Coherent presentations of super plactic monoids of type A by insertions

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Abstract – In this paper, we study by rewriting methods the presentations of the super plactic monoid of type A which is related to the representations of the general linear Lie superalgebra. We construct a convergent presentation of this monoid whose generators are super columns and whose rules are defined by insertions on super tableaux over a signed alphabet. We extend this presentation into a coherent one whose syzygies are defined as relations among insertion algorithms. Finally, we reduce this coherent presentation to a Tietze equivalent one over the initial signed alphabet. Such coherent presentations are used for representations of super plactic monoids by describing their actions on categories.

Keywords – Super plactic monoids, syzygies, super tableaux, insertions, string rewriting.

M.S.C. 2010 – Primary: 20M05, 16S15. **Secondary:** 68Q42, 20M35.

1. Introduction

We study the presentations of the super version of the plactic monoid of type A over a signed alphabet by rewriting methods using combinatorial properties of super tableaux. Study presentations by rewriting consists in the orientation of the relations, then called *reduction rules*. Two words are equal in a monoid presented by a rewriting system if they are related by a zig-zag sequence of applications of reductions rules. A rewriting system is *convergent* if the reduction relation induced by the rules is well-founded and any reductions starting on a same word can be extended to end on a same reduced word. Starting with a presentation of a monoid, we are interested in the computation of all *syzygies* for this presentation and in particular to compute a family of generators for the syzygies. In commutative algebra, the theory of Gröbner bases gives algorithms to compute bases for linear syzygies. By a similar method, the syzygy problem for presentation of monoids can be algorithmically solved using *coherent convergent presentations*. Such presentations extend the notion of a convergent presentation of the monoid by globular *homotopy generators* taking into account the relations amongst the relations. Study the syzygies in a monoid produces in higher dimensions free objects that are homotopically equivalent to the original monoid and then allows us to compute its *homological invariants*. Indeed, this study provides the first two steps in the computation of a *polygraphic resolution* of the monoid, that is, a categorical cofibrant replacement of the monoid in a free $(\omega, 1)$-category, whose acyclicity is proved by an iterative construction of a normalization reduction strategy, [15]. Moreover, coherent presentations are used to describe the notion of *actions of the monoid on categories*, [12].

This paper is a part of a broader project that consists of studying, by a rewriting approach, families of *plactic-like monoids* defined from combinatorial objects constructed using insertion algorithms. For instance, plactic monoids are related to Young tableaux, [25, 26], Chinese monoids to Chinese staircases, [9], hypoplactic monoids to quasi-ribbon tableaux, [30], patience sorting monoids to patience sorting tableaux, [8], and stalactic monoids to stalactic tableaux [31]. Moreover, binary search trees

are used to describe normal forms for sylvester, taiga and Baxter monoids, [13, 21, 31]. Coherent presentations are constructed for Artin monoids in [12], for plactic monoids of type A in [19], for plactic monoids of type C in [29], and for Chinese monoids in [20].

A *signed alphabet* is a finite or countable totally ordered set $\mathcal{S}$ which is disjoint union of two subsets $\mathcal{S}_0$ and $\mathcal{S}_1$. The *super plactic monoid* over a signed alphabet $\mathcal{S}$, denoted by $\mathbf{P}(\mathcal{S})$, is presented by the rewriting system $\mathrm{Knuth}_2(\mathcal{S})$ on $\mathcal{S}$ submitted to the following family of super *Knuth-like relations*, [24]:

$$\begin{aligned} zxy &\Rightarrow xzy, \quad \text{with } x = y \text{ only if } y \in \mathcal{S}_0 \text{ and } y = z \text{ only if } y \in \mathcal{S}_1, \\ yzx &\Rightarrow yxz, \quad \text{with } x = y \text{ only if } y \in \mathcal{S}_1 \text{ and } y = z \text{ only if } y \in \mathcal{S}_0, \end{aligned}$$

oriented with respect the lexicographic order, for any $x \leqslant y \leqslant z$ of elements of $\mathcal{S}$. The congruence generated by this rewriting system, denoted by $\sim_{\mathbf{P}(\mathcal{S})}$, also relates those words that yield the same *super tableau* as the result of the *super Schensted-like insertion* algorithm, [24]. This is the *cross-section property* of super tableaux with respect to the congruence relation $\sim_{\mathbf{P}(\mathcal{S})}$. Note that when all the elements of $\mathcal{S}_0$ are less than the ones of $\mathcal{S}_1$, the congruence $\sim_{\mathbf{P}(\mathcal{S})}$ is defined by $u \sim_{\mathbf{P}(\mathcal{S})} v$ if and only if there is a *crystal isomorphism* between connected components of the crystal graph of the vector representation of the *general linear Lie superalgebra*, that map u to v, [1]. Note also that the monoid $\mathbf{P}(\mathcal{S})$ appeared in [27] as a deformation of the *parastatistics algebra* which is is a superalgebra with even parafermi and odd parabose creation and annihilation operators. Moreover, super algebraic structures have found many applications as combinatorial tools in the study of the invariant theory of superalgebras, the representation theory of general Lie super algebras, and algebras satisfying identities, [2, 3, 5, 14]. When $\mathcal{S} = \mathcal{S}_0 = \{1 < \ldots < n\}$, we recover the notion of the *plactic monoid of type A*, [25], which emerged from the works of Schensted [32] and Knuth [22] on the combinatorial study of Young tableaux. Plactic monoids have found several applications in algebraic combinatorics, representation theory and probabilistic combinatorics, [10, 11, 28], and they were recently investigated by rewriting methods. Indeed, for rank $n > 3$, the *Knuth presentation* of the plactic monoid of type A does not admit a finite completion with respect the lexicographic order, [23]. Then convergent presentations are constructed by adding *column generators* and *rows generators* to the Knuth presentation, [4, 6], and similar convergent presentations are constructed for the plactic monoids of classical types, [7, 17]. The author and Malbos extend in [19] the *column presentation* of the plactic monoid of type A into a coherent presentation of this monoid and we reduce it into a smaller one having Knuth's generators.

Let $\mathcal{S}$ be a signed alphabet. A *super tableau* over $\mathcal{S}$ is a collection of boxes in left-justified rows filled by elements of $\mathcal{S}$ such that the entries in each row are weakly increasing from left to right with respect $\mathcal{S}_0$ and the ones in each column are weakly increasing from top to bottom with respect $\mathcal{S}_1$. Note that when $\mathcal{S} = \mathcal{S}_0$ and $\mathcal{S} = \mathcal{S}_1$, we recover the notion of *row-strict* and *column-strict semistandard tableaux* of type A, [11]. We will denote by $\mathrm{Yt}(\mathcal{S})$ the set of all super tableaux over $\mathcal{S}$ and by R_{row} the map on $\mathrm{Yt}(\mathcal{S})$ that reads the entries of a super tableau row-wise from bottom to top and from left to right. A *super column* is a word $x_1 \ldots x_k$ over $\mathcal{S}$ such that $x_{i+1} \leqslant x_i$ with $x_i = x_{i+1}$ only if $x_i \in \mathcal{S}_1$. We will denote by $\mathrm{col}(\mathcal{S})$ the set of all super columns over $\mathcal{S}$. Super Schensted-like left and right insertion algorithms are introduced in [24], and consist in inserting elements of $\mathcal{S}$ into super tableaux by rows and columns respectively. Define the map $[\![.]\!]_r$ on the set of words over $\mathcal{S}$ sending a word to the corresponding super tableau by inserting its letters iteratively from left to right using the right insertion starting from the empty tableau. Following the cross-section property, we deduce that the internal product $\star_r$ defined on $\mathrm{Yt}(\mathcal{S})$ by setting $t \star_r t' := [\![R_{row}(t)R_{row}(t')]\!]_r$, for all t and t' in $\mathrm{Yt}(\mathcal{S})$, is associative, and then the set $(\mathrm{Yt}(\mathcal{S}), \star_r)$ is isomorphic to the super plactic monoid $\mathbf{P}(\mathcal{S})$. We show in

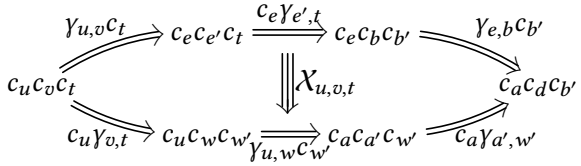
Lemma 2.6 that for all u and v in $\mathrm{col}(\mathcal{S})$ such that the topmost juxtaposition of $[\![u]\!]_r$ and $[\![v]\!]_r$ does not form a super tableau, the super tableau $[\![uv]\!]_r$ contains at most two columns, and if it contains exactly two columns then the left one contains more elements than u.

We construct in Section 2 a convergent presentation of the monoid $\mathbf{P}(\mathcal{S})$, denoted by $\mathrm{Col}_2(\mathcal{S})$, whose set of generators is $\mathrm{Col}_1(\mathcal{S}) := \{c_u \mid u \in \mathrm{col}(\mathcal{S})\}$ and whose rules are

$$\gamma_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

for every c_u, c_v in $\mathrm{Col}_1(\mathcal{S})$ such that the topmost juxtaposition of $[\![u]\!]_r$ and $[\![v]\!]_r$ does not form a super tableau and where w and w' denote the readings of the left and right columns of $[\![uv]\!]_r$ respectively. We show that $\mathrm{Col}_2(\mathcal{S})$ can be obtained from $\mathrm{Knuth}_2(\mathcal{S})$ by applying *Tietze transformations* that consist in adding or removing derivable generators and in adding or removing derivable relations on a presentation of a monoid in such a way that they do not change the presented monoid. We also show that the confluence of $\mathrm{Col}_2(\mathcal{S})$ is a direct consequence of the associativity of the product $\star_r$. Theorem 2.12 states that $\mathrm{Col}_2(\mathcal{S})$ is a convergent presentation of the monoid $\mathbf{P}(\mathcal{S})$, called the *super column presentation.*

We extend in Section 3 the super column presentation into coherent presentations of the monoid $\mathbf{P}(\mathcal{S})$ using the *homotopical completion-reduction procedure*, [12]. Denote by $\mathrm{Col}_3(\mathcal{S})$ the extended presentation of $\mathbf{P}(\mathcal{S})$ obtained from $\mathrm{Col}_2(\mathcal{S})$ by adjunction of one family of syzygies of the following form



for every c_u, c_v and c_t in $\mathrm{Col}_1(\mathcal{S})$ such that the juxtapositions of $[\![u]\!]_r$ and $[\![v]\!]_r$, and of $[\![v]\!]_r$ and $[\![t]\!]_r$ do not form super tableaux. Following Squier's coherence theorem, [33], Theorem 3.3 states that $\mathrm{Col}_3(\mathcal{S})$ is a coherent presentation of the monoid $\mathbf{P}(\mathcal{S})$, called the *super column coherent presentation.* We then apply the homotopical reduction procedure on $\mathrm{Col}_3(\mathcal{S})$ in order to reduce it into a smaller one. As a first step, we apply a homotopical reduction on $\mathrm{Col}_3(\mathcal{S})$ with a *collapsible part* defined by some of the *generating triple confluences* of $\mathrm{Col}_2(\mathcal{S})$ and we reduce it into the coherent presentation $\overline{\mathrm{Col}_3}(\mathcal{S})$ of $\mathbf{P}(\mathcal{S})$, whose underlying rewriting system is $\mathrm{Col}_2(\mathcal{S})$ and the syzygies $\mathcal{X}_{x,v,t}$ are those of $\mathrm{Col}_3(\mathcal{S})$, but with x in $\mathcal{S}$. In a second step, we reduce $\overline{\mathrm{Col}_3}(\mathcal{S})$ into a coherent presentation $\mathrm{PreCol}_3(\mathcal{S})$ of $\mathbf{P}(\mathcal{S})$, using a collapsible part defined by a set of syzygies of $\overline{\mathrm{Col}_3}(\mathcal{S})$. In a final step, we reduce $\mathrm{PreCol}_3(\mathcal{S})$ into an extended presentation of $\mathbf{P}(\mathcal{S})$, denoted by $\mathrm{Knuth}_3(\mathcal{S})$, whose underlying rewriting system is $\mathrm{Knuth}_2(\mathcal{S})$. Theorem 3.12 states that $\mathrm{Knuth}_3(\mathcal{S})$ is a coherent presentation of the monoid $\mathbf{P}(\mathcal{S})$. Finally, we use in Subsection 3.13 this coherent presentation in order to describe the actions of super plactic monoids on categories.

Notation. Let $\mathcal{A}$ be a totally ordered alphabet. We will denote by $\mathcal{A}^*$ the free monoid of *words over* $\mathcal{A}$, the product being concatenation of words, and the identity being the empty word. We will denote by $w = x_1 \dots x_k$ a word in $\mathcal{A}^*$ of *length* k, where $x_1, \dots, x_k$ belong to $\mathcal{A}$. The length of a word w will be denoted by $|w|$. Let $w = x_1 \dots x_k$ be a word in $\mathcal{A}^*$. We denote by $\ell(w)$ the leftmost letter of w and by $\mathrm{Rem}(w)$ the subword of w such that $w = \ell(w)\,\mathrm{Rem}(w)$. A word w' is a *subsequence* of w if $w' = x_{i_1} \dots x_{i_l}$ with $1 \leqslant i_1 < \dots < i_l < k$. We will denote by $[n]$ the ordered set $\{1 < \dots < n\}$ for n

in $\mathbb{Z}_{>0}$. Let $\mathcal{S}$ be a finite or countable totally ordered set and $||.|| : \mathcal{S} \to \mathbb{Z}_2$ be any map, where $\mathbb{Z}_2 = \{0, 1\}$ denotes the additive cyclic group of order 2. The ordered pair $(\mathcal{S}, ||.||)$ is called a *signed alphabet*, and we denote $\mathcal{S}_0 = \{a \in \mathcal{S} \mid ||a|| = 0\}$ and $\mathcal{S}_1 = \{a \in \mathcal{S} \mid ||a|| = 1\}$. A monoid $\mathbf{M}$ is said a $\mathbb{Z}_2$-*graded monoid* or a *supermonoid* if a map $||.|| : \mathbf{M} \to \mathbb{Z}_2$ is given such that $||u.v|| = ||u|| + ||v||$, for all u and v in $\mathbf{M}$. We call $||u||$ the $\mathbb{Z}_2$-degree of the element u. The free monoid $\mathcal{S}^*$ over $\mathcal{S}$ is $\mathbb{Z}_2$-graded by considering $||w|| := ||x_1|| + \ldots + ||x_k||$, for any word $w = x_1 \ldots x_k$ in $\mathcal{S}^*$. In the rest of this article, and if there is no possible confusion, $\mathcal{S}$ denotes a signed alphabet.

2. Super column presentation of the super plactic monoid

In this article, rewriting methods are presented in the language of polygraphs, that we recall in this section and we refer the reader to [12, 16] for a deeper presentation. We also recall the notions of super plactic monoids and super tableaux from [18, 24]. We end this section by constructing a convergent presentation of the super plactic monoid by adding super columns generators and using combinatorial properties of super tableaux.

2.1. Presentations of monoids by 2-polygraphs. In this article, we deal with presentations of monoids by rewriting systems, described by 2-polygraphs with only 0-cell denoted by $\bullet$. Such a 2-*polygraph* Σ is given by a pair (Σ_1, Σ_2), where Σ_1 is a set and Σ_2 is a *globular extension* of the free monoid Σ_1^*, that is a set of 2-cells $\alpha : s_1(\alpha) \Rightarrow t_1(\alpha)$ relating 1-cells in Σ_1^*, where $s_1(\alpha)$ and $t_1(\alpha)$ denote the *source* and the *target* of α respectively. If there is no possible confusion, Σ_2 will denote the 2-polygraph itself. Recall that a 2-*category* (resp. $(2,1)$-*category*) is a category enriched in categories (resp. in groupoids). When two 1-cells, or 2-cells, f and g of a 2-category are 0-composable (resp. 1-composable), we denote by fg (resp. $f \star_1 g$) their 0-composite (resp. 1-composite). We will denote by Σ_2^* (resp. $\Sigma_2^\top$) the 2-category (resp. $(2,1)$-category) freely generated by the 2-polygraph Σ, see [16] for more information. The *monoid presented by a 2-polygraph* Σ, denoted by $\overline{\Sigma}$, is defined as the quotient of the free monoid Σ_1^* by the congruence generated by the set of 2-cells Σ_2. A *presentation* of a monoid $\mathbf{M}$ is a 2-polygraph whose presented monoid is isomorphic to $\mathbf{M}$. Two 2-polygraphs are *Tietze equivalent* if they present isomorphic monoids. A 2-cell α of a 2-polygraph Σ is *collapsible*, if $t_1(\alpha)$ is a 1-cell of Σ_1 and $s_1(\alpha)$ does not contain $t_1(\alpha)$, then $t_1(\alpha)$ is called *redundant*. An *elementary Tietze transformation* of a 2-polygraph Σ is a 2-functor with domain $\Sigma_2^\top$ that belongs to one of the following transformations, [12]:

i) *adjunction* $\iota_\alpha^1 : \Sigma_2^\top \to \Sigma_2^\top[x](\alpha)$ (resp. $\iota_\alpha : \Sigma_2^\top \to \Sigma_2^\top(\alpha)$) of a redundant 1-cell x with its collapsible 2-cell α (resp. of a redundant 2-cell α),

ii) *elimination* $\pi_\alpha : \Sigma_2^\top \to (\Sigma_1 \setminus \{x\}, \Sigma_2 \setminus \{\alpha\})^\top$ (resp. $\pi_{(\mu,\alpha)} : \Sigma_2^\top \to \Sigma_2^\top/(\mu, \alpha)$) of a redundant 1-cell x with its collapsible 2-cell α (resp. of a redundant 2-cell α).

If Σ and Υ are 2-polygraphs, a *Tietze transformation* from Σ to Υ is a 2-functor $F : \Sigma^\top \to \Upsilon^\top$ that decomposes into sequence of elementary Tietze transformations. Two 2-polygraphs are Tietze equivalent if and only if there exists a Tietze transformation between them, [12]. Given a 2-polygraph Σ and a 2-cell $\mu_1 \star_1 \mu \star_1 \mu_2$ in $\Sigma_2^\top$, the *Nielsen transformation* $\kappa_{\mu \leftarrow \alpha}$ is the Tietze transformation that replaces in $\Sigma_2^\top$ the 2-cell μ by a 2-cell $\alpha : s_1(\mu_1) \Rightarrow t_1(\mu_2)$. When μ_2 is identity, we denote by $\kappa'_{\mu \leftarrow \alpha}$ the Nielsen transformation which, given a 2-cell $\mu_1 \star_1 \mu$ in $\Sigma_2^\top$, replaces the 2-cell μ by a 2-cell $\alpha : s_1(\mu_1) \Rightarrow t_1(\mu)$.

A *rewriting step* of a 2-polygraph Σ is a 2-cell of Σ_2^* with shape $w\alpha w'$, where α is a 2-cell of Σ_2 and w and w' are 1-cells of Σ_1^*. A *rewriting sequence* of Σ is a finite or infinite sequence of rewriting steps. A 1-cell u of Σ_1^* is a *normal form* if there is no rewriting step with source u. The 2-polygraph Σ *terminates* if it has no infinite rewriting sequence. A *branching* of Σ is a non ordered pair (f, g) of 2-cells of Σ_2^* such that $s_1(f) = s_1(g)$. A branching (f, g) is *local* if f and g are rewriting steps. A branching is *aspherical* if it is of the form (f, f), for a rewriting step f and *Peiffer* when it is of the form (fv, ug) for rewriting steps f and g with $s_1(f) = u$ and $s_1(g) = v$. The *overlapping* branchings are the remaining local branchings. An overlapping local branching is *critical* if it is minimal for the order $\sqsubseteq$ generated by the relations $(f, g) \sqsubseteq (wfw', wgw')$, given for any local branching (f, g) and any possible 1-cells w and w' of the category Σ_1^*. A branching (f, g) is *confluent* if there exist 2-cells f' and g' in Σ_2^* such that $s_1(f') = t_1(f)$, $s_1(g') = t_1(g)$ and $t_1(f') = t_1(g')$. A 2-polygraph Σ is *confluent* if all of its branchings are confluent. It is *convergent* if it terminates and it is confluent. In that case, every 1-cell u of Σ_1^* has a unique normal form.

2.2. Super Young tableaux and insertions. A *partition* of a positive integer n, is a weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$ such that $\sum \lambda_i = n$. The integer k is called *number of parts* or *height* of λ. The *Young diagram* of a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is the set $\mathcal{Y}(\lambda) := \{(i, j) \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \lambda_i\}$, that can be represented by a diagram by drawing a box for each pair (i, j). The transposed diagram $\{(j, i) \mid (i, j) \in \mathcal{Y}(\lambda)\}$ defines another partition, called the *conjugate partition* of λ, whose parts are the lengths of the columns of $\mathcal{Y}(\lambda)$. Let λ be a partition. A *super semistandard Young tableau*, or *super tableau* for short, over $\mathcal{S}$ is a pair $t := (\lambda, \mathcal{T})$ where $\mathcal{T} : \mathcal{Y}(\lambda) \to \mathcal{S}$ is a map satisfying $\mathcal{T}(i, j) \leqslant \mathcal{T}(i, j+1)$, with $\mathcal{T}(i, j) = \mathcal{T}(i, j+1)$ only if $||\mathcal{T}(i, j)|| = 0$, and $\mathcal{T}(i, j) \leqslant \mathcal{T}(i+1, j)$, with $\mathcal{T}(i, j) = \mathcal{T}(i+1, j)$ only if $||\mathcal{T}(i, j)|| = 1$. We will call $\mathcal{Y}(\lambda)$, $\mathcal{T}$ and λ, the *frame*, the *filing* and the *shape* of the super tableau t respectively. We will denote by $\mathrm{Yt}(\mathcal{S})$ the set of all super tableaux over $\mathcal{S}$.

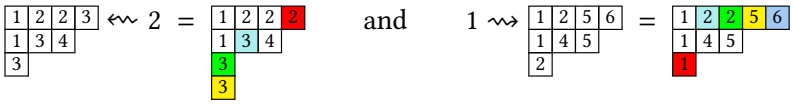
Denote by R_{row} (resp. R_{col}) the reading map on $\mathrm{Yt}(\mathcal{S})$ that reads a super tableau row-wise (column-wise) from bottom to top and from left to right. For instance, consider the alphabet $\mathcal{S} = \{1, 2, 3, 4, 5\}$ with signature given by $\mathcal{S}_0 = \{1, 2, 4\}$ and $\mathcal{S}_1$ defined consequently. The following diagram is a super tableau over $\mathcal{S}$:

$$t = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 4 & 4 \\ \hline 5 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array} \qquad \text{with } R_{row}(t) = 55344112 \quad \text{and } R_{col}(t) = 55314142.$$

A *super row* (resp. *super column*) is a word $x_1 \dots x_k$ in $\mathcal{S}^*$ such that $x_i \leqslant x_{i+1}$ (resp. $x_{i+1} \leqslant x_i$) with $x_i = x_{i+1}$ only if $||x_i|| = 0$ (resp. $||x_i|| = 1$). In other words, a super row (resp. super column) is the reading of a super tableau whose shape is a row (resp. column). We will denote by $\mathrm{col}(\mathcal{S})$ the set of all super columns over $\mathcal{S}$. Denote by $\leqslant_{\mathrm{deglex}}$ (resp. $\leqslant_{\mathrm{rev}}$) the *length lexicographic order* (resp. *length reverse lexicographic order*) on $\mathrm{col}(\mathcal{S})$ defined by $u \leqslant_{\mathrm{deglex}} v$ (resp. $u \leqslant_{\mathrm{rev}} v$) if $|u| < |v|$ (resp. $|u| > |v|$) or $|u| = |v|$ and $u <_{lex} v$, for all u and v in $\mathrm{col}(\mathcal{S})$, where $<_{\mathrm{lex}}$ denotes the lexicographic order on $\mathcal{S}$.

Recall from [24] the right and left insertion algorithms on $\mathrm{Yt}(\mathcal{S})$ that insert an element x in $\mathcal{S}$ into a super tableau t of $\mathrm{Yt}(\mathcal{S})$. The *right (or row) insertion*, denoted by $\leftsquigarrow$, computes a super tableau $t \leftsquigarrow x$ as follows. If $x \in \mathcal{S}_o$ (resp. $x \in \mathcal{S}_1$) is at least as large as (resp. larger than) the last element of the top row of t, then put x in a box to the right of this row. Otherwise, let y be the smallest element of the top row of t such that $y > x$ (resp. $y \geqslant x$). Then x replaces y in this row and y is bumped into the next row

where the process is repeated. The algorithm terminates when the element which is bumped is at least as large as (resp. larger than) the last element of the next row. Then it is placed in a box at the right of that row. The *left (or column) insertion*, denoted by $\rightsquigarrow$, computes a super tableau $x \rightsquigarrow t$ as follows. If $x \in \mathcal{S}_o$ (resp. $x \in \mathcal{S}_1$) is larger than (resp. at least as large as) the bottom element of the leftmost column of t, then put x in a box to the bottom of this column. Otherwise, let y be the smallest element of the leftmost column of t such that $y \geqslant x$ (resp. $y > x$). Then x replaces y in this column and y is bumped into the next column where the process is repeated. The algorithm terminates when the element which is bumped is greater than (resp. at least as large as) all the elements of the next column. Then it is placed in a box at the bottom of that column. For instance, consider $\mathcal{S} = \mathbb{N}$ with signature given by $\mathcal{S}_0$ the set of even numbers and $\mathcal{S}_1$ defined consequently. We have



Note that when $\mathcal{S} = \mathcal{S}_0 = [n]$, the right (resp. left) insertion corresponds to the Schensted's right (resp. left) insertion introduced in [32] on row-strict semistandard tableaux over $[n]$. For any word $w = x_1 \dots x_k$ over $\mathcal{S}$, denote by $[\![w]\!]_r$ the super tableau obtained from w by inserting its letters iteratively from left to right using the right insertion starting from the empty tableau:

$$[\![w]\!]_r \;:=\; (\emptyset \leftsquigarrow w) \;=\; ((\dots(\emptyset \leftsquigarrow x_1) \leftsquigarrow \dots) \leftsquigarrow x_k).$$

Note that for any super tableau t in $\mathrm{Yt}(\mathcal{S})$, the equality $[\![R_{row}(t)]\!]_r = t$ holds in $\mathrm{Yt}(\mathcal{S})$, [24]. We define an internal product $\star_r$ on $\mathrm{Yt}(\mathcal{S})$ by setting

$$t \star_r t' := (t \leftsquigarrow R_{row}(t'))$$

for all t, t' in $\mathrm{Yt}(\mathcal{S})$. By definition the relations $t \star_r \emptyset = t$ and $\emptyset \star_r t = t$ hold, showing that the product $\star_r$ is unitary with respect to $\emptyset$.

2.3. The super plactic monoid. The *super plactic monoid* over $\mathcal{S}$, denoted by $\mathbf{P}(\mathcal{S})$, is presented by the 2-polygraph $\mathrm{Knuth}_2(\mathcal{S})$, whose set of 1-cells is $\mathcal{S}$ and whose 2-cells are, [24] :

$$\begin{array}{l} \eta_{x,y,z} : zxy \Rightarrow xzy, \;\; \text{with} \;\; x = y \;\; \text{only if} \;\; ||y|| = 0 \;\; \text{and} \;\; y = z \;\; \text{only if} \;\; ||y|| = 1, \\ \varepsilon_{x,y,z} : yzx \Rightarrow yxz, \;\; \text{with} \;\; x = y \;\; \text{only if} \;\; ||y|| = 1 \;\; \text{and} \;\; y = z \;\; \text{only if} \;\; ||y|| = 0, \end{array} \tag{1}$$

oriented with respect the lexicographic order, for any $x \leqslant y \leqslant z$ of elements of $\mathcal{S}$. The congruence generated by this 2-polygraph, denoted by $\sim_{\mathbf{P}(\mathcal{S})}$, is called the *super plactic congruence*. Note that since the relations (1) are $\mathbb{Z}_2$-homogeneous we have that $\mathbf{P}(\mathcal{S})$ is a supermonoid. Moreover, for any w in $\mathcal{S}^*$, we have $w \sim_{\mathbf{P}(\mathcal{S})} R_{row}([\![w]\!]_r)$, [24], and for any t in $\mathrm{Yt}(\mathcal{S})$, we have $R_{row}(t) \sim_{\mathbf{P}(\mathcal{S})} R_{col}(t)$, [18]. Note finally that super tableaux satisfy the *cross-section property* for $\sim_{\mathbf{P}(\mathcal{S})}$, that is, for all w and w' in $\mathcal{S}^*$, $w \sim_{\mathbf{P}(\mathcal{S})} w'$ if and only if $[\![w]\!]_r = [\![w']\!]_r$, [24]. As a consequence of the cross-section property, we deduce that the product $\star_r$ is associative and the following equality

$$y \rightsquigarrow (t \leftsquigarrow x) \;=\; (y \rightsquigarrow t) \leftsquigarrow x \tag{2}$$

holds in $\mathrm{Yt}(\mathcal{S})$, for all t in $\mathrm{Yt}(\mathcal{S})$ and x, y in $\mathcal{S}$. In particular, for any word $w = x_1 \dots x_k$ in $\mathcal{S}^*$ the super tableau $[\![w]\!]_r$ can be also computed by inserting its elements iteratively from right to left using the left insertion starting from the empty tableau:

$$[\![w]\!]_r \;=\; (w \rightsquigarrow \emptyset) \;:=\; (x_1 \rightsquigarrow (\dots \rightsquigarrow (x_k \rightsquigarrow \emptyset)\dots)).$$

Note that the associativity of the product $\star_r$ can be obtained using the properties of the *super jeu de taquin* introduced in [18]. Note also that we can show the equality (2) using only the definitions of the insertion algorithms and without supposing the cross-section property, and then the associativity of the product $\star_r$ and the cross-section property will be consequences of this equality.

Let w be in $\mathcal{S}^*$. For any $k \geqslant 0$, denote by $l_k(w)$ (resp. $\widetilde{l_k}(w)$) the maximal number which can be obtained as the sum of the lengths of k super rows (resp. super columns) that are disjoint subsequences of w. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be the shape of the super tableau $[\![w]\!]_r$ and $\widetilde{\lambda} = (\widetilde{\lambda_1}, \dots, \widetilde{\lambda_l})$ be the conjugate partition of λ. For any $k \geqslant 0$, we have $l_k(w) = \lambda_1 + \dots + \lambda_k$ and $\widetilde{l_k}(w) = \widetilde{\lambda_1} + \dots + \widetilde{\lambda_l}$, [24]. In particular, we deduce the following result.

2.4. Lemma ([24]). *Let w be a word over $\mathcal{S}$. The number of columns in the super tableau $[\![w]\!]_r$ is equal to the length of the longest super row word that is a subsequence of w, and the number of rows in $[\![w]\!]_r$ is equal to the length of the longest super column word that is a subsequence of w.*

2.5. Graphical notations. Given $u = x_1 \dots x_p$ and $v = y_1 \dots y_q$ in $\mathrm{col}(\mathcal{S})$. We will use the following notations depending on whether the juxtaposition of $[\![u]\!]_r$ and $[\![v]\!]_r$ forms a super tableau:

i) we will denote $u^\frown v$ if the topmost juxtaposition of $[\![u]\!]_r$ and $[\![v]\!]_r$ forms a super tableau, that is, $|u| \geqslant |v|$ and $x_i \leqslant y_i$, for any $i \leqslant |v|$, with $x_i = y_i$ only if $||x_i|| = 0$,

ii) we will denote $u^\times v$ in all the other cases, that is, when $|u| < |v|$ or $x_i \geqslant y_i$, for some $i \leqslant |v|$, with $x_i = y_i$ only if $||x_i|| = 1$.

2.6. Lemma. *Let u and v be in $\mathrm{col}(\mathcal{S})$ such that $u^\times v$. The super tableau $[\![uv]\!]_r$ contains at most two columns. Moreover, if $[\![uv]\!]_r$ contains exactly two columns, the left column contains more elements than u.*

Proof. Consider $u = x_1 \dots x_p$ and $v = y_1 \dots y_q$ in $\mathrm{col}(\mathcal{S})$ such that $u^\times v$. If $x_p \geqslant y_1$ with $x_p = y_1$ only if $||x_p|| = 1$, then the super tableau $[\![uv]\!]_r$ consists of only column whose reading is uv. Otherwise, suppose $x_p \leqslant y_1$ with $x_p = y_1$ only if $||x_p|| = 0$. Since the words u and v are decreasing with respect to $\mathcal{S}_1$, the longest super row word that is a subsequence of uv contains one element from each of u and v. Then its length is equal to 2, showing by Lemma 2.4 that $[\![uv]\!]_r$ contains two columns. Suppose now that $[\![uv]\!]_r$ contains exactly two columns. Then its leftmost column is obtained by inserting some elements of v into u, and by keeping the elements of u unchangeable, showing that it contains more elements than u. □

When $u^\times v$, we will denote $u^{\times 1} v$ if the super tableau $[\![uv]\!]_r$ consists of one column and by $u^{\times 2} v$ if it consists of two columns.

2.7. Super columns as generators. Suppose $\mathcal{S} = \{x_1, x_2, \ldots\}$. Let $\mathrm{Col}_1(\mathcal{S}) = \{c_u \mid u \in \mathrm{col}(\mathcal{S})\}$ be the set of *super column generators* of the super plactic monoid $\mathbf{P}(\mathcal{S})$ and

$$\mathrm{C}_2(\mathcal{S}) \;=\; \{\, \mu_u : c_{x_p} \ldots c_{x_1} \Rightarrow c_u \;\mid u = x_p \ldots x_1 \in \mathrm{col}(\mathcal{S}) \;\text{ with } |u| \geqslant 2\}$$

be the set of the defining relations for the super column generators. We will denote by $\mathrm{Knuth}_2^{\mathrm{c}}(\mathcal{S})$ the 2-polygraph whose set of 1-cells is $\{c_{x_1}, c_{x_2}, \ldots\}$ and whose 2-cells are

$$\begin{aligned} \eta^c_{x,y,z} &: c_z c_x c_y \Rightarrow c_x c_z c_y, \;\text{ with } x = y \text{ only if } ||y|| = 0 \text{ and } y = z \text{ only if } ||y|| = 1, \\ \varepsilon^c_{x,y,z} &: c_y c_z c_x \Rightarrow c_y c_x c_z, \;\text{ with } x = y \text{ only if } ||y|| = 1 \text{ and } y = z \text{ only if } ||y|| = 0, \end{aligned}$$

for any $x \leqslant y \leqslant z$ of elements of $\mathcal{S}$. By definition, this 2-polygraph is Tietze equivalent to $\mathrm{Knuth}_2(\mathcal{S})$. In the sequel, if there is no possible confusion, we will identify the 2-polygraphs $\mathrm{Knuth}_2^{\mathrm{c}}(\mathcal{S})$ and $\mathrm{Knuth}_2(\mathcal{S})$. Denote by $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$ the 2-polygraph whose set of 1-cells is $\{c_{x_1}, c_{x_2}, \ldots\}$ and whose set of 2-cells is $\mathrm{C}_2(\mathcal{S}) \cup \mathrm{Knuth}_2^{\mathrm{c}}(\mathcal{S})$.

2.8. Proposition. *The 2-polygraph* $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$ *is a presentation of the super plactic monoid* $\mathbf{P}(\mathcal{S})$.

Proof. By adding to the 2-polygraph $\mathrm{Knuth}_2^{\mathrm{c}}(\mathcal{S})$ all the column generators c_u, for all $u = x_p \ldots x_1$ in $\mathrm{col}(\mathcal{S})$ such that $|u| \geqslant 2$, and the corresponding collapsible 2-cell $\mu_u : c_{x_p} \ldots c_{x_1} \Rightarrow c_u$, we obtain that the 2-polygraphs $\mathrm{Knuth}_2^{\mathrm{c}}(\mathcal{S})$ and $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$ are Tietze-equivalent, showing the claim. □

2.9. Super pre-column presentation. Denote by $\mathrm{PC}_2(\mathcal{S})$ the set of 2-cells of the following form:

$$\begin{aligned} \gamma'_{x,zy} &: c_x c_{zy} \Rightarrow c_{zx} c_y, \;\text{ with } x = y \text{ only if } ||y|| = 0 \text{ and } y = z \text{ only if } ||y|| = 1, \\ \gamma'_{y,zx} &: c_y c_{zx} \Rightarrow c_{yx} c_z, \;\text{ with } x = y \text{ only if } ||y|| = 1 \text{ and } y = z \text{ only if } ||y|| = 0, \end{aligned}$$

for any $x \leqslant y \leqslant z$ elements of $\mathcal{S}$. Define the 2-polygraph $\mathrm{PreCol}_2(\mathcal{S})$ whose set of 1-cells is $\mathrm{Col}_1(\mathcal{S})$ and whose set of 2-cells is $\mathrm{PC}_2(\mathcal{S}) \cup \{\gamma'_{x,u} : c_x c_u \Rightarrow c_{xu} \mid xu \in \mathrm{col}(\mathcal{S}) \text{ and } x \in \mathcal{S}\}$.

2.10. Proposition. *The 2-polygraph* $\mathrm{PreCol}_2(\mathcal{S})$ *is a presentation of the super plactic monoid* $\mathbf{P}(\mathcal{S})$.

Proof. We first prove that 2-polygraph $\mathrm{CPC}_2(\mathcal{S})$ whose set of 1-cells is $\mathrm{Col}_1(\mathcal{S})$ and set of 2-cells is $\mathrm{C}_2(\mathcal{S}) \cup \mathrm{PC}_2(\mathcal{S})$, is Tietze equivalent to $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$. We consider the following critical branching

$$\begin{array}{ccc} & \overset{\eta^c_{x,y,z}}{\nearrow} c_x c_z c_y & \overset{c_x \mu_{zy}}{\Longrightarrow} c_x c_{zy} \\ c_z c_x c_y & \underset{\mu_{zx} c_y}{\Longrightarrow} & c_{zx} c_y \end{array}$$

of the 2-polygraph $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$, for any $x \leqslant y \leqslant z$ with $x = y$ only if $||y|| = 0$ and $y = z$ only if $||y|| = 1$, and the Tietze transformation $\kappa_{\eta^c_{x,y,z} \leftarrow \gamma'_{x,zy}} : \mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})^{\top} \longrightarrow \mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})^{\top} / (\eta^c_{x,y,z} \leftarrow \gamma'_{x,zy})$, that substitutes the 2-cell $\gamma'_{x,zy}$ for the 2-cell $\eta^c_{x,y,z}$, and denote by $T_{\eta \leftarrow \gamma'}$ the successive applications of $\kappa_{\eta^c_{x,y,z} \leftarrow \gamma'_{x,zy}}$, for any $x \leqslant y \leqslant z$ with $x = y$ only if $||y|| = 0$ and $y = z$ only if $||y|| = 1$, with respect to the lexicographic order on the triples (x, y, z) induced by the total order on $\mathcal{S}$. Similarly, we study the critical branching $(\varepsilon^c_{x,y,z}, c_y \mu_{zx})$ of $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$, for any $x \leqslant y \leqslant z$ with $x = y$ only if $||y|| = 1$ and $y = z$ only if $||y|| = 0$, by introducing the Tietze transformation $\kappa_{\varepsilon^c_{x,y,z} \leftarrow \gamma'_{y,zx}}$ from $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})^{\top}$

to $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})^\top/(\varepsilon^c_{x,y,z} \leftarrow \gamma'_{y,zx})$. Denote by $T_{\varepsilon\leftarrow\gamma'}$ the successive applications of this Tietze transformation with respect to the lexicographic order on the triples (x,y,z) induced by the total order on $\mathcal{S}$, and by $T_{\eta,\varepsilon\leftarrow\gamma'}$ the Tietze transformation from $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})^\top$ to $\mathrm{CPC}_2(\mathcal{S})^\top$ given by $T_{\eta\leftarrow\gamma'} \circ T_{\varepsilon\leftarrow\gamma'}$.

Finally, we prove that the 2-polygraphs $\mathrm{CPC}_2(\mathcal{S})$ and $\mathrm{PreCol}_2(\mathcal{S})$ are Tietze equivalents. Let $x_p \dots x_1$ be in $\mathrm{col}(\mathcal{S})$ with $|x_p \dots x_1| > 2$ and define $\gamma'_{y,x} := \mu_{yx} : c_y c_x \Rightarrow c_{yx}$, for any $x \leqslant y$ with $x = y$ only if $||x|| = 1$. We consider the following critical branching

$$\begin{array}{ccc} & c_{x_p}\mu_{x_{p-1}\dots x_1} \Rightarrow & c_{x_p}c_{x_{p-1}\dots x_1} \\ c_{x_p}\dots c_{x_1} & \Rightarrow & \\ & \mu_{x_p\dots x_1} \Rightarrow & c_{x_p\dots x_1} \end{array}$$

of the 2-polygraph $\mathrm{CPC}_2(\mathcal{S})$ and the following Tietze transformation

$$\kappa'_{\mu_{x_p\dots x_1}\leftarrow\gamma'_{x_p,x_{p-1}\dots x_1}} : \mathrm{CPC}_2(\mathcal{S})^\top \longrightarrow \mathrm{CPC}_2(\mathcal{S})^\top/(\mu_{x_p\dots x_1} \leftarrow \gamma'_{x_p,x_{p-1}\dots x_1}),$$

that substitutes the 2-cell $\gamma'_{x_p,x_{p-1}\dots x_1}$ for the 2-cell $\mu_{x_p\dots x_1}$, for each column $x_p \dots x_1$ such that $p > 2$. Starting from $\mathrm{CPC}_2(\mathcal{S})$, we apply successively the Tietze transformation $\kappa'_{\mu_{x_p\dots x_1}\leftarrow\gamma'_{x_p,x_{p-1}\dots x_1}}$, from the bigger column to the smaller one with respect to $\leqslant_{\mathrm{deglex}}$. The composite $T_{\mu\leftarrow\gamma'} = \kappa'_{\mu_{x_3x_2x_1}\leftarrow\gamma'_{x_3,x_2x_1}} \circ \dots \circ \kappa'_{\mu_{x_p\dots x_1}\leftarrow\gamma'_{x_p,x_{p-1}\dots x_1}}$, defines a Tietze transformation from $\mathrm{CPC}_2(\mathcal{S})^\top$ to $\mathrm{PreCol}_2(\mathcal{S})^\top$. □

2.11. Super column presentation. Consider the 2-polygraph $\mathrm{Col}_2(\mathcal{S})$ whose set of 1-cells is $\mathrm{Col}_1(\mathcal{S})$ and whose 2-cells are

$$\gamma_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

for every u and v in $\mathrm{col}(\mathcal{S})$ such that $u \times v$, where w and w' are respectively the readings of the left and right columns of the super tableau $[\![uv]\!]_r$, if $u \times^2 v$, and $w = uv$ and $c_{w'} = 1$, if $u \times^1 v$.

2.12. Theorem. *The 2-polygraph $\mathrm{Col}_2(\mathcal{S})$ is a convergent presentation of the super plactic monoid $\mathbf{P}(\mathcal{S})$.*

The rest of this section is devoted to prove this result. We first show in Lemma 2.13 that the 2-polygraph $\mathrm{Col}_2(\mathcal{S})$ is terminating and confluent. We then show in Lemma 2.14 that it is a presentation of the super plactic monoid by showing that it is Tietze equivalent to the 2-polygraph $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$.

2.13. Lemma. *The 2-polygraph $\mathrm{Col}_2(\mathcal{S})$ is convergent.*

Proof. The termination of $\mathrm{Col}_2(\mathcal{S})$ is proved using the length-lexicographic order $\ll$ on $\mathrm{Col}_1(\mathcal{S})^*$ with respect to $\leqslant_{\mathrm{rev}}$ defined by setting $c_{u_1} \dots c_{u_k} \ll c_{v_1} \dots c_{v_l}$, if $k < l$ or $k = l$ and there exists $i \in \{1, \dots, k\}$ such that $u_i \leqslant_{\mathrm{rev}} v_i$ and $c_{u_j} = c_{v_j}$, for any $j < i$. We prove that $\ll$ is a well-ordering on $\mathrm{Col}_1(\mathcal{S})^*$ that is compatible with rules in $\mathrm{Col}_2(\mathcal{S})$, that is if $Uc_uc_vV \Rightarrow Uc_wc_{w'}V$ then $Uc_uc_vV \ll Uc_wc_{w'}V$, for all Uc_uc_vV and $Uc_wc_{w'}V$ in $\mathrm{Col}_1(\mathcal{S})^*$. If $c_{w'} = 1$, then Uc_uc_vV contains more elements than Uc_wV, showing that $Uc_wV \ll Uc_uc_vV$. On the other hand, suppose that the super tableau $[\![uv]\!]_r$ consists of two columns whose readings of are denoted by w and w' respectively. By Lemma 2.6, the word w contains more elements than u. Then, we obtain $w \leqslant_{\mathrm{rev}} u$, showing that $Uc_wV \ll Uc_uc_vV$. Since every application of a rule yield a $\ll$-preceding word, it follows that every sequence of rewriting using the 2-cells of $\mathrm{Col}_2(\mathcal{S})$ must terminate.

We show that the 2-polygraph $\mathrm{Col}_2(\mathcal{S})$ is locally confluent and thus confluent by the termination hypothesis. Any critical branching of the 2-polygraph $\mathrm{Col}_2(\mathcal{S})$ has the following form:

$$\begin{array}{ccc} & \overset{\gamma_{u,v}c_t}{\Rightarrow} & c_e c_{e'} c_t \\ c_u c_v c_t & & \\ & \underset{c_u\gamma_{v,t}}{\Rightarrow} & c_u c_w c_{w'} \end{array} \tag{3}$$

for every u, v and t in $\mathrm{Col}(\mathcal{S})$ such that $u \overset{\times}{} v \overset{\times}{} t$, where e and e' (resp. w and w') are the readings of the two columns of the super tableau $[\![uv]\!]_r$ (resp. $[\![vt]\!]_r$). Following the definition of the right insertion algorithm, and by applying the leftmost (resp. rightmost) rewriting path with respect to $\mathrm{Col}_2(\mathcal{S})$ on $c_u c_v c_t$, we yield to the three super columns generators corresponding to the columns of the super tableau $([\![u]\!]_r \star_r [\![v]\!]_r) \star_r [\![t]\!]_r$ (resp. $[\![u]\!]_r \star_r ([\![v]\!]_r \star_r [\![t]\!]_r)$). By the associativity of the product $\star_r$, we have $([\![u]\!]_r \star_r [\![v]\!]_r) \star_r [\![t]\!]_r = [\![u]\!]_r \star_r ([\![v]\!]_r \star_r [\![t]\!]_r)$, showing that the critical branching (3) is confluent. □

2.14. Lemma. *The 2-polygraph* $\mathrm{Col}_2(\mathcal{S})$ *is a presentation of the super plactic monoid* $\mathbf{P}(\mathcal{S})$.

Proof. Prove that the 2-polygraph $\mathrm{Col}_2(\mathcal{S})$ is Tietze equivalent to the 2-polygraph $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$. Any 2-cell in $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$ can be deduced from a 2-cell in $\mathrm{Col}_2(\mathcal{S})$ as follows. For any $x \leqslant y \leqslant z$ in $\mathcal{S}$, with $x = y$ only if $||y|| = 0$ and $y = z$ only if $||y|| = 1$ (resp. $x = y$ only if $||y|| = 1$ and $y = z$ only if $||y|| = 0$), the 2-cells $\eta^c_{x,y,z}$ (resp. $\varepsilon^c_{x,y,z}$) can be deduced by the following composition

$$\begin{array}{ccc} c_z c_x c_y & \overset{\eta^c_{x,y,z}}{\Rightarrow} & c_x c_z c_y \\ {\scriptstyle \gamma_{z,x}c_y}\Downarrow & & \Downarrow{\scriptstyle c_x\gamma_{z,y}} \\ c_{zx} c_y & \underset{\gamma_{x,zy}}{\Leftarrow} & c_x c_{zy} \end{array} \qquad \text{(resp.} \qquad \begin{array}{ccc} c_y c_z c_x & \overset{\varepsilon^c_{x,y,z}}{\Rightarrow} & c_y c_x c_z \\ {\scriptstyle c_y\gamma_{z,x}}\Downarrow & & \Downarrow{\scriptstyle \gamma_{y,x}c_z} \\ c_y c_{zx} & \underset{\gamma_{y,zx}}{\Rightarrow} & c_{yx} c_z \end{array} \text{)}.$$

Moreover, for any super column $x_p \dots x_1$, the 2-cell $\mu_{x_p \dots x_1}$ can be deduced by the following composition

$$\begin{array}{ccccc} & & c_{x_p} \dots c_{x_1} & \overset{\mu_{x_p \dots x_1}}{\Rightarrow} & c_{x_p \dots x_1} \\ & {\scriptstyle \gamma_{x_p,x_{p-1}}c_{x_{p-2}} \dots c_{x_1}}\Downarrow & & & \Uparrow{\scriptstyle \gamma_{x_p \dots x_2,x_1}} \\ & & c_{x_p x_{p-1}} c_{x_{p-2}} \dots c_{x_1} & \Rightarrow (\dots) \Rightarrow & c_{x_p \dots x_2} c_{x_1} \end{array}$$

As a consequence, if the words w and w' in $\mathrm{Col}_1(\mathcal{S})^*$ are equal modulo relations in $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$, then they are equal modulo relations in $\mathrm{Col}_2(\mathcal{S})$. Conversely, following Subsection 2.3, the following equivalence

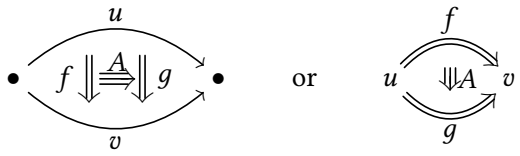
$$uv \sim_{\mathbf{P}(\mathcal{S})} R_{row}([\![uv]\!]_r) \sim_{\mathbf{P}(\mathcal{S})} R_{col}([\![uv]\!]_r)$$

holds, for all u, v in $\mathrm{Col}(\mathcal{S})$ such that $u \overset{\times}{} v$. In this way, if two words in $\mathrm{Col}_1(\mathcal{S})^*$ are equal modulo relations in $\mathrm{Col}_2(\mathcal{S})$, they are super plactic congruent and hence they are equal modulo $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$, showing the claim. □

3. Coherent presentations of the super plactic monoid

In this section, we begin by recalling the notion of coherent presentations of monoids from [12] and we extend the super column presentation into a coherent presentation of the super plactic monoid. In a second part, we recall the homotopical reduction procedure from [12] and we reduce the super coherent column presentation into a smaller one over the initial signed alphabet. We follow for this aim the approach developed in [19] for the non-signed case.

3.1. Coherent presentations of monoids. A $(3,1)$-*polygraph* is a pair (Σ_2, Σ_3) made of a 2-polygraph Σ_2 and a globular extension Σ_3 of the $(2,1)$-category $\Sigma_2^\top$, that is a set of 3-cells $A : f \Rrightarrow g$ relating 2-cells f and g in $\Sigma_2^\top$, respectively denoted by $s_2(A)$ and $t_2(A)$ and satisfying the globular relations $s_1s_2(A) = s_1t_2(A)$ and $t_1s_2(A) = t_1t_2(A)$. Such a 3-cell can be represented with the following globular shape:



We will denote by $\Sigma_3^\top$ the free $(3,1)$-category generated by the $(3,1)$-polygraph (Σ_2, Σ_3). A pair (f, g) of 2-cells of $\Sigma_2^\top$ such that $s_1(f) = s_1(g)$ and $t_1(f) = t_1(g)$ is called a 2-*sphere* of $\Sigma_2^\top$. An *extended presentation* of a monoid $\mathbf{M}$ is a $(3,1)$-polygraph whose underlying 2-polygraph is a presentation of the monoid $\mathbf{M}$. A *coherent presentation of* $\mathbf{M}$ is an extended presentation Σ of $\mathbf{M}$ such that the cellular extension Σ_3 is a *homotopy basis* of the $(2,1)$-category $\Sigma_2^\top$, that is, for every 2-sphere μ of $\Sigma_2^\top$, there exists a 3-cell in $\Sigma_3^\top$ with boundary μ. The elements in $\Sigma_3^\top$ are called *syzygies* of the presentation Σ.

Let Σ be a $(3,1)$-polygraph. A 3-cell A of Σ is called *collapsible* if $t_2(A)$ is in Σ_2 and $s_2(A)$ is a 2-cell of the free $(2,1)$-category over $(\Sigma_2 \setminus \{t_2(A)\})^\top$, then $t_2(A)$ is called *redundant*. An *elementary Tietze transformation* of a $(3,1)$-polygraph Σ is a 3-functor with domain $\Sigma_3^\top$ that belongs to one of the following operations, [12]:

i) *adjunction* ι_α^1 and *elimination* π_α of a 2-cell α as described in Subsection 2.1,

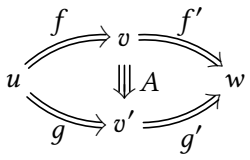
ii) *coherent adjunction* $\iota_A^2 : \Sigma_3^\top \to \Sigma_3^\top(\alpha)(A)$ (resp. $\iota_A : \Sigma_3^\top \to \Sigma_3^\top(A)$) of a redundant 2-cell α with its collapsible 3-cell A (resp. of a redundant 3-cell A),

iii) *coherent elimination* $\pi_A : \Sigma_3^\top \to \Sigma_3^\top/A$ (resp. $\pi_{(B,A)} : \Sigma_3^\top \to \Sigma_3^\top/(B,A)$) of a redundant 2-cell α with its collapsible 3-cell A (resp. of a redundant 3-cell A, that maps A to B).

For $(3,1)$-polygraphs Σ and Υ, a *Tietze transformation from* Σ *to* Υ is a 3-functor $F : \Sigma_3^\top \to \Upsilon_3^\top$ that decomposes into a sequence of elementary Tietze transformations. Two $(3,1)$-polygraphs Σ and Υ are *Tietze-equivalent* if there exists an equivalence of 2-categories $F : \Sigma_2^\top/\Sigma_3 \to \Upsilon_2^\top/\Upsilon_3$ and the presented monoids $\overline{\Sigma}_2$ and $\overline{\Upsilon}_2$ are isomorphic. Two $(3,1)$-polygraphs are Tietze equivalent if, and only if, there exists a Tietze transformation between them, [12].

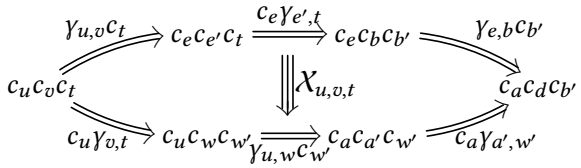
Recall that Squier's coherence theorem, [33], states that, any convergent 2-polygraph Σ presenting a

monoid $\mathbf{M}$ can be extended into a coherent presentation of $\mathbf{M}$ having a generating syzygy



for every critical branching (f, g) of Σ, where f' and g' are chosen confluent rewriting paths.

3.2. Super column coherent presentation. Denote by $\mathrm{Col}_3(\mathcal{S})$ the extended presentation of the monoid $\mathbf{P}(\mathcal{S})$ obtained from $\mathrm{Col}_2(\mathcal{S})$ by adjunction of one family of syzygies $\mathcal{X}_{u,v,t}$ of the following form



(4)

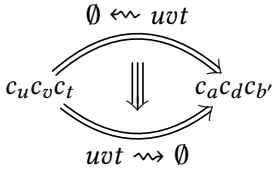
for every u, v and t in $\mathrm{Col}(\mathcal{S})$ such that $u \overset{\times}{} v \overset{\times}{} t$, where e and e' (resp. w and w') are the readings of the two columns of the super tableau $[\![uv]\!]_r$ (resp. $[\![vt]\!]_r$), and where a and a' (resp. b and b') denote the readings of the two columns of the super tableau $[\![uw]\!]_r$ (resp. $[\![e't]\!]_r$) and a, d, b' are the readings of the three columns of the super tableau $[\![uvt]\!]_r$. Note that one or further columns e', w', a' and b' can be empty as illustrated in Figure 1. In those cases some indicated 2-cells γ in the confluence diagram correspond to identities.

3.3. Theorem. *The* $(3,1)$*-polygraph* $\mathrm{Col}_3(\mathcal{S})$ *is a coherent presentation of the monoid* $\mathbf{P}(\mathcal{S})$.

Proof. Consider the 2-polygraph whose set of 1-cells is $\mathrm{Col}_1(\mathcal{S})$ and whose 2-cells are

$$c_u c_v \Rightarrow c_w c_{w'}$$

for all c_u, c_v in $\mathrm{Col}_1(\mathcal{S})$ such that $u \overset{\times}{} v$ and where w and w' denote respectively the readings of the left and right column of the super tableau $(uv \rightsquigarrow \emptyset)$ obtained by applying the left insertion. Following Subsection 2.3, by the associativity of the product $\star_r$, the super tableaux $(uv \rightsquigarrow \emptyset)$ and $[\![uv]\!]_r$ are equal, thus this 2-polygraph coincides with $\mathrm{Col}_2(\mathcal{S})$. Hence, the generating syzygy of the extended presentation of the 2-polygraph $\mathrm{Col}_2(\mathcal{S})$ has the following form



for all c_u, c_v, c_t in $\mathrm{Col}_1(\mathcal{S})$ such that $u \overset{\times}{} v \overset{\times}{} t$, where a, d, b' are the readings of the three columns of the super tableau $[\![uvt]\!]_r$, and where the 2-source (resp. 2-target) of the syzygy corresponds to the application of the right (resp. left) insertion $(\emptyset \leftsquigarrow uvt)$ (resp. $(uvt \rightsquigarrow \emptyset)$) on the word $c_u c_v c_t$. Finally, by definition of the right (resp. left) insertion algorithm $\leftsquigarrow$ (resp. $\rightsquigarrow$), we can lead to the normal form $c_a c_d c_{b'}$ by applying at most three steps of reduction on the initial word $c_u c_v c_t$, showing the claim. □

3.3.1. Remark. It is worth noting that there are fives forms for the generating syzygy (4) depending on the following four cases: $u^{\times 1}v^{\times 1}t$, $u^{\times 2}v^{\times 1}t$, $u^{\times 1}v^{\times 2}t$ and $u^{\times 2}v^{\times 2}t$, as shown in Figure 1. These forms are obtained by a case-by-case analysis and they will be used in the sequel in order to reduce the super column coherent presentation $\mathrm{Col}_3(\mathcal{S})$ into a smaller one over the initial signed alphabet.

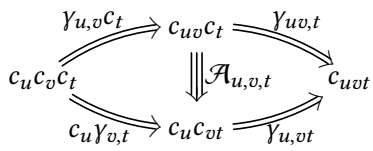
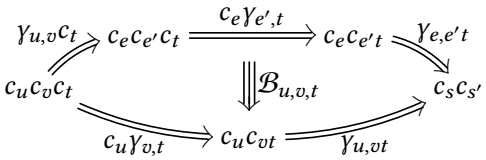
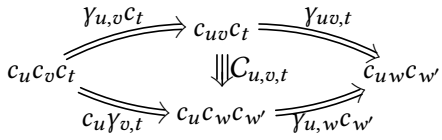
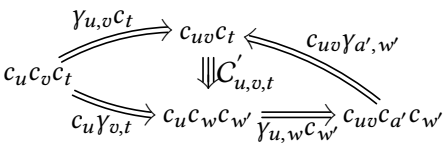
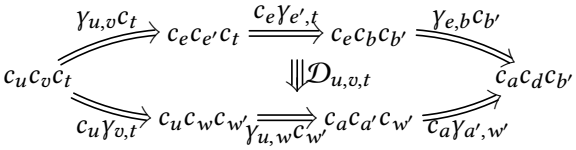
	Confluence diagrams
$u^{\times 1}v^{\times 1}t$	
$u^{\times 2}v^{\times 1}t$	
$u^{\times 1}v^{\times 2}t$ with $uv^{\times}t$	
$u^{\times 1}v^{\times 2}t$ with $u\widehat{v}t$	
$u^{\times 2}v^{\times 2}t$	

Figure 1: confluence diagrams

3.4. Homotopical reduction procedure. Let Σ be a $(3,1)$-polygraph. A 3-*sphere* of the $(3,1)$-category $\Sigma_3^\top$ is a pair (f,g) of 3-cells of $\Sigma_3^\top$ such that $s_2(f) = s_2(g)$ and $t_2(f) = t_2(g)$. A *collapsible part of* Σ is a triple $(\Gamma_2, \Gamma_3, \Gamma_4)$ made of a family Γ_2 of 2-cells of Σ, a family Γ_3 of 3-cells of Σ and a family Γ_4 of 3-spheres of $\Sigma_3^\top$, such that the following conditions are satisfied, [12]:

i) every μ of every Γ_k is collapsible, that is, $t_{k-1}(\mu)$ is in Σ_{k-1} and $s_{k-1}(\mu)$ does not contain $t_{k-1}(\mu)$,

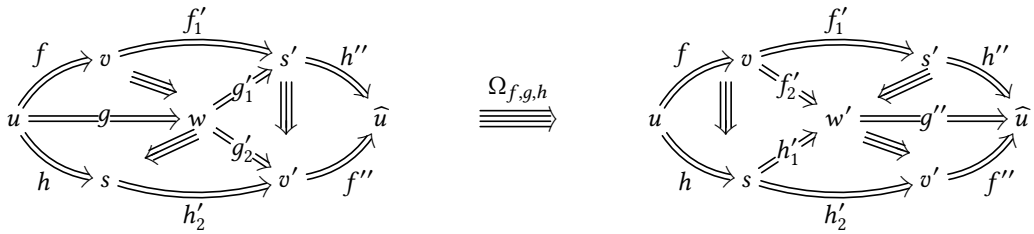
ii) no cell of Γ_2 (resp. Γ_3) is the target of a collapsible 3-cell of Γ_3 (resp. 3-sphere of Γ_4),

iii) there exists a well-founded order on the cells of Σ such that, for every μ in every Γ_k, $t_{k-1}(\mu)$ is strictly greater than every generating $(k-1)$-cell that occurs in the source of μ.

The *homotopical reduction* of the $(3,1)$-polygraph Σ with respect to a collapsible part Γ is the Tietze transformation, denoted by R_Γ, from the $(3,1)$-category $\Sigma_3^\top$ to the $(3,1)$-category freely generated by

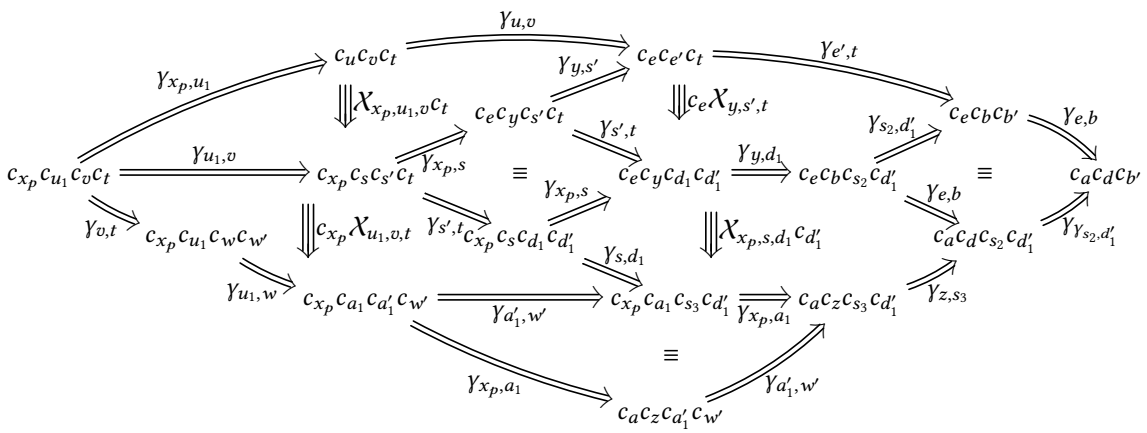
the $(3,1)$-polygraph obtained from Σ by removing the cells of Γ and all the corresponding redundant cells. For any μ in Γ, we have $R_\Gamma(t(\mu)) = R_\Gamma(s(\mu))$ and $R_\Gamma(\mu) = 1_{R_\Gamma(s(\mu))}$. In any other cases, the transformation R_Γ acts as an identity, see [12] for more details.

A *local triple branching* of a 2-polygraph Σ is a triple (f,g,h) of rewriting steps of Σ with a common source. An *aspherical* triple branchings have two of their 2-cells equal. A *Peiffer* triple branchings have at least one of their 2-cells that form a Peiffer branching with the other two. The *overlap* triple branchings are the remaining local triple branchings. Local triple branchings are ordered by inclusion of their sources and a minimal overlap triple branching is called *critical*. If Σ is a coherent and convergent $(3,1)$-polygraph, a *triple generating confluence of* Σ is a 3-sphere

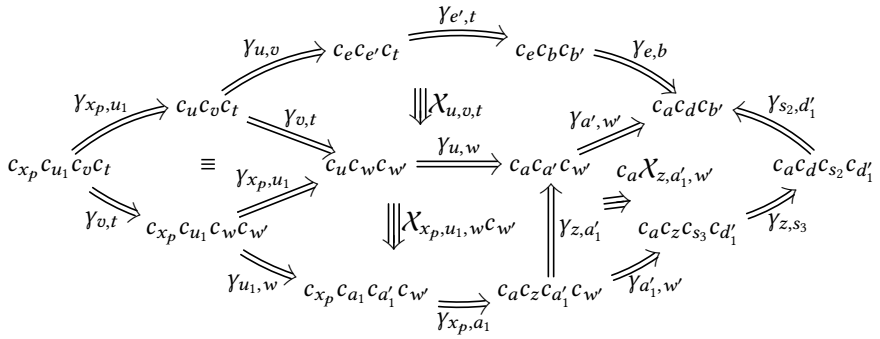


where (f,g,h) is a triple critical branching of the 2-polygraph Σ_2 and the other cells are obtained by confluence, [12].

3.5. Reduced super coherent column presentation. We apply the homotopical reduction procedure on the $(3,1)$-polygraph $\mathrm{Col}_3(\mathcal{S})$ using the collapsible part made of generating triple confluences. Following Theorem 3.3, the family of syzygies $\mathcal{X}_{u,v,t}$ given in (4) and indexed by columns u, v and t in $\mathrm{col}(\mathcal{S})$ such that $u \overset{\times}{} v \overset{\times}{} t$ forms a homotopy basis of the $(2,1)$-category $\mathrm{Col}_2(\mathcal{S})^\top$. Consider such a triple (u,v,t) with $|u| \geqslant 2$, and let x_p be in $\mathcal{S}$ and u_1 be in $\mathrm{col}(\mathcal{S})$ such that $x_p = \ell(u)$ and $u_1 = \mathrm{Rem}(u)$. We will show that the confluence diagram induced by the critical triple branching with source $c_{x_p}c_{u_1}c_vc_t$ is represented by the 3-sphere $\Omega_{x_p,u_1,v,t}$ whose source is the following 3-cell



and whose target is the following 3-cell



In order to facilitate the reading of the generating triple confluence, we have omitted the context of the 2-cells γ. Note also that some super columns may be empty and thus the indicated 2-cells γ may be identities.

The 3-sphere $\Omega_{x_p,u_1,v,t}$ is constructed as follows. Since $x_p{}^{\times 1}u_1$ and $u_1{}^{\times}w$, then $\mathcal{X}_{x_p,u_1,w}$ is either of the form $\mathcal{A}_{x_p,u_1,w}$, $C_{x_p,u_1,w}$ or $C'_{x_p,u_1,w}$. Denote a_1 and a'_1 the readings of the two columns of $[\![u_1w]\!]_r$. The 3-cell $\mathcal{X}_{x_p,u_1,w}$ being confluent, we have $R_{col}([\![x_pa_1]\!]_r) = az$ with z in $\mathcal{S}$ and $R_{col}([\![za'_1]\!]_r) = a'$. Since $z^{\times 1}a'_1$ and $a'_1{}^{\times}w'$ (resp. $x_p{}^{\times 1}u_1$ and $u_1{}^{\times}v$), we deduce that $\mathcal{X}_{z,a'_1,w'}$ (resp. $\mathcal{X}_{x_p,u_1,v}$) is either of the form $\mathcal{A}_{z,a'_1,w'}$, $C_{z,a'_1,w'}$ or $C'_{z,a'_1,w'}$ (resp. $\mathcal{A}_{x_p,u_1,v}$, $C_{x_p,u_1,v}$ or $C'_{x_p,u_1,v}$). Denote by s and s' the readings of the two columns of $[\![u_1v]\!]_r$. The 3-cell $\mathcal{X}_{x_p,u_1,v}$ being confluent, we obtain that $R_{col}([\![x_ps]\!]_r) = ey$ with y in $\mathcal{S}$ and $R_{col}([\![ys']\!]_r) = e'$. Since $y^{\times 1}s'$ and $s'{}^{\times}t$, we deduce that $\mathcal{X}_{y,s',t}$ is either of the form $\mathcal{A}_{y,s',t}$, $C_{y,s',t}$ or $C'_{y,s',t}$. Denote by d_1 and d'_1 the readings of the two columns of $[\![s't]\!]_r$. The 3-cell $\mathcal{X}_{y,s',t}$ being confluent and $R_{col}([\![e't]\!]_r) = bb'$, we have $R_{col}([\![yd_1]\!]_r) = bs_2$ and $R_{col}([\![s_2d'_1]\!]_r) = b'$. On the other hand, the 3-cell $\mathcal{X}_{u_1,v,t}$ being confluent, we have $R_{col}([\![sd_1]\!]_r) = a_1s_3$ and $R_{col}([\![a'_1w']\!]_r) = s_3d'_1$. Finally, since $\mathcal{X}_{x_p,s,d_1}$ is confluent, we deduce that $R_{col}([\![zs_3]\!]_r) = ds_2$.

Denote by $\overline{\mathrm{Col}_3}(\mathcal{S})$ the extended presentation of the monoid $\mathbf{P}(\mathcal{S})$ obtained from $\mathrm{Col}_2(\mathcal{S})$ by adjunction of one family of 3-cells $\mathcal{X}_{x,v,t}$ of the form (4), for all x in $\mathcal{S}$ and v, t in $\mathrm{col}(\mathcal{S})$ such that $x^{\times}v^{\times}t$. Let Γ_4 be the collapsible part made of the family of 3-sphere $\Omega_{x_p,u_1,v,t}$, indexed by x_p in $\mathcal{S}$ and u_1, v, t in $\mathrm{col}(\mathcal{S})$ such that $u^{\times}v^{\times}t$ and $u = x_pu_1$. We consider the order $\lhd$ on the 3-cells of $\mathrm{Col}_3(\mathcal{S})$ defined by $\mathcal{A}_{u,v,t} \lhd C'_{u,v,t} \lhd C_{u,v,t} \lhd \mathcal{B}_{u,v,t} \lhd \mathcal{D}_{u,v,t}$, and if $\mathcal{X}_{u,v,t} \in \{\mathcal{A}_{u,v,t}, \mathcal{B}_{u,v,t}, C_{u,v,t}, C'_{u,v,t}, \mathcal{D}_{u,v,t}\}$ with $u' \preccurlyeq_{\mathrm{deglex}} u$, then $\mathcal{X}_{u',v',t'} \lhd \mathcal{X}_{u,v,t}$, for all u, v, t in $\mathrm{col}(\mathcal{S})$ such that $u^{\times}v^{\times}t$. By construction of the 3-sphere $\Omega_{x_p,u_1,v,t}$, its source contains the 3-cell $\mathcal{X}_{u_1,v,t}$ and its target contains the 3-cell $\mathcal{X}_{u,v,t}$ with $|u_1| < |u|$. Up to a Nielsen transformation, by applying the reduction R_{Γ_4} on the $(3,1)$-polygraph $\mathrm{Col}_3(\mathcal{S})$ with respect to Γ_4 and the order $\lhd$, we obtain the $(3,1)$-polygraph $\overline{\mathrm{Col}_3}(\mathcal{S})$. Hence, we deduce the following result.

3.6. Proposition. *The $(3,1)$-polygraph $\overline{\mathrm{Col}_3}(\mathcal{S})$ is a coherent presentation of the monoid $\mathbf{P}(\mathcal{S})$.*

3.7. Super pre-column coherent presentation. We reduce the coherent presentation $\overline{\mathrm{Col}_3}(\mathcal{S})$ into a coherent presentation whose underlying 2-polygraph is $\underline{\mathrm{Pre}}\mathrm{Col}_2(\mathcal{S})$. This reduction is obtained using the homotopical reduction R_{Γ_3} on the $(3,1)$-polygraph $\overline{\mathrm{Col}_3}(\mathcal{S})$ whose collapsible part Γ_3 is the set of 3-cells $\mathcal{A}_{x,v,t}$, $\mathcal{B}_{x,v,t}$ and $C_{x,v,t}$ with $x \in \mathcal{S}$ and $v, t \in \mathrm{col}(\mathcal{S})$ such that $x^{\times 1}v^{\times 1}t$, $x^{\times 2}v^{\times 1}t$ and $x^{\times 1}v^{\times 2}t$

with $xv^{\times}t$ respectively, and the order $\triangleleft$ on the 2-cells of $\mathrm{Col}_2(\mathcal{S})$ defined by $\gamma_{u',v'} \triangleleft \gamma_{u,v}$ if $|uv| > |u'v'|$, or $|uv| = |u'v'|$ and $|u| > |R_{col}([\![u'v']\!]_r)|$ or $|u| \leqslant |R_{col}([\![u'v']\!]_r)|$ and $u' \preccurlyeq_{\mathrm{rev}} u$, for all $u, v, u', v' \in \mathrm{col}(\mathcal{S})$ such that $u^{\times}v$ and $u'^{\times}v'$.

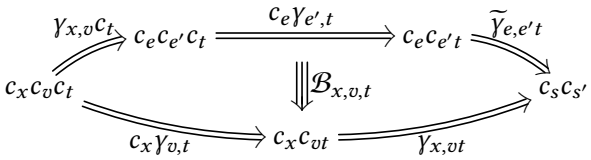
The reduction R_{Γ_3} induced by the order $\triangleleft$ on the 2-cells of $\mathrm{Col}_2(\mathcal{S})$ and the order $\lhd$ on 3-cells can be decomposed as follows. For all x in $\mathcal{S}$ and v, t in $\mathrm{col}(\mathcal{S})$ such that $x^{\times 1}v^{\times 1}t$, the 2-cells $\gamma_{x,v}$, $\gamma_{v,t}$ and $\gamma_{x,vt}$ are smaller then $\gamma_{xv,t}$ for the order $\triangleleft$. Then the reduction R_{Γ_3} removes the 2-cell $\gamma_{xv,t}$ together with the 3-cell $\mathcal{A}_{x,v,t}$. By iterating this reduction on the length of v, we reduce all the 2-cells of $\mathrm{Col}_2(\mathcal{S})$ to the following set of 2-cells

$$\{\ \gamma_{u,v} \mid |u| \geqslant 1,\ |v| \geqslant 2 \text{ and } u^{\times 2}v\ \} \cup \{\ \gamma_{u,v} \mid |u| = 1,\ |v| \geqslant 1 \text{ and } u^{\times 1}v\ \}. \tag{5}$$

For all x in $\mathcal{S}$ and v, t in $\mathrm{col}(\mathcal{S})$ such that $x^{\times 1}v^{\times 2}t$ with $xv^{\times}t$, the 2-cells $\gamma_{x,v}$, $\gamma_{v,t}$, $\gamma_{x,w}$ and $\gamma_{a',w'}$ are smaller than $\gamma_{xv,t}$ for the order $\triangleleft$. Then, the reduction R_{Γ_3} removes the 2-cell $\gamma_{xv,t}$ together with the 3-cell $C_{x,v,t}$. By iterating this reduction on the length of v, we reduce the set (5) to the following one:

$$\{\ \gamma_{u,v} \mid |u| = 1,\ |v| \geqslant 2 \text{ and } u^{\times 2}v\ \} \cup \{\ \gamma_{u,v} \mid |u| = 1,\ |v| \geqslant 1 \text{ and } u^{\times 1}v\ \}. \tag{6}$$

For all x in $\mathcal{S}$ and v, t in $\mathrm{col}(\mathcal{S})$ such that $x^{\times 2}v^{\times 1}t$, consider the following 3-cell:



where $\widetilde{\gamma}_{e,e't}$ is the 2-cell in (6) obtained from the 2-cell $\gamma_{e,e't}$ by the previous step of the homotopical reduction by the 3-cell $C_{x,v,t}$. Since x is in $\mathcal{S}$, we have also that e' is in $\mathcal{S}$ by definition of γ. The 2-cells $\gamma_{x,v}$, $\gamma_{e',t}$, $\gamma_{v,t}$ and $\widetilde{\gamma}_{e,e't}$ being smaller than $\gamma_{x,vt}$ for the order $\triangleleft$, we can remove the 2-cells $\gamma_{x,vt}$ together with the 3-cell $\mathcal{B}_{x,v,t}$. By iterating this reduction on the length of t, we reduce the set (6) to the following set:

$$\{\ \gamma_{u,v} \mid |u| = 1,\ |v| = 2 \text{ and } u^{\times 2}v\ \} \cup \{\ \gamma_{u,v} \mid |u| = 1,\ |v| \geqslant 1 \text{ and } u^{\times 1}v\ \}. \tag{7}$$

Let us show that the set (7) is equal to the set of 2-cells $\mathrm{PreCol}_2(\mathcal{S})$ defined in Subsection 2.9. It is sufficient to prove that $\mathrm{PC}_2(\mathcal{S})$ is equal to the set of 2-cells $\gamma_{u,v}$ in $\mathrm{Col}_2(\mathcal{S})$ such that $|u| = 1$, $|v| = 2$ and $u^{\times 2}v$. Suppose that $v = xx'$ with $x \geqslant x'$ in $\mathcal{S}$ with $x = x'$ only if $||x|| = 1$. Since $u^{\times 2}v$, we obtain that $u \leqslant x$ with $u = x$ only if $||u|| = 0$. If $u \leqslant x'$ with $u = x'$ only if $||u|| = 0$, then $R_{col}([\![uv]\!]_r) = (xu)x'$ and the 2-cell $\gamma_{u,v}$ is equal to the 2-cell $\gamma'_{u,xx'} : c_uc_{xx'} \Rightarrow c_{xu}c_{x'}$. If $x' \leqslant u$ with $u = x'$ only if $||u|| = 1$, then $R_{col}([\![uv]\!]_r) = (ux')x$, and the 2-cell $\gamma_{u,v}$ is equal to $\gamma'_{u,xx'} : c_uc_{xx'} \Rightarrow c_{ux'}c_x$, showing the claim.

Hence, the homotopical reduction R_{Γ_3} reduces $\overline{\mathrm{Col}_3}(\mathcal{S})$ into a coherent presentation of the monoid $\mathbf{P}(\mathcal{S})$ whose set of 2-cells is $\mathrm{PreCol}_2(\mathcal{S})$. Denote by $\mathrm{PreCol}_3(\mathcal{S})$ the extended presentation of the monoid $\mathbf{P}(\mathcal{S})$ obtained from $\mathrm{PreCol}_2(\mathcal{S})$ by adjunction of the 3-cells $R_{\Gamma_3}(C'_{x,v,t})$ for $x^{\times 1}v^{\times 2}t$ with $\widehat{xv\ t}$, and $R_{\Gamma_3}(\mathcal{D}_{x,v,t})$ for $x^{\times 2}v^{\times 2}t$. Since the homotopical reduction R_{Γ_3} eliminates the 3-cells of $\overline{\mathrm{Col}_3}(\mathcal{S})$ of the form $\mathcal{A}_{x,v,t}$, $\mathcal{B}_{x,v,t}$ and $C_{x,v,t}$, we obtain the following result.

3.8. Proposition. *The* $(3,1)$*-polygraph* $\mathrm{PreCol}_3(\mathcal{S})$ *is a coherent presentation of the monoid* $\mathbf{P}(\mathcal{S})$.

3.9. Super Knuth's coherent presentation. We reduce the coherent presentation $\mathrm{PreCol}_3(\mathcal{S})$ into a coherent presentation of the super plactic monoid $\mathbf{P}(\mathcal{S})$ whose underlying 2-polygraph is $\mathrm{Knuth}_2(\mathcal{S})$. The Tietze transformation $T_{\mu\leftarrow\gamma'} : \mathrm{CPC}_2(\mathcal{S})^\top \to \mathrm{PreCol}_2(\mathcal{S})^\top$ defined in Proposition 2.10 substitutes a 2-cell $\gamma'_{x_p,x_{p-1}\dots x_1}$ for the 2-cell $\mu_{x_p\dots x_1}$ in $C_2(\mathcal{S})$, from the bigger column to the smaller one with respect to the order $\preccurlyeq_{\mathrm{deglex}}$. Consider the inverse of this Tietze transformation $T^{-1}_{\mu\leftarrow\gamma'}$ that substitutes the 2-cell $\mu_{x_p\dots x_1}$ for the 2-cell $\gamma'_{x_p,x_{p-1}\dots x_1}$, for each column $x_p\dots x_1$ such that $|x_p\dots x_1| > 2$ with respect to $\preccurlyeq_{\mathrm{deglex}}$. Denote by $\mathrm{CPC}_3(\mathcal{S})$ the $(3,1)$-polygraph whose underlying 2-polygraph is $\mathrm{CPC}_2(\mathcal{S})$ and whose 3-cells are $T^{-1}_{\mu\leftarrow\gamma'}(R_{\Gamma_3}(C'_{x,v,t}))$ for $x^{\times 1}v^{\times 2}t$ with $x\widehat{v}t$ and $T^{-1}_{\mu\leftarrow\gamma'}(R_{\Gamma_3}(\mathcal{D}_{x,v,t}))$ for $x^{\times 2}v^{\times 2}t$. In this way, we extend $T^{-1}_{\mu\leftarrow\gamma'}$ into a Tietze transformation between $\mathrm{PreCol}_3(\mathcal{S})$ and $\mathrm{CPC}_3(\mathcal{S})$. The $(3,1)$-polygraph $\mathrm{PreCol}_3(\mathcal{S})$ being a coherent presentation of $\mathbf{P}(\mathcal{S})$, we deduce the following result.

3.10. Lemma. *The* $(3,1)$*-polygraph* $\mathrm{CPC}_3(\mathcal{S})$ *is a coherent presentation of the monoid* $\mathbf{P}(\mathcal{S})$.

The Tietze transformation $T_{\eta,\varepsilon\leftarrow\gamma'} : \mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})^\top \to \mathrm{CPC}_2(\mathcal{S})^\top$ defined in the proof of Proposition 2.10 replaces the 2-cells $\eta^c_{x,y,z}$ and $\varepsilon^c_{x,y,z}$ in $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$ by composite of 2-cells in $\mathrm{CPC}_2(\mathcal{S})$. Consider the inverse of this Tietze transformation $T^{-1}_{\eta,\varepsilon\leftarrow\gamma'} : \mathrm{CPC}_2(\mathcal{S})^\top \longrightarrow \mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})^\top$, that substitutes the 2-cell $\eta^c_{x,y,z}$ (resp. $\varepsilon^c_{x,y,z}$) for the 2-cell $\gamma'_{x,zy}$ (resp. $\gamma'_{y,zx}$). Denote by $\mathrm{Knuth}_3^{\mathrm{cc}}(\mathcal{S})$ the $(3,1)$-polygraph whose underlying 2-polygraph is $\mathrm{Knuth}_2^{\mathrm{cc}}(\mathcal{S})$ and whose set of 3-cells is

$$\{\ T^{-1}_{\eta,\varepsilon\leftarrow\gamma'}(T^{-1}_{\mu\leftarrow\gamma'}(R_{\Gamma_3}(C'_{x,v,t}))) \text{ for } x^{\times 1}v^{\times 2}t \text{ with } x\widehat{v}t\ \} \cup \{\ T^{-1}_{\eta,\varepsilon\leftarrow\gamma'}(T^{-1}_{\mu\leftarrow\gamma'}(R_{\Gamma_3}(\mathcal{D}_{x,v,t}))) \text{ for } x^{\times 2}v^{\times 2}t\ \}.$$

We extend $T^{-1}_{\eta,\varepsilon\leftarrow\gamma'}$ into a Tietze transformation between the $(3,1)$-polygraphs $\mathrm{CPC}_3(\mathcal{S})^\top$ and $\mathrm{Knuth}_3^{\mathrm{cc}}(\mathcal{S})^\top$. Since $\mathrm{CPC}_3(\mathcal{S})$ is a coherent presentation of $\mathbf{P}(\mathcal{S})$, we deduce the following result:

3.11. Lemma. *The (3,1)-polygraph* $\mathrm{Knuth}_3^{\mathrm{cc}}(\mathcal{S})$ *is a coherent presentation of the monoid* $\mathbf{P}(\mathcal{S})$.

Finally, we perform the homotopical reduction R_{Γ_2} on the $(3,1)$-polygraph $\mathrm{Knuth}_3^{\mathrm{cc}}(\mathcal{S})$ whose collapsible part Γ_2 is defined by the 2-cells μ_u of $\mathrm{C}_2(\mathcal{S})$ and the order $\preccurlyeq_{\mathrm{deglex}}$. Thus, for every 2-cell $\mu_{x_p\dots x_1} : c_{x_p}\dots c_{x_1} \Rightarrow c_{x_p\dots x_1}$ in $\mathrm{C}_2(\mathcal{S})$, we eliminate $c_{x_p\dots x_1}$ together with $\mu_{x_p\dots x_1}$, from the bigger column to the smaller one with respect to the order $\preccurlyeq_{\mathrm{deglex}}$.

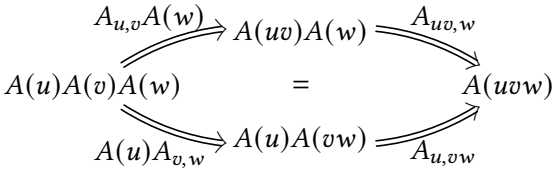
Consider the following composite of Tietze transformations $\mathcal{R} := R_{\Gamma_2}\circ T^{-1}_{\eta,\varepsilon\leftarrow\gamma'}\circ T^{-1}_{\mu\leftarrow\gamma'}\circ R_{\Gamma_3}$, defined from $\overline{\mathrm{Col}}_3(\mathcal{S})^\top$ to $\mathrm{Knuth}_3^{\mathrm{cc}}(\mathcal{S})^\top$ as follows. Firstly, the transformation $\mathcal{R}$ eliminates the 3-cells of $\overline{\mathrm{Col}}_3(\mathcal{S})$ of the form $\mathcal{A}_{x,v,t}$, $\mathcal{B}_{x,v,t}$ and $\mathcal{C}_{x,v,t}$ and reduces its set of 2-cells to $\mathrm{PreCol}_2(\mathcal{S})$. Secondly, this transformation coherently replaces the 2-cells $\mu_{x_p\dots x_1}$ by the 2-cells $\gamma'_{x_p,x_{p-1}\dots x_1}$, for each column $x_p\dots x_1$ such that $|x_p\dots x_1| > 2$, the 2-cells $\gamma'_{x,zy}$ by $\eta^c_{x,y,z}$, for any $x\preccurlyeq y\preccurlyeq z$, with $x = y$ only if $||y|| = 0$ and $y = z$ only if $||y|| = 1$ and the 2-cells $\gamma'_{y,zx}$ by $\varepsilon^c_{x,y,z}$ for any $x\preccurlyeq y\preccurlyeq z$, with $x = y$ only if $||y|| = 1$ and $y = z$ only if $||y|| = 0$. Finally, for each column u, the transformation $\mathcal{R}$ eliminates the generator c_u together with the 2-cell μ_u with respect to the order $\preccurlyeq_{\mathrm{deglex}}$.

Let $\mathrm{Knuth}_3(\mathcal{S})$ be the extended presentation of the monoid $\mathbf{P}(\mathcal{S})$ obtained from $\mathrm{Knuth}_2(\mathcal{S})$ by adjunction of the 3-cells $\mathcal{R}(C'_{x,v,t})$ for $x^{\times 1}v^{\times 2}t$ with $x\widehat{v}t$ and $\mathcal{R}(\mathcal{D}_{x,v,t})$ for $x^{\times 2}v^{\times 2}t$. The transformation $\mathcal{R}$ being a composite of Tietze transformations, we deduce the following result.

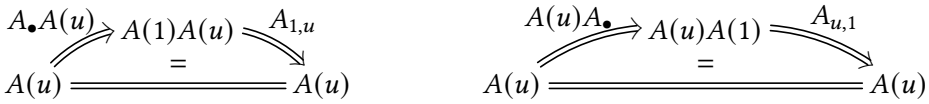
3.12. Theorem. *The* $(3,1)$*-polygraph* $\mathrm{Knuth}_3(\mathcal{S})$ *is a coherent presentation of the monoid* $\mathbf{P}(\mathcal{S})$.

3.13. Actions of super plactic monoids on categories. A monoid $\mathbf{M}$ can be seen as a 2-category with exactly one 0-cell $\bullet$, with the elements of the monoid $\mathbf{M}$ as 1-cells and with identity 2-cells only. The category of *actions of* $\mathbf{M}$ *on categories* is the category $\mathrm{Act}(\mathbf{M})$ of 2-representations of $\mathbf{M}$ in the category Cat of categories. The full subcategory of $\mathrm{Act}(\mathbf{M})$ whose objects are the 2-functors is denoted by $2\mathrm{Cat}(\mathbf{M}, \mathrm{Cat})$. We refer the reader to [12] for a full introduction on the category of 2-representations of 2-categories. More explicitly, an action A of the monoid $\mathbf{M}$ is specified by a category $\mathbf{C} = A(\bullet)$, an endofunctor $A(u) : \mathbf{C} \to \mathbf{C}$ for every u in $\mathbf{M}$, a natural isomorphism $A_{u,v} : A(u)A(v) \Rightarrow A(uv)$ for every elements u and v of $\mathbf{M}$, and a natural isomorphism $A_\bullet : 1_\mathbf{C} \Rightarrow A(1)$ such that:

i) for every triple (u, v, w) of elements of the monoid $\mathbf{M}$, the following diagram commutes



ii) for every element u of the monoid $\mathbf{M}$, the following diagrams commute



Let $\mathbf{M}$ be a monoid and let Σ be an extended presentation of $\mathbf{M}$. The $(3, 1)$-polygraph Σ is a coherent presentation of $\mathbf{M}$ if, and only if, for every 2-category C, there is an equivalence of categories between $\mathrm{Act}(\mathbf{M})$ and $2\mathrm{Cat}(\Sigma_1^*/\Sigma_2, C)$, that is natural in C, [12]. In this way, up to equivalence, the actions of a monoid $\mathbf{M}$ on categories are the same as the 2-functors from Σ_1^*/Σ_2 to Cat.

As a consequence, Theorem 3.12 allows us to present actions of super plactic monoids on categories as follows:

3.14. Theorem. *The category* $\mathrm{Act}(\mathbf{P}(\mathcal{S}))$ *of actions of the monoid* $\mathbf{P}(\mathcal{S})$ *on categories is equivalent to the category of* 2*-functors from the free* $(2, 1)$*-category* $\mathrm{Knuth}_2(\mathcal{S})^\top$ *generated by the* 2*-polygraph* $\mathrm{Knuth}_2(\mathcal{S})$ *to the category* Cat *of categories, sending the* 3*-cells of* $\mathrm{Knuth}_3(\mathcal{S})$ *to commutative diagrams in* Cat.

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