Hermitian matrix $X \in M_n$, we shall denote

$$\lambda(X) = (\lambda_1(X), \lambda_2(X), \dots, \lambda_n(X))^t$$

which is clearly a real vector of order $n$.

A norm is said to be *unitarily invariant* norm if for all $A \in \mathbb{M}_n$, we have $||UAV|| = ||A||$ for all $U, V \in \mathbb{M}_n$ unitary matrices and it is denoted by $|||\cdot|||$.

Majorization relations are great tools for deriving determinantal inequalities, see [8, Chapter 2] for details on this subject. If $\lambda(A), \lambda(B) \in \mathbb{R}^n$, then by $\lambda(A) \prec_{wlog} \lambda(B)$, we mean that $\lambda(A)$ is *weakly log-majorized* by $\lambda(B)$, that is

$$\prod_{i=1}^{k} \lambda_i(A) \le \prod_{i=1}^{k} \lambda_i(B) \qquad \text{for all } k = 1, 2, ..., n. \tag{4}$$

In addition, we shall write $\lambda(A) \prec_{log} \lambda(B)$ and we will say that $\lambda(A)$ is *log-majorized* by $\lambda(B)$ if (4) is true and equality holds for $k = n$.

The rest of the paper is organized as follows. In the next section, we present a proof of Theorem 1.1. In the final section, we present a determinantal inequality which is motivated by some recent work in [6]. More explicitly, we shall show that with the same setting, the inequalities in Theorem 1.1 are reversed when replacing $A^k$ by $A^{-k}$ provided that $A$ is invertible.

## 2. Proof of Theorem 1.1

We shall start with the following auxiliary results. The first one is the well known Lowner-Heinz inequality.

**Lemma 2.1.** *Let $X$ and $Y$ be two positive semi-definite matrices such that $X \le Y$. Then*

$$X^r \le Y^r, \qquad 0 \le r \le 1.$$

Next, we need the following result which can be found in [4, Theorem IX.2.10].

**Lemma 2.2.** *Let $X$ and $Y$ be two positive semi-definite matrices. Then, for every unitarily invariant norm, we have*

$$|||X^pY^pX^p||| \leq |||(XYX)^p||| \quad \text{for } 0 \leq p \leq 1.$$

*In particular,*

$$\lambda_1(X^pY^pX^p) \leq \lambda_1((XYX)^p) \quad \text{for } 0 \leq p \leq 1.$$

In order to prove our main result, the following lemma is essential and its proof can be found in [5].

**Lemma 2.3.** *Let $Y$ be a positive definite matrix and $X$ be a Hermitian matrix. Then for all $p, q \in [0, \infty[$*

$$\lambda(XY^pXY^{-q}) \succ_{wlog} \lambda(X^2Y^{p-q}).$$

The next lemma is also needed for our purposes and it shows a close connection between log-majorization and determinantal inequalities and can be found in [7, (P2)].

**Lemma 2.4.** *Let $X$ and $Y$ be two matrices in $M_n$. If $\lambda(X), \lambda(Y)$ are in $\mathbb{R}^n_+$ such that $\lambda(X) \prec_{wlog} \lambda(Y)$, then*

$$\det(I_n + X) \leq \det(I_n + Y).$$

For positive definite matrices $A$ and $B$, the authors in [1] proved the following log majorization inequality:

$$\lambda(A^{\frac{k't'}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{t'}A^{\frac{k't'}{2}}) \prec_{wlog} \lambda(A^{k't'-t'}B^{t'}), \qquad 0 \leq t' \leq 1 \leq k'. \tag{5}$$

By taking $A = A^s,\ B = B^s,\ k' = \frac{k}{t}$ and $t' = \frac{t}{s}$ in (5), we obtain

$$\lambda(A^{\frac{k}{2}}(A^{-\frac{s}{2}}B^sA^{-\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \prec_{wlog} \lambda(A^{k-t}B^t), \qquad 0 \leq t \leq s \text{ and } k \geq t. \tag{6}$$

Now we are in a position to present our first result which gives a further generalization of the determinantal inequalities (1) and (2).

**Theorem 2.1.** *Let $A$ and $B$ be two positive semi-definite matrices. Then, for all $0 \leq t \leq s$ and $k \geq t$,*

$$\det(A^k + |B^{\frac{s}{2}}A^{\frac{s}{2}}|^{\frac{2t}{s}}) \leq \det(A^k + A^tB^t).$$

*Proof.* Without loss of generality, we shall assume that $A$ and $B$ are positive definite matrices as the general case would then follow by a standard continuity argument. Replacing $A$ with $A^{-1}$ in (6) gives

$$\lambda(A^{-\frac{k}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{-\frac{k}{2}}) \prec_{wlog} \lambda(A^{t-k}B^t), \qquad 0 \leq t \leq s \text{ and } k \geq t. \tag{7}$$

Applying Lemma 2.4 on (7) yields

$$\det(I_n + A^{-\frac{k}{2}}(A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{t}{s}}A^{-\frac{k}{2}}) \leq \det(I_n + A^{t-k}B^t), \quad 0 \leq t \leq s \text{ and } k \geq t.$$

Multiplying both sides with $\det(A^k) > 0$ implies the desired result. $\square$

Before presenting the proof of Theorem 1.1, we shall establish the following log-majorization relation.

**Lemma 2.5.** *Let $A, B$ be two positive definite matrices. Then for all $0 \leq t \leq s \leq k$,*

$$\lambda(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \succ_{wlog} \lambda(A^{k-t}B^t).$$

*Proof.* As usual in such situation, by a standard anti-symmetric tensor product argument, it is enough to prove that for all $0 \leq t \leq s \leq k$

$$\lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \geq \lambda_1(A^{k-t}B^t).$$

Without loss of generality, we assume

$$\lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) = 1.$$

This is equivalent to

$$A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}} \leq I_n$$

which in turn gives

$$(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}} \leq A^{-k}. \tag{8}$$

Now obviously proving our claim is equivalent to showing that

$$\lambda_1(A^{k-t}B^t) \leq 1.$$

Clearly, inequality (8) gives

$$A^k \leq (B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}})^{\frac{t}{s}}.$$

Now using Lemma 2.1 for $0 \leq \frac{k-t}{k} \leq 1$, we obtain

$$A^{k-t} \leq (B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}})^{\frac{t(k-t)}{sk}}. \tag{9}$$

Thus, we can write

$$\begin{aligned}
\lambda_1(A^{k-t}B^t) &= \lambda_1(B^{\frac{t}{2}}A^{k-t}B^{\frac{t}{2}}) \\
&\leq \lambda_1(B^{\frac{t}{2}}(B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}})^{\frac{t(k-t)}{sk}}B^{\frac{t}{2}}) && \text{(Using (9))} \\
&\leq \lambda_1(B^{\frac{s}{2}}(B^{-\frac{s}{2}}A^sB^{-\frac{s}{2}})^{\frac{k-t}{k}}B^{\frac{s}{2}})^{\frac{t}{s}} && \text{(Using Lemma 2.2 for } 0 \leq p = \frac{t}{s} \leq 1) \\
&= \lambda_1(B^{\frac{s}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{k}-1}B^{\frac{s}{2}})^{\frac{t}{s}} \\
&\leq \lambda_1(B^{\frac{s}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{-1}B^{\frac{s}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{k}})^{\frac{t}{s}} && \text{(Using Lemma 2.3 for } p = \frac{t}{k} \text{ and } q = 1) \\
&= \lambda_1(A^{\frac{s}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{k}}A^{\frac{s}{2}})^{\frac{t}{s}} \\
&\leq \lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^{-s}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}})^{\frac{t}{k}} && \text{(Using Lemma 2.2 for } 0 \leq p = \frac{s}{k} \leq 1) \\
&= 1.
\end{aligned}$$

□

Now we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1:** In order to prove the right inequality, we shall assume again that $A$ and $B$ are positive definite matrices as the general case can be obtained by a continuity argument.

With this in mind, replacing now $A$ with $A^{-1}$ in Lemma 2.5 gives

$$\lambda(A^{-\frac{k}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{-\frac{k}{2}}) \succ_{wlog} \lambda(A^{t-k}B^t), \quad 0 \leq t \leq s \leq k. \tag{10}$$

Applying Lemma 2.4 to (10) yields

$$\det(I_n + A^{-k/2}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{-k/2}) \geq \det(I_n + A^{t-k}B^t), \quad 0 \leq t \leq s \leq k.$$

Finally, multiplying both sides with $\det(A^k) > 0$ implies

$$\det(A^k + |A^{\frac{s}{2}}B^{\frac{s}{2}}|^{\frac{2t}{s}}) \geq \det(A^k + A^tB^t), \quad 0 \leq t \leq s \leq k.$$

In order to complete the proof, it suffices to see that the left inequality is a particular case of Theorem 2.1. $\square$

## 3. Yet another related Determinantal Inequalities

Motivated by some recent work in [6], we will show a determinantal inequality related to Theorem 1.1. More explicitly, our main goal here is to show that with the same setting, the inequalities in Theorem 1.1 are reversed when replacing $A^k$ by $A^{-k}$ provided that $A$ is invertible.

The starting point here is the following lemma.

**Lemma 3.1.** *Let $A$ and $B$ be two positive definite matrices. Then for all $0 \leq t \leq s \leq k$,*

$$\lambda(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \prec_{wlog} \lambda(A^{k+t}B^t).$$

*Proof.* As in similar situations, it is enough to prove that for all $0 \leq t \leq s \leq k$,

$$\lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \leq \lambda_1(A^{k+t}B^t).$$

Again, without loss of generality, we shall assume that $\lambda_1(A^{k+t}B^t) = 1$. As mentioned earlier, our task now is equivalent to proving

$$B^{\frac{t}{2}}A^{k+t}B^{\frac{t}{2}} \leq I_n,$$

which is in turn equivalent to showing that

$$A^{k+t} \leq B^{-t}.$$

By appealing to Lemma 2.1 for $0 \leq \frac{k}{k+t} \leq 1$, we obtain

$$A^k \leq B^{-\frac{tk}{k+t}}. \tag{11}$$

Then, using again a power $\frac{s}{k}$ where $0 \leq \frac{s}{k} \leq 1$ on (11), we get

$$A^s \leq B^{-\frac{ts}{k+t}}. \tag{12}$$

Now, we can write

$$\begin{aligned}
\lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) &\leq \lambda_1(A^{\frac{k}{2}}(B^{\frac{s}{2}}B^{-\frac{st}{k+t}}B^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \qquad \text{(Using (12))} \\
&= \lambda_1(A^{\frac{k}{2}}B^{\frac{tk}{k+t}}A^{\frac{k}{2}}) \\
&= \lambda_1(B^{\frac{tk}{2(k+t)}}A^kB^{\frac{tk}{2(k+t)}}) \\
&\leq \lambda_1(B^{\frac{tk}{2(k+t)}}B^{-\frac{tk}{(k+t)}}B^{\frac{tk}{2(k+t)}}) \qquad \text{(Using (11))} \\
&= \lambda_1(I_n) = 1.
\end{aligned}$$

As before, applying anti-symmetric tensor product argument gives for all $0 \le t \le s \le k$,

$$\lambda(A^{\frac{k}{2}}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{\frac{k}{2}}) \prec_{wlog} \lambda(A^{k+t}B^t).$$

□

Next, we have the following lemma which is a specialization of Proposition 4.1 in [6].

**Lemma 3.2.** *Let $A$ and $B$ be two $n \times n$ complex matrices such that $A$ is positive definite and $B$ is positive semi-definite matrix. Then, for all $0 \le t \le s$, and for all $k \ge 0$, it holds that*

$$\det(A^{-k} + A^tB^t) \le \det\left(A^{-k} + \left(A^{\frac{s}{2}}B^sA^{\frac{s}{2}}\right)^{\frac{t}{s}}\right).$$

As a result, we have the following theorem.

**Theorem 3.1.** *Let $A$ and $B$ be two $n \times n$ complex matrices such that $A$ is positive definite and $B$ is positive semi-definite matrix. Then for all $0 \le t \le s \le k$, it holds that*

$$\det(A^{-k} + |A^{\frac{s}{2}}B^{\frac{s}{2}}|^{\frac{2t}{s}}) \le \det(A^{-k} + A^tB^t) \le \det(A^{-k} + |B^{\frac{s}{2}}A^{\frac{s}{2}}|^{\frac{2t}{s}}).$$

*Proof.* For the left inequality, as usual we shall assume that $B$ is a positive definite matrix. Applying Lemma 2.4 to the majorization inequality of Lemma 3.1 gives

$$\det(I_n + A^{k/2}(B^{\frac{s}{2}}A^sB^{\frac{s}{2}})^{\frac{t}{s}}A^{k/2}) \le \det(I_n + A^{t+k}B^t), \quad 0 \le t \le s \le k.$$

Multiplying both sides with $\det(A^{-k}) > 0$ yields

$$\det(A^{-k} + |A^{\frac{s}{2}}B^{\frac{s}{2}}|^{\frac{2t}{s}}) \le \det(A^{-k} + A^tB^t), \qquad 0 \le t \le s \le k.$$

Finally, to complete the proof it is enough to notice that the right inequality is valid in view of the preceding lemma. □

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## Declaration of Competing Interest

The authors declare that they have no competing interests.

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