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MINIMAL BV-LIFTINGS OF $W^{1,1}(\Omega, \mathbb{S}^1)$ MAPS IN 2D ARE “OFTEN” UNIQUE

EDUARD CURCĂ

ABSTRACT. Let \mathbb{S}^1 be the unit circle, Ω a smooth, bounded and simply connected domain in \mathbb{R}^2 , and k a positive integer. We prove that the set of configurations $a = (a_1, \dots, a_k) \in \Omega^k$ for which each $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$ admits a unique (*mod* 2π) minimal BV-lifting $\varphi \in \text{BV}(\Omega, \mathbb{R})$ is of full measure in Ω^k .

In particular, this implies that the set of those $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ that admit a unique (*mod* 2π) minimal BV-lifting is dense in $W^{1,1}(\Omega, \mathbb{S}^1)$. This answers a question of Brezis and Mironescu.

1. INTRODUCTION

Suppose Ω is a smooth, bounded and simply connected domain in \mathbb{R}^2 . It is known (see [6, Section 6.2], [4], [8] and [2, Theorem 2.4]) that for each $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ there exists a BV-lifting of u on Ω , i.e., there exists $\varphi \in \text{BV}(\Omega, \mathbb{R})$ such that $u = e^{i\varphi}$ on Ω . This is “the best one can get”, since such u need not have a lifting $\varphi \in W^{1,1}(\Omega, \mathbb{R})$. For example, if $a \in \Omega$, then $u(x) = \frac{x-a}{|x-a|}$ belongs to $W^{1,1}(\Omega, \mathbb{S}^1)$, but has no lifting $\varphi \in W^{1,1}(\Omega, \mathbb{R})$ [5], [2, Remark 2.6].¹

Clearly, φ is not unique; if φ is a BV-lifting, then so is $\varphi + 2k\pi$, $k \in \mathbb{Z}$. Actually, non-uniqueness is much “richer”: if $\omega \subset \Omega$ is a finite perimeter set and φ is a BV-lifting of u , then so is $\varphi + 2k\pi \mathbb{1}_\omega$, with $k \in \mathbb{Z}$.

“Minimal” liftings are distinguished BV-liftings. By definition, φ is a minimal BV-lifting of u if

$$(1.1) \quad |\varphi|_{\text{BV}} = \inf_{u=e^{i\phi}} |\phi|_{\text{BV}},$$

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¹Actually, one may prove that a BV lifting of $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ belongs to SBV (but, in general, not to $W^{1,1}$).

where

$$\begin{aligned} |\phi|_{\text{BV}} &:= \|D\phi\|_{\mathcal{M}(\Omega, \mathbb{R}^2)} \\ &= \sup \left\{ \int_{\Omega} \varphi \operatorname{div} \zeta \mid \zeta \in C_c^\infty(\Omega, \mathbb{R}^2), \|\nabla \zeta\|_{L^\infty} \leq 1 \right\}. \end{aligned}$$

Minimal liftings appear naturally in connection with the “relaxed energy” associated with a map $u \in W^{1,1}(\Omega, \mathbb{S}^1)$. More specifically, set

$$(1.2) \quad E_{\text{rel}}(u) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla u_n| \mid (u_n) \subset C^\infty(\bar{\Omega}, \mathbb{S}^1), u_n \rightarrow u \right\}.$$

Then $E_{\text{rel}}(u)$ equals the right-hand side of (1.1). Moreover, if (u_n) is a minimizing sequence in (1.2), then, possibly up to a subsequence, $u_n \rightarrow e^{i\varphi}$ in $L^1(\Omega)$, where φ is a minimal BV-lifting of u [2, Section 2.7].

Clearly, the infimum in (1.1) is attained, and thus minimal BV-liftings do exist. In general, such a minimal BV-lifting is not unique, even (*mod* 2π). For example, the following functions have more than one minimal BV-lifting (*mod* 2π):

- a) $u(z) := \frac{z}{|z|}$, on $\Omega := B(0, 1)$ (the unit disc);
- b) $u(z) := \frac{2\bar{z} - 1}{|2\bar{z} - 1|} \frac{2z + 1}{|2z + 1|}$, on $\Omega := (-1, 1)^2$.

(See Remark 10 below.)

In order to simplify the presentation, in what follows, uniqueness of liftings is meant (*mod* 2π). We do not specify this anymore.

We are going to answer the following question raised in [2]: is the set of functions $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ which admit a unique minimal BV-lifting, *residual*² in $W^{1,1}(\Omega, \mathbb{S}^1)$?

The answer is positive. More specifically, we have the following result.

Theorem 1. *Suppose Ω is a smooth, bounded and simply connected domain in \mathbb{R}^2 . Consider the set*

$$U := \{u \in W^{1,1}(\Omega, \mathbb{S}^1) \mid u \text{ has a unique minimal BV-lifting}\}.$$

Then U is a G_δ dense subset of $W^{1,1}(\Omega, \mathbb{S}^1)$.

²Recall that a residual (or comeagre) set is a countable intersection of sets with dense interiors.

Theorem 1 will be proved by using the geometrical description of the minimal liftings given in [2], combined with some “generic” geometric properties of k -tuples in Ω^k , where k is a positive integer. In fact, our proof will give a somewhat more precise result. Consider $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$, where a_1, \dots, a_k are distinct points in Ω . It is easy to see (see Remark 9 below) that whether or not u admits a unique minimal BV-lifting, depends only on the vector of singularities $a = (a_1, \dots, a_k) \in \Omega^k$ and the vector of degrees $d = (d_1, \dots, d_k)$, with $d_j := \deg(u, a_j)$ (the topological degree of u on a small circle around a_j). We have that, “with probability 1”, the minimal BV-lifting of such u is unique:

Theorem 2. *Suppose Ω is a smooth, bounded and simply connected domain in \mathbb{R}^2 . Let k be a positive integer. The set of configurations $a = (a_1, \dots, a_k) \in \Omega^k$ for which every $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$ admits a unique minimal BV-lifting (regardless the choice of $d_1, \dots, d_k \in \mathbb{Z}$) is of full measure in Ω^k .*

As we will see in Section 3 (see Remark 12), Theorem 2 implies (and is actually equivalent to) the following geometric result. Consider $a = (a_1, \dots, a_k) \in \Omega^k$ and (arbitrary) $(d_1, \dots, d_k) \in \mathbb{Z}^k$. Then, with probability 1, there exists exactly one rectifiable curve $\mathcal{C} \subset \Omega$ of minimal length among the rectifiable curves with boundary $\sum_{j=1}^k d_j \delta_{a_j}$. Here, the boundary is considered in the sense of geometric measure theory, i.e.,

$$\int_{\mathcal{C}} d\xi = \sum_{j=1}^k d_j \xi(a_j), \quad \forall \xi \in C_c^\infty(\Omega).$$

Theorem 1 follows from Theorem 2 combined with a density result in [1] (see Lemma 13 in Section 3).

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2. “GENERIC” PROPERTIES OF k -TUPLES IN Ω^k

In this part, Ω is an open subset of \mathbb{R}^2 such that $\Omega \neq \emptyset, \mathbb{R}^2$.

We start by fixing some notation. Given a point $x \in \Omega$, we will denote by P_x its set of projections on the boundary of Ω , i.e.,

$$P_x := \{y \in \partial\Omega \mid \text{dist}(x, \partial\Omega) = |x - y|\}.$$

We say that $x \in \Omega$ has a unique projection on $\partial\Omega$ if P_x contains only one point. Also, given a set $A \subset \mathbb{R}^2$ we denote by $\text{diam } A$ its diameter.

For the convenience of the reader we mention some elementary geometric facts.

Fact 1. Consider $r > 0$. Suppose P is a point in the open ball $B(O, r) \subset \mathbb{R}^2$, which is not its center. Consider $\alpha \in [0, 2\pi]$ and let $Q_\alpha \in \partial B(O, r)$ be such that the angle $\angle POQ_\alpha$ equals α . Then the distance $|PQ_\alpha|$ is a strictly increasing function of α , for $\alpha \in [0, \pi]$.

Fact 2. Suppose P is a point in the open ball $B(O, r) \subset \mathbb{R}^2$, which is not its center. Consider $\alpha < \beta$ two angles in $[0, \pi]$. Suppose Q_α is as above and $Q'_\beta \in \mathbb{R}^2 \setminus B(O, r)$ is a point such that that the angle $\angle POQ'_\beta$ equals β . Then $|PQ_\alpha| < |PQ'_\beta|$.

Fact 1 is a direct consequence of the cosine formula. Fact 2 is a direct consequence of Fact 1 and the cosine formula. Indeed, with the above notation, we have from Fact 1 that $|PQ_\alpha| < |PQ_\beta|$. Now, since the function $x \rightarrow x^2 - 2x|OP|\cos\beta$ is increasing on $(|OP|, \infty)$ and $|OQ'_\beta| \geq |OQ_\beta| = r > |OP|$, we have

$$\begin{aligned} |PQ'_\beta|^2 &= |OQ'_\beta|^2 - 2|OQ'_\beta||OP|\cos\beta + |OP|^2 \\ &> |OQ_\beta|^2 - 2|OQ_\beta||OP|\cos\beta + |OP|^2 \\ &= |PQ_\beta|^2 > |PQ_\alpha|^2. \end{aligned}$$

Using these facts we prove the following geometric lemma.

Lemma 3. Let Ω be an open subset of \mathbb{R}^2 such that $\Omega \neq \emptyset, \mathbb{R}^2$. Suppose that $B(x_0, r) \subset \Omega$. Then, for any $\varepsilon > 0$, there exist two numbers $\alpha, \delta > 0$ depending only on ε , and a cone C_α of angle α , with vertex x_0 , such that for any $x \in C_\alpha \cap B(x_0, \delta r)$ we have $\text{diam } P_x < \varepsilon$.

Proof. Choose $x_1 \in P_{x_0}$. We can suppose without loss of generality that $r = |x_1 - x_0|$. For each $0 < \beta < 2\pi$ we consider the open cone C_β of angle β with vertex x_0 and axis determined by the vector $x_1 - x_0$.

Let $0 < \alpha < \pi/4$ be an angle that will be chosen later. Fact 2 implies that

$$(2.1) \quad \overline{B}(x, |x - x_1|) \setminus B(x_0, |x_0 - x_1|) \subset \overline{C}_{2\alpha}$$

for any $x \in C_\alpha$. Indeed, suppose by contradiction that there exists $y \in \overline{B}(x, |x - x_1|) \setminus B(x_0, |x_0 - x_1|)$ such that $y \notin \overline{C}_{2\alpha}$. In particular, we have $y \in \mathbb{R}^2 \setminus B(x_0, r)$ and

$$|\angle(y - x_0, x - x_0)| > \alpha/2 > |\angle(x_1 - x_0, x - x_0)|.$$

Fact 2 gives now that $|y - x| > |x_1 - x|$, which contradicts the fact that $y \in \overline{B}(x, |x - x_1|)$.

Now, for any $\varepsilon' > 0$ there exists a $\delta > 0$ depending only on ε' , such that, if $|x - x_0| < \delta r$, then

$$(2.2) \quad \overline{B}(x, |x - x_1|) \subset \overline{B}(x_0, (1 + \varepsilon')|x_0 - x_1|).$$

Fix $\varepsilon' > 0$ and choose $\delta > 0$ as above. From (2.1) and (2.2) we get that, for any $x \in C_\alpha$ with $|x - x_0| < \delta r$, we have the inclusion

$$(2.3) \quad \overline{B}(x, |x - x_1|) \setminus B(x_0, |x_0 - x_1|) \subset A_{\alpha, \varepsilon'},$$

where

$$A_{\alpha, \varepsilon'} := (\overline{C}_{2\alpha} \cap \overline{B}(x_0, (1 + \varepsilon')|x_0 - x_1|)) \setminus B(x_0, |x_0 - x_1|).$$

If $x' \in P_x$, then $|x - x'| \leq |x - x_1|$, and hence $P_x \subseteq \overline{B}(x, |x - x_1|)$. Also, we have $P_x \subseteq \partial\Omega$, and since $B(x_0, |x_0 - x_1|)$ contains no point from $\partial\Omega$, it follows that $P_x \subseteq \overline{B}(x, |x - x_1|) \setminus B(x_0, |x_0 - x_1|)$. Hence, thanks to (2.3), we get $P_x \subset A_{\alpha, \varepsilon'}$.

It remains to observe that, if α and ε' are sufficiently small, then $\text{diam } A_{\alpha, \varepsilon'} < \varepsilon$. This implies

$$\text{diam } P_x \leq \text{diam } A_{\alpha, \varepsilon'} < \varepsilon$$

for any $x \in C_\alpha \cap B(x_0, \delta r)$. \square

The above lemma implies the following proposition concerning the smallness of the set of points with non-unique projections on the boundary.

Proposition 4. Let $\Omega \subset \mathbb{R}^2$ be an open set such that $\Omega \neq \emptyset, \mathbb{R}^2$. If M is the set of the points of Ω which have unique projection on $\partial\Omega$, then $M^c := \Omega \setminus M$ is a Lebesgue null set.

Proof. First we note that

$$M = \bigcap_{n=1}^{\infty} M_n \text{ where } M_n := \{x \in \Omega \mid \text{diam } P_x < 1/n\}.$$

We will show that each M_n contains a Lebesgue measurable set of full measure and hence the exterior measure of each M_n^c is 0. This will show in particular that each M_n is measurable, M is measurable and

$$m(M^c) \leq \sum_{n=1}^{\infty} m(M_n^c) = 0.$$

Fix $n \geq 1$. With $\varepsilon = 1/n$, let α and δ be as in Lemma 3. If $B(x_0, r) \subset \Omega$ and Q is a square centred at x_0 and such that $Q \subset B(x_0, \delta r)$, by applying Lemma 3, we can find a cone C of angle α with vertex x_0 such that $C \cap Q \subset M_n$. Note that

$$(2.4) \quad \frac{m(C \cap Q)}{m(Q)} \geq \eta,$$

where $0 < \eta < 1$ only depends on α and hence it only depends on n .

Consider a nonempty open set $V \subset \Omega$. We claim that we may write

$$V = \bigcup_{j=1}^{\infty} Q_j,$$

with Q_j essentially disjoint squares such that, for each j , there exists some ball $B(x_j, r_j) \subset \Omega$ (where x_j is the center of Q_j) with $Q_j \subset B(x_j, \delta r_j)$. Indeed, it suffices to consider first the Whitney decomposition

$$V = \bigcup_{k=1}^{\infty} \tilde{Q}_k$$

of V , then cut each \tilde{Q}_k into a finite number of squares of size $< \delta r_0$, where r_0 is the distance from \tilde{Q}_k to $\partial\Omega$.

Applying (2.4), we get a collection of cones C^1, C^2, \dots such that $C^j \cap Q_j$ are essentially disjoint and $m(C^j \cap Q_j) \geq \eta m(Q_j)$ for all $j \geq 1$. Now, for $\mathcal{A} := \bigcup_{j \geq 1} (C^j \cap Q_j)$ we have

$$m(\mathcal{A}) = \sum_{j=1}^{\infty} m(C^j \cap Q_j) \geq \eta \sum_{j=1}^{\infty} m(Q_j) = \eta m(V).$$

Note that, since each $C^j \cap Q_j$ is included in M_n , we have $\mathcal{A} \subset M_n$. This implies that, for any nonempty open set $V \subset \Omega$ (of finite measure) and any $\theta > 0$, there exists a closed set $A \subset V \cap M_n$ such that

$$(2.5) \quad \frac{m(A)}{m(V)} \geq \eta - \theta.$$

We now introduce the following quantity

$$R := \inf_{V \subset \Omega} \sup_{A \subset M_n \cap V} \frac{m(A)}{m(V)},$$

where inf is taken over all nonempty open sets $V \subset \Omega$ and sup is taken over all closed sets $A \subset V \cap M_n$. By (2.5), we have $\eta \leq R \leq 1$. We show that $R = 1$.

Let V be as above. Choose $0 < \theta < R$. We can find a closed set $A_0 \subset V \cap M_n$ such that $m(A_0)/m(V) > R - \theta$. The set $V \setminus A_0$ is nonempty and open. Hence, by (2.5) we can find $A_1 \subset (V \setminus A_0) \cap M_n$ such that $m(A_1)/m(V \setminus A_0) > R - \theta$. We have that $A_0 \cup A_1 \subset V \cap M_n$ and

$$\begin{aligned} \frac{m(A_0 \cup A_1)}{m(V)} &= \frac{m(A_0)}{m(V)} + \frac{m(A_1)}{m(V)} \\ &\geq \frac{m(A_0)}{m(V)} + (R - \theta) \frac{m(V \setminus A_0)}{m(V)} \\ &= \frac{m(A_0)}{m(V)} + (R - \theta) \left(1 - \frac{m(A_0)}{m(V)}\right) \\ &= (1 - R + \theta) \frac{m(A_0)}{m(V)} + R - \theta \\ &\geq (1 - R + \theta)(R - \theta) + R - \theta. \end{aligned}$$

Since θ can be chosen arbitrarily small, we get

$$R \geq (1 - R)R + R.$$

Hence, we have $R = 0$ or $R = 1$. Since $R \geq \eta > 0$, we get $R = 1$.

This shows that M_n has full measure in Ω , concluding the proof of the Proposition 4. \square

A shorter proof of this proposition can be given by using Rademacher's differentiation theorem. The following proof was suggested to the author by P. Bousquet.

Another proof of Proposition 4. Consider the function $\varphi : \Omega \rightarrow \mathbb{R}$ defined by $\varphi(x) := (\text{dist}(x, \partial\Omega))^2$. Choose $x \in \Omega$ such that φ is differentiable in x . Fix $v \in \mathbb{R}^2$. If $x' \in P_x$, then

$$\begin{aligned} \varphi(x + tv) &\leq |x + tv - x'|^2 \\ &= |x - x'|^2 + 2t \langle v, x - x' \rangle + t^2 |v|^2 \\ &= \varphi(x) + 2t \langle v, x - x' \rangle + t^2 |v|^2, \end{aligned}$$

for any $t \in \mathbb{R}$ with $x + tv \in \Omega$. Hence, if $t > 0$ is as above, we get

$$\frac{\varphi(x + tv) - \varphi(x)}{t} \leq 2 \langle v, x - x' \rangle + t |v|^2,$$

and letting $t \rightarrow 0$, we obtain $\langle \nabla \varphi(x), v \rangle \leq 2 \langle v, x - x' \rangle$. By a similar argument (considering $t < 0$) we get $\langle \nabla \varphi(x), v \rangle \geq 2 \langle v, x - x' \rangle$. Since v is arbitrary, we get $\nabla \varphi(x) = 2(x - x')$. In particular, we obtain that $P_x = \{x'\}$ (x has unique projection on $\partial\Omega$). (This argument is taken from [7, p. 14].)

Since φ is locally Lipschitz, the set of points $x \in \Omega$ such that φ is differentiable in x is of full measure in Ω . By the above observation we get Proposition 4. \square

Lemma 5. Suppose $d_1, d_2 \in \mathbb{N}^*$ and $K \subset (0, 1)^{d_1} \times (0, 1)^{d_2}$ is a closed set with $m(K) > 0$. For any $y \in (0, 1)^{d_2}$ define

$$K_y := \left\{ x \in (0, 1)^{d_1} \mid (x, y) \in K \right\}.$$

Then, there exists a measurable set $A \subset (0, 1)^{d_2} \times (0, 1)^{d_2}$, with $m(A) > 0$, such that for all the pairs $(y_1, y_2) \in A$, $m(K_{y_1} \cap K_{y_2}) > 0$. In particular, there exists a point $P = (y_1, y_2) \in A$ such that all of its $2d_2$ coordinates are pairwise distinct and $m(K_{y_1} \cap K_{y_2}) > 0$.

Proof. For $(y_1, y_2) \in (0, 1)^{d_2} \times (0, 1)^{d_2}$ we write

$$\begin{aligned} m(K_{y_1} \cap K_{y_2}) &= \int_{(0,1)^{d_1}} \mathbb{1}_{K_{y_1}}(x) \mathbb{1}_{K_{y_2}}(x) dx \\ &= \int_{(0,1)^{d_1}} \mathbb{1}_K(x, y_1) \mathbb{1}_K(x, y_2) dx. \end{aligned}$$

Integrating on $(0, 1)^{d_2} \times (0, 1)^{d_2}$, and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} &\int_{(0,1)^{d_2}} \int_{(0,1)^{d_2}} m(K_{y_1} \cap K_{y_2}) dy_1 dy_2 \\ &= \int_{(0,1)^{d_1}} \left(\int_{(0,1)^{d_2}} \mathbb{1}_K(x, y_1) dy_1 \right) \left(\int_{(0,1)^{d_2}} \mathbb{1}_K(x, y_2) dy_2 \right) dx \\ &= \int_{(0,1)^{d_1}} \left(\int_{(0,1)^{d_2}} \mathbb{1}_K(x, y) dy \right)^2 dx \\ &\geq \left(\int_{(0,1)^{d_1}} \int_{(0,1)^{d_2}} \mathbb{1}_K(x, y) dy dx \right)^2 \\ &= (m(K))^2 > 0, \end{aligned}$$

whence the first claim.

To get the second claim we observe that the set of the points in $(0, 1)^{d_2} \times (0, 1)^{d_2}$ for which at least two of the $2d_2$ real coordinates coincide, is contained in a finite union of hyperplanes, and hence is a Lebesgue null set. Hence, its complement is of full measure and intersects A . \square

We now use the above lemma to prove the following.

Lemma 6. Let $\Omega \subset \mathbb{R}^2$ be an open set such that $\Omega \neq \emptyset, \mathbb{R}^2$, and $k \in \mathbb{N}^*$. Consider some real numbers $a_i, 1 \leq i \leq k, \alpha_{ij}, 1 \leq i < j \leq k$ not all zero, and $c \in \mathbb{R}$. Almost everywhere on Ω^k , we have

$$\sum_{1 \leq i \leq k} a_i \operatorname{dist}(x_i, \partial\Omega) + \sum_{1 \leq i < j \leq k} \alpha_{ij} |x_i - x_j| \neq c.$$

Proof. Consider the function $f : \Omega^k \mapsto \mathbb{R}$ defined by

$$f(X) = \sum_{1 \leq i \leq k} a_i \operatorname{dist}(x_i, \partial\Omega) + \sum_{1 \leq i < j \leq k} \alpha_{ij} |x_i - x_j|,$$

where $X := (x_1, \dots, x_k) \in \Omega^k$.

Suppose by contradiction that the set $M := \{x \in \Omega^k \mid f(X) = c\}$ has nonzero measure. Since f is continuous, the set M is closed. By Lemma 5, we can find some $Y_1 = (x_2^1, \dots, x_k^1), Y_2 = (x_2^2, \dots, x_k^2) \in \Omega^{k-1}$ such that the elements $x_2^1, \dots, x_k^1, x_2^2, \dots, x_k^2$ are pairwise distinct and $m(M_{Y_1} \cap M_{Y_2}) > 0$. We have that, for any $x \in M' := M_{Y_1} \cap M_{Y_2} \subset \Omega$,

$$(2.6) \quad a_1 \operatorname{dist}(x, \partial\Omega) + \sum_{j=2}^k \alpha_{1j} |x - x_j^1| = c_1,$$

$$(2.7) \quad a_1 \operatorname{dist}(x, \partial\Omega) + \sum_{j=2}^k \alpha_{1j} |x - x_j^2| = c_2$$

where c_1 and c_2 are some constants. By subtracting the above equalities, we get for any $x \in M'$,

$$\sum_{j=2}^k \alpha_{1j} |x - x_j^1| - \sum_{j=2}^k \alpha_{1j} |x - x_j^2| = c_3$$

for some constant c_3 . The function $g : \Omega \setminus \{x_2^1, \dots, x_d^1, x_2^2, \dots, x_d^2\} \rightarrow \mathbb{R}$ defined by

$$g(x) = \sum_{j=2}^k \alpha_{1j} |x - x_j^1| - \sum_{j=2}^k \alpha_{1j} |x - x_j^2|$$

(which is real analytic) is constant on M' . Since $m(M') > 0$, it follows that $g \equiv c_3$ on Ω .

Suppose now that $\alpha_{1j_0} \neq 0$ for some $j_0 \geq 2$. We can write

$$(2.8) \quad \alpha_{1j_0} |x - x_{j_0}^1| = - \sum_{\substack{j=2 \\ j \neq j_0}}^k \alpha_{1j} |x - x_j^1| + \sum_{j=2}^k \alpha_{1j} |x - x_j^2| + c_3$$

on Ω . However, in a neighborhood of $x_{j_0}^1$, the right hand side of (2.8) is a C^1 function, while the left hand side is not. Hence, we must have $\alpha_{1j} = 0$ for all $j \geq 2$.

By a similar argument we get that all the coefficients α_{ij} are zero.

The relation (2.6) reads now as $a_1 \operatorname{dist}(x, \partial\Omega) = c_1$ on M' . Suppose $a_1 \neq 0$ and consider the set

$$\mathcal{S} := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) = c_1/a_1\}.$$

Since $M' \subset \mathcal{S}$, the set \mathcal{S} has positive measure. Hence, there exists a Lebesgue point x_0 in \mathcal{S} , i.e., some $x_0 \in \mathcal{S}$ satisfying

$$(2.9) \quad \lim_{r \rightarrow 0} \frac{m(\mathcal{S} \cap B(x_0, r))}{m(B(x_0, r))} = 1.$$

Let $x_1 \in \partial\Omega$ such that $|x_0 - x_1| = \operatorname{dist}(x_0, \partial\Omega)$. Using the notation from the proof of Lemma 3, we have that $C_{2\pi/3} \cap \mathcal{S} \cap B(x_0, |x_0 - x_1|) = \emptyset$. Indeed, if $x \in C_{2\pi/3} \cap B(x_0, |x_0 - x_1|)$, then $\operatorname{dist}(x, \partial\Omega) < c_1/a_1$. Hence,

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{m(\mathcal{S} \cap B(x_0, r))}{m(B(x_0, r))} &= \lim_{r \rightarrow 0} \frac{m((\mathcal{S} \cap B(x_0, r)) \setminus C_{2\pi/3})}{m(B(x_0, r))} \\ &\leq \frac{2\pi - 2\pi/3}{2\pi} = \frac{2}{3}, \end{aligned}$$

which contradicts (2.9).

Hence $a_1 = c_1 = 0$. By a similar argument we get also that all the coefficients a_i are zero, obtaining a contradiction. \square

With this results we can easily prove the following

Proposition 7. Let $\Omega \subset \mathbb{R}^2$ be an open set such that $\Omega \neq \emptyset, \mathbb{R}^2$, and $k \in \mathbb{N}^*$. For almost all $X = (x_1, \dots, x_k) \in \Omega^k$ we have that the numbers $\text{dist}(x_i, \partial\Omega)$, $1 \leq i \leq k$, $|x_i - x_j|$, $1 \leq i < j \leq k$ are linearly independent over \mathbb{Z} and each x_i has a unique projection on $\partial\Omega$.

(We will say that a point X as above has the property (P).)

Proof. Let v_1, v_2, \dots , be an enumeration of the set $\mathbb{Z}^N \setminus \{0\}$, where $N := k + \binom{k}{2}$, and for each $X = (x_1, \dots, x_k) \in \Omega^k$ consider the vector

$$\Delta(X) := \left((\text{dist}(x_i, \partial\Omega))_{1 \leq i \leq k}, (|x_i - x_j|)_{1 \leq i < j \leq k} \right) \in \mathbb{R}^N.$$

Let $\Lambda_n := \{X \in \Omega^k \mid \langle v_n, \Delta(X) \rangle = 0\}$ for $n \geq 1$. By Lemma 6 we have that $m(\Lambda_n) = 0$ for all $n \geq 1$. Hence, the set

$$\begin{aligned} \Lambda &:= \{X \in \Omega^k \mid \text{there exists } v \in \mathbb{Z}^N \setminus \{0\} \text{ with } \langle v, \Delta(X) \rangle = 0\} \\ &= \bigcup_{n=1}^{\infty} \Lambda_n \end{aligned}$$

is Lebesgue null.

This fact combined with Proposition 4 gives the result. \square

Remark 8. It is easy to see that Lemma 3, Proposition 4, Lemma 6 and Proposition 7 remain true in \mathbb{R}^d for $d \geq 3$. The adaptations of the above proofs are obvious.

3. GEOMETRIC PROPERTIES OF LIFTINGS IN 2D

From now on we suppose that Ω is a smooth, bounded and simply connected domain in \mathbb{R}^2 . We are going to apply the Proposition 7 in order to obtain the prevalence of the set of those $u \in W^{1,1}(\Omega, \mathbb{S}^1)$, with a finite number of singularities, that admit a unique minimal BV-lifting. We will use the conventions and several facts from [2, Chapter 3] to describe the minimal liftings (and the minimal configurations) of a given $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ with a finite number of singularities. We quickly recall these conventions and facts.

Consider a function $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$ where $a_1, \dots, a_k \in \Omega$ are pairwise distinct points. To the vector of singularities $a = (a_1, a_2, \dots, a_k)$ we associate the vector of degrees $d = (d_1, d_2, \dots, d_k)$ where $d_j := \deg(u, a_j)$ is the degree of u computed on a small circle around a_j . We consider a fictitious point $a_{k+1} \in \partial\Omega$, of degree

$$d_{k+1} = - \sum_{j=1}^k d_j.$$

We split the family of points $a_1, a_2, \dots, a_k, a_{k+1}$ in two disjoint parts: the family of “positive points” whose degree is positive and the family of “negative points” whose degree is negative. We omit the points of zero degree. The points from the first family will be denoted P_l and those from the second family N_l . With these points we create a list $\{P_l, N_l\}_{1 \leq l \leq m}$ by repeating $|d_j|$ times each point of degree d_j . It is easy to see that there are as many positive and negative points, and therefore these points can be matched in pairs.

We introduce the following pseudometric on $\overline{\Omega}$:

$$\text{dist}_s(A_1, A_2) := \min \{|A_1 - A_2|, \text{dist}(A_1, \partial\Omega) + \text{dist}(A_2, \partial\Omega)\},$$

for $A_1, A_2 \in \overline{\Omega}$.

With this we define the quantity:

$$(3.1) \quad L(a, d) := \min_{\sigma \in S_m} \sum_{l=1}^m \text{dist}_s(P_l, N_{\sigma(l)}).$$

We recall that ([2, Chapter 3, Lemma 3.4]) we can further add points from $\partial\Omega$ to the collection $\{P_l, N_l\}_{1 \leq l \leq m}$, to obtain a possibly larger collection $\{P_l, N_l\}_{1 \leq l \leq n}$ satisfying the properties:

$$(3.2) \quad \sum_{l=1}^n \delta_{P_l} = \sum_{\substack{j=1 \\ d_j > 0}}^k d_j \delta_{a_j}, \quad \sum_{l=1}^n \delta_{N_l} = \sum_{\substack{j=1 \\ d_j < 0}}^k d_j \delta_{a_j} \quad \text{in } \mathcal{D}'(\Omega),$$

and

$$(3.3) \quad L(a, d) = \sum_{l=1}^n |P_l - N_l|.$$

We will say that a collection of oriented segments $(P_l, N_l)_{1 \leq l \leq n}$ (counted with multiplicities) formed with points satisfying (3.2) and (3.3) is a *minimal configuration* associated with (a, d) . Note that, in general there is more than one minimal configurations for a given u .

A *connection* associated with (a, d) is an \mathbb{R}^2 -valued measure μ on Ω of the form

$$\mu = \sum_{i=1}^{\infty} \nu_i \mathcal{H}^1 \llcorner (S_i \cap \Omega),$$

where S_i are Borel subsets of C^1 oriented curves in \mathbb{R}^2 of normal vectors ν_i , with

$$\sum_{i=1}^{\infty} \mathcal{H}^1[(S_i \cap \Omega)] < \infty,$$

and satisfying

$$\operatorname{curl} \mu = \sum_{j=1}^k d_j \delta_{a_j}.$$

A *minimal connection* (associated with (a, d)) is a connection μ (associated with (a, d)) such that $\|\mu\|_{\mathcal{M}} = L(a, d)$.

It is known (see [2, Chapter 3]) that there is a one-to-one correspondence between the minimal connections and the minimal liftings of a given $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$. (Recall that, by our convention, two minimal liftings are equal if they differ by an integer multiple of 2π .)

Remark 9. The above one-to-one correspondence between the minimal liftings and the minimal connections gives us that the property that $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$ admits a unique minimal BV-lifting depends only on the vector of singularities $a = (a_1, \dots, a_k) \in \Omega^k$ and the vector of degrees $d = (d_1, \dots, d_k)$.

Remark 10. Let us discuss the examples, presented in the introduction, of maps having several minimal BV-liftings.

a) In the case of $u(z) := \frac{z}{|z|}$, on $\Omega = B(0, 1)$, we have one singularity at the origin, of degree $+1$. The minimal configurations are given by the pairs (P_1, N_1) where $P_1 = 0$ and N_1 is any point on $\partial D(0, 1)$ (considered with the degree -1). Hence, there are infinitely many minimal configurations. Each one of these configurations corresponds to a minimal connection, hence we have an infinite number of minimal BV-liftings for this u .

b) In the case of $u(z) := \frac{2\bar{z} - 1}{|2\bar{z} - 1|} \frac{2z + 1}{|2z + 1|}$, on $\Omega = (-1, 1)^2$, we have two singularities, $a_1 = -1/2$, respectively $a_2 = 1/2$, of degrees $d_1 = +1$, respectively $d_2 = -1$. We have in this case exactly two minimal configurations. One configuration is given by the collection of oriented segments (P_1, N_1) , (P_2, N_2) , where $P_1 := -1/2$ (of degree $+1$), $N_1 := -1$ (of degree -1), $N_2 := 1/2$ (of degree $+1$), $P_2 := 1$ (of degree $+1$). Another minimal configuration is given by the oriented segment (P_1, N_2) (the same notation). Each one of these configurations

corresponds to a minimal connection, hence we have two minimal BV-liftings for this u .

Remark 11. In order to prove Theorem 2, we will use a property weaker than the bijective correspondence between the minimal connections and the minimal liftings. More specifically, we rely on the fact that there is a surjective correspondence between the minimal configurations and the minimal liftings of a given $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$. In particular, if there exists only one minimal configuration for u as above, then, there exists only one minimal lifting of u . (See [2, Chapter 3, Remark 3.8].)

We need to introduce some new notation. Let $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$ and (a, d) be given as above, and suppose the vector $a = (a_1, a_2, \dots, a_k) \in \Omega^k$ has the property (P) described in Proposition 7, namely, the numbers $\text{dist}(a_i, \partial\Omega)$, $1 \leq i \leq k$, $|a_i - a_j|$, $1 \leq i < j \leq k$ are linearly independent over \mathbb{Z} and each a_i has unique projection on $\partial\Omega$. Let P be a positive point and N a negative point as above. We observe that one and only one of the following may happen:

- (i) $\text{dist}_s(P, N) = |P - N|$;
- (ii) $\text{dist}_s(P, N) = |P - N'| + |P' - N|$ for some $P', N' \in \partial\Omega$ with $\text{dist}(P, \partial\Omega) = |P - N'|$ and $\text{dist}(N, \partial\Omega) = |P' - N|$. Thanks to property (P), the points N' and P' are unique.

Indeed, the definition of d_s ensures that the pair (P, N) is in at least one of the above cases. Also, thanks to the fact that $|P - N|$, $|P - N'|$, $|P' - N|$ are linearly independent over \mathbb{Z} , we have that only one of the above situations is possible.

Consider the set of oriented segments

$$\begin{aligned} M & : = \{(P_i, N_j) \mid 1 \leq i, j \leq m, (P_i, N_j) \text{ is in case (i)}\} \\ & \cup \{(P_i, N'_j) \mid 1 \leq i, j \leq m, (P_i, N_j) \text{ is in case (ii)}\} \\ & \cup \{(P'_i, N_j) \mid 1 \leq i, j \leq m, (P_i, N_j) \text{ is in case (ii)}\}, \end{aligned}$$

respectively the set of numbers

$$\begin{aligned} M_d & : = \{|P_i - N_j| \mid 1 \leq i, j \leq m, (P_i, N_j) \text{ is in case (i)}\} \\ & \cup \{|P_i - N'_j| \mid 1 \leq i, j \leq m, (P_i, N_j) \text{ is in case (ii)}\} \\ & \cup \{|P'_i - N_j| \mid 1 \leq i, j \leq m, (P_i, N_j) \text{ is in case (ii)}\}. \end{aligned}$$

Clearly, the function $\delta : M \mapsto M_d$, defined by $\delta(P, N) := |P - N|$, is a bijection.

Fix $\sigma \in S_m$. Consider the sum

$$(3.4) \quad L_\sigma := \sum_{l=1}^m \text{dist}_s(P_l, N_{\sigma(l)}).$$

Note that, thanks to the definition of dist_s , this is a sum with elements from M_d . Proposition 7 allows us to define the set

$$\mathfrak{C}_\sigma := \left\{ (\delta^{-1}(r), n) \mid (r, n) \in M_d \times \mathbb{N}, r \text{ appears exactly } n \text{ times in (3.4)} \right\}.$$

If

$$\mathfrak{C}_\sigma = \left\{ (\delta^{-1}(r_1), n_1), \dots, (\delta^{-1}(r_p), n_p) \right\},$$

let C_σ be the collection

$$\underbrace{\delta^{-1}(r_1), \dots, \delta^{-1}(r_1)}_{n_1 \text{ times}}, \dots, \underbrace{\delta^{-1}(r_p), \dots, \delta^{-1}(r_p)}_{n_p \text{ times}}.$$

Thanks to Proposition 7, we immediately see that if $\sigma_1, \sigma_2 \in S_m$ are such that $L_{\sigma_1} = L_{\sigma_2}$, then $C_{\sigma_1} = C_{\sigma_2}$. If σ is minimal, i.e, $L_\sigma = L(a, d)$, then C_σ is a minimal configuration. In particular, it follows that there is only one minimal configuration. Hence, we get Theorem 2 (see Remark 11). \square

Remark 12. Consider a connection μ associated with (a, d) as above. We can associate with μ a unique 1-rectifiable current given by

$$\mathcal{C} := \sum_{i=1}^{\infty} \tau_i \mathcal{H}^1 \llcorner (S_i \cap \Omega),$$

where τ_i is obtained from ν_i by a rotation of $-\pi/2$ (hence τ_i is tangent to the C^1 curve that supports S_i). We have

$$(3.5) \quad \partial \mathcal{C} = \sum_{j=1}^k d_j \delta_{a_j}.$$

Also to each 1-rectifiable current satisfying (3.5) we can associate a unique connection μ . In case where μ is a minimal connection, \mathcal{C} is a mass minimizing 1-rectifiable current.

In the language of geometric measure theory, Remark 9 and Remark 11 give the following: if there exists only one minimal configuration for (a, d) as above, then, there exists only one mass minimizing 1-rectifiable current (i.e., “least length curve”) with (measure geometric) boundary

$\sum_{j=1}^k d_j \delta_{a_j}$. (See [2, Chapter 3, Section 3.9.4] for details.) Thus the proof

of Theorem 2 implies the following: for a.e. $(a_1, \dots, a_k) \in \Omega^k$, and for every $(d_1, \dots, d_k) \in \mathbb{Z}^k$, there exists exactly one least length curve with

boundary $\sum_{j=1}^k d_j \delta_{a_j}$.

We next explain how Theorem 1 follows from Theorem 2. From now on, we consider domains Ω which are bounded, simply connected and smooth.

Fix $k \in \mathbb{N}$. Let $d = (d_1, \dots, d_k) \in (\mathbb{Z} \setminus \{0\})^k$ and consider the set W_d of those $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ for which there exist some distinct $a_1, \dots, a_k \in \Omega$ such that $u \in W^{1,1}(\Omega, \mathbb{S}^1) \cap C(\Omega \setminus \{a_1, \dots, a_k\})$ and $\deg(u, a_j) = d_j$ for all $1 \leq j \leq k$. The set W_d is a metric space with the norm induced by $W^{1,1}(\Omega, \mathbb{S}^1)$.

It is easy to see that each $u \in W_d$ can be written as $u = u_a e^{i\psi}$ with $a = (a_1, \dots, a_k)$ as above, with u_a given by the formula

$$u_a(z) := \prod_{j=1}^k \left(\frac{z - a_j}{|z - a_j|} \right)^{d_j}, \quad z \in \Omega,$$

and $\psi \in W^{1,1}(\Omega, \mathbb{R})$.

This can be proved as follows. From [2, Chapter 3], we have

$$J(u) = J(u_a) = \pi \sum_{j=1}^k d_j \delta_{a_j}$$

where $J(u) := \text{curl}(u \wedge \nabla u) / 2$ is the *distributional Jacobian* of u . Hence, if $v := u_a^{-1} u = \bar{u}_a u$, then $v \in W^{1,1}(\Omega, \mathbb{S}^1)$ and

$$(3.6) \quad J(v) = J(\bar{u}_a) + J(u) = -J(u_a) + J(u) = 0.$$

Here, we have used the properties

$$J(uv) = Ju + Jv, \quad J(\bar{u}) = -Ju, \quad \forall u, v \in W^{1,1}(\Omega, \mathbb{S}^1);$$

see [2, Section 2.2].

Combining (3.6) with [2, Chapter 2, Lemma 2.8], we find that there exists some $\psi \in W^{1,1}(\Omega, \mathbb{R})$ such that $v = e^{i\psi}$.

We have the following.

Lemma 13. Fix $k \in \mathbb{N}$. For each $d \in (\mathbb{Z} \setminus \{0\})^k$, the set $U_d := U \cap W_d$ is dense in W_d .

Proof. Let $\varepsilon > 0$ and $u \in W_d$. From the above observation, we can write $u = u_a e^{i\psi}$ for some $a = (a_1, \dots, a_k) \in \Omega^k$, and $\psi \in W^{1,1}(\Omega, \mathbb{R})$. If $a' \in \Omega^k$ and the distance $|a - a'|$ is sufficiently small, then $\|\nabla(u_a - u_{a'})\|_{L^1} < \varepsilon/2$ and

$$\int_{\Omega} |u_a - u_{a'}| |\nabla\psi| dx < \varepsilon / (2 + 2\|\nabla\psi\|_{L^1}).$$

For $u' := u_{a'} e^{i\psi}$ we have

$$\|\nabla(u - u')\|_{L^1} \leq \|\nabla(u_a - u_{a'})\|_{L^1} + \int_{\Omega} |u_a - u_{a'}| |\nabla\psi| dx < \varepsilon.$$

Note that Theorem 2 allows us to choose $a' \in \Omega^k$ as above and such that $u' \in U_d$ admits a unique minimal BV-lifting. \square

Note that, since the set of those $u \in W^{1,1}(\Omega, \mathbb{S}^1)$ with a finite number of singularities is dense in $W^{1,1}(\Omega, \mathbb{S}^1)$ (see [1]), Lemma 13 immediately implies that U is dense in $W^{1,1}(\Omega, \mathbb{S}^1)$. This gives the density part in Theorem 1.

To complete the proof of Theorem 1, it remains to prove that U is a G_δ set in $W^{1,1}(\Omega, \mathbb{S}^1)$. We present below the argument.

For each $u \in W^{1,1}(\Omega, \mathbb{S}^1)$, we consider the set $\mathcal{L}(u)$ of all minimal BV-liftings ϕ of u satisfying

$$(3.7) \quad \left| \frac{1}{m(\Omega)} \int_{\Omega} \phi(x) dx \right| \leq \pi.$$

We endow $\mathcal{L}(u)$ with the L^1 metric and we consider $\rho : \mathcal{L}(u) \times \mathcal{L}(u) \mapsto [0, \infty)$ defined by

$$\rho(\varphi_1, \varphi_2) := \inf_{k \in \mathbb{Z}} \|\varphi_1 - \varphi_2 + 2\pi k\|_{L^1}, \quad (\varphi_1, \varphi_2) \in \mathcal{L}(u) \times \mathcal{L}(u).$$

Define

$$(3.8) \quad \text{diam}_\rho \mathcal{L}(u) := \sup_{\phi_1, \phi_2 \in \mathcal{L}(u)} \rho(\phi_1, \phi_2),$$

and consider the sets

$$D_n := \{u \in W^{1,1}(\Omega, \mathbb{S}^1) \mid \text{diam}_\rho \mathcal{L}(u) < 1/n\}, \quad n \geq 1.$$

We easily check that $U = \bigcap_{n \geq 1} D_n$ and hence it suffices to prove that each D_n is open in $W^{1,1}(\Omega, \mathbb{S}^1)$.

For this purpose we start by establishing some useful properties.

First, let $(u_m)_{m \geq 1}$ be a sequence in $W^{1,1}(\Omega, \mathbb{S}^1)$ converging to some $u \in W^{1,1}(\Omega, \mathbb{S}^1)$, and let $(\varphi^m)_{m \geq 1}$ be a sequence in $\text{BV}(\Omega, \mathbb{R})$ such that φ^m is a minimal lifting of u_m for each $m \geq 1$. If φ^m converges to some $\varphi \in \text{BV}(\Omega, \mathbb{R})$ in the L^1 norm, then φ is a minimal lifting of u .

Indeed,

$$\|u_m - e^{i\varphi}\|_{L^1(\Omega)} = \|e^{i\varphi^m} - e^{i\varphi}\|_{L^1(\Omega)} \leq \|\varphi^m - \varphi\|_{L^1(\Omega)} \rightarrow 0,$$

when $m \rightarrow \infty$. It follows that $u_m \rightarrow e^{i\varphi} \in \text{BV}(\Omega, \mathbb{S}^1)$ in the sense of distributions and hence $u = e^{i\varphi}$, i.e., φ is a BV-lifting of u .

Define, for $u \in W^{1,1}(\Omega, \mathbb{S}^1)$,

$$\Sigma(u) := \inf_{\phi \in \text{BV}(\Omega, \mathbb{R})} \|u \wedge \nabla u - D\phi\|_{\mathcal{M}}.$$

The relevance of $\Sigma(u)$ in the study of minimal liftings is provided by the following result [2, Corollary 2.4] (see also [3]): if $\varphi \in \text{BV}(\Omega, \mathbb{R})$ is a BV-lifting of u , then u is a minimal lifting if and only if

$$(3.9) \quad \|u \wedge \nabla u - D\varphi\|_{\mathcal{M}} = \Sigma(u).$$

(We emphasize the fact that (3.9) by itself does not suffice to deduce that φ is a minimal lifting of u . We also need to know that φ is a BV-lifting of u .)

By (3.9), we have

$$(3.10) \quad \Sigma(u_m) = \|u_m \wedge \nabla u_m - D\varphi^m\|_{\mathcal{M}}.$$

Since φ is a BV-lifting of u , in order to show the minimality of φ , it suffices to prove that

$$\|u \wedge \nabla u - D\varphi\|_{\mathcal{M}} \leq \|u \wedge \nabla u - D\psi\|_{\mathcal{M}},$$

for any $\psi \in \text{BV}(\Omega, \mathbb{R})$.

Fix $\psi \in \text{BV}(\Omega, \mathbb{R})$. By (3.10) we have, for all $m \geq 1$,

$$(3.11) \quad \|u_m \wedge \nabla u_m - D\varphi^m\|_{\mathcal{M}} \leq \|u_m \wedge \nabla u_m - D\psi\|_{\mathcal{M}}.$$

Since $u_m \wedge \nabla u_m \rightarrow u \wedge \nabla u$ in L^1 , we immediately see that

$$\|u_m \wedge \nabla u_m - D\psi\|_{\mathcal{M}} \rightarrow \|u \wedge \nabla u - D\psi\|_{\mathcal{M}}.$$

Also, $D\varphi^m \rightarrow D\varphi$ in the sense of distributions and hence, from (3.11) we get

$$\begin{aligned} \|u \wedge \nabla u - D\varphi\|_{\mathcal{M}} &\leq \liminf_{m \rightarrow \infty} \|u_m \wedge \nabla u_m - D\varphi^m\|_{\mathcal{M}} \\ &\leq \|u \wedge \nabla u - D\psi\|_{\mathcal{M}}. \end{aligned}$$

A second observation is that the supremum in (3.8) is attained. Indeed, by the above observation, $\mathcal{L}(u)$ is compact in $L^1(\Omega)$. Since $\mathcal{L}(u) \times \mathcal{L}(u)$ is compact in $L^1(\Omega) \times L^1(\Omega)$ and ρ is continuous, there exist $\varphi_1, \varphi_2 \in \mathcal{L}(u)$ such that

$$(3.12) \quad \rho(\varphi_1, \varphi_2) = \text{diam}_\rho \mathcal{L}(u).$$

Going back to the proof Theorem 1, it remains to prove that D_n^c is a closed set. We have that:

$$D_n^c = \{u \in W^{1,1}(\Omega, \mathbb{S}^1) \mid \text{diam}_\rho \mathcal{L}(u) \geq 1/n\}.$$

Suppose that $(u_m)_{m \geq 1}$ is a sequence in D_n^c converging to some $u \in W^{1,1}(\Omega, \mathbb{S}^1)$. From (3.12), there exist two sequences $(\varphi_1^m)_{m \geq 1}, (\varphi_2^m)_{m \geq 1}$ with $\varphi_1^m, \varphi_2^m \in \mathcal{L}(u_m)$ for all $m \geq 1$, such that $u_m = e^{i\varphi_1^m} = e^{i\varphi_2^m}$ and

$$(3.13) \quad \rho(\varphi_1^m, \varphi_2^m) = \text{diam}_\rho \mathcal{L}(u_m) \geq 1/n.$$

Since $(u_m)_{m \geq 1}$ is bounded in $W^{1,1}(\Omega, \mathbb{S}^1)$, we get $\|\varphi_1^m\|_{\text{BV}}, \|\varphi_2^m\|_{\text{BV}} \lesssim \|u_m\|_{W^{1,1}} \lesssim 1$. Hence, there exist $\varphi_1, \varphi_2 \in \text{BV}(\Omega, \mathbb{R})$ such that $\varphi_1^m \rightarrow \varphi_1, \varphi_2^m \rightarrow \varphi_2$ in L^1 , possibly up to a subsequence. According to our observation, φ_1 and φ_2 are minimal liftings of u . We have from (3.13) and the continuity of ρ that $\rho(\varphi_1, \varphi_2) \geq 1/n$. Also, φ_1, φ_2 satisfy (3.7). Therefore, we have $u \in D_n^c$.

The proof of Theorem 1 is complete. \square

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