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Automorphism groups of universal diversities

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Abstract

We prove that the automorphism group of the Urysohn diversity is a universal Polish group. Furthermore we show that the automorphism group of the rational Urysohn diversity has ample generics, a dense conjugacy class and that it embeds densely into the automorphism group of the (full) Urysohn diversity. It follows that this latter group also has a dense conjugacy class.

Keywords: Diversities, Polish groups, Fraïssé structures, ample generics, universality, dense conjugacy class.

1 Introduction

Diversities were introduced by Bryant and Tupper in [6] and further developed in [7] in order to generalise applications of metric space theory to combinatorial optimisation and graph theory to the hypergraph setting. The idea is very simple: instead of only assigning real numbers to pairs of elements, a diversity assigns a real number to every finite subset of the space. This turns out to generalise metric spaces quite nicely, and in [6] and [7] the authors prove diversity versions of a number of results concerning or using metric spaces. The term *diversity* comes from a special example of a diversity that appears in *phylogenetics* and *ecological* diversities demonstrating the broad variety of applications of diversities and of mathematics in general of course. The precise definition of a diversity is as follows:

Definition 1.1. *A* **diversity** *is a set X equipped with a map δ, the* **diversity map***, defined on the finite subsets of X to $\mathbb{R}$ such that for all finite $A, B, C \subseteq X$ we have*

(D1) $\delta(A) \geq 0$ *and* $\delta(A) = 0$ *if and only if* $|A| \leq 1$.

(D2) If $B \neq \emptyset$ *then* $\delta(A \cup B) + \delta(B \cup C) \geq \delta(A \cup C)$.

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As an abuse of language we will follow [4, 5] and from time to time refer to the diversity map as a diversity as well. Hopefully this confusion of names will not cause confusion for the reader.

The following observation is useful and is easy to verify.

Lemma 1.2. *(D2) holds for δ if and only if the following two conditions hold:*

(D2') **Monotonicity**, *i.e. if $A \subseteq B$ then $\delta(A) \leq \delta(B)$.*
(D2") **Connected sublinearity**, *i.e. if $A \cap B \neq \emptyset$ then*

$$\delta(A \cup B) \leq \delta(A) + \delta(B).$$

Another general observation to make is that any diversity is automatically a metric space since the map $d(a, b) = \delta(\{a, b\})$ defines a metric. We refer to this metric as the **induced metric**. A diversity is **complete**, respectively **separable**, if the induced metric is complete, respectively separable. A bijective map $f\colon X \to Y$ between two diversities that preserves the values of the diversity map will be called an **isoversity**. If $Y = X$ we will call f an **autoversity** or simply an automorphism of X. The group of all autoversities of a diversity X is denoted by $\mathrm{Aut}(X)$. As usual for automorphism groups we consider $\mathrm{Aut}(X)$ as a topological group using the topology of pointwise convergence on X with the induced metric.

An important observation to make is that for each $n \in \mathbb{N}$ the diversity map induces a uniformly continuous map δ^n on X^n given by

$$\delta^n(x_0, \ldots, x_{n-1}) = \delta(\{x_0, \ldots, x_n\}).$$

We will use this fact several times, so we include it here for the convenience of the reader. The proof can be found both in [4] and in [5, Lemma 21].

Lemma 1.3 ([5, Lemma 21])**.** *Let (X, δ) be a diversity and for $n \in \mathbb{N}$ let δ^n denote the map on X^n that δ induces. Then δ^n is 1-Lipschitz in each argument. It follows that for all $\bar{x}, \bar{y} \in X^n$ we have*

$$|\delta^n(\bar{x}) - \delta^n(\bar{y})| \leq \sum \delta(x_i, y_i).$$

In particular, δ^n is uniformly continuous.

It follows that if (X, δ) is a separable complete diversity, then $\mathrm{Aut}(X)$ is a Polish group, since it is a closed subgroup of the isometry group of the induced metric space.

To ease our notation we will hardly discern between the maps δ^n and the diversity map itself. Hence, we will from time to time write $\delta(\bar{a})$ for an ordered tuple $\bar{a} = (a_0, \ldots, a_n)$ instead of writing $\delta(\{a_0, \ldots, a_n\})$. It will be clear from the context what is meant, so this causes no confusion.

The interest in diversities from the viewpoint of Polish group theory began with the paper [4]. There the authors construct a diversity analogue of the Urysohn metric space by adapting Katětov's construction to the diversity setting. They call the resulting space the *Urysohn diversity*, denoted by $\mathbb{U}$, and show, among other things, that the metric space it induces is the Urysohn metric space. The existence of such a universal object among diversities gives rise to a plethora of questions concerning its automorphism group, $\mathrm{Aut}(\mathbb{U})$, since this group is virtually unstudied. The first main result of this paper is the following (cf. Theorem 2.11 below):

Theorem 1. $\mathrm{Aut}(\mathbb{U})$ *is a universal Polish group.*

The proof of this theorem follows Uspenskij's proof of the corresponding fact for the Urysohn metric space in [31]. It uses the Katětov-like construction of the Urysohn diversity done in [4]. We have therefore, for the convenience of the reader, included the main ingredients of this construction.

The other main results of the paper are the following (cf. Corollary 4.9 and Theorem below 5.11):

Theorem 2. $\mathrm{Aut}(\mathbb{U})$ *has a dense conjugacy class.*

Theorem 3. *The automorphism group of the the rational Urysohn diversity has ample generics.*

Here a *rational* diversity simply means that the diversity map only takes rational values. We will denote the rational Urysohn diversity by $\mathbb{U}_{\mathbb{Q}}$. Of course the first thing we need to do is to show that $\mathbb{U}_{\mathbb{Q}}$ actually exists. We do this by showing that the class of all finite rational diversities is a *Fraïssé class*. It follows that this class has a *Fraïssé limit* and this limit is the rational Urysohn diversity. Moreover, we show that the completion of $\mathbb{U}_{\mathbb{Q}}$ is the Urysohn diversity, thus providing a new proof of the existence of $\mathbb{U}$. For the convenience of the reader we have included a short introduction to Fraïssé theory in Section 3 where we also define a useful amalgamation of diversities that generalises the free amalgamation of metric spaces.

Once the existence of $\mathbb{U}_{\mathbb{Q}}$ is established, it is easy to show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has a dense conjugacy class by applying a theorem of Kechris and Rosendal from [19]. Furthermore, we will show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ embeds densely into $\mathrm{Aut}(\mathbb{U})$ (cf. Theorem 4.8 below) from which Theorem 2 follows immediately.

Afterwards we show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has *ample generics*. Ample generics is a property with many strong implications such as the *automatic continuity property*, the *small index property* and the fact that the group cannot be the union of countably many non-open subgroups. All of these notions will be explained below. Theorem 3 follows from an extension theorem for diversities inspired by a result of Solecki in [27] and another theorem of Kechris and Rosendal from [19].

2 Universality of Aut($\mathbb{U}$)

The first thing we need is to introduce some terminology and definitions from [4]. In that paper the authors construct the Urysohn diversity by adapting Katětov's construction of the ditto metric space to the diversity setting. Following [31], we shall use this construction to prove that Aut($\mathbb{U}$) is a universal Polish group. The first thing we need to introduce is the diversity analogue of Katětov functions. These are the so-called *admissible* maps. First we let $[X]^{<\omega}$ denote the finite subsets of X.

Definition 2.1. *Let (X, δ_X) be a diversity. A map $f\colon [X]^{<\omega} \to \mathbb{R}$ is* **admissible** *if the following holds:*

(i) $f(\emptyset) = 0$,
(ii) $f(A) \geq \delta_X(A)$ *for every* A,
(iii) $f(A \cup C) + \delta_X(B \cup C) \geq f(A \cup B)$ *for all* A, B, C *with* $C \neq \emptyset$,
(iv) $f(A) + f(B) \geq f(A \cup B)$.

The set of all admissible maps on (X, δ_X) is denoted $E(X)$.

The reason why these maps are called admissible is because they define diversity extensions as the lemma below tells us.

Lemma 2.2 ([4, Lemma 2])**.** *Let (X, δ) be a diversity and let $f\colon [X]^{<\omega} \to \mathbb{R}$. Then $f \in E(X)$ if and only if for some y the map $\widehat{\delta}\colon [X \cup \{y\}]^{<\omega} \to \mathbb{R}$ given by*

$$\widehat{\delta}(A) = \delta(A), \quad \widehat{\delta}(A \cup \{y\}) = f(A)$$

for $A \subseteq X$ finite, defines a diversity map on $X \cup \{y\}$.

Similarly to the metric setting, we can define a diversity map on the set of admissible maps. This is defined as follows.

Definition 2.3 ([4, Page 5])**.** *Let (X, δ) be a diversity. On $[E(X)]^{<\omega}$ we define a map $\widehat{\delta}$ by*

$$\widehat{\delta}(\{f_1, \ldots, f_n\}) = \max_{j \leq k} \sup \Big\{ f_j\big(\bigcup_{i \neq j} A_i\big) - \sum_{i \neq j} f_i(A_i) \;\Big|\; A_i \subseteq X \text{ finite} \Big\}$$

whenever $n \geq 2$ and $\widehat{\delta}(f) = \widehat{\delta}(\emptyset) = 0$.

Observe that

$$\widehat{\delta}(f_1, f_2) = \sup_{B \text{ finite}} |f_1(B) - f_2(B)|.$$

Moreover, as the notation suggests, $\widehat{\delta}$ is a diversity map on $E(X)$.

Theorem 2.4 ([4, Theorem 3]). *Let (X, δ) be a diversity. Then $(E(X), \widehat{\delta})$ is a diversity and (X, δ) embeds into $(E(X), \widehat{\delta})$ via the map $x \mapsto \kappa_x$ where $\kappa_x(A) = \delta(A \cup \{x\})$.*

Unfortunately, just like in the metric setting, $E(X)$ need not be separable even if X is. Therefore we need to restrict ourselves to a subspace of $E(X)$ to maintain separability. This is the subspace of the *finitely supported admissible maps.* These are defined as follows:

Definition 2.5. *Let (X, δ) be a diversity and let $S \subseteq X$ be any subset. If $f \in E(S)$ then we define the extension of f to X by*

$$f_S^X(A) = \inf \Big\{ f(B) + \sum_{b \in B} \delta(A_b \cup \{b\}) \;\Big|\; B \subseteq S \text{ finite}, \bigcup_{b \in B} A_b = A \Big\},$$

where $A \subseteq X$ is finite. We say that S is the **support** *of f_S^X.*

The set of all **finitely supported admissible maps** *on X is the set of all those $h \in E(X)$ such that for some finite $S \subseteq X$ and some $f \in E(S)$ we have $h = f_S^X$. This set will be denoted by $E(X, \omega)$.*

Of course one needs to check that the extension map f_S^X is in fact admissible. We refer the reader to [4, Lemma 6] for the details. It is also easy to check that κ_x is supported on $\{x\}$ for any $x \in X$ and hence that X embeds into $E(X, \omega)$. Therefore $(E(X, \omega), \widehat{\delta})$ is a diversity extension of X. Moreover, $E(X, \omega)$ is separable.

Theorem 2.6 ([4, Theorem 9]). *Let (X, δ) be a separable diversity. Then $(E(X, \omega), \widehat{\delta})$ is a separable diversity as well.*

One can then iterate this construction and obtain a *Katětov tower* on a given diversity X consisting of a sequence (X_n, δ_n) where X_i embeds into X_{i+1}. The union X_ω of all of these diversities turns out to have an analogue of the extension property for metric spaces that characterises the Urysohn diversity. This extension property is defined as follows:

Definition 2.7. *A diversity (X, δ_X) has the* **approximate extension property** *if for any finite subset $F \subseteq X$, any admissible map f defined on F and any $\varepsilon > 0$, there is $x \in X$ such that $|f(A) - \delta_X(A \cup \{x\})| \leq \varepsilon$ for $A \subseteq F$.*

If the above holds for $\varepsilon = 0$, (X, δ_X) has the **extension property**.

Just like in the metric setting, it turns out that complete diversities with the approximate extension property actually has the extension property. Moreover, the completion of any separable diversity with the approximate extension property has the approximate extension property. Therefore we have:

Proposition 2.8 ([4, Lemmas 16 and 17]). *Suppose (X, δ_X) is a separable diversity with the approximate extension property. Then its completion has the extension property.*

Furthermore, as mentioned, this property characterises the Urysohn diversity, meaning that any two *Polish* diversities, i.e. with complete and separable induced metrics, that have the extension property are isomorphic. This is one of the main results of [4].

Theorem 2.9 ([4, Theorems 14 and 22])**.** *Any two Polish diversities both having the extension property are isomorphic. In particular, any Polish diversity with the extension property is isomorphic to the Urysohn diversity.*

With these preliminaries established we move on to show that $\mathrm{Aut}(\mathbb{U})$ is a universal Polish group. The strategy to show this is the following: any Polish group G can be embedded into the automorphism group of a separable diversity (X, δ_X). Denote the diversity Katětov tower on X by X_ω. Then $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(X_\omega)$, which in turn embeds into $\mathrm{Aut}(\mathbb{U})$ because the completion of X_ω is isomorphic to $\mathbb{U}$. Moreover, these embeddings are all continuous with continuous inverses. Below we elaborate each of these steps. First, a lemma:

Lemma 2.10. *Let (X, δ_X) be a separable diversity and let $X_1 := E(X, \omega)$ denote the diversity of admissible maps on X with finite support. Then $\mathrm{Aut}(X)$ embeds as a topological group into $\mathrm{Aut}(X_1)$.*

Proof. Let $\Phi\colon \mathrm{Aut}(X) \to \mathrm{Aut}(X_1)$ be the map defined by $\Phi(g)(f_S^X) = f'^X_{g(S)}$ where $f'(g(A)) = f(A)$ for $A \subseteq S$. It is straightforward to check that $\Phi(g)$ is a bijection of X_1 extending g. Moreover, we note that

$$\Phi(g)(f_S^X)(A) = f_S^X(g^{-1}A) \tag{1}$$

for any finite $A \subseteq X$. Using this it is straightforward to verify that $\Phi(g)$ is an automorphism of X_1 and that Φ is injective. Furthermore, continuity of Φ follows either from Pettis' theorem (cf. [23]) or simply by a direct argument using (1). Finally, continuity of the inverse of Φ can be seen as follows:

Suppose $(\Phi(g_n))_n$ is a sequence converging to $\Phi(g)$ and let $x \in X$ be given. We must show that $g_n(x) \to g(x)$. For this, let $\kappa_x \in X_1$ denote the image of x under the embedding of X into X_1. Then $\Phi(g_n)(\kappa_x) \to \Phi(g)(\kappa_x)$ which means that

$$\sup_{B \text{ finite}} |\Phi(g_n)(\kappa_x)(B) - \Phi(g)(\kappa_x)(B)| \to 0.$$

In particular, this is true for $B = \{gx\}$. Therefore we have

$$\begin{aligned} |\Phi(g_n)(\kappa_x)(\{gx\}) - \Phi(g)(\kappa_x)(\{gx\})| &= |\delta(\{g_n^{-1}gx, x\}) - \delta(\{g^{-1}gx, x\})| \\ &= \delta(\{gx, g_n x\}) \to 0, \end{aligned}$$

where we have used that $\Phi(h)(\kappa_x) = \kappa_{hx}$ for any $h \in \mathrm{Aut}(X)$, which easily follows from (1) above. We conclude that $g_n \to g$ in $\mathrm{Aut}(X)$. $\square$

With this lemma we can now show that $\mathrm{Aut}(\mathbb{U})$ is a universal Polish group.

Theorem 2.11. $\mathrm{Aut}(\mathbb{U})$ *is a universal Polish group.*

Proof. First, any Polish group G can be embedded into the isometry group of its left completion (G_L, d_L) equipped with a left-invariant metric d_L (cf. [24] for details on completions of Polish groups). We turn G_L into a diversity by using the diameter diversity, denoted here by δ_L, associated to d_L, i.e. $\delta_L(A)$ is simply the diameter of A. Then $\mathrm{Aut}(G_L, \delta_L)$ is still just $\mathrm{Iso}(G_L, d_L)$ so G embeds into this group.

Given any separable diversity X, we let X_1 denote $E(X, \omega)$ and for any $n \in \mathbb{N}$ we let X_n denote $E(X_{n-1}, \omega)$. By X_ω we denote the union $\bigcup X_n$. In the lemma above we saw that $\mathrm{Aut}(X_i)$ embeds into $\mathrm{Aut}(X_{i+1})$ for every i. Hence we obtain a chain of embeddings

$$\mathrm{Aut}(X) \xhookrightarrow{\varphi_0} \mathrm{Aut}(X_1) \xhookrightarrow{\varphi_1} \mathrm{Aut}(X_2) \xhookrightarrow{\varphi_2} \dots,$$

where $\varphi_i(g)$ extends $g \in \mathrm{Aut}(X_i)$. It follows that we obtain a map

$$\Phi\colon\ \mathrm{Aut}(X) \to \mathrm{Aut}(X_\omega),$$

where $\Phi(g) \in \mathrm{Aut}(X_\omega)$ is defined by

$$\Phi(g)(a) = \Phi_n(g)(a),$$

where Φ_n denotes $\varphi_{n-1} \circ \dots \circ \varphi_0$ for $n \in \mathbb{N}$ such that $a \in X_n$ for $a \in X_\omega$. Note that this is well defined since if $a \in X_m$ as well for $m < n$ we must have $\Phi_m(g)(a) = \Phi_n(g)(a)$ because

$$\begin{aligned}\Phi_n(g)(a) &= (\varphi_{n-1} \circ \dots \circ \varphi_0)(g)(a)\\ &= (\varphi_{n-1} \circ \dots \circ \varphi_m \circ \Phi_m)(g)(a)\\ &= \Phi_m(g)(a).\end{aligned}$$

It is easy to check that Φ is a topological group embedding and therefore it follows that $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(X_\omega)$.

Finally, by [4, Theorem 19], the completion of X_ω is isomorphic to $\mathbb{U}$. Therefore, it follows from uniform continuity of δ (cf. Lemma 1.3) that $\mathrm{Aut}(X_\omega)$ embeds into $\mathrm{Aut}(\mathbb{U})$.

In conclusion, we have seen that given any Polish group G, we can embed G into $\mathrm{Aut}(G_L, \delta_L)$, which in turn may be embedded into $\mathrm{Aut}(\mathbb{U})$ using the construction above. Hence, $\mathrm{Aut}(\mathbb{U})$ is a universal Polish group, which was what we wanted. $\square$

3 Fraïssé theory

In this section we briefly recall the Fraïssé theory that we will need to construct the rational Urysohn diversity as the Fraïssé limit of the class of all finite rational diversities. We will also define a useful *free* amalgamation of diversities, that generalises the usual free amalgamation of metric spaces.

First let us fix some notation. Given two structures A and B in some signature $\mathcal{L}$, we denote by $A \preceq B$ that A **embeds** into B, i.e. that there is an injective map $f\colon A \to B$ that preserves the structure on A. Fraïssé classes for relational signatures are then defined as follows:

Definition 3.1. *Let $\mathcal{L}$ be a countable relational signature for a first-order language and let $\mathcal{K}$ be a class of finite $\mathcal{L}$-structures. Then $\mathcal{K}$ is a* **Fraïssé class** *if it has the following properties:*

(*i*) *(HP) $\mathcal{K}$ is* **hereditary***, i.e. if $B \in \mathcal{K}$ and $A \preceq B$ then $A \in \mathcal{K}$.*

(*ii*) *(JEP) $\mathcal{K}$ has the* **joint embedding property***, i.e. if $A, B \in \mathcal{K}$ then there is some $C \in \mathcal{K}$ such that $A, B \preceq C$.*

(*iii*) *(AP) $\mathcal{K}$ has the* **amalgamation property***, i.e. if $A, B, C \in \mathcal{K}$ and $f\colon A \to B$ and $g\colon A \to C$ are embeddings, then there is $D \in \mathcal{K}$ and embeddings $h_B\colon B \to D$ and $h_C\colon C \to D$ such that $h_B \circ f = h_C \circ g$. In diagram form:*

$$\begin{tikzcd}
 & \forall B \arrow[dr, dashed, "\exists h_B"] & \\
A \arrow[ur, "\forall f"] \arrow[dr, "\forall g"'] & \circlearrowleft & \exists D \\
 & \forall C \arrow[ur, dashed, "\exists h_C"'] &
\end{tikzcd}$$

We call such a structure D an **amalgam of B and C over A**.

(*iv*) *$\mathcal{K}$ contains countably many structures (up to isomorphism), and contains structures of arbitrarily large (finite) cardinality.*

Remark 3.2. Note that if the class $\mathcal{K}$ above contains the empty structure, then AP implies JEP.

The main reason for studying Fraïssé classes is that any Fraïssé class $\mathcal{K}$ has a so-called *Fraïssé limit* $\mathbb{K}$, which is *universal* and *ultrahomogeneous*. Universality in this case means that the class of all finite structures that embeds into $\mathbb{K}$ equals $\mathcal{K}$. This class is the so-called **age** of $\mathbb{K}$ and is denoted $\mathrm{Age}(\mathbb{K})$. Ultrahomogeneity is defined as follows:

Definition 3.3. *A structure $\mathcal{A}$ is* **ultrahomogeneous** *if any isomorphism between finite substructures of $\mathcal{A}$ extends to an automorphism of $\mathcal{A}$.*

Fraïssé's theorem then reads:

Theorem 3.4 (Fraïssé, [11, 10], cf. also [13, Theorem 7.1.2])**.** *Let L be a countable relational signature and let $\mathcal{K}$ be a Fraïssé class of L-structures. Then there exists a unique (up to isomorphism) countable structure $\mathbb{K}$ satisfying:*

(*i*) *$\mathbb{K}$ is ultrahomogeneous.*
(*ii*) $\mathrm{Age}(\mathbb{K}) = \mathcal{K}$.

The structure $\mathbb{K}$ in the theorem above is the **Fraïssé limit** of $\mathcal{K}$. Using this theorem we will show that there is a universal ultrahomogeneous countable rational diversity. First we need a couple of definitions and an amalgamation lemma to make it simpler for us to verify the AP for the class of finite rational diversities.

Definition 3.5. *Let Y be a set and let $X \subseteq Y$. A* **connected cover** *of X is a collection $\{E_i\}$ of subsets of Y such that $X \subseteq \bigcup E_i$ and such that the* **intersection graph** *$\mathcal{G}$ defined on $\{E_i\}$ by $E_i \mathcal{G} E_j \iff E_i \cap E_j \neq \emptyset$ is connected.*

Remark 3.6. If (Y, δ) is a diversity and $X \subseteq Y$ is finite, then for any finite connected cover $\{E_i\}$ of X with each E_i finite, we have that $\delta(X) \leq \sum \delta(E_i)$. This inequality is the main reason why we are interested in connected covers.

With this terminology established we can define a *free* amalgamation of two diversities B and C sharing a common **subdiversity** A, i.e. where $A \subseteq B, C$ and where the inclusion is an embedding. This is a diversity version of the free amalgamation of metric spaces.

Definition 3.7. *Let (A, δ_A), (B, δ_B) and (C, δ_C) be non-empty finite diversities such that $A = B \cap C$ and such that A is a subdiversity of B and C. The* **free amalgam of B and C over A** *is the diversity (D, δ_D) where $D = B \cup C$ and where $\delta_D(X)$ is given by the minimum over all sums $\sum_i \delta(E_i)$ for $\{E_i : i \leq n\}$ a connected cover of $X \subseteq D$ such that for each i either $E_i \subseteq B$ or $E_i \subseteq C$.*

Remark 3.8. If X has elements from both B and C, the definition of $\delta_D(X)$ above requires the connected cover to include elements from A. Hence, if we restrict δ_D to pairs we obtain the usual free amalgamation of metric spaces, i.e.

$$\delta_D(b, c) = \min_{a \in A} \{\delta_B(b, a) + \delta_D(a, c)\}$$

for $b \in B$ and $c \in C$.

Of course it is not necessarily evident that δ_D above defines a diversity map and that both (B, δ_B) and (C, δ_C) embeds into (D, δ_D). We proceed to verify this.

Lemma 3.9. *δ_D defined above is a diversity map on $B \cup C$ extending both δ_B and δ_C. It follows that (D, δ_D) is an amalgam of B and C over A.*

Proof. First we show δ_D agrees with δ_B and δ_C on B and C, respectively. Suppose therefore $X \subseteq B$ (the other case is similar). Then $\{X\}$ is a connected cover of X so $\delta_D(X) \leq \delta_B(X)$. To show equality, let $\{E_i\}$ be a connected cover of X. Then we can assume $E_i \subseteq B$ as well. By monotonicity of δ_B we have

$$\delta_B(X) \leq \delta_B(\bigcup E_i).$$

By connectivity we have

$$\delta_B(\bigcup E_i) \leq \sum \delta_B(E_i).$$

We conclude that $\delta_B(X) \leq \delta_D(X)$ as well, so in fact $\delta_D(X) = \delta_B(X)$. In particular $\delta_D(X) = 0$ if $|X| \leq 1$.

Next we show monotonicity. Let therefore $X \subseteq Y$ be given. Then any connected cover of Y whose elements are contained in either B or C must also cover X. Hence $\delta_D(X) \leq \delta_D(Y)$.

Lastly we show connected sublinearity. Suppose therefore that $X \cap Y \neq \emptyset$. Let $\{E_i\}$ and $\{F_j\}$ be connected covers realising $\delta_D(X)$ and $\delta_D(Y)$, respectively. Then, since $X$and Y intersect, we have that $\{E_i, F_j\}$ is a connected cover of $X \cup Y$ whose elements are either contained in B or C. Hence we must have

$$\delta_D(X \cup Y) \leq \sum \delta(E_i) + \sum \delta(F_j) = \delta_D(X) + \delta_D(Y).$$

It follows that δ_D is a diversity map. □

Observe that if the diversities A, B and C above are all rational, then the amalgam D will also be a rational diversity. It follows that the class of finite rational diversities, denoted $\mathscr{D}$, has the AP and hence that this class is a Fraïssé class.

Proposition 3.10. *$\mathscr{D}$ is a Fraïssé class with limit $\mathbb{U}_{\mathbb{Q}}$. Moreover, the completion of $\mathbb{U}_{\mathbb{Q}}$ is (isomorphic to) the Urysohn diversity.*

Proof. We first note that clearly there are rational diversities of arbitrarily large finite cardinality. Moreover, up to isomorphism, there are only countably many possible finite rational diversities. Hence $\mathscr{D}$ has property (iv) of Definition 3.1 above. We verify that $\mathscr{D}$ has the three other properties: HP, JEP and AP.

HP is clearly satisfied and JEP follows from AP since clearly the empty diversity is rational. Therefore we proceed to show AP.

We claim that AP follows from Lemma 3.9 above. To see this, suppose we are given $A, B, C \in \mathscr{D}$ with $A \preceq B, C$ via embeddings f_B and f_C. Then

we let $D = B \cup_A C$ be the union of B and C where we identify $f_B(A)$ with $f_C(A)$ while leaving $B \setminus f_B(A)$ and $C \setminus f_C(A)$ disjoint. Identifying A with its image inside D we now have that $A = B \cap C$. Therefore Definition 3.7 applies, and we obtain an amalgam (D, δ_D) of B and C over A.

Thus $\mathscr{D}$ is a Fraïssé class and hence that it has a Fraïssé limit which we will denote by $\mathbb{U}_\mathbb{Q}$.

The "moreover" part follows since $\mathbb{U}_\mathbb{Q}$ has the approximate extension property: if $F \subseteq \mathbb{U}_\mathbb{Q}$ is finite, $f \in E(F)$ is admissible and $\varepsilon > 0$, we can find an admissible map f' with rational values such that $|f'(A) - f(A)| < \varepsilon$. Then f' defines a rational diversity on $F \cup \{z\}$ for some new element z. By universality and ultrahomogeneity of $\mathbb{U}_\mathbb{Q}$ we find $x \in \mathbb{U}_\mathbb{Q}$ such that for all $A \subseteq F$ we have $|\delta(A \cup \{x\}) - f(A)| = |f'(A) - f(A)| < \varepsilon$. It now follows from Proposition 2.8 above that the completion of $\mathbb{U}_\mathbb{Q}$ has the extension property. Moreover, from Theorem 2.9 it follows that this completion is isomorphic to $\mathbb{U}$ as claimed. $\square$

4 A dense conjugacy class

With the existence of $\mathbb{U}_\mathbb{Q}$ established we set out to show that $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ and $\mathrm{Aut}(\mathbb{U})$ have a dense conjugacy class. First recall that the *conjugacy action* of a group on itself is given by $g \cdot h := ghg^{-1}$. Having a dense conjugacy class is then defined as follows.

Definition 4.1. *A Polish group G is said to have a* **dense conjugacy class** *if there is some element of G whose orbit under the conjugacy action of G on itself is dense.*

In [19] Kechris and Rosendal characterise when the automorphism group of a Fraïssé limit of a class $\mathcal{K}$ has a dense conjugacy class. They do this in terms of the JEP not for $\mathcal{K}$ itself, but for the class of all $\mathcal{K}$*-systems*. Below, $A \subsetneq B$ denotes that A is a **substructure** of B, i.e. that $A \subseteq B$ and that the inclusion is an embedding of A into B.

Definition 4.2. *Let $\mathcal{K}$ be a Fraïssé class. A* **$\mathcal{K}$-system** *consists of a structure A in $\mathcal{K}$ together with a substructure $A_0 \subsetneq A$ and a partial automorphism $f\colon A_0 \to A$. Such a system is denoted $\mathcal{A} = (A, (f, A_0))$. The class of all $\mathcal{K}$-systems is denoted $\mathcal{K}_p$.*

An **embedding** *of a $\mathcal{K}$-system $\mathcal{A} = (A, (f, A_0))$ into another $\mathcal{K}$-system $\mathcal{B} = (B, (g, B_0))$ is a map $\Phi\colon A \to B$ that embeds A into B, A_0 into B_0 and $f(A_0)$ into $g(B_0)$ such that $\Phi \circ f \subseteq g \circ \Phi$. In diagram form:*

$$\begin{tikzcd} A_0 \arrow[r, "\Phi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & B_0 \arrow[d, "g"] \\ f(A_0) \arrow[r, "\Phi"'] & g(B_0) \end{tikzcd}$$

Kechris and Rosendal then obtain the following characterisation of having a dense conjugacy class.

Theorem 4.3 ([19, Theorem 2.1])**.** *Let $\mathcal{K}$ be a Fraïssé class with limit $\mathbb{K}$. Then the following are equivalent:*

(*i*) *There is a dense conjugacy class in* $\mathrm{Aut}(\mathbb{K})$.
(*ii*) *$\mathcal{K}_p$ has the JEP.*

As an immediate corollary to this, we obtain that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has a dense conjugacy class.

Corollary 4.4. *$\mathscr{D}_p$ has the JEP. Hence* $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ *has a dense conjugacy class.*

Proof. Let $\mathcal{A} = (A, (f, A_0))$ and $\mathcal{B} = (B, (g, B_0))$ be $\mathscr{D}$-systems. Then let $\mathcal{C} = (C, (h, C_0))$ be the system where $C = A \sqcup B$, $C_0 = A_0 \sqcup B_0$ and $h = f \cup g$ and where the diversity map δ_C is defined to be δ_A on A, δ_B on B and on subsets with elements from both A and B, δ_C is constant, equal to some rational $N > \delta_A(A), \delta_B(B)$. It is easy to check that $\mathcal{C}$ is in $\mathcal{K}_p$ and that both $\mathcal{A}$ and $\mathcal{B}$ embeds into $\mathcal{C}$. □

We now wish to show the same thing for the automorphism group of the full Urysohn diversity. In order to do that, we will show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ embeds densely into $\mathrm{Aut}(\mathbb{U})$. This will follow from a homogeneity-like property that the rational and complete Urysohn diversities and metric spaces all share. In short, the property says that if two finite subspaces are close to being isomorphic, then we can find an isomorphic copy of one space close to the other space. In [32] the author refers to this property for metric spaces as *pair propinquity*. To emphasise that we are working with diversities we will call this property *diversity propinquity*. First we need to define what we mean by "close to being isomorphic".

Definition 4.5. *Let (X, δ_X) be a diversity and let $\bar{a} = (a_i)_{i \in I}$ and $\bar{b} = (b_i)_{i \in I}$ be two tuples of elements of X. For $\varepsilon > 0$ we say that $\bar{a}$ and $\bar{b}$ are* **ε-isomorphic** *if we have*

$$|\delta_X(\bar{a}_J) - \delta_X(\bar{b}_J)| < \varepsilon$$

for all $J \subseteq I$ where $\bar{b}_J := (b_j)_{j \in J}$.

Diversity propinquity is then defined as follows:

Definition 4.6. *Let (X, δ_X) be a diversity. We say that (X, δ_X) has* **diversity propinquity** *if for all $\varepsilon > 0$ there is an $\varepsilon' > 0$ such that for all ε'-isomorphic tuples $\bar{a}$ and $\bar{b}$ in X there is some $\bar{a}'$ isomorphic to $\bar{a}$ and pointwise within ε of $\bar{b}$, i.e.* $\max_i \delta_X(a'_i, b_i) < \varepsilon$.

We now have the following lemma, the proof of which is modelled on the proof of the corresponding fact for the Urysohn metric space in [25, Lemma 6.5].

Lemma 4.7. *The Urysohn diversity and the rational Urysohn diversity both have diversity propinquity. Moreover the ε' of the definition may simply be chosen to be the given ε.*

Proof. The proof for the two diversities is the same. In the rational case all one needs to check is that the diversity maps defined below are rational, but since we are dealing with finite sets this is easily verified.

Let $n \in \mathbb{N}$ and let $\varepsilon > 0$. The first thing we need, is to introduce some notation for dealing with the various diversities one may assign to an n-tuple. Thus let $D_{\bar{x}}$ be the set of all *diversity assignments* to the n-tuple $\bar{x} = (x_0, \ldots, x_{n-1})$. That is, if we denote $\{x_i : i \in I\}$ by $\bar{x}_I$, then $D_{\bar{x}}$ is the set of those maps on the power set of $\bar{x}$, $\bar{r} \colon \mathbb{P}(\bar{x}) \to \mathbb{R}$ (or into $\mathbb{Q}$ for the rational case), such that

(*i*) $\bar{r}(\emptyset) = 0$ and $\bar{r}(\bar{x}_I) = 0$ if and only if $|I| \leq 1$,

(*ii*) For all I_1, I_2 and all $I \neq \emptyset$ we have $\bar{r}(\bar{x}_{I_1} \cup \bar{x}_{I_2}) \leq \bar{r}(\bar{x}_{I_1} \cup \bar{x}_I) + \bar{r}(\bar{x}_I \cup \bar{x}_{I_2})$.

Of course any $\bar{r} \in D_{\bar{x}}$ corresponds to an element of $\mathbb{R}^{2^n}$ that we will also denote by $\bar{r}$. Thus we will use the notation $\bar{r}(I)$ for $\bar{r}(\bar{x}_I)$ which will be convenient below.

Let now d_∞ denote the maximum metric on $D_{\bar{x}}$, i.e.

$$d_\infty(\bar{r}, \bar{r}') = \sup_{I \subseteq n} \{|\bar{r}(I) - \bar{r}'(I)|\}.$$

Next we define another metric on $D_{\bar{x}}$ that measures how close together we can embed two diversities with n elements into a third diversity. To define this metric, let $\bar{y}$ be another n-tuple of elements disjoint from $\bar{x}$. Then define d_1 to be the metric given by

$$d_1(\bar{r}_1, \bar{r}_2) = \inf_{\bar{r}} \{\max_{i \leq n} \{\bar{r}(x_i, y_i)\} : \bar{r} \in D_{\bar{x} \cup \bar{y}}, \bar{r} \restriction \bar{x} = \bar{r}_1, \bar{r} \restriction \bar{y} = \bar{r}_2\}$$

where $\bar{r}_1, \bar{r}_2 \in D_{\bar{x}}$ are two different diversity assignments. If $\bar{r}_1 = \bar{r}_2$ we set $d_1(\bar{r}_1, \bar{r}_1) = 0$. Of course here $\bar{r} \restriction \bar{y} = \bar{r}_2$ means that the diversity assignment on $\bar{y}$ given by $\bar{r}_2$ (i.e. $\bar{y}_I \mapsto \bar{r}_2(\bar{x}_I)$) is equal to $\bar{r} \restriction \bar{y}$. That d_1 is in fact a metric follows from Lemma 1.3. We now claim that $d_1(\bar{r}_1, \bar{r}_2) \leq d_\infty(\bar{r}_1, \bar{r}_2)$. Moreover we claim that this will imply the lemma, but let's do one thing at a time.

Let therefore $\bar{r}_1, \bar{r}_2 \in D_{\bar{x}}$ be two different diversity assignments and set $c := d_\infty(\bar{r}_1, \bar{r}_2)$. We need to define some $\bar{r} \in D_{\bar{x} \cup \bar{y}}$ such that $\bar{r} \restriction \bar{x} = \bar{r}_1$, $\bar{r} \restriction \bar{y} = \bar{r}_2$ and such that $\max \bar{r}(x_i, y_i) \leq c$. In order to define such an $\bar{r}$, we need to introduce some notation. Given a subset $s = \{y_{i_1}, \ldots, y_{i_k}\} \subseteq \bar{y}$, we

denote the corresponding set $\{x_{i_1}, \ldots, x_{i_k}\} \subseteq \bar{x}$ by s'. A collection of subsets $\{E_i\}$ of $\bar{x}$ or $\bar{y}$ is said to be *connected* if the intersection graph on $\{E_i\}$ forms a connected graph. Let now $\bar{r}$ be the diversity assignment where for each $s \subseteq \bar{x} \cup \bar{y}$, $\bar{r}(s)$ is defined to be the minimum over sums of the form

$$\sum_i \bar{r}_1(E_i) + \sum_j \bar{r}_2(F'_j) + c/2,$$

where

- $E_i \subseteq \bar{x}$,
- $F_j \subseteq \bar{y}$,
- $\{E_i, F'_j\}$ is connected,
- $s \cap \bar{x} \subseteq \bigcup E_i$,
- $s \cap \bar{y} \subseteq \bigcup F_j$.

Let us argue why $\bar{r}$ is a diversity assignment. If $s_1 \subseteq s_2$ then any collection satisfying the properties of the minimum above for s_2 will also satisfy the properties for s_1. Hence $\bar{r}(s_1) \leq \bar{r}(s_2)$. If $s_1 \cap s_2 \neq \emptyset$ we let $\{E^1_i\}, \{F^1_j\}$ realise $\bar{r}(s_1)$ and $\{E^2_l\}, \{F^2_k\}$ realise $\bar{r}(s_2)$. Then it is easy to check that $\{E^1_i, E^2_l\}, \{F^1_j, F^2_k\}$ satisfiy the properties of the minimum for $s_1 \cup s_2$. Therefore $\bar{r}(s_1 \cup s_2) \leq \bar{r}(s_1) + \bar{r}(s_2)$ as required. We conclude that $\bar{r}$ is in fact a diversity assignment. Moreover, we see that $\sup_i \bar{r}(x_i, y_i) = c/2$ since the singletons $\{x_i\}$ and $\{y_i\}$ satisfy the properties of the minimum. This shows that $d_1(\bar{r}_1, \bar{r}_2) \leq c/2 < d_\infty(r_1, r_2)$ as we claimed.

It now follows that both $\mathbb{U}$ and $\mathbb{U}_\mathbb{Q}$ have diversity propinquity. Since the argument for both diversities is the same, we only provide it for $\mathbb{U}$. Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then we claim that ε works as the ε' of Definition 4.6. To see this, let $\bar{a}$ and $\bar{b}$ be n-tuples of elements of $\mathbb{U}$ and suppose $\sup_{I \subseteq n} |\delta(\bar{a}_I) - \delta(\bar{b}_I)| < \varepsilon$. Let $\bar{r}_{\bar{a}}$ and $\bar{r}_{\bar{b}}$ be the diversity assignments corresponding to $\bar{a}$ and $\bar{b}$. Then $d_\infty(\bar{r}_{\bar{a}}, \bar{r}_{\bar{b}}) < \varepsilon$ and so $d_1(\bar{r}_{\bar{a}}, \bar{r}_{\bar{b}}) < \varepsilon$ as well. Therefore we find a diversity assignment $\bar{r}$ on $\bar{a} \cup \bar{b}$ such that restricted to $\bar{a}$ we get $\bar{r}_{\bar{a}}$ and restricted to $\bar{b}$ we get $\bar{r}_{\bar{b}}$ and such that $\sup_i \bar{r}(a_i, b_i) < \varepsilon$. By universality of $\mathbb{U}$ we find $\bar{a}', \bar{b}' \in \mathbb{U}^n$ isomorphic as diversities to $\bar{a}$ and $\bar{b}$, respectively, such that $\sup \delta(a'_i, b'_i) < \varepsilon$. By ultrahomogeneity we find an automorphism g of $\mathbb{U}$ such that $\delta(a_i, g{\cdot}b_i) < \varepsilon$ which was what we wanted. □

We can now show that $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ embeds densely into $\mathrm{Aut}(\mathbb{U})$.

Theorem 4.8. $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ *continuously embeds into* $\mathrm{Aut}(\mathbb{U})$ *as a dense subgroup.*

Proof. Recall that $\mathbb{U}_\mathbb{Q}$ is dense in $\mathbb{U}$ by Proposition 3.10. Furthermore, since the diversity map defines uniformly continuous maps on finite powers of $\mathbb{U}$ (cf. Lemma 1.3 above), it follows that any $g \in \mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ uniquely extends to an autoversity of $\mathbb{U}$. Thus $\mathrm{Aut}(\mathbb{U}_\mathbb{Q})$ embeds into $\mathrm{Aut}(\mathbb{U})$. Moreover, this embedding must be continuous by Pettis' theorem (cf. [23]).

We move on to show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ is dense in $\mathrm{Aut}(\mathbb{U})$. Recall that the topology on $\mathrm{Aut}(\mathbb{U})$ is the pointwise convergence topology generated at the identity by sets of the form

$$U_{\bar{a},r} := \{g \in \mathrm{Aut}(\mathbb{U}) : \delta(g(\bar{a}), \bar{a}) < r\}$$

for a tuple $\bar{a} = (a_1, \ldots, a_n)$ of elements of $\mathbb{U}$ and some $r > 0$. In each of these sets we must find an autoversity extending a rational autoversity. Let therefore $U_{\bar{a},r}$ be given and let $g \in U_{\bar{a},r}$. Set $\varepsilon := r - \max_i \delta(g(a_i), a_i) > 0$ and find a tuple $\bar{x}$ of n elements of $\mathbb{U}_{\mathbb{Q}}$ with $\max_i \delta(a_i, x_i) < \varepsilon/4$. Let moreover $\bar{y}$ be an n-tuple of elements of $\mathbb{U}_{\mathbb{Q}}$ such that $\max_i \delta(y_i, g(x_i)) < \varepsilon/(4n)$. Note that $g(x_i)$ is not necessarily in $\mathbb{U}_{\mathbb{Q}}$ – hence this approximation. By Lemma 1.3 it follows that $(\bar{y}, \delta)$ is $\varepsilon/4$-isomorphic to $(\bar{x}, \delta)$ and therefore, by propinquity and ultrahomogeneity of $\mathbb{U}_{\mathbb{Q}}$, we find an autoversity g_0 of $\mathbb{U}_{\mathbb{Q}}$ such that $\max_i \delta(g_0(x_i), y_i) < \varepsilon/4$. We claim that the extension of g_0 to $\mathbb{U}$ is in $U_{\bar{a},r}$. Let therefore $\tilde{g}_0$ denote this extension. We have

$$\begin{aligned}
\delta(a_i, \tilde{g}_0(a_i)) &\leq \delta(a_i, g(a_i)) + \delta(g(a_i), g(x_i)) + \delta(g(x_i), y_i) + \delta(y_i, \tilde{g}_0(x_i)) \\
&\quad + \delta(\tilde{g}_0(x_i), \tilde{g}_0(a_i)) \\
&< r - \varepsilon + \varepsilon/4 + \varepsilon/(4n) + \varepsilon/4 + \varepsilon/4 \\
&\leq r.
\end{aligned}$$

We conclude that $\tilde{g}_0 \in U_{\bar{a},r}$ and hence that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ is a dense subgroup of $\mathrm{Aut}(\mathbb{U})$. □

As an immediate corollary we obtain that $\mathrm{Aut}(\mathbb{U})$ has a dense conjugacy class.

Corollary 4.9. $\mathrm{Aut}(\mathbb{U})$ *has a dense conjugacy class.*

Proof. This follows easily since $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has a dense conjugacy class and is densely embedded into $\mathrm{Aut}(\mathbb{U})$. □

5 Ample generics

We move on to our next endeavour: ample generics of $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$. Let us begin by defining this notion.

Definition 5.1. *A Polish group G has* **ample generics** *if for each $n \in \mathbb{N}$ there is a comeagre orbit for the diagonal conjugacy action of G on G^n defined by*

$$g \cdot (g_1, \ldots, g_n) = (gg_1g^{-1}, \ldots, gg_ng^{-1}).$$

Ample generics turns out to be a very powerful property with many interesting consequences. Before explaining some of these consequences, we mention a few examples of groups that are known to have ample generics.

Examples. The following groups have ample generics.

- The automorphism group of the random graph, [15], cf. also [14].
- The free group on countably many generators, [8].
- The group of measure preserving homeomorphisms of the Cantor space, [19].
- The automorphism group of $\mathbb{N}^{<\omega}$ seen as the infinitely splitting regular rooted tree, [19].
- The isometry group of the rational Urysohn metric space, [27].

In [19], where these examples are taken from, Kechris and Rosendal show, as mentioned, a number of powerful consequences of ample generics. We have collected the most important ones in the theorem below.

Theorem 5.2. *Let G be a Polish group with ample generics. Then G has the following properties:*

(1) **Automatic continuity property**, *i.e. any homomorphism from G to a separable group H is continuous.*
(2) **Small index property**, *i.e. any subgroup of G of index $< 2^{\aleph_0}$ is open.*
(3) G cannot be the union of countably many non-open subgroups.
(4) G has a unique Polish group topology.

Another important result from [19] is a characterisation of when the automorphism group of a Fraïssé limit has ample generics in terms of the JEP and a weak form of the AP. This weaker form of amalgamation is, naturally enough, called the *weak amalgamation property* (or *WAP* for short) and is defined as follows:

Definition 5.3. *Let $\mathcal{K}$ be a class of finite structures. Then $\mathcal{K}$ has the* **weak amalgamation property** *(WAP) if for any $A_0 \in \mathcal{K}$ there is $A \in \mathcal{K}$ and an embedding $f_0 \colon A_0 \to A$ such that whenever $g_B \colon A \to B$ and $g_C \colon A \to C$ are embeddings into $B, C \in \mathcal{K}$, there is $D \in \mathcal{K}$ and embeddings $h_B \colon B \to D$ and $h_C \colon C \to D$ such that $h_B \circ g_B \circ f_0 = h_C \circ g_C \circ f_0$. In diagram form:*

$$\begin{tikzcd}
 & & \forall B \arrow[dr, dashed, "\exists h_B"] & \\
A_0 \arrow[r, dashed, "\exists f_0"] & \exists A \arrow[ur, "\forall g_B"] \arrow[dr, "\forall g_C"'] \arrow[rr, phantom, "\circlearrowleft"] & & \exists D \\
 & & \forall C \arrow[ur, dashed, "\exists h_C"'] &
\end{tikzcd}$$

However, it is not the Fraïssé class itself that must have the WAP and the JEP in order for the automorphism group to have ample generics, but the class of so-called *n-systems* for $n \in \mathbb{N}$. This is the class of finite structures A, together with n substructures of A and n partial automorphisms of A defined on these substructures. The exact definition is as follows:

Definition 5.4. *Let $\mathcal{K}$ be a Fraïssé class and let $n \geq 1$ be given. An* **n-system in $\mathcal{K}$** *consists of a structure A in $\mathcal{K}$ together with n substructures $A_1, \ldots, A_n \subsetneq A$ and n partial automorphisms $f_1 \colon A_1 \to A, \ldots, f_n \colon A_n \to A$. We denote such a system by $\mathcal{A} = (A, (f_i, A_i)_{i \leq n})$. The class of all n-systems in $\mathcal{K}$ is denoted $\mathcal{K}_p^n$.*

An **embedding** *of an n-system $\mathcal{A} = (A, (f_i, A_i))$ into another n-system $\mathcal{B} = (B, (g_i, B_i))$ is a map $\Phi \colon A \to B$ that embeds A into B, A_i into B_i and $f_i(A_i)$ into $g_i(B_i)$ and such that $\Phi \circ f_i \subseteq g_i \circ \Phi$ for each $i \leq n$. In diagram form, for each $i \leq n$:*

$$\begin{tikzcd} A_i \arrow[r, "\Phi"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & B_i \arrow[d, "g_i"] \\ f_i(A_i) \arrow[r, "\Phi"'] & g_i(B_i) \end{tikzcd}$$

Note that since we have defined embeddings between n-systems, we can talk about the WAP and the JEP for the class $\mathcal{K}_p^n$. In [19] Kechris and Rosendal show that these two properties for $\mathcal{K}_p^n$ actually characterise ample generics.

Theorem 5.5 ([19, Theorem 6.2])**.** *Let $\mathcal{K}$ be a Fraïssé class and let $\mathbb{K}$ denote its limit. Then the following are equivalent:*

(*i*) $\mathrm{Aut}(\mathbb{K})$ *has ample generics.*
(*ii*) *For all $n \geq 1$, $\mathcal{K}_n^p$ has the JEP and the WAP.*

Using this theorem, we can show that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has ample generics. The proof uses the following extension result inspired by [27, Theorem 2.1]. Below, a *partial isoversity* of a diversity A is simply an isoversity $f \colon A_0 \to B$ defined on a subset $A_0 \subseteq A$ into some diversity B. A *partial autoversity* of A is a partial isoversity into A. Our extension theorem for diversities now reads:

Theorem 5.6. *Let (A, δ_A) be a finite diversity. Then there is a finite diversity (B, δ_B) containing A as a subdiversity and such that any partial autoversity of A extends to a full autoversity of B.*

The proof of this theorem uses a theorem due to Herwig and Lascar from [12]. For the convenience of the reader, we have included it here. However, before we can state it properly, we need to make a few definitions.

Definition 5.7. *A class of structures $\mathcal{K}$ has the* **extension property for partial automorphisms** *(EPPA for short) if for any finite $A \in \mathcal{K}$ and any $B \in \mathcal{K}$ such that $A \subsetneq B$ and any partial automorphisms $g_1, \ldots, g_n$ of A that extend to automorphisms of B, there is a finite $C \in \mathcal{K}$ and automorphisms $h_1, \ldots, h_n$ of C such that h_i extends g_i.*

Loosely speaking, this property says that if it is possible to extend partial automorphisms to full automorphisms of *some* structure, then there is a *finite* structure where the partial automorphisms also extend.

Definition 5.8. *Let L be a finite relational signature. A map between two L-structures $h\colon A \to B$ is a* **weak homomorphism** *if for all predicates $R \in L$ and all tuples $\bar{a}$ of elements from A such that $A \vDash R(\bar{a})$, we have $B \vDash R(h(\bar{a}))$.*

Definition 5.9. *Let L be a finite relational signature and let $\mathcal{T}$ be a set of L-structures. An L-structure A is said to be* **$\mathcal{T}$-free** *if there is no structure $T \in \mathcal{T}$ and weak homomorphism $h\colon T \to A$.*

Herwig and Lascar's theorem then reads:

Theorem 5.10 ([12, Theorem 3.2])**.** *Let L be a finite signature and $\mathcal{T}$ a finite set of finite L-structures. Then the class of $\mathcal{T}$-free L-structures has the EPPA.*

We are now ready to prove Theorem 5.6.

Proof of Theorem 5.6. We can of course assume without loss of generality that $|A| \geq 2$. Let D be the set

$$D := \{(\delta_A(X), |X|) : X \subseteq A\} \setminus \{(0,1),(0,0)\}.$$

That is, D is all pairs of the non-zero values of δ_A together with the size of the set the value comes from. For each $(r,n) \in D$, we let $R_{(r,n)}$ be an n-ary relation symbol and let L be the (finite) relational language consisting of these symbols. We call a tuple of elements of D, $\alpha = ((r_0,n_0),\dots,(r_k,n_k))$, a *configuration* if we have that

$$\sum_{i=1}^{k} r_i < r_0 \quad \text{and} \quad 1 + \sum_{i=1}^{k}(n_i - 1) \geq n_0.$$

Given a configuration $\alpha = ((r_i,n_i))$ let $Y_0, Y_1, \dots, Y_k$ be sets such that

(*i*) $Y_0 \subseteq \bigcup_{i=1}^{k} Y_i$,

(*ii*) $|Y_i| = n_i$,

(*iii*) The intersection graph on $\{Y_1,\dots,Y_k\}$ is connected.

We call such a family of sets $\{Y_i : 0 \leq i \leq k\}$ an α-family. Note that since α is a configuration it is always possible to find at least one α-family. Moreover, we note that there are only finitely many α-families.

Given a configuration $\alpha = ((r_i, n_i) : 0 \leq i \leq k)$ and an α-family $\beta = \{Y_i\}$, we define an L-structure $\mathcal{M}_{\alpha,\beta}$ with universe $\bigcup Y_i$ by declaring that the only relations satisfied by $\mathcal{M}_{\alpha,\beta}$ are the following:

$$\mathcal{M}_{\alpha,\beta} \vDash R_{(r_i,n_i)}(\sigma(Y_i))$$

for any permutation σ of the elements of Y_i (considered here as an ordered tuple and not just a set). The permutations merely ensure that the relations are symmetric and do not really serve any other purpose. Let $\mathcal{T}$ denote the family of all $\mathcal{M}_{\alpha,\beta}$ for all configurations α and all α-families β. Note that $\mathcal{T}$ is finite.

Any diversity (X, δ_X) is naturally also an L-structure by letting

$$X \vDash R_{(r,n)}(Y) \iff \delta_X(Y) = r \text{ and } |Y| = n$$

for any finite subset/tuple Y of elements of X, meaning that we are considering Y as a subset on the right-hand side and as an ordered tuple on the left-hand side above. Note, however, that the order we choose on Y is not important. Observe that any partial autoversity of X is also a partial *automorphism* of X as an L-structure.

Suppose now we are given a configuration $\alpha = ((r_i, n_i) : 0 \leq i \leq k)$ and an α-family $\beta = \{Y_0, \ldots, Y_k\}$. We then claim that X is $\mathcal{T}$-free, i.e. that there are no weak homomorphisms $h\colon \mathcal{M}_{\alpha,\beta} \to X$. To see this, suppose h is such a map. Then since β is an α-family, we have that

$$h(Y_0) \subseteq \bigcup_{i=1}^{k} h(Y_i)$$

and so, by the monotonicity of the diversity map,

$$\delta_X(h(Y_0)) \leq \delta_X(\bigcup_{i=1}^{k} h(Y_i)).$$

Since the intersection graph on $\{Y_i : 1 \leq i \leq k\}$ is connected, it follows that the intersection graph on the images $\{h(Y_i) : 1 \leq i \leq k\}$ is connected too. Therefore, we can find Y_{i_0} such that the intersection graph on the family $\{h(Y_i) : i \neq i_0\}$ remains connected (this is always possible for finite connected graphs). Hence, by connected sublinearity of the diversity map, it follows that

$$\delta_X(\bigcup_{i=1}^{k} h(Y_i)) \leq \delta_X(h(Y_{i_0})) + \delta_X(\bigcup_{i \neq i_0} h(Y_i)).$$

By induction, we obtain that

$$\delta_X(\bigcup_{i=1}^{k} h(Y_i)) \leq \sum_{i=1}^{k} \delta_X(h(Y_i)).$$

However, since h is a weak homomorphism and β is an α-family, we have that $X \vDash R_{(r_i,n_i)}(h(Y_i))$. Hence, $\delta_X(h(Y_i)) = r_i$ for each i, and so we have that

$$r_0 = \delta_X(h(Y_0)) \leq \sum_{i=1}^{k} \delta_X(h(Y_i)) = \sum_{i=1}^{k} r_i < r_0,$$

which is of course a contradiction.

Next, by universality of $(\mathbb{U}, \delta_{\mathbb{U}})$, we can embed (A, δ_A) into $(\mathbb{U}, \delta_{\mathbb{U}})$. By ultrahomogeneity, we can extend each partial autoversity of A to an autoversity of $\mathbb{U}$. Note that since D includes all values of δ_A, any partial L-automorphism of A is a partial autoversity of A (and vice versa of course). Hence, we can extend any partial L-automorphism of A to a full L-automorphism of $\mathbb{U}$ viewed as an L-structure. By Theorem 5.10 above, we can find a finite $\mathcal{T}$-free L-structure C containing A as a substructure such that each partial L-automorphism of A extends to an automorphism of C. Given a partial automorphism g of A, we will denote its extension to C by $\tilde{g}$. By convention, we will assume that the empty map is extended to the identity map.

A sequence of subsets $e_1, \ldots, e_k \subseteq C$ is called a *connection* if the intersection graph on $\{e_i\}$ is connected and if there are $(r_1, n_1), \ldots, (r_n, n_k) \in D$ such that for each $i \leq k$

(*i*) $|e_i| = n_i$,

(*ii*) $C \vDash R_{(r_i,n_i)}(\sigma(e_i))$ for any permutation σ of e_i considered as an ordered tuple.

Given $c, c' \in C$, we say that they are *connected* if there is a connection $e_1, \ldots, e_k$ such that $c \in e_1$ and $c' \in e_k$. Let $B \subseteq C$ be those $b \in C$ that are connected to some $a \in A$. Note that any $b \in B$ is connected to all $a \in A$, since if b is connected to $a' \in A$ via the connection $e_1, \ldots, e_k$, then $\{a, a'\}, e_1, \ldots, e_k$ is a connection between a and b. Moreover, clearly $A \subseteq B$, since given $a \in A$, we pick $a' \in A \setminus \{a\}$ (remember that we have assumed $|A| \geq 2$) and see that $\{a, a'\}$ is a connection between a and a'.

Given a partial automorphism g of A, we claim that $\tilde{g}(B) = B$. To show this, it is enough to show that $\tilde{g}(B) \subseteq B$, since we are dealing with finite sets. If g is the empty map, then we extend it to the identity and there is nothing to show. If not, pick a in the domain of g and let $b \in B$. Then, as noted above, we can find a connection between a and b. Let $e_1, \ldots, e_k$ denote such a connection. Since $\tilde{g}$ is an automorphism, it follows that $\tilde{g}(e_1), \ldots, \tilde{g}(e_k)$ is a connection between $\tilde{g}(a) = g(a) \in A$ and $\tilde{g}(b)$, because clearly the intersection graph on $\{\tilde{g}(e_i) : 1 \leq i \leq k\}$ is connected and $\tilde{g}$ preserves the relations. We conclude that $\tilde{g}(b) \in B$ as we claimed.

Define now a diversity δ_B on B by letting $\delta_B(X)$ be 0 if $|X| \leq 1$ and otherwise letting it be the minimum over all sums $\sum_{i=1}^{k} r_i$, where for some connection $e_1, \ldots, e_k$ with $e_i \subseteq B$, we have $C \vDash R_{(r_i,|e_i|)}(\sigma(e_i))$ for any

permutation of e_i considered as a tuple and where $X \subseteq \bigcup e_i$. Note that since $X \subseteq B$, each element of X is connected to the *same* element of $a \in A$. Hence, the collection of all these connections, one for each $x \in X$, forms a connection containing X. Therefore, this minimum is not over the empty set and so δ_B is well-defined.

We must argue why δ_B is a diversity map, i.e. we must show that for each $X, Y, Z \subseteq B$ with $Z \neq \emptyset$ we have

$$\delta_B(X \cup Y) \leq \delta_B(X \cup Z) + \delta_B(Z \cup Y).$$

Let $\{e_i\}$ and $\{r_i\}$ realise $\delta_B(X \cup Z)$ and let $\{f_j\}$ and $\{s_j\}$ realise $\delta_B(Z \cup Y)$. Then since $Z \subseteq X \cup Z \subseteq \bigcup e_i$ and $Z \subseteq Z \cup Y \subseteq \bigcup f_j$, it follows that the intersection graph on $\{e_i\} \cup \{f_j\}$ is connected. Hence, $\{e_i\} \cup \{f_j\}$ forms a connection. Moreover, this connection covers $X \cup Y$. Therefore, we have that

$$\delta_B(X \cup Y) \leq \sum r_i + \sum s_i = \delta_B(X \cup Z) + \delta_B(Z \cup Y)$$

as we wanted. Moreover, if g is a partial autoversity of A, it follows that the extension $\tilde{g}$ and its inverse $\tilde{g}^{-1}$ maps connections to connections. Therefore, we must have that $\tilde{g} \colon B \to B$ is an autoversity with respect to δ_B.

Finally, we must show that δ_B extends δ_A. First of all, it is clear that we must have $\delta_B(X) \leq \delta_A(X)$ for all $X \subseteq A$, since $\{X\}$ is itself a connection covering X as $C \vDash R_{(\delta_A(X), |X|)}(X)$.

Suppose towards a contradiction that we have $\delta_B(X) < \delta_A(X)$. Then let $e_1, \ldots, e_k$ be a connection with corresponding values $r_1, \ldots, r_k$ witnessing this, i.e. $\sum r_i < \delta_A(X)$. It follows that

$$(\delta_A(X), |X|), (r_1, |e_1|), \ldots, (r_k, |e_k|)$$

is a configuration because $X \subseteq \bigcup e_i$, so

$$\begin{aligned} |X| &\leq |e_1| + |e_2 \setminus e_1| + \ldots + |e_k \setminus (\bigcup_{i=1}^{k-1} e_i)| \\ &\leq 1 + \sum (|e_i| - 1), \end{aligned}$$

where the second inequality follows since the first sum counts each element of $\bigcup e_i$ exactly once and the second sum counts each element at least once given that the intersection graph on $\{e_i\}$ is connected. If we denote this configuration by α, then $\{e_i, X : 1 \leq i \leq k\}$ is an α-family β. Therefore, $\mathcal{M}_{\alpha,\beta}$ is in $\mathcal{T}$ and the identity map on $\mathcal{M}_{\alpha,\beta}$ is a weak homomorphism into C. This contradicts that C is $\mathcal{T}$-free. We conclude that $\delta_B(X) = \delta_A(X)$.

All in all, we have extended each partial autoversity of A to an autoversity of (B, δ_B), and this diversity contains (A, δ_A) as a subdiversity. This was what we wanted. $\square$

We are now ready to prove that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has ample generics.

Theorem 5.11. $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ *has ample generics.*

Proof. We show that for each $n \in \mathbb{N}$, the class $\mathscr{D}_p^n$ of n-systems in $\mathscr{D}$ has the WAP. Since it clearly has the JEP, it follows from Kechris and Rosendal's Theorem 5.5 above that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has ample generics.

Let therefore $\mathcal{A} = (A, (f_i, A_i))$ be an n-system in $\mathscr{D}_p^n$. By the extension theorem above we find a rational diversity B containing A where the partial isoversities of A extend to autoversities of B. Let $\tilde{f}_i$ denote the extension of f_i to B and let $\mathcal{B}$ denote the resulting n-system in $\mathscr{D}_p^n$. Suppose now that we are given n-systems $\mathcal{C}_1 = (C_1, (g_1^i, C_1^i))$ and $\mathcal{C}_2 = (C_2, (g_2^i, C_2^i))$ and embeddings $\Phi_j \colon \mathcal{B} \to \mathcal{C}_j$, $j = 1, 2$. We need to construct an amalgam of $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathcal{B}$. To do that we apply the extension theorem to both C_1 and C_2 and get $\widetilde{C}_1$ and $\widetilde{C}_2$ where the partial isoversities g_1^i and g_2^i extend to full autoversities $\tilde{g}_1^i$ and $\tilde{g}_2^i$ of $\widetilde{C}_1$ and $\widetilde{C}_2$, respectively. Denote the resulting n-systems by $\widetilde{\mathcal{C}}_1$ and $\widetilde{\mathcal{C}}_2$. As usual we can assume that $B = \widetilde{C}_1 \cap \widetilde{C}_2$. Therefore we can construct the free amalgam D of $\widetilde{C}_1$ and $\widetilde{C}_2$ over B. Moreover, we can define an n-system using D by letting h_i be $\tilde{g}_1^i \cup \tilde{g}_2^i$, which is an autoversity of D. Denote the resulting n-system by $\mathcal{D}$. In diagram form for the n-systems:

$$\begin{tikzcd}
 & & \mathcal{C}_1 \arrow[r] & \widetilde{\mathcal{C}}_1 \arrow[dr] & \\
\mathcal{A} \arrow[r] & \mathcal{B} \arrow[ur] \arrow[dr] & \circlearrowleft & & \mathcal{D} \\
 & & \mathcal{C}_2 \arrow[r] & \widetilde{\mathcal{C}}_2 \arrow[ur] &
\end{tikzcd}$$

and in diagram form for $j = 1, 2$ and each i:

$$\begin{tikzcd}
A_i \arrow[r] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[r] \arrow[d, "\tilde{f}_i"'] \arrow[dr, phantom, "\circlearrowleft"] & C_j^i \arrow[r] \arrow[d, "g_j^i"'] \arrow[dr, phantom, "\circlearrowleft"] & \widetilde{C}_j \arrow[r] \arrow[d, "\tilde{g}_j^i"'] \arrow[dr, phantom, "\circlearrowleft"] & D \arrow[d, "h_i"] \\
f_i(A_i) \arrow[r] & B \arrow[r] & g_j^i(C_j^i) \arrow[r] & \widetilde{C}_j \arrow[r] & D
\end{tikzcd}$$

It is easy to check that $\mathcal{D}$ is an amalgam of $\mathcal{C}_1$ and $\mathcal{C}_2$ over $\mathcal{B}$. Therefore, we conclude that $\mathscr{D}_p^n$ has the WAP and hence that $\mathrm{Aut}(\mathbb{U}_{\mathbb{Q}})$ has ample generics. $\square$

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