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# COHERENT PRESENTATIONS OF SUPER PLACTIC MONOIDS OF TYPE $A$ BY INSERTIONS

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NOHRA HAGE

**Abstract** – In this paper, we study by rewriting methods the presentations of the super plactic monoid of type  $A$  which is related to the representations of the general linear Lie superalgebra. We construct a convergent presentation of this monoid whose generators are super columns and whose rules are defined by insertions on super tableaux over a signed alphabet. We extend this presentation into a coherent one whose syzygies are defined as relations among insertion algorithms. Finally, we reduce this coherent presentation to a Tietze equivalent one over the initial signed alphabet. Such coherent presentations are used for representations of super plactic monoids by describing their actions on categories.

**Keywords** – Super plactic monoids, syzygies, super tableaux, insertions, string rewriting.

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## 1. INTRODUCTION

We study the presentations of the super version of the plactic monoid of type  $A$  over a signed alphabet by rewriting methods using combinatorial properties of super tableaux. Study presentations by rewriting consists in the orientation of the relations, then called *reduction rules*. Two words are equal in a monoid presented by a rewriting system if they are related by a zig-zag sequence of applications of reductions rules. A rewriting system is *convergent* if the reduction relation induced by the rules is well-founded and any reductions starting on a same word can be extended to end on a same reduced word. Starting with a presentation of a monoid, we are interested in the computation of all *syzygies* for this presentation and in particular to compute a family of generators for the syzygies. In commutative algebra, the theory of Gröbner bases gives algorithms to compute bases for linear syzygies. By a similar method, the syzygy problem for presentation of monoids can be algorithmically solved using *coherent convergent presentations*. Such presentations extend the notion of a convergent presentation of the monoid by globular *homotopy generators* taking into account the relations amongst the relations. Study the syzygies in a monoid produces in higher dimensions free objects that are homotopically equivalent to the original monoid and then allows us to compute its *homological invariants*. Indeed, this study provides the first two steps in the computation of a *polygraphic resolution* of the monoid, that is, a categorical cofibrant replacement of the monoid in a free  $(\omega, 1)$ -category, whose acyclicity is proved by an iterative construction of a normalization reduction strategy, [15]. Moreover, coherent presentations are used to describe the notion of *actions of the monoid on categories*, [12].

This paper is a part of a broader project that consists of studying, by a rewriting approach, families of *plactic-like monoids* defined from combinatorial objects constructed using insertion algorithms. For instance, plactic monoids are related to Young tableaux, [25, 26], Chinese monoids to Chinese staircases, [9], hypoplactic monoids to quasi-ribbon tableaux, [30], patience sorting monoids to patience sorting tableaux, [8], and stalactic monoids to stalactic tableaux [31]. Moreover, binary search trees

## 1. Introduction

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are used to describe normal forms for sylvester, taiga and Baxter monoids, [13, 21, 31]. Coherent presentations are constructed for Artin monoids in [12], for plactic monoids of type A in [19], for plactic monoids of type C in [29], and for Chinese monoids in [20].

A *signed alphabet* is a finite or countable totally ordered set  $\mathcal{S}$  which is disjoint union of two subsets  $\mathcal{S}_0$  and  $\mathcal{S}_1$ . The *super plactic monoid* over a signed alphabet  $\mathcal{S}$ , denoted by  $\mathbf{P}(\mathcal{S})$ , is presented by the rewriting system  $\text{Knuth}_2(\mathcal{S})$  on  $\mathcal{S}$  submitted to the following family of super *Knuth-like relations*, [24]:

$$\begin{aligned} zxy &\Rightarrow xzy, & \text{with } x = y & \text{ only if } y \in \mathcal{S}_0 \text{ and } y = z \text{ only if } y \in \mathcal{S}_1, \\ yzx &\Rightarrow yxz, & \text{with } x = y & \text{ only if } y \in \mathcal{S}_1 \text{ and } y = z \text{ only if } y \in \mathcal{S}_0, \end{aligned}$$

oriented with respect the lexicographic order, for any  $x \leq y \leq z$  of elements of  $\mathcal{S}$ . The congruence generated by this rewriting system, denoted by  $\sim_{\mathbf{P}(\mathcal{S})}$ , also relates those words that yield the same *super tableau* as the result of the *super Schensted-like insertion* algorithm, [24]. This is the *cross-section property* of super tableaux with respect to the congruence relation  $\sim_{\mathbf{P}(\mathcal{S})}$ . Note that when all the elements of  $\mathcal{S}_0$  are less than the ones of  $\mathcal{S}_1$ , the congruence  $\sim_{\mathbf{P}(\mathcal{S})}$  is defined by  $u \sim_{\mathbf{P}(\mathcal{S})} v$  if and only if there is a *crystal isomorphism* between connected components of the crystal graph of the vector representation of the *general linear Lie superalgebra*, that map  $u$  to  $v$ , [1]. Note also that the monoid  $\mathbf{P}(\mathcal{S})$  appeared in [27] as a deformation of the *parastatistics algebra* which is a superalgebra with even parafermi and odd parabose creation and annihilation operators. Moreover, super algebraic structures have found many applications as combinatorial tools in the study of the invariant theory of superalgebras, the representation theory of general Lie super algebras, and algebras satisfying identities, [2, 3, 5, 14]. When  $\mathcal{S} = \mathcal{S}_0 = \{1 < \dots < n\}$ , we recover the notion of the *plactic monoid of type A*, [25], which emerged from the works of Schensted [32] and Knuth [22] on the combinatorial study of Young tableaux. Plactic monoids have found several applications in algebraic combinatorics, representation theory and probabilistic combinatorics, [10, 11, 28], and they were recently investigated by rewriting methods. Indeed, for rank  $n > 3$ , the *Knuth presentation* of the plactic monoid of type A does not admit a finite completion with respect the lexicographic order, [23]. Then convergent presentations are constructed by adding *column generators* and *rows generators* to the Knuth presentation, [4, 6], and similar convergent presentations are constructed for the plactic monoids of classical types, [7, 17]. The author and Malbos extend in [19] the *column presentation* of the plactic monoid of type A into a coherent presentation of this monoid and we reduce it into a smaller one having Knuth's generators.

Let  $\mathcal{S}$  be a signed alphabet. A *super tableau* over  $\mathcal{S}$  is a collection of boxes in left-justified rows filled by elements of  $\mathcal{S}$  such that the entries in each row are weakly increasing from left to right with respect  $\mathcal{S}_0$  and the ones in each column are weakly increasing from top to bottom with respect  $\mathcal{S}_1$ . Note that when  $\mathcal{S} = \mathcal{S}_0$  and  $\mathcal{S} = \mathcal{S}_1$ , we recover the notion of *row-strict* and *column-strict semistandard tableaux* of type A, [11]. We will denote by  $\text{Yt}(\mathcal{S})$  the set of all super tableaux over  $\mathcal{S}$  and by  $R_{\text{row}}$  the map on  $\text{Yt}(\mathcal{S})$  that reads the entries of a super tableau row-wise from bottom to top and from left to right. A *super column* is a word  $x_1 \dots x_k$  over  $\mathcal{S}$  such that  $x_{i+1} \leq x_i$  with  $x_i = x_{i+1}$  only if  $x_i \in \mathcal{S}_1$ . We will denote by  $\text{col}(\mathcal{S})$  the set of all super columns over  $\mathcal{S}$ . Super Schensted-like left and right insertion algorithms are introduced in [24], and consist in inserting elements of  $\mathcal{S}$  into super tableaux by rows and columns respectively. Define the map  $\llbracket \cdot \rrbracket_r$  on the set of words over  $\mathcal{S}$  sending a word to the corresponding super tableau by inserting its letters iteratively from left to right using the right insertion starting from the empty tableau. Following the cross-section property, we deduce that the internal product  $\star_r$  defined on  $\text{Yt}(\mathcal{S})$  by setting  $t \star_r t' := \llbracket R_{\text{row}}(t)R_{\text{row}}(t') \rrbracket_r$ , for all  $t$  and  $t'$  in  $\text{Yt}(\mathcal{S})$ , is associative, and then the set  $(\text{Yt}(\mathcal{S}), \star_r)$  is isomorphic to the super plactic monoid  $\mathbf{P}(\mathcal{S})$ . We show in

Lemma 2.6 that for all  $u$  and  $v$  in  $\text{col}(\mathcal{S})$  such that the topmost juxtaposition of  $\llbracket u \rrbracket_r$  and  $\llbracket v \rrbracket_r$  does not form a super tableau, the super tableau  $\llbracket uv \rrbracket_r$  contains at most two columns, and if it contains exactly two columns then the left one contains more elements than  $u$ .

We construct in Section 2 a convergent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ , denoted by  $\text{Col}_2(\mathcal{S})$ , whose set of generators is  $\text{Col}_1(\mathcal{S}) := \{c_u \mid u \in \text{col}(\mathcal{S})\}$  and whose rules are

$$\gamma_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

for every  $c_u, c_v$  in  $\text{Col}_1(\mathcal{S})$  such that the topmost juxtaposition of  $\llbracket u \rrbracket_r$  and  $\llbracket v \rrbracket_r$  does not form a super tableau and where  $w$  and  $w'$  denote the readings of the left and right columns of  $\llbracket uv \rrbracket_r$  respectively. We show that  $\text{Col}_2(\mathcal{S})$  can be obtained from  $\text{Knuth}_2(\mathcal{S})$  by applying *Tietze transformations* that consist in adding or removing derivable generators and in adding or removing derivable relations on a presentation of a monoid in such a way that they do not change the presented monoid. We also show that the confluence of  $\text{Col}_2(\mathcal{S})$  is a direct consequence of the associativity of the product  $\star_r$ . Theorem 2.12 states that  $\text{Col}_2(\mathcal{S})$  is a convergent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ , called the *super column presentation*.

We extend in Section 3 the super column presentation into coherent presentations of the monoid  $\mathbf{P}(\mathcal{S})$  using the *homotopical completion-reduction procedure*, [12]. Denote by  $\text{Col}_3(\mathcal{S})$  the extended presentation of  $\mathbf{P}(\mathcal{S})$  obtained from  $\text{Col}_2(\mathcal{S})$  by adjunction of one family of syzygies of the following form

$$\begin{array}{ccccc} & \gamma_{u,v} c_t & \xrightarrow{\quad} & c_e c_{e'} c_t & \xrightarrow{c_e \gamma_{e',t}} & c_e c_b c_{b'} & \xrightarrow{\quad} & \gamma_{e,b} c_{b'} \\ c_u c_v c_t & \xrightarrow{\quad} & & & \Downarrow \mathcal{X}_{u,v,t} & & & c_a c_d c_{b'} \\ & c_u \gamma_{v,t} & \xrightarrow{\quad} & c_u c_w c_{w'} & \xrightarrow{\gamma_{u,w} c_{w'}} & c_a c_{a'} c_{w'} & \xrightarrow{\quad} & c_a \gamma_{a',w} \end{array}$$

for every  $c_u, c_v$  and  $c_t$  in  $\text{Col}_1(\mathcal{S})$  such that the juxtapositions of  $\llbracket u \rrbracket_r$  and  $\llbracket v \rrbracket_r$ , and of  $\llbracket v \rrbracket_r$  and  $\llbracket t \rrbracket_r$  do not form super tableaux. Following Squier's coherence theorem, [33], Theorem 3.3 states that  $\text{Col}_3(\mathcal{S})$  is a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ , called the *super column coherent presentation*. We then apply the homotopical reduction procedure on  $\text{Col}_3(\mathcal{S})$  in order to reduce it into a smaller one. As a first step, we apply a homotopical reduction on  $\text{Col}_3(\mathcal{S})$  with a *collapsible part* defined by some of the *generating triple confluences* of  $\text{Col}_2(\mathcal{S})$  and we reduce it into the coherent presentation  $\overline{\text{Col}}_3(\mathcal{S})$  of  $\mathbf{P}(\mathcal{S})$ , whose underlying rewriting system is  $\text{Col}_2(\mathcal{S})$  and the syzygies  $\mathcal{X}_{x,v,t}$  are those of  $\text{Col}_3(\mathcal{S})$ , but with  $x$  in  $\mathcal{S}$ . In a second step, we reduce  $\overline{\text{Col}}_3(\mathcal{S})$  into a coherent presentation  $\text{PreCol}_3(\mathcal{S})$  of  $\mathbf{P}(\mathcal{S})$ , using a collapsible part defined by a set of syzygies of  $\overline{\text{Col}}_3(\mathcal{S})$ . In a final step, we reduce  $\text{PreCol}_3(\mathcal{S})$  into an extended presentation of  $\mathbf{P}(\mathcal{S})$ , denoted by  $\text{Knuth}_3(\mathcal{S})$ , whose underlying rewriting system is  $\text{Knuth}_2(\mathcal{S})$ . Theorem 3.12 states that  $\text{Knuth}_3(\mathcal{S})$  is a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ . Finally, we use in Subsection 3.13 this coherent presentation in order to describe the actions of super plactic monoids on categories.

**Notation.** Let  $\mathcal{A}$  be a totally ordered alphabet. We will denote by  $\mathcal{A}^*$  the free monoid of *words over  $\mathcal{A}$* , the product being concatenation of words, and the identity being the empty word. We will denote by  $w = x_1 \dots x_k$  a word in  $\mathcal{A}^*$  of *length  $k$* , where  $x_1, \dots, x_k$  belong to  $\mathcal{A}$ . The length of a word  $w$  will be denoted by  $|w|$ . Let  $w = x_1 \dots x_k$  be a word in  $\mathcal{A}^*$ . We denote by  $\ell(w)$  the leftmost letter of  $w$  and by  $\text{Rem}(w)$  the subword of  $w$  such that  $w = \ell(w) \text{Rem}(w)$ . A word  $w'$  is a *subsequence* of  $w$  if  $w' = x_{i_1} \dots x_{i_l}$  with  $1 \leq i_1 < \dots < i_l < k$ . We will denote by  $[n]$  the ordered set  $\{1 < \dots < n\}$  for  $n$

## 2. Super column presentation of the super plactic monoid

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in  $\mathbb{Z}_{>0}$ . Let  $\mathcal{S}$  be a finite or countable totally ordered set and  $\|\cdot\| : \mathcal{S} \rightarrow \mathbb{Z}_2$  be any map, where  $\mathbb{Z}_2 = \{0, 1\}$  denotes the additive cyclic group of order 2. The ordered pair  $(\mathcal{S}, \|\cdot\|)$  is called a *signed alphabet*, and we denote  $\mathcal{S}_0 = \{a \in \mathcal{S} \mid \|a\| = 0\}$  and  $\mathcal{S}_1 = \{a \in \mathcal{S} \mid \|a\| = 1\}$ . A monoid  $\mathbf{M}$  is said a  $\mathbb{Z}_2$ -graded monoid or a *supermonoid* if a map  $\|\cdot\| : \mathbf{M} \rightarrow \mathbb{Z}_2$  is given such that  $\|u.v\| = \|u\| + \|v\|$ , for all  $u$  and  $v$  in  $\mathbf{M}$ . We call  $\|u\|$  the  $\mathbb{Z}_2$ -degree of the element  $u$ . The free monoid  $\mathcal{S}^*$  over  $\mathcal{S}$  is  $\mathbb{Z}_2$ -graded by considering  $\|w\| := \|x_1\| + \dots + \|x_k\|$ , for any word  $w = x_1 \dots x_k$  in  $\mathcal{S}^*$ . In the rest of this article, and if there is no possible confusion,  $\mathcal{S}$  denotes a signed alphabet.

### 2. SUPER COLUMN PRESENTATION OF THE SUPER PLACTIC MONOID

In this article, rewriting methods are presented in the language of polygraphs, that we recall in this section and we refer the reader to [12, 16] for a deeper presentation. We also recall the notions of super plactic monoids and super tableaux from [18, 24]. We end this section by constructing a convergent presentation of the super plactic monoid by adding super columns generators and using combinatorial properties of super tableaux.

**2.1. Presentations of monoids by 2-polygraphs.** In this article, we deal with presentations of monoids by rewriting systems, described by 2-polygraphs with only 0-cell denoted by  $\bullet$ . Such a 2-polygraph  $\Sigma$  is given by a pair  $(\Sigma_1, \Sigma_2)$ , where  $\Sigma_1$  is a set and  $\Sigma_2$  is a *globular extension* of the free monoid  $\Sigma_1^*$ , that is a set of 2-cells  $\alpha : s_1(\alpha) \Rightarrow t_1(\alpha)$  relating 1-cells in  $\Sigma_1^*$ , where  $s_1(\alpha)$  and  $t_1(\alpha)$  denote the *source* and the *target* of  $\alpha$  respectively. If there is no possible confusion,  $\Sigma_2$  will denote the 2-polygraph itself. Recall that a 2-category (resp. (2, 1)-category) is a category enriched in categories (resp. in groupoids). When two 1-cells, or 2-cells,  $f$  and  $g$  of a 2-category are 0-composable (resp. 1-composable), we denote by  $fg$  (resp.  $f \star_1 g$ ) their 0-composite (resp. 1-composite). We will denote by  $\Sigma_2^*$  (resp.  $\Sigma_2^\top$ ) the 2-category (resp. (2, 1)-category) freely generated by the 2-polygraph  $\Sigma$ , see [16] for more information. The *monoid presented by a 2-polygraph*  $\Sigma$ , denoted by  $\bar{\Sigma}$ , is defined as the quotient of the free monoid  $\Sigma_1^*$  by the congruence generated by the set of 2-cells  $\Sigma_2$ . A *presentation* of a monoid  $\mathbf{M}$  is a 2-polygraph whose presented monoid is isomorphic to  $\mathbf{M}$ . Two 2-polygraphs are *Tietze equivalent* if they present isomorphic monoids. A 2-cell  $\alpha$  of a 2-polygraph  $\Sigma$  is *collapsible*, if  $t_1(\alpha)$  is a 1-cell of  $\Sigma_1$  and  $s_1(\alpha)$  does not contain  $t_1(\alpha)$ , then  $t_1(\alpha)$  is called *redundant*. An *elementary Tietze transformation* of a 2-polygraph  $\Sigma$  is a 2-functor with domain  $\Sigma_2^\top$  that belongs to one of the following transformations, [12]:

- i) *adjunction*  $\iota_\alpha^1 : \Sigma_2^\top \rightarrow \Sigma_2^\top[x](\alpha)$  (resp.  $\iota_\alpha : \Sigma_2^\top \rightarrow \Sigma_2^\top(\alpha)$ ) of a redundant 1-cell  $x$  with its collapsible 2-cell  $\alpha$  (resp. of a redundant 2-cell  $\alpha$ ),
- ii) *elimination*  $\pi_\alpha : \Sigma_2^\top \rightarrow (\Sigma_1 \setminus \{x\}, \Sigma_2 \setminus \{\alpha\})^\top$  (resp.  $\pi_{(\mu, \alpha)} : \Sigma_2^\top \rightarrow \Sigma_2^\top / (\mu, \alpha)$ ) of a redundant 1-cell  $x$  with its collapsible 2-cell  $\alpha$  (resp. of a redundant 2-cell  $\alpha$ ).

If  $\Sigma$  and  $\Upsilon$  are 2-polygraphs, a *Tietze transformation* from  $\Sigma$  to  $\Upsilon$  is a 2-functor  $F : \Sigma^\top \rightarrow \Upsilon^\top$  that decomposes into sequence of elementary Tietze transformations. Two 2-polygraphs are Tietze equivalent if and only if there exists a Tietze transformation between them, [12]. Given a 2-polygraph  $\Sigma$  and a 2-cell  $\mu_1 \star_1 \mu \star_1 \mu_2$  in  $\Sigma_2^\top$ , the *Nielsen transformation*  $\kappa_{\mu \leftarrow \alpha}$  is the Tietze transformation that replaces in  $\Sigma_2^\top$  the 2-cell  $\mu$  by a 2-cell  $\alpha : s_1(\mu_1) \Rightarrow t_1(\mu_2)$ . When  $\mu_2$  is identity, we denote by  $\kappa'_{\mu \leftarrow \alpha}$  the Nielsen transformation which, given a 2-cell  $\mu_1 \star_1 \mu$  in  $\Sigma_2^\top$ , replaces the 2-cell  $\mu$  by a 2-cell  $\alpha : s_1(\mu_1) \Rightarrow t_1(\mu)$ .

A *rewriting step* of a 2-polygraph  $\Sigma$  is a 2-cell of  $\Sigma_2^*$  with shape  $w\alpha w'$ , where  $\alpha$  is a 2-cell of  $\Sigma_2$  and  $w$  and  $w'$  are 1-cells of  $\Sigma_1^*$ . A *rewriting sequence* of  $\Sigma$  is a finite or infinite sequence of rewriting steps. A 1-cell  $u$  of  $\Sigma_1^*$  is a *normal form* if there is no rewriting step with source  $u$ . The 2-polygraph  $\Sigma$  *terminates* if it has no infinite rewriting sequence. A *branching* of  $\Sigma$  is a non ordered pair  $(f, g)$  of 2-cells of  $\Sigma_2^*$  such that  $s_1(f) = s_1(g)$ . A branching  $(f, g)$  is *local* if  $f$  and  $g$  are rewriting steps. A branching is *aspherical* if it is of the form  $(f, f)$ , for a rewriting step  $f$  and *Peiffer* when it is of the form  $(fv, ug)$  for rewriting steps  $f$  and  $g$  with  $s_1(f) = u$  and  $s_1(g) = v$ . The *overlapping* branchings are the remaining local branchings. An overlapping local branching is *critical* if it is minimal for the order  $\sqsubseteq$  generated by the relations  $(f, g) \sqsubseteq (wfw', wgw')$ , given for any local branching  $(f, g)$  and any possible 1-cells  $w$  and  $w'$  of the category  $\Sigma_1^*$ . A branching  $(f, g)$  is *confluent* if there exist 2-cells  $f'$  and  $g'$  in  $\Sigma_2^*$  such that  $s_1(f') = t_1(f)$ ,  $s_1(g') = t_1(g)$  and  $t_1(f') = t_1(g')$ . A 2-polygraph  $\Sigma$  is *confluent* if all of its branchings are confluent. It is *convergent* if it terminates and it is confluent. In that case, every 1-cell  $u$  of  $\Sigma_1^*$  has a unique normal form.

**2.2. Super Young tableaux and insertions.** A *partition* of a positive integer  $n$ , is a weakly decreasing sequence  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$  such that  $\sum \lambda_i = n$ . The integer  $k$  is called *number of parts* or *height* of  $\lambda$ . The *Young diagram* of a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  is the set  $\mathcal{Y}(\lambda) := \{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}$ , that can be represented by a diagram by drawing a box for each pair  $(i, j)$ . The transposed diagram  $\{(j, i) \mid (i, j) \in \mathcal{Y}(\lambda)\}$  defines another partition, called the *conjugate partition* of  $\lambda$ , whose parts are the lengths of the columns of  $\mathcal{Y}(\lambda)$ . Let  $\lambda$  be a partition. A *super semistandard Young tableau*, or *super tableau* for short, over  $\mathcal{S}$  is a pair  $t := (\lambda, \mathcal{T})$  where  $\mathcal{T} : \mathcal{Y}(\lambda) \rightarrow \mathcal{S}$  is a map satisfying  $\mathcal{T}(i, j) \leq \mathcal{T}(i, j + 1)$ , with  $\mathcal{T}(i, j) = \mathcal{T}(i, j + 1)$  only if  $\|\mathcal{T}(i, j)\| = 0$ , and  $\mathcal{T}(i, j) \leq \mathcal{T}(i + 1, j)$ , with  $\mathcal{T}(i, j) = \mathcal{T}(i + 1, j)$  only if  $\|\mathcal{T}(i, j)\| = 1$ . We will call  $\mathcal{Y}(\lambda)$ ,  $\mathcal{T}$  and  $\lambda$ , the *frame*, the *filing* and the *shape* of the super tableau  $t$  respectively. We will denote by  $\text{Yt}(\mathcal{S})$  the set of all super tableaux over  $\mathcal{S}$ .

Denote by  $R_{\text{row}}$  (resp.  $R_{\text{col}}$ ) the reading map on  $\text{Yt}(\mathcal{S})$  that reads a super tableau row-wise (column-wise) from bottom to top and from left to right. For instance, consider the alphabet  $\mathcal{S} = \{1, 2, 3, 4, 5\}$  with signature given by  $\mathcal{S}_0 = \{1, 2, 4\}$  and  $\mathcal{S}_1$  defined consequently. The following diagram is a super tableau over  $\mathcal{S}$ :

$$t = \begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 3 & 4 & 4 \\ \hline 5 & & \\ \hline 5 & & \\ \hline \end{array} \quad \text{with } R_{\text{row}}(t) = 55344112 \quad \text{and } R_{\text{col}}(t) = 55314142.$$

A *super row* (resp. *super column*) is a word  $x_1 \dots x_k$  in  $\mathcal{S}^*$  such that  $x_i \leq x_{i+1}$  (resp.  $x_{i+1} \leq x_i$ ) with  $x_i = x_{i+1}$  only if  $\|x_i\| = 0$  (resp.  $\|x_i\| = 1$ ). In other words, a super row (resp. super column) is the reading of a super tableau whose shape is a row (resp. column). We will denote by  $\text{col}(\mathcal{S})$  the set of all super columns over  $\mathcal{S}$ . Denote by  $\leq_{\text{deglex}}$  (resp.  $\leq_{\text{rev}}$ ) the *length lexicographic order* (resp. *length reverse lexicographic order*) on  $\text{col}(\mathcal{S})$  defined by  $u \leq_{\text{deglex}} v$  (resp.  $u \leq_{\text{rev}} v$ ) if  $|u| < |v|$  (resp.  $|u| > |v|$ ) or  $|u| = |v|$  and  $u <_{\text{lex}} v$ , for all  $u$  and  $v$  in  $\text{col}(\mathcal{S})$ , where  $<_{\text{lex}}$  denotes the lexicographic order on  $\mathcal{S}$ .

Recall from [24] the right and left insertion algorithms on  $\text{Yt}(\mathcal{S})$  that insert an element  $x$  in  $\mathcal{S}$  into a super tableau  $t$  of  $\text{Yt}(\mathcal{S})$ . The *right (or row) insertion*, denoted by  $\llcorner$ , computes a super tableau  $t \llcorner x$  as follows. If  $x \in \mathcal{S}_0$  (resp.  $x \in \mathcal{S}_1$ ) is at least as large as (resp. larger than) the last element of the top row of  $t$ , then put  $x$  in a box to the right of this row. Otherwise, let  $y$  be the smallest element of the top row of  $t$  such that  $y > x$  (resp.  $y \geq x$ ). Then  $x$  replaces  $y$  in this row and  $y$  is bumped into the next row

## 2. Super column presentation of the super plactic monoid

where the process is repeated. The algorithm terminates when the element which is bumped is at least as large as (resp. larger than) the last element of the next row. Then it is placed in a box at the right of that row. The *left (or column) insertion*, denoted by  $\leftarrow$ , computes a super tableau  $x \leftarrow t$  as follows. If  $x \in \mathcal{S}_0$  (resp.  $x \in \mathcal{S}_1$ ) is larger than (resp. at least as large as) the bottom element of the leftmost column of  $t$ , then put  $x$  in a box to the bottom of this column. Otherwise, let  $y$  be the smallest element of the leftmost column of  $t$  such that  $y \geq x$  (resp.  $y > x$ ). Then  $x$  replaces  $y$  in this column and  $y$  is bumped into the next column where the process is repeated. The algorithm terminates when the element which is bumped is greater than (resp. at least as large as) all the elements of the next column. Then it is placed in a box at the bottom of that column. For instance, consider  $\mathcal{S} = \mathbb{N}$  with signature given by  $\mathcal{S}_0$  the set of even numbers and  $\mathcal{S}_1$  defined consequently. We have

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 1 & 3 & 4 & \\ \hline 3 & & & \\ \hline \end{array} \leftarrow 2 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 2 \\ \hline 1 & 3 & 4 & \\ \hline 3 & & & \\ \hline 3 & & & \\ \hline \end{array} \quad \text{and} \quad 1 \rightsquigarrow \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 5 & 6 \\ \hline 1 & 4 & 5 & \\ \hline 2 & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 2 & 5 & 6 \\ \hline 1 & 4 & 5 & & \\ \hline & & & & \\ \hline 1 & & & & \\ \hline \end{array}$$

Note that when  $\mathcal{S} = \mathcal{S}_0 = [n]$ , the right (resp. left) insertion corresponds to the Schensted's right (resp. left) insertion introduced in [32] on row-strict semistandard tableaux over  $[n]$ . For any word  $w = x_1 \dots x_k$  over  $\mathcal{S}$ , denote by  $\llbracket w \rrbracket_r$  the super tableau obtained from  $w$  by inserting its letters iteratively from left to right using the right insertion starting from the empty tableau:

$$\llbracket w \rrbracket_r := (\emptyset \leftarrow w) = ((\dots (\emptyset \leftarrow x_1) \leftarrow \dots) \leftarrow x_k).$$

Note that for any super tableau  $t$  in  $\text{Yt}(\mathcal{S})$ , the equality  $\llbracket R_{\text{row}}(t) \rrbracket_r = t$  holds in  $\text{Yt}(\mathcal{S})$ , [24]. We define an internal product  $\star_r$  on  $\text{Yt}(\mathcal{S})$  by setting

$$t \star_r t' := (t \leftarrow R_{\text{row}}(t'))$$

for all  $t, t'$  in  $\text{Yt}(\mathcal{S})$ . By definition the relations  $t \star_r \emptyset = t$  and  $\emptyset \star_r t = t$  hold, showing that the product  $\star_r$  is unitary with respect to  $\emptyset$ .

**2.3. The super plactic monoid.** The *super plactic monoid* over  $\mathcal{S}$ , denoted by  $\mathbf{P}(\mathcal{S})$ , is presented by the 2-polygraph  $\text{Knuth}_2(\mathcal{S})$ , whose set of 1-cells is  $\mathcal{S}$  and whose 2-cells are, [24] :

$$\begin{aligned} \eta_{x,y,z} : zxy &\Rightarrow xzy, \quad \text{with } x = y \text{ only if } \|y\| = 0 \text{ and } y = z \text{ only if } \|y\| = 1, \\ \varepsilon_{x,y,z} : yzx &\Rightarrow yxz, \quad \text{with } x = y \text{ only if } \|y\| = 1 \text{ and } y = z \text{ only if } \|y\| = 0, \end{aligned} \quad (1)$$

oriented with respect the lexicographic order, for any  $x \leq y \leq z$  of elements of  $\mathcal{S}$ . The congruence generated by this 2-polygraph, denoted by  $\sim_{\mathbf{P}(\mathcal{S})}$ , is called the *super plactic congruence*. Note that since the relations (1) are  $\mathbb{Z}_2$ -homogeneous we have that  $\mathbf{P}(\mathcal{S})$  is a supermonoid. Moreover, for any  $w$  in  $\mathcal{S}^*$ , we have  $w \sim_{\mathbf{P}(\mathcal{S})} R_{\text{row}}(\llbracket w \rrbracket_r)$ , [24], and for any  $t$  in  $\text{Yt}(\mathcal{S})$ , we have  $R_{\text{row}}(t) \sim_{\mathbf{P}(\mathcal{S})} R_{\text{col}}(t)$ , [18]. Note finally that super tableaux satisfy the *cross-section property* for  $\sim_{\mathbf{P}(\mathcal{S})}$ , that is, for all  $w$  and  $w'$  in  $\mathcal{S}^*$ ,  $w \sim_{\mathbf{P}(\mathcal{S})} w'$  if and only if  $\llbracket w \rrbracket_r = \llbracket w' \rrbracket_r$ , [24]. As a consequence of the cross-section property, we deduce that the product  $\star_r$  is associative and the following equality

$$y \rightsquigarrow (t \leftarrow x) = (y \rightsquigarrow t) \leftarrow x \quad (2)$$

holds in  $\text{Yt}(\mathcal{S})$ , for all  $t$  in  $\text{Yt}(\mathcal{S})$  and  $x, y$  in  $\mathcal{S}$ . In particular, for any word  $w = x_1 \dots x_k$  in  $\mathcal{S}^*$  the super tableau  $\llbracket w \rrbracket_r$  can be also computed by inserting its elements iteratively from right to left using the left insertion starting from the empty tableau:

$$\llbracket w \rrbracket_r = (w \rightsquigarrow \emptyset) := (x_1 \rightsquigarrow (\dots \rightsquigarrow (x_k \rightsquigarrow \emptyset) \dots)).$$

Note that the associativity of the product  $\star_r$  can be obtained using the properties of the *super jeu de taquin* introduced in [18]. Note also that we can show the equality (2) using only the definitions of the insertion algorithms and without supposing the cross-section property, and then the associativity of the product  $\star_r$  and the cross-section property will be consequences of this equality.

Let  $w$  be in  $\mathcal{S}^*$ . For any  $k \geq 0$ , denote by  $l_k(w)$  (resp.  $\tilde{l}_k(w)$ ) the maximal number which can be obtained as the sum of the lengths of  $k$  super rows (resp. super columns) that are disjoint subsequences of  $w$ . Let  $\lambda = (\lambda_1, \dots, \lambda_k)$  be the shape of the super tableau  $\llbracket w \rrbracket_r$  and  $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_l)$  be the conjugate partition of  $\lambda$ . For any  $k \geq 0$ , we have  $l_k(w) = \lambda_1 + \dots + \lambda_k$  and  $\tilde{l}_k(w) = \tilde{\lambda}_1 + \dots + \tilde{\lambda}_l$ , [24]. In particular, we deduce the following result.

**2.4. Lemma ([24]).** *Let  $w$  be a word over  $\mathcal{S}$ . The number of columns in the super tableau  $\llbracket w \rrbracket_r$  is equal to the length of the longest super row word that is a subsequence of  $w$ , and the number of rows in  $\llbracket w \rrbracket_r$  is equal to the length of the longest super column word that is a subsequence of  $w$ .*

**2.5. Graphical notations.** Given  $u = x_1 \dots x_p$  and  $v = y_1 \dots y_q$  in  $\text{col}(\mathcal{S})$ . We will use the following notations depending on whether the juxtaposition of  $\llbracket u \rrbracket_r$  and  $\llbracket v \rrbracket_r$  forms a super tableau:

- i) we will denote  $u \widehat{v}$  if the topmost juxtaposition of  $\llbracket u \rrbracket_r$  and  $\llbracket v \rrbracket_r$  forms a super tableau, that is,  $|u| \geq |v|$  and  $x_i \leq y_i$ , for any  $i \leq |v|$ , with  $x_i = y_i$  only if  $||x_i|| = 0$ ,
- ii) we will denote  $u^\times v$  in all the other cases, that is, when  $|u| < |v|$  or  $x_i \geq y_i$ , for some  $i \leq |v|$ , with  $x_i = y_i$  only if  $||x_i|| = 1$ .

**2.6. Lemma.** *Let  $u$  and  $v$  be in  $\text{col}(\mathcal{S})$  such that  $u^\times v$ . The super tableau  $\llbracket uv \rrbracket_r$  contains at most two columns. Moreover, if  $\llbracket uv \rrbracket_r$  contains exactly two columns, the left column contains more elements than  $u$ .*

*Proof.* Consider  $u = x_1 \dots x_p$  and  $v = y_1 \dots y_q$  in  $\text{col}(\mathcal{S})$  such that  $u^\times v$ . If  $x_p \geq y_1$  with  $x_p = y_1$  only if  $||x_p|| = 1$ , then the super tableau  $\llbracket uv \rrbracket_r$  consists of only column whose reading is  $uv$ . Otherwise, suppose  $x_p \leq y_1$  with  $x_p = y_1$  only if  $||x_p|| = 0$ . Since the words  $u$  and  $v$  are decreasing with respect to  $\mathcal{S}_1$ , the longest super row word that is a subsequence of  $uv$  contains one element from each of  $u$  and  $v$ . Then its length is equal to 2, showing by Lemma 2.4 that  $\llbracket uv \rrbracket_r$  contains two columns. Suppose now that  $\llbracket uv \rrbracket_r$  contains exactly two columns. Then its leftmost column is obtained by inserting some elements of  $v$  into  $u$ , and by keeping the elements of  $u$  unchangeable, showing that it contains more elements than  $u$ .  $\square$

When  $u^\times v$ , we will denote  $u^{\times 1}v$  if the super tableau  $\llbracket uv \rrbracket_r$  consists of one column and by  $u^{\times 2}v$  if it consists of two columns.



## 2. Super column presentation of the super plactic monoid

**2.7. Super columns as generators.** Suppose  $\mathcal{S} = \{x_1, x_2, \dots\}$ . Let  $\text{Col}_1(\mathcal{S}) = \{c_u \mid u \in \text{col}(\mathcal{S})\}$  be the set of *super column generators* of the super plactic monoid  $\mathbf{P}(\mathcal{S})$  and

$$C_2(\mathcal{S}) = \{ \mu_u : c_{x_p} \dots c_{x_1} \Rightarrow c_u \mid u = x_p \dots x_1 \in \text{col}(\mathcal{S}) \text{ with } |u| \geq 2 \}$$

be the set of the defining relations for the super column generators. We will denote by  $\text{Knuth}_2^c(\mathcal{S})$  the 2-polygraph whose set of 1-cells is  $\{c_{x_1}, c_{x_2}, \dots\}$  and whose 2-cells are

$$\begin{aligned} \eta_{x,y,z}^c : c_z c_x c_y &\Rightarrow c_x c_z c_y, \text{ with } x = y \text{ only if } \|y\| = 0 \text{ and } y = z \text{ only if } \|y\| = 1, \\ \varepsilon_{x,y,z}^c : c_y c_z c_x &\Rightarrow c_y c_x c_z, \text{ with } x = y \text{ only if } \|y\| = 1 \text{ and } y = z \text{ only if } \|y\| = 0, \end{aligned}$$

for any  $x \leq y \leq z$  of elements of  $\mathcal{S}$ . By definition, this 2-polygraph is Tietze equivalent to  $\text{Knuth}_2(\mathcal{S})$ . In the sequel, if there is no possible confusion, we will identify the 2-polygraphs  $\text{Knuth}_2^c(\mathcal{S})$  and  $\text{Knuth}_2(\mathcal{S})$ . Denote by  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$  the 2-polygraph whose set of 1-cells is  $\{c_{x_1}, c_{x_2}, \dots\}$  and whose set of 2-cells is  $C_2(\mathcal{S}) \cup \text{Knuth}_2^c(\mathcal{S})$ .

**2.8. Proposition.** *The 2-polygraph  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$  is a presentation of the super plactic monoid  $\mathbf{P}(\mathcal{S})$ .*

*Proof.* By adding to the 2-polygraph  $\text{Knuth}_2^c(\mathcal{S})$  all the column generators  $c_u$ , for all  $u = x_p \dots x_1$  in  $\text{col}(\mathcal{S})$  such that  $|u| \geq 2$ , and the corresponding collapsible 2-cell  $\mu_u : c_{x_p} \dots c_{x_1} \Rightarrow c_u$ , we obtain that the 2-polygraphs  $\text{Knuth}_2^c(\mathcal{S})$  and  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$  are Tietze-equivalent, showing the claim.  $\square$

**2.9. Super pre-column presentation.** Denote by  $\text{PC}_2(\mathcal{S})$  the set of 2-cells of the following form:

$$\begin{aligned} \gamma'_{x,zy} : c_x c_z c_y &\Rightarrow c_{zx} c_y, \text{ with } x = y \text{ only if } \|y\| = 0 \text{ and } y = z \text{ only if } \|y\| = 1, \\ \gamma'_{y,zx} : c_y c_z c_x &\Rightarrow c_{yx} c_z, \text{ with } x = y \text{ only if } \|y\| = 1 \text{ and } y = z \text{ only if } \|y\| = 0, \end{aligned}$$

for any  $x \leq y \leq z$  elements of  $\mathcal{S}$ . Define the 2-polygraph  $\text{PreCol}_2(\mathcal{S})$  whose set of 1-cells is  $\text{Col}_1(\mathcal{S})$  and whose set of 2-cells is  $\text{PC}_2(\mathcal{S}) \cup \{\gamma'_{x,u} : c_x c_u \Rightarrow c_{xu} \mid xu \in \text{col}(\mathcal{S}) \text{ and } x \in \mathcal{S}\}$ .

**2.10. Proposition.** *The 2-polygraph  $\text{PreCol}_2(\mathcal{S})$  is a presentation of the super plactic monoid  $\mathbf{P}(\mathcal{S})$ .*

*Proof.* We first prove that 2-polygraph  $\text{CPC}_2(\mathcal{S})$  whose set of 1-cells is  $\text{Col}_1(\mathcal{S})$  and set of 2-cells is  $C_2(\mathcal{S}) \cup \text{PC}_2(\mathcal{S})$ , is Tietze equivalent to  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$ . We consider the following critical branching

$$\begin{array}{ccc} & & c_x \mu_{zy} \\ & \nearrow \eta_{x,y,z}^c & \longrightarrow c_x c_z c_y \longrightarrow c_x c_z c_y \\ & \searrow & \\ c_z c_x c_y & \xrightarrow{\mu_{zx}^c} & c_{zx} c_y \end{array}$$

of the 2-polygraph  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$ , for any  $x \leq y \leq z$  with  $x = y$  only if  $\|y\| = 0$  and  $y = z$  only if  $\|y\| = 1$ , and the Tietze transformation  $\kappa_{\eta_{x,y,z}^c \leftarrow \gamma'_{x,zy}} : \text{Knuth}_2^{\text{cc}}(\mathcal{S})^\top \longrightarrow \text{Knuth}_2^{\text{cc}}(\mathcal{S})^\top / (\eta_{x,y,z}^c \leftarrow \gamma'_{x,zy})$ , that substitutes the 2-cell  $\gamma'_{x,zy}$  for the 2-cell  $\eta_{x,y,z}^c$ , and denote by  $T_{\eta \leftarrow \gamma'}$  the successive applications of  $\kappa_{\eta_{x,y,z}^c \leftarrow \gamma'_{x,zy}}$ , for any  $x \leq y \leq z$  with  $x = y$  only if  $\|y\| = 0$  and  $y = z$  only if  $\|y\| = 1$ , with respect to the lexicographic order on the triples  $(x, y, z)$  induced by the total order on  $\mathcal{S}$ . Similarly, we study the critical branching  $(\varepsilon_{x,y,z}^c, c_y \mu_{zx})$  of  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$ , for any  $x \leq y \leq z$  with  $x = y$  only if  $\|y\| = 1$  and  $y = z$  only if  $\|y\| = 0$ , by introducing the Tietze transformation  $\kappa_{\varepsilon_{x,y,z}^c \leftarrow \gamma'_{y,zx}}$  from  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})^\top$

to  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})^\top / (\varepsilon_{x,y,z}^c \leftarrow \gamma'_{y,zx})$ . Denote by  $T_{\varepsilon \leftarrow \gamma'}$  the successive applications of this Tietze transformation with respect to the lexicographic order on the triples  $(x, y, z)$  induced by the total order on  $\mathcal{S}$ , and by  $T_{\eta, \varepsilon \leftarrow \gamma'}$  the Tietze transformation from  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})^\top$  to  $\text{CPC}_2(\mathcal{S})^\top$  given by  $T_{\eta \leftarrow \gamma'} \circ T_{\varepsilon \leftarrow \gamma'}$ .

Finally, we prove that the 2-polygraphs  $\text{CPC}_2(\mathcal{S})$  and  $\text{PreCol}_2(\mathcal{S})$  are Tietze equivalents. Let  $x_p \dots x_1$  be in  $\text{col}(\mathcal{S})$  with  $|x_p \dots x_1| > 2$  and define  $\gamma'_{y,x} := \mu_{yx} : c_y c_x \Rightarrow c_{yx}$ , for any  $x \leq y$  with  $x = y$  only if  $\|x\| = 1$ . We consider the following critical branching

$$\begin{array}{c} c_{x_p} \mu_{x_{p-1} \dots x_1} \xrightarrow{\quad} c_{x_p} c_{x_{p-1} \dots x_1} \\ \searrow \quad \nearrow \\ c_{x_p} \dots c_{x_1} \xrightarrow{\mu_{x_p \dots x_1}} c_{x_p \dots x_1} \end{array}$$

of the 2-polygraph  $\text{CPC}_2(\mathcal{S})$  and the following Tietze transformation

$$\kappa'_{\mu_{x_p \dots x_1} \leftarrow \gamma'_{x_p, x_{p-1} \dots x_1}} : \text{CPC}_2(\mathcal{S})^\top \longrightarrow \text{CPC}_2(\mathcal{S})^\top / (\mu_{x_p \dots x_1} \leftarrow \gamma'_{x_p, x_{p-1} \dots x_1}),$$

that substitutes the 2-cell  $\gamma'_{x_p, x_{p-1} \dots x_1}$  for the 2-cell  $\mu_{x_p \dots x_1}$ , for each column  $x_p \dots x_1$  such that  $p > 2$ . Starting from  $\text{CPC}_2(\mathcal{S})$ , we apply successively the Tietze transformation  $\kappa'_{\mu_{x_p \dots x_1} \leftarrow \gamma'_{x_p, x_{p-1} \dots x_1}}$ , from the bigger column to the smaller one with respect to  $\leq_{\text{deglex}}$ . The composite  $T_{\mu \leftarrow \gamma'} = \kappa'_{\mu_{x_3 x_2 x_1} \leftarrow \gamma'_{x_3, x_2 x_1}} \circ \dots \circ \kappa'_{\mu_{x_p \dots x_1} \leftarrow \gamma'_{x_p, x_{p-1} \dots x_1}}$ , defines a Tietze transformation from  $\text{CPC}_2(\mathcal{S})^\top$  to  $\text{PreCol}_2(\mathcal{S})^\top$ .  $\square$

**2.11. Super column presentation.** Consider the 2-polygraph  $\text{Col}_2(\mathcal{S})$  whose set of 1-cells is  $\text{Col}_1(\mathcal{S})$  and whose 2-cells are

$$\gamma_{u,v} : c_u c_v \Rightarrow c_w c_{w'}$$

for every  $u$  and  $v$  in  $\text{col}(\mathcal{S})$  such that  $u \times v$ , where  $w$  and  $w'$  are respectively the readings of the left and right columns of the super tableau  $\llbracket uv \rrbracket_r$ , if  $u \times^2 v$ , and  $w = uv$  and  $c_{w'} = 1$ , if  $u \times^1 v$ .

**2.12. Theorem.** *The 2-polygraph  $\text{Col}_2(\mathcal{S})$  is a convergent presentation of the super plactic monoid  $\mathbf{P}(\mathcal{S})$ .*

The rest of this section is devoted to prove this result. We first show in Lemma 2.13 that the 2-polygraph  $\text{Col}_2(\mathcal{S})$  is terminating and confluent. We then show in Lemma 2.14 that it is a presentation of the super plactic monoid by showing that it is Tietze equivalent to the 2-polygraph  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$ .

**2.13. Lemma.** *The 2-polygraph  $\text{Col}_2(\mathcal{S})$  is convergent.*

*Proof.* The termination of  $\text{Col}_2(\mathcal{S})$  is proved using the length-lexicographic order  $\ll$  on  $\text{Col}_1(\mathcal{S})^*$  with respect to  $\leq_{\text{rev}}$  defined by setting  $c_{u_1} \dots c_{u_k} \ll c_{v_1} \dots c_{v_l}$ , if  $k < l$  or  $k = l$  and there exists  $i \in \{1, \dots, k\}$  such that  $u_i \leq_{\text{rev}} v_i$  and  $c_{u_j} = c_{v_j}$ , for any  $j < i$ . We prove that  $\ll$  is a well-ordering on  $\text{Col}_1(\mathcal{S})^*$  that is compatible with rules in  $\text{Col}_2(\mathcal{S})$ , that is if  $U c_u c_v V \Rightarrow U c_w c_{w'} V$  then  $U c_u c_v V \ll U c_w c_{w'} V$ , for all  $U c_u c_v V$  and  $U c_w c_{w'} V$  in  $\text{Col}_1(\mathcal{S})^*$ . If  $c_{w'} = 1$ , then  $U c_u c_v V$  contains more elements than  $U c_w V$ , showing that  $U c_w V \ll U c_u c_v V$ . On the other hand, suppose that the super tableau  $\llbracket uv \rrbracket_r$  consists of two columns whose readings of are denoted by  $w$  and  $w'$  respectively. By Lemma 2.6, the word  $w$  contains more elements than  $u$ . Then, we obtain  $w \leq_{\text{rev}} u$ , showing that  $U c_w V \ll U c_u c_v V$ . Since every application of a rule yield a  $\ll$ -preceding word, it follows that every sequence of rewriting using the 2-cells of  $\text{Col}_2(\mathcal{S})$  must terminate.

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We show that the 2-polygraph  $\text{Col}_2(\mathcal{S})$  is locally confluent and thus confluent by the termination hypothesis. Any critical branching of the 2-polygraph  $\text{Col}_2(\mathcal{S})$  has the following form:

$$\begin{array}{ccc} & \gamma_{u,v}c_t & \longrightarrow c_e c_{e'} c_t \\ c_u c_v c_t & \xrightarrow{\quad} & c_u c_w c_{w'} \\ & \xleftarrow{c_u \gamma_{v,t}} & \end{array} \quad (3)$$

for every  $u, v$  and  $t$  in  $\text{Col}(\mathcal{S})$  such that  $u \times v \times t$ , where  $e$  and  $e'$  (resp.  $w$  and  $w'$ ) are the readings of the two columns of the super tableau  $\llbracket uv \rrbracket_r$  (resp.  $\llbracket vt \rrbracket_r$ ). Following the definition of the right insertion algorithm, and by applying the leftmost (resp. rightmost) rewriting path with respect to  $\text{Col}_2(\mathcal{S})$  on  $c_u c_v c_t$ , we yield to the three super columns generators corresponding to the columns of the super tableau  $(\llbracket u \rrbracket_r \star_r \llbracket v \rrbracket_r) \star_r \llbracket t \rrbracket_r$  (resp.  $\llbracket u \rrbracket_r \star_r (\llbracket v \rrbracket_r \star_r \llbracket t \rrbracket_r)$ ). By the associativity of the product  $\star_r$ , we have  $(\llbracket u \rrbracket_r \star_r \llbracket v \rrbracket_r) \star_r \llbracket t \rrbracket_r = \llbracket u \rrbracket_r \star_r (\llbracket v \rrbracket_r \star_r \llbracket t \rrbracket_r)$ , showing that the critical branching (3) is confluent.  $\square$

**2.14. Lemma.** *The 2-polygraph  $\text{Col}_2(\mathcal{S})$  is a presentation of the super plactic monoid  $\mathbf{P}(\mathcal{S})$ .*

*Proof.* Prove that the 2-polygraph  $\text{Col}_2(\mathcal{S})$  is Tietze equivalent to the 2-polygraph  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$ . Any 2-cell in  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$  can be deduced from a 2-cell in  $\text{Col}_2(\mathcal{S})$  as follows. For any  $x \leq y \leq z$  in  $\mathcal{S}$ , with  $x = y$  only if  $\|y\| = 0$  and  $y = z$  only if  $\|y\| = 1$  (resp.  $x = y$  only if  $\|y\| = 1$  and  $y = z$  only if  $\|y\| = 0$ ), the 2-cells  $\eta_{x,y,z}^c$  (resp.  $\varepsilon_{x,y,z}^c$ ) can be deduced by the following composition

$$\begin{array}{ccc} c_z c_x c_y \xrightarrow{\eta_{x,y,z}^c} c_x c_z c_y & & c_y c_z c_x \xrightarrow{\varepsilon_{x,y,z}^c} c_y c_x c_z \\ \gamma_{z,x} c_y \Downarrow & \Downarrow c_x \gamma_{z,y} & c_y \gamma_{z,x} \Downarrow & \Downarrow \gamma_{y,x} c_z \\ c_{zx} c_y \xleftarrow{\gamma_{x,zy}} c_x c_{zy} & \text{(resp. } & c_y c_{zx} \xrightarrow{\gamma_{y,zx}} c_{yx} c_z \end{array} \quad \text{).}$$

Moreover, for any super column  $x_p \dots x_1$ , the 2-cell  $\mu_{x_p \dots x_1}$  can be deduced by the following composition

$$\begin{array}{ccc} c_{x_p} \dots c_{x_1} \xrightarrow{\mu_{x_p \dots x_1}} c_{x_p \dots x_1} & & \\ \gamma_{x_p, x_{p-1}} c_{x_{p-2}} \dots c_{x_1} \Downarrow & & \Uparrow \gamma_{x_p, \dots, x_2, x_1} \\ c_{x_p x_{p-1}} c_{x_{p-2}} \dots c_{x_1} \Rightarrow (\dots) \Rightarrow c_{x_p \dots x_2} c_{x_1} & & \end{array}$$

As a consequence, if the words  $w$  and  $w'$  in  $\text{Col}_1(\mathcal{S})^*$  are equal modulo relations in  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$ , then they are equal modulo relations in  $\text{Col}_2(\mathcal{S})$ . Conversely, following Subsection 2.3, the following equivalence

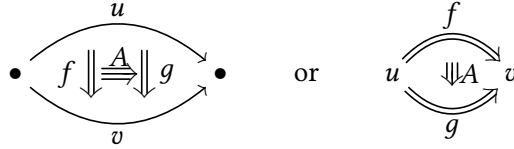
$$uv \sim_{\mathbf{P}(\mathcal{S})} R_{\text{row}}(\llbracket uv \rrbracket_r) \sim_{\mathbf{P}(\mathcal{S})} R_{\text{col}}(\llbracket uv \rrbracket_r)$$

holds, for all  $u, v$  in  $\text{Col}(\mathcal{S})$  such that  $u \times v$ . In this way, if two words in  $\text{Col}_1(\mathcal{S})^*$  are equal modulo relations in  $\text{Col}_2(\mathcal{S})$ , they are super plactic congruent and hence they are equal modulo  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$ , showing the claim.  $\square$

### 3. COHERENT PRESENTATIONS OF THE SUPER PLACTIC MONOID

In this section, we begin by recalling the notion of coherent presentations of monoids from [12] and we extend the super column presentation into a coherent presentation of the super plactic monoid. In a second part, we recall the homotopical reduction procedure from [12] and we reduce the super coherent column presentation into a smaller one over the initial signed alphabet. We follow for this aim the approach developed in [19] for the non-signed case.

**3.1. Coherent presentations of monoids.** A  $(3, 1)$ -polygraph is a pair  $(\Sigma_2, \Sigma_3)$  made of a 2-polygraph  $\Sigma_2$  and a globular extension  $\Sigma_3$  of the  $(2, 1)$ -category  $\Sigma_2^\top$ , that is a set of 3-cells  $A : f \Rrightarrow g$  relating 2-cells  $f$  and  $g$  in  $\Sigma_2^\top$ , respectively denoted by  $s_2(A)$  and  $t_2(A)$  and satisfying the globular relations  $s_1 s_2(A) = s_1 t_2(A)$  and  $t_1 s_2(A) = t_1 t_2(A)$ . Such a 3-cell can be represented with the following globular shape:



We will denote by  $\Sigma_3^\top$  the free  $(3, 1)$ -category generated by the  $(3, 1)$ -polygraph  $(\Sigma_2, \Sigma_3)$ . A pair  $(f, g)$  of 2-cells of  $\Sigma_2^\top$  such that  $s_1(f) = s_1(g)$  and  $t_1(f) = t_1(g)$  is called a *2-sphere* of  $\Sigma_2^\top$ . An *extended presentation* of a monoid  $\mathbf{M}$  is a  $(3, 1)$ -polygraph whose underlying 2-polygraph is a presentation of the monoid  $\mathbf{M}$ . A *coherent presentation* of  $\mathbf{M}$  is an extended presentation  $\Sigma$  of  $\mathbf{M}$  such that the cellular extension  $\Sigma_3$  is a *homotopy basis* of the  $(2, 1)$ -category  $\Sigma_2^\top$ , that is, for every 2-sphere  $\mu$  of  $\Sigma_2^\top$ , there exists a 3-cell in  $\Sigma_3^\top$  with boundary  $\mu$ . The elements in  $\Sigma_3^\top$  are called *syzygies* of the presentation  $\Sigma$ .

Let  $\Sigma$  be a  $(3, 1)$ -polygraph. A 3-cell  $A$  of  $\Sigma$  is called *collapsible* if  $t_2(A)$  is in  $\Sigma_2$  and  $s_2(A)$  is a 2-cell of the free  $(2, 1)$ -category over  $(\Sigma_2 \setminus \{t_2(A)\})^\top$ , then  $t_2(A)$  is called *redundant*. An *elementary Tietze transformation* of a  $(3, 1)$ -polygraph  $\Sigma$  is a 3-functor with domain  $\Sigma_3^\top$  that belongs to one of the following operations, [12]:

- i) *adjunction*  $\iota_\alpha^1$  and *elimination*  $\pi_\alpha$  of a 2-cell  $\alpha$  as described in Subsection 2.1,
- ii) *coherent adjunction*  $\iota_A^2 : \Sigma_3^\top \rightarrow \Sigma_3^\top(\alpha)(A)$  (resp.  $\iota_A : \Sigma_3^\top \rightarrow \Sigma_3^\top(A)$ ) of a redundant 2-cell  $\alpha$  with its collapsible 3-cell  $A$  (resp. of a redundant 3-cell  $A$ ),
- iii) *coherent elimination*  $\pi_A : \Sigma_3^\top \rightarrow \Sigma_3^\top/A$  (resp.  $\pi_{(B,A)} : \Sigma_3^\top \rightarrow \Sigma_3^\top/(B,A)$ ) of a redundant 2-cell  $\alpha$  with its collapsible 3-cell  $A$  (resp. of a redundant 3-cell  $A$ , that maps  $A$  to  $B$ ).

For  $(3, 1)$ -polygraphs  $\Sigma$  and  $\Upsilon$ , a *Tietze transformation from  $\Sigma$  to  $\Upsilon$*  is a 3-functor  $F : \Sigma_3^\top \rightarrow \Upsilon_3^\top$  that decomposes into a sequence of elementary Tietze transformations. Two  $(3, 1)$ -polygraphs  $\Sigma$  and  $\Upsilon$  are *Tietze-equivalent* if there exists an equivalence of 2-categories  $F : \Sigma_2^\top/\Sigma_3 \rightarrow \Upsilon_2^\top/\Upsilon_3$  and the presented monoids  $\bar{\Sigma}_2$  and  $\bar{\Upsilon}_2$  are isomorphic. Two  $(3, 1)$ -polygraphs are Tietze equivalent if, and only if, there exists a Tietze transformation between them, [12].

Recall that Squier's coherence theorem, [33], states that, any convergent 2-polygraph  $\Sigma$  presenting a

### 3. Coherent presentations of the super plactic monoid

monoid  $\mathbf{M}$  can be extended into a coherent presentation of  $\mathbf{M}$  having a generating syzygy

$$\begin{array}{ccc} & f & \\ & \rightarrow & \\ u & \xrightarrow{f} & v \\ & \searrow & \nearrow \\ & g & \\ & \rightarrow & \\ & v' & \\ & \xrightarrow{g'} & w \end{array}$$

$\Downarrow A$

for every critical branching  $(f, g)$  of  $\Sigma$ , where  $f'$  and  $g'$  are chosen confluent rewriting paths.

**3.2. Super column coherent presentation.** Denote by  $\text{Col}_3(\mathcal{S})$  the extended presentation of the monoid  $\mathbf{P}(\mathcal{S})$  obtained from  $\text{Col}_2(\mathcal{S})$  by adjunction of one family of syzygies  $\lambda_{u,v,t}$  of the following form

$$\begin{array}{ccccc} & \gamma_{u,v}c_t & \xrightarrow{c_e c_{e'} c_t} & c_e \gamma_{e',t} & \xrightarrow{c_e c_b c_{b'}} & \gamma_{e,b} c_{b'} \\ c_u c_v c_t & \xrightarrow{\gamma_{u,v,t}} & c_u c_w c_{w'} & \xrightarrow{\gamma_{u,w} c_{w'}} & c_a c_d c_{b'} & \\ & c_u \gamma_{v,t} & \xrightarrow{c_u c_w c_{w'}} & c_a c_{a'} c_{w'} & \xrightarrow{c_a \gamma_{a',w'}} & \end{array} \quad (4)$$

$\Downarrow \lambda_{u,v,t}$

for every  $u, v$  and  $t$  in  $\text{Col}(\mathcal{S})$  such that  $u \times v \times t$ , where  $e$  and  $e'$  (resp.  $w$  and  $w'$ ) are the readings of the two columns of the super tableau  $\llbracket uv \rrbracket_r$  (resp.  $\llbracket vt \rrbracket_r$ ), and where  $a$  and  $a'$  (resp.  $b$  and  $b'$ ) denote the readings of the two columns of the super tableau  $\llbracket uv \rrbracket_r$  (resp.  $\llbracket e't \rrbracket_r$ ) and  $a, d, b'$  are the readings of the three columns of the super tableau  $\llbracket uvt \rrbracket_r$ . Note that one or further columns  $e', w', a'$  and  $b'$  can be empty as illustrated in Figure 1. In those cases some indicated 2-cells  $\gamma$  in the confluence diagram correspond to identities.

**3.3. Theorem.** *The  $(3, 1)$ -polygraph  $\text{Col}_3(\mathcal{S})$  is a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ .*

*Proof.* Consider the 2-polygraph whose set of 1-cells is  $\text{Col}_1(\mathcal{S})$  and whose 2-cells are

$$c_u c_v \Rightarrow c_w c_{w'}$$

for all  $c_u, c_v$  in  $\text{Col}_1(\mathcal{S})$  such that  $u \times v$  and where  $w$  and  $w'$  denote respectively the readings of the left and right column of the super tableau  $(uv \rightsquigarrow \emptyset)$  obtained by applying the left insertion. Following Subsection 2.3, by the associativity of the product  $\star_r$ , the super tableaux  $(uv \rightsquigarrow \emptyset)$  and  $\llbracket uv \rrbracket_r$  are equal, thus this 2-polygraph coincides with  $\text{Col}_2(\mathcal{S})$ . Hence, the generating syzygy of the extended presentation of the 2-polygraph  $\text{Col}_2(\mathcal{S})$  has the following form

$$\begin{array}{ccc} & \emptyset \leftarrow uvt & \\ & \xrightarrow{c_u c_v c_t} & \\ & \searrow & \nearrow \\ & uvt \rightsquigarrow \emptyset & \\ & \xrightarrow{c_a c_d c_{b'}} & \end{array}$$

$\Downarrow$

for all  $c_u, c_v, c_t$  in  $\text{Col}_1(\mathcal{S})$  such that  $u \times v \times t$ , where  $a, d, b'$  are the readings of the three columns of the super tableau  $\llbracket uvt \rrbracket_r$ , and where the 2-source (resp. 2-target) of the syzygy corresponds to the application of the right (resp. left) insertion  $(\emptyset \leftarrow uvt)$  (resp.  $(uvt \rightsquigarrow \emptyset)$ ) on the word  $c_u c_v c_t$ . Finally, by definition of the right (resp. left) insertion algorithm  $\leftarrow$  (resp.  $\rightsquigarrow$ ), we can lead to the normal form  $c_a c_d c_{b'}$  by applying at most three steps of reduction on the initial word  $c_u c_v c_t$ , showing the claim.  $\square$

**3.3.1. Remark.** It is worth noting that there are five forms for the generating syzygy (4) depending on the following four cases:  $u^{\times 1}v^{\times 1}t$ ,  $u^{\times 2}v^{\times 1}t$ ,  $u^{\times 1}v^{\times 2}t$  and  $u^{\times 2}v^{\times 2}t$ , as shown in Figure 1. These forms are obtained by a case-by-case analysis and they will be used in the sequel in order to reduce the super column coherent presentation  $\text{Col}_3(\mathcal{S})$  into a smaller one over the initial signed alphabet.

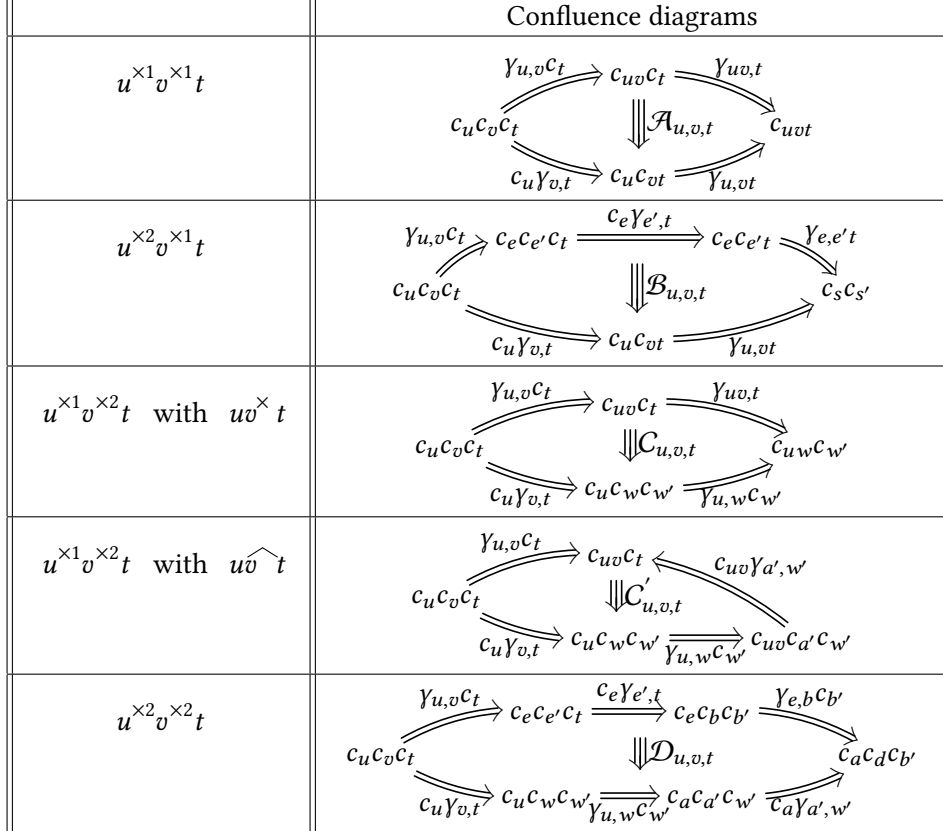


Figure 1: confluence diagrams

**3.4. Homotopical reduction procedure.** Let  $\Sigma$  be a  $(3, 1)$ -polygraph. A 3-sphere of the  $(3, 1)$ -category  $\Sigma_3^\top$  is a pair  $(f, g)$  of 3-cells of  $\Sigma_3^\top$  such that  $s_2(f) = s_2(g)$  and  $t_2(f) = t_2(g)$ . A *collapsible part* of  $\Sigma$  is a triple  $(\Gamma_2, \Gamma_3, \Gamma_4)$  made of a family  $\Gamma_2$  of 2-cells of  $\Sigma$ , a family  $\Gamma_3$  of 3-cells of  $\Sigma$  and a family  $\Gamma_4$  of 3-spheres of  $\Sigma_3^\top$ , such that the following conditions are satisfied, [12]:

- i) every  $\mu$  of every  $\Gamma_k$  is collapsible, that is,  $t_{k-1}(\mu)$  is in  $\Sigma_{k-1}$  and  $s_{k-1}(\mu)$  does not contain  $t_{k-1}(\mu)$ ,
- ii) no cell of  $\Gamma_2$  (resp.  $\Gamma_3$ ) is the target of a collapsible 3-cell of  $\Gamma_3$  (resp. 3-sphere of  $\Gamma_4$ ),
- iii) there exists a well-founded order on the cells of  $\Sigma$  such that, for every  $\mu$  in every  $\Gamma_k$ ,  $t_{k-1}(\mu)$  is strictly greater than every generating  $(k-1)$ -cell that occurs in the source of  $\mu$ .

The *homotopical reduction* of the  $(3, 1)$ -polygraph  $\Sigma$  with respect to a collapsible part  $\Gamma$  is the Tietze transformation, denoted by  $R_\Gamma$ , from the  $(3, 1)$ -category  $\Sigma_3^\top$  to the  $(3, 1)$ -category freely generated by

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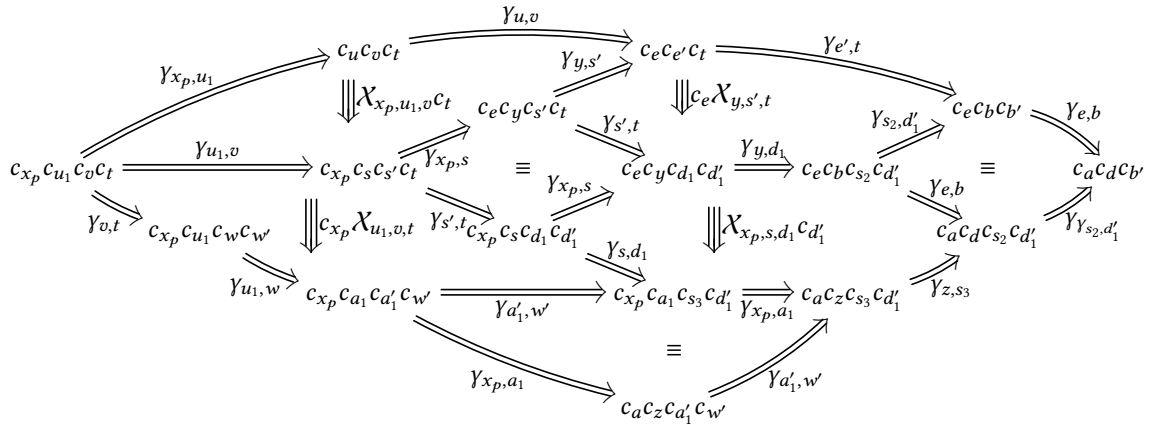
the  $(3, 1)$ -polygraph obtained from  $\Sigma$  by removing the cells of  $\Gamma$  and all the corresponding redundant cells. For any  $\mu$  in  $\Gamma$ , we have  $R_\Gamma(t(\mu)) = R_\Gamma(s(\mu))$  and  $R_\Gamma(\mu) = 1_{R_\Gamma(s(\mu))}$ . In any other cases, the transformation  $R_\Gamma$  acts as an identity, see [12] for more details.

A *local triple branching* of a 2-polygraph  $\Sigma$  is a triple  $(f, g, h)$  of rewriting steps of  $\Sigma$  with a common source. An *aspherical* triple branchings have two of their 2-cells equal. A *Peiffer* triple branchings have at least one of their 2-cells that form a Peiffer branching with the other two. The *overlap* triple branchings are the remaining local triple branchings. Local triple branchings are ordered by inclusion of their sources and a minimal overlap triple branching is called *critical*. If  $\Sigma$  is a coherent and convergent  $(3, 1)$ -polygraph, a *triple generating confluence* of  $\Sigma$  is a 3-sphere

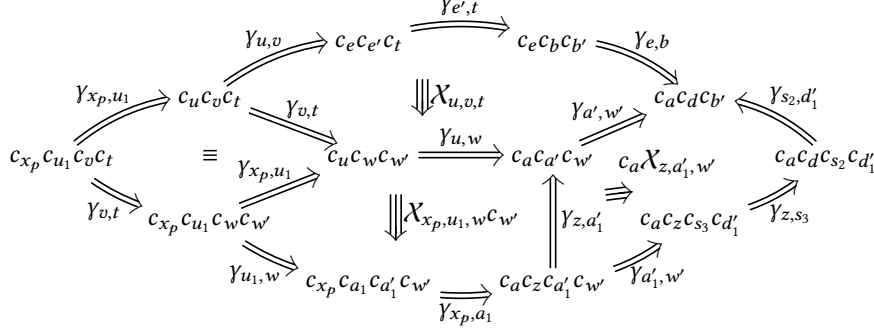


where  $(f, g, h)$  is a triple critical branching of the 2-polygraph  $\Sigma_2$  and the other cells are obtained by confluence, [12].

**3.5. Reduced super coherent column presentation.** We apply the homotopical reduction procedure on the  $(3, 1)$ -polygraph  $\text{Col}_3(\mathcal{S})$  using the collapsible part made of generating triple confluences. Following Theorem 3.3, the family of syzygies  $\mathcal{X}_{u,v,t}$  given in (4) and indexed by columns  $u, v$  and  $t$  in  $\text{col}(\mathcal{S})$  such that  $u \times v \times t$  forms a homotopy basis of the  $(2, 1)$ -category  $\text{Col}_2(\mathcal{S})^\top$ . Consider such a triple  $(u, v, t)$  with  $|u| \geq 2$ , and let  $x_p$  be in  $\mathcal{S}$  and  $u_1$  be in  $\text{col}(\mathcal{S})$  such that  $x_p = \ell(u)$  and  $u_1 = \text{Rem}(u)$ . We will show that the confluence diagram induced by the critical triple branching with source  $c_{x_p}c_{u_1}c_v c_t$  is represented by the 3-sphere  $\Omega_{x_p, u_1, v, t}$  whose source is the following 3-cell



and whose target is the following 3-cell



In order to facilitate the reading of the generating triple confluence, we have omitted the context of the 2-cells  $\gamma$ . Note also that some super columns may be empty and thus the indicated 2-cells  $\gamma$  may be identities.

The 3-sphere  $\Omega_{x_p, u_1, v, t}$  is constructed as follows. Since  $x_p^{\times 1} u_1$  and  $u_1^{\times} w$ , then  $\mathcal{X}_{x_p, u_1, w}$  is either of the form  $\mathcal{A}_{x_p, u_1, w}$ ,  $C_{x_p, u_1, w}$  or  $C'_{x_p, u_1, w}$ . Denote  $a_1$  and  $a_1'$  the readings of the two columns of  $\llbracket u_1 w \rrbracket_r$ . The 3-cell  $\mathcal{X}_{x_p, u_1, w}$  being confluent, we have  $R_{col}(\llbracket x_p a_1 \rrbracket_r) = az$  with  $z$  in  $\mathcal{S}$  and  $R_{col}(\llbracket x_p a_1' \rrbracket_r) = a'$ . Since  $z^{\times} a_1'$  and  $a_1'^{\times} w'$  (resp.  $x_p^{\times 1} u_1$  and  $u_1^{\times} v$ ), we deduce that  $\mathcal{X}_{z, a_1', w'}$  (resp.  $\mathcal{X}_{x_p, u_1, v}$ ) is either of the form  $\mathcal{A}_{z, a_1', w'}$ ,  $C_{z, a_1', w'}$  or  $C'_{z, a_1', w'}$  (resp.  $\mathcal{A}_{x_p, u_1, v}$ ,  $C_{x_p, u_1, v}$  or  $C'_{x_p, u_1, v}$ ). Denote by  $s$  and  $s'$  the readings of the two columns of  $\llbracket u_1 v \rrbracket_r$ . The 3-cell  $\mathcal{X}_{x_p, u_1, v}$  being confluent, we obtain that  $R_{col}(\llbracket x_p s \rrbracket_r) = ey$  with  $y$  in  $\mathcal{S}$  and  $R_{col}(\llbracket y s' \rrbracket_r) = e'$ . Since  $y^{\times 1} s'$  and  $s'^{\times} t$ , we deduce that  $\mathcal{X}_{y, s', t}$  is either of the form  $\mathcal{A}_{y, s', t}$ ,  $C_{y, s', t}$  or  $C'_{y, s', t}$ . Denote by  $d_1$  and  $d_1'$  the readings of the two columns of  $\llbracket s' t \rrbracket_r$ . The 3-cell  $\mathcal{X}_{y, s', t}$  being confluent and  $R_{col}(\llbracket e' t \rrbracket_r) = bb'$ , we have  $R_{col}(\llbracket y d_1 \rrbracket_r) = bs_2$  and  $R_{col}(\llbracket s_2 d_1' \rrbracket_r) = b'$ . On the other hand, the 3-cell  $\mathcal{X}_{u_1, v, t}$  being confluent, we have  $R_{col}(\llbracket s d_1 \rrbracket_r) = a_1 s_3$  and  $R_{col}(\llbracket a_1' w' \rrbracket_r) = s_3 d_1'$ . Finally, since  $\mathcal{X}_{x_p, s, d_1}$  is confluent, we deduce that  $R_{col}(\llbracket z s_3 \rrbracket_r) = ds_2$ .

Denote by  $\overline{\text{Col}}_3(\mathcal{S})$  the extended presentation of the monoid  $\mathbf{P}(\mathcal{S})$  obtained from  $\text{Col}_2(\mathcal{S})$  by adjunction of one family of 3-cells  $\mathcal{X}_{x, v, t}$  of the form (4), for all  $x$  in  $\mathcal{S}$  and  $v, t$  in  $\text{col}(\mathcal{S})$  such that  $x^{\times} v^{\times} t$ . Let  $\Gamma_4$  be the collapsible part made of the family of 3-sphere  $\Omega_{x_p, u_1, v, t}$ , indexed by  $x_p$  in  $\mathcal{S}$  and  $u_1, v, t$  in  $\text{col}(\mathcal{S})$  such that  $u^{\times} v^{\times} t$  and  $u = x_p u_1$ . We consider the order  $\triangleleft$  on the 3-cells of  $\overline{\text{Col}}_3(\mathcal{S})$  defined by  $\mathcal{A}_{u, v, t} \triangleleft C'_{u, v, t} \triangleleft C_{u, v, t} \triangleleft \mathcal{B}_{u, v, t} \triangleleft \mathcal{D}_{u, v, t}$ , and if  $\mathcal{X}_{u, v, t} \in \{\mathcal{A}_{u, v, t}, \mathcal{B}_{u, v, t}, C_{u, v, t}, C'_{u, v, t}, \mathcal{D}_{u, v, t}\}$  with  $u' \leq_{\text{deglex}} u$ , then  $\mathcal{X}_{u', v', t'} \triangleleft \mathcal{X}_{u, v, t}$ , for all  $u, v, t$  in  $\text{col}(\mathcal{S})$  such that  $u^{\times} v^{\times} t$ . By construction of the 3-sphere  $\Omega_{x_p, u_1, v, t}$ , its source contains the 3-cell  $\mathcal{X}_{u_1, v, t}$  and its target contains the 3-cell  $\mathcal{X}_{u, v, t}$  with  $|u_1| < |u|$ . Up to a Nielsen transformation, by applying the reduction  $R_{\Gamma_4}$  on the (3, 1)-polygraph  $\overline{\text{Col}}_3(\mathcal{S})$  with respect to  $\Gamma_4$  and the order  $\triangleleft$ , we obtain the (3, 1)-polygraph  $\overline{\text{Col}}_3(\mathcal{S})$ . Hence, we deduce the following result.

**3.6. Proposition.** *The (3, 1)-polygraph  $\overline{\text{Col}}_3(\mathcal{S})$  is a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ .*

**3.7. Super pre-column coherent presentation.** We reduce the coherent presentation  $\overline{\text{Col}}_3(\mathcal{S})$  into a coherent presentation whose underlying 2-polygraph is  $\text{PreCol}_2(\mathcal{S})$ . This reduction is obtained using the homotopical reduction  $R_{\Gamma_3}$  on the (3, 1)-polygraph  $\overline{\text{Col}}_3(\mathcal{S})$  whose collapsible part  $\Gamma_3$  is the set of 3-cells  $\mathcal{A}_{x, v, t}$ ,  $\mathcal{B}_{x, v, t}$  and  $C_{x, v, t}$  with  $x \in \mathcal{S}$  and  $v, t \in \text{col}(\mathcal{S})$  such that  $x^{\times 1} v^{\times 1} t$ ,  $x^{\times 2} v^{\times 1} t$  and  $x^{\times 1} v^{\times 2} t$



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with  $xv^{\times}t$  respectively, and the order  $\triangleleft$  on the 2-cells of  $\text{Col}_2(\mathcal{S})$  defined by  $\gamma_{u',v'} \triangleleft \gamma_{u,v}$  if  $|uv| > |u'v'|$ , or  $|uv| = |u'v'|$  and  $|u| > |R_{\text{col}}(\llbracket u'v' \rrbracket_r)\rangle|$  or  $|u| \leq |R_{\text{col}}(\llbracket u'v' \rrbracket_r)\rangle|$  and  $u' \leq_{\text{rev}} u$ , for all  $u, v, u', v' \in \text{col}(\mathcal{S})$  such that  $u^{\times}v$  and  $u'^{\times}v'$ .

The reduction  $R_{\Gamma_3}$  induced by the order  $\triangleleft$  on the 2-cells of  $\text{Col}_2(\mathcal{S})$  and the order  $\triangleleft$  on 3-cells can be decomposed as follows. For all  $x$  in  $\mathcal{S}$  and  $v, t$  in  $\text{col}(\mathcal{S})$  such that  $x^{\times 1}v^{\times 1}t$ , the 2-cells  $\gamma_{x,v}$ ,  $\gamma_{v,t}$  and  $\gamma_{x,vt}$  are smaller than  $\gamma_{xv,t}$  for the order  $\triangleleft$ . Then the reduction  $R_{\Gamma_3}$  removes the 2-cell  $\gamma_{xv,t}$  together with the 3-cell  $\mathcal{A}_{x,v,t}$ . By iterating this reduction on the length of  $v$ , we reduce all the 2-cells of  $\text{Col}_2(\mathcal{S})$  to the following set of 2-cells

$$\{ \gamma_{u,v} \mid |u| \geq 1, |v| \geq 2 \text{ and } u^{\times 2}v \} \cup \{ \gamma_{u,v} \mid |u| = 1, |v| \geq 1 \text{ and } u^{\times 1}v \}. \quad (5)$$

For all  $x$  in  $\mathcal{S}$  and  $v, t$  in  $\text{col}(\mathcal{S})$  such that  $x^{\times 1}v^{\times 2}t$  with  $xv^{\times}t$ , the 2-cells  $\gamma_{x,v}$ ,  $\gamma_{v,t}$ ,  $\gamma_{x,w}$  and  $\gamma_{a',w}$  are smaller than  $\gamma_{xv,t}$  for the order  $\triangleleft$ . Then, the reduction  $R_{\Gamma_3}$  removes the 2-cell  $\gamma_{xv,t}$  together with the 3-cell  $C_{x,v,t}$ . By iterating this reduction on the length of  $v$ , we reduce the set (5) to the following one:

$$\{ \gamma_{u,v} \mid |u| = 1, |v| \geq 2 \text{ and } u^{\times 2}v \} \cup \{ \gamma_{u,v} \mid |u| = 1, |v| \geq 1 \text{ and } u^{\times 1}v \}. \quad (6)$$

For all  $x$  in  $\mathcal{S}$  and  $v, t$  in  $\text{col}(\mathcal{S})$  such that  $x^{\times 2}v^{\times 1}t$ , consider the following 3-cell:

$$\begin{array}{ccccc} & \gamma_{x,v}c_t & \xrightarrow{c_e c_{e'} c_t} & c_e \gamma_{e',t} & \xrightarrow{c_e c_{e'} t} & c_e c_{e'} t & \xrightarrow{\tilde{\gamma}_{e',t}} & c_s c_{s'} \\ & \searrow & & \Downarrow \mathcal{B}_{x,v,t} & & & & \\ c_x c_v c_t & \xrightarrow{c_x \gamma_{v,t}} & c_x c_{vt} & \xrightarrow{\gamma_{x,vt}} & c_s c_{s'} & & & \end{array}$$

where  $\tilde{\gamma}_{e',t}$  is the 2-cell in (6) obtained from the 2-cell  $\gamma_{e',t}$  by the previous step of the homotopical reduction by the 3-cell  $C_{x,v,t}$ . Since  $x$  is in  $\mathcal{S}$ , we have also that  $e'$  is in  $\mathcal{S}$  by definition of  $\gamma$ . The 2-cells  $\gamma_{x,v}$ ,  $\gamma_{e',t}$ ,  $\gamma_{v,t}$  and  $\tilde{\gamma}_{e',t}$  being smaller than  $\gamma_{x,vt}$  for the order  $\triangleleft$ , we can remove the 2-cells  $\gamma_{x,vt}$  together with the 3-cell  $\mathcal{B}_{x,v,t}$ . By iterating this reduction on the length of  $t$ , we reduce the set (6) to the following set:

$$\{ \gamma_{u,v} \mid |u| = 1, |v| = 2 \text{ and } u^{\times 2}v \} \cup \{ \gamma_{u,v} \mid |u| = 1, |v| \geq 1 \text{ and } u^{\times 1}v \}. \quad (7)$$

Let us show that the set (7) is equal to the set of 2-cells  $\text{PreCol}_2(\mathcal{S})$  defined in Subsection 2.9. It is sufficient to prove that  $\text{PC}_2(\mathcal{S})$  is equal to the set of 2-cells  $\gamma_{u,v}$  in  $\text{Col}_2(\mathcal{S})$  such that  $|u| = 1$ ,  $|v| = 2$  and  $u^{\times 2}v$ . Suppose that  $v = xx'$  with  $x \geq x'$  in  $\mathcal{S}$  with  $x = x'$  only if  $\|x\| = 1$ . Since  $u^{\times 2}v$ , we obtain that  $u \leq x$  with  $u = x$  only if  $\|u\| = 0$ . If  $u \leq x'$  with  $u = x'$  only if  $\|u\| = 0$ , then  $R_{\text{col}}(\llbracket uv \rrbracket_r) = (xu)x'$  and the 2-cell  $\gamma_{u,v}$  is equal to the 2-cell  $\gamma'_{u,xx'} : c_u c_{xx'} \Rightarrow c_{xu} c_{x'}$ . If  $x' \leq u$  with  $u = x'$  only if  $\|u\| = 1$ , then  $R_{\text{col}}(\llbracket uv \rrbracket_r) = (ux')x$ , and the 2-cell  $\gamma_{u,v}$  is equal to  $\gamma'_{u,xx'} : c_u c_{xx'} \Rightarrow c_{ux'} c_x$ , showing the claim.

Hence, the homotopical reduction  $R_{\Gamma_3}$  reduces  $\overline{\text{Col}}_3(\mathcal{S})$  into a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$  whose set of 2-cells is  $\text{PreCol}_2(\mathcal{S})$ . Denote by  $\text{PreCol}_3(\mathcal{S})$  the extended presentation of the monoid  $\mathbf{P}(\mathcal{S})$  obtained from  $\text{PreCol}_2(\mathcal{S})$  by adjunction of the 3-cells  $R_{\Gamma_3}(C'_{x,v,t})$  for  $x^{\times 1}v^{\times 2}t$  with  $x\widehat{v}t$ , and  $R_{\Gamma_3}(\mathcal{D}_{x,v,t})$  for  $x^{\times 2}v^{\times 2}t$ . Since the homotopical reduction  $R_{\Gamma_3}$  eliminates the 3-cells of  $\overline{\text{Col}}_3(\mathcal{S})$  of the form  $\mathcal{A}_{x,v,t}$ ,  $\mathcal{B}_{x,v,t}$  and  $C_{x,v,t}$ , we obtain the following result.

**3.8. Proposition.** *The (3, 1)-polygraph  $\text{PreCol}_3(\mathcal{S})$  is a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ .*

**3.9. Super Knuth's coherent presentation.** We reduce the coherent presentation  $\text{PreCol}_3(\mathcal{S})$  into a coherent presentation of the super plactic monoid  $\mathbf{P}(\mathcal{S})$  whose underlying 2-polygraph is  $\text{Knuth}_2(\mathcal{S})$ . The Tietze transformation  $T_{\mu \leftarrow \gamma'} : \text{CPC}_2(\mathcal{S})^\top \rightarrow \text{PreCol}_2(\mathcal{S})^\top$  defined in Proposition 2.10 substitutes a 2-cell  $\gamma'_{x_p, x_{p-1} \dots x_1}$  for the 2-cell  $\mu_{x_p \dots x_1}$  in  $C_2(\mathcal{S})$ , from the bigger column to the smaller one with respect to the order  $\leq_{\text{deglex}}$ . Consider the inverse of this Tietze transformation  $T_{\mu \leftarrow \gamma'}^{-1}$  that substitutes the 2-cell  $\mu_{x_p \dots x_1}$  for the 2-cell  $\gamma'_{x_p, x_{p-1} \dots x_1}$ , for each column  $x_p \dots x_1$  such that  $|x_p \dots x_1| > 2$  with respect to  $\leq_{\text{deglex}}$ . Denote by  $\text{CPC}_3(\mathcal{S})$  the (3, 1)-polygraph whose underlying 2-polygraph is  $\text{CPC}_2(\mathcal{S})$  and whose 3-cells are  $T_{\mu \leftarrow \gamma'}^{-1}(R_{\Gamma_3}(C'_{x,v,t}))$  for  $x^{\times 1} v^{\times 2} t$  with  $x \widehat{v} t$  and  $T_{\mu \leftarrow \gamma'}^{-1}(R_{\Gamma_3}(\mathcal{D}_{x,v,t}))$  for  $x^{\times 2} v^{\times 2} t$ . In this way, we extend  $T_{\mu \leftarrow \gamma'}^{-1}$  into a Tietze transformation between  $\text{PreCol}_3(\mathcal{S})$  and  $\text{CPC}_3(\mathcal{S})$ . The (3, 1)-polygraph  $\text{PreCol}_3(\mathcal{S})$  being a coherent presentation of  $\mathbf{P}(\mathcal{S})$ , we deduce the following result.

**3.10. Lemma.** *The (3, 1)-polygraph  $\text{CPC}_3(\mathcal{S})$  is a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ .*

The Tietze transformation  $T_{\eta, \varepsilon \leftarrow \gamma'} : \text{Knuth}_2^{\text{cc}}(\mathcal{S})^\top \rightarrow \text{CPC}_2(\mathcal{S})^\top$  defined in the proof of Proposition 2.10 replaces the 2-cells  $\eta_{x,y,z}^c$  and  $\varepsilon_{x,y,z}^c$  in  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$  by composite of 2-cells in  $\text{CPC}_2(\mathcal{S})$ . Consider the inverse of this Tietze transformation  $T_{\eta, \varepsilon \leftarrow \gamma'}^{-1} : \text{CPC}_2(\mathcal{S})^\top \rightarrow \text{Knuth}_2^{\text{cc}}(\mathcal{S})^\top$ , that substitutes the 2-cell  $\eta_{x,y,z}^c$  (resp.  $\varepsilon_{x,y,z}^c$ ) for the 2-cell  $\gamma'_{x,zy}$  (resp.  $\gamma'_{y,zx}$ ). Denote by  $\text{Knuth}_3^{\text{cc}}(\mathcal{S})$  the (3, 1)-polygraph whose underlying 2-polygraph is  $\text{Knuth}_2^{\text{cc}}(\mathcal{S})$  and whose set of 3-cells is

$$\{ T_{\eta, \varepsilon \leftarrow \gamma'}^{-1}(T_{\mu \leftarrow \gamma'}^{-1}(R_{\Gamma_3}(C'_{x,v,t}))) \text{ for } x^{\times 1} v^{\times 2} t \text{ with } x \widehat{v} t \} \cup \{ T_{\eta, \varepsilon \leftarrow \gamma'}^{-1}(T_{\mu \leftarrow \gamma'}^{-1}(R_{\Gamma_3}(\mathcal{D}_{x,v,t}))) \text{ for } x^{\times 2} v^{\times 2} t \}.$$

We extend  $T_{\eta, \varepsilon \leftarrow \gamma'}^{-1}$  into a Tietze transformation between the (3, 1)-polygraphs  $\text{CPC}_3(\mathcal{S})^\top$  and  $\text{Knuth}_3^{\text{cc}}(\mathcal{S})^\top$ . Since  $\text{CPC}_3(\mathcal{S})$  is a coherent presentation of  $\mathbf{P}(\mathcal{S})$ , we deduce the following result:

**3.11. Lemma.** *The (3, 1)-polygraph  $\text{Knuth}_3^{\text{cc}}(\mathcal{S})$  is a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ .*

Finally, we perform the homotopical reduction  $R_{\Gamma_2}$  on the (3, 1)-polygraph  $\text{Knuth}_3^{\text{cc}}(\mathcal{S})$  whose collapsible part  $\Gamma_2$  is defined by the 2-cells  $\mu_u$  of  $C_2(\mathcal{S})$  and the order  $\leq_{\text{deglex}}$ . Thus, for every 2-cell  $\mu_{x_p \dots x_1} : c_{x_p} \dots c_{x_1} \Rightarrow c_{x_p \dots x_1}$  in  $C_2(\mathcal{S})$ , we eliminate  $c_{x_p \dots x_1}$  together with  $\mu_{x_p \dots x_1}$ , from the bigger column to the smaller one with respect to the order  $\leq_{\text{deglex}}$ .

Consider the following composite of Tietze transformations  $\mathcal{R} := R_{\Gamma_2} \circ T_{\eta, \varepsilon \leftarrow \gamma'}^{-1} \circ T_{\mu \leftarrow \gamma'}^{-1} \circ R_{\Gamma_3}$ , defined from  $\overline{\text{Col}}_3(\mathcal{S})^\top$  to  $\text{Knuth}_3^{\text{cc}}(\mathcal{S})^\top$  as follows. Firstly, the transformation  $\mathcal{R}$  eliminates the 3-cells of  $\overline{\text{Col}}_3(\mathcal{S})$  of the form  $\mathcal{A}_{x,v,t}$ ,  $\mathcal{B}_{x,v,t}$  and  $C_{x,v,t}$  and reduces its set of 2-cells to  $\text{PreCol}_2(\mathcal{S})$ . Secondly, this transformation coherently replaces the 2-cells  $\mu_{x_p \dots x_1}$  by the 2-cells  $\gamma'_{x_p, x_{p-1} \dots x_1}$ , for each column  $x_p \dots x_1$  such that  $|x_p \dots x_1| > 2$ , the 2-cells  $\gamma'_{x,zy}$  by  $\eta_{x,y,z}^c$ , for any  $x \leq y \leq z$ , with  $x = y$  only if  $\|y\| = 0$  and  $y = z$  only if  $\|y\| = 1$  and the 2-cells  $\gamma'_{y,zx}$  by  $\varepsilon_{x,y,z}^c$  for any  $x \leq y \leq z$ , with  $x = y$  only if  $\|y\| = 1$  and  $y = z$  only if  $\|y\| = 0$ . Finally, for each column  $u$ , the transformation  $\mathcal{R}$  eliminates the generator  $c_u$  together with the 2-cell  $\mu_u$  with respect to the order  $\leq_{\text{deglex}}$ .

Let  $\text{Knuth}_3(\mathcal{S})$  be the extended presentation of the monoid  $\mathbf{P}(\mathcal{S})$  obtained from  $\text{Knuth}_2(\mathcal{S})$  by adjunction of the 3-cells  $\mathcal{R}(C'_{x,v,t})$  for  $x^{\times 1} v^{\times 2} t$  with  $x \widehat{v} t$  and  $\mathcal{R}(\mathcal{D}_{x,v,t})$  for  $x^{\times 2} v^{\times 2} t$ . The transformation  $\mathcal{R}$  being a composite of Tietze transformations, we deduce the following result.

**3.12. Theorem.** *The (3, 1)-polygraph  $\text{Knuth}_3(\mathcal{S})$  is a coherent presentation of the monoid  $\mathbf{P}(\mathcal{S})$ .*

## REFERENCES

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**3.13. Actions of super plactic monoids on categories.** A monoid  $\mathbf{M}$  can be seen as a 2-category with exactly one 0-cell  $\bullet$ , with the elements of the monoid  $\mathbf{M}$  as 1-cells and with identity 2-cells only. The category of *actions of  $\mathbf{M}$  on categories* is the category  $\text{Act}(\mathbf{M})$  of 2-representations of  $\mathbf{M}$  in the category  $\text{Cat}$  of categories. The full subcategory of  $\text{Act}(\mathbf{M})$  whose objects are the 2-functors is denoted by  $2\text{Cat}(\mathbf{M}, \text{Cat})$ . We refer the reader to [12] for a full introduction on the category of 2-representations of 2-categories. More explicitly, an action  $A$  of the monoid  $\mathbf{M}$  is specified by a category  $\mathbf{C} = A(\bullet)$ , an endofunctor  $A(u) : \mathbf{C} \rightarrow \mathbf{C}$  for every  $u$  in  $\mathbf{M}$ , a natural isomorphism  $A_{u,v} : A(u)A(v) \Rightarrow A(uv)$  for every elements  $u$  and  $v$  of  $\mathbf{M}$ , and a natural isomorphism  $A_\bullet : 1_{\mathbf{C}} \Rightarrow A(1)$  such that:

i) for every triple  $(u, v, w)$  of elements of the monoid  $\mathbf{M}$ , the following diagram commutes

$$\begin{array}{ccc}
 & A_{u,v}A(w) \xrightarrow{\quad} & A(uv)A(w) \xrightarrow{\quad} A_{u,v,w} \\
 A(u)A(v)A(w) & \xrightarrow{\quad} & A(uvw) \\
 & \xrightarrow{\quad} & \\
 & A(u)A_{v,w} \xrightarrow{\quad} & A(u)A(vw) \xrightarrow{\quad} A_{u,vw}
 \end{array}
 =$$

ii) for every element  $u$  of the monoid  $\mathbf{M}$ , the following diagrams commute

$$\begin{array}{ccc}
 A_\bullet A(u) \xrightarrow{\quad} & A(1)A(u) \xrightarrow{\quad} A_{1,u} \\
 \xrightarrow{\quad} & = & \xrightarrow{\quad} \\
 A(u) & \xrightarrow{\quad} & A(u)
 \end{array}
 \qquad
 \begin{array}{ccc}
 A(u)A_\bullet \xrightarrow{\quad} & A(u)A(1) \xrightarrow{\quad} A_{u,1} \\
 \xrightarrow{\quad} & = & \xrightarrow{\quad} \\
 A(u) & \xrightarrow{\quad} & A(u)
 \end{array}$$

Let  $\mathbf{M}$  be a monoid and let  $\Sigma$  be an extended presentation of  $\mathbf{M}$ . The  $(3, 1)$ -polygraph  $\Sigma$  is a coherent presentation of  $\mathbf{M}$  if, and only if, for every 2-category  $\mathbf{C}$ , there is an equivalence of categories between  $\text{Act}(\mathbf{M})$  and  $2\text{Cat}(\Sigma_1^*/\Sigma_2, \mathbf{C})$ , that is natural in  $\mathbf{C}$ , [12]. In this way, up to equivalence, the actions of a monoid  $\mathbf{M}$  on categories are the same as the 2-functors from  $\Sigma_1^*/\Sigma_2$  to  $\text{Cat}$ .

As a consequence, Theorem 3.12 allows us to present actions of super plactic monoids on categories as follows:

**3.14. Theorem.** *The category  $\text{Act}(\mathbf{P}(\mathcal{S}))$  of actions of the monoid  $\mathbf{P}(\mathcal{S})$  on categories is equivalent to the category of 2-functors from the free  $(2, 1)$ -category  $\text{Knuth}_2(\mathcal{S})^\top$  generated by the 2-polygraph  $\text{Knuth}_2(\mathcal{S})$  to the category  $\text{Cat}$  of categories, sending the 3-cells of  $\text{Knuth}_3(\mathcal{S})$  to commutative diagrams in  $\text{Cat}$ .*

## REFERENCES

- [1] Georgia Benkart, Seok-Jin Kang, and Masaki Kashiwara. Crystal bases for the quantum superalgebra  $U_q(\mathfrak{gl}(m, n))$ . *Journal of the American Mathematical Society*, 13(2):295–331, 2000.
- [2] A Berele and A Regev. Hook young diagrams with applications to combinatorics and to representations of lie superalgebras. *Advances in Mathematics*, 64(2):118–175, 1987.
- [3] A. Berele and J.B. Remmel. Hook flag characters and their combinatorics. *Journal of Pure and Applied Algebra*, 35:225–245, 1985.
- [4] Leonid A. Bokut, Yuqun Chen, Weiping Chen, and Jing Li. New approaches to plactic monoid via Gröbner–Shirshov bases. *J. Algebra*, 423:301–317, 2015.

- 
- [5] F. Bonetti, D. Senato, and A. Venezia. The Robinson–Schensted correspondence for the fourfold algebra. *Boll. Un. Mat. Ital. B (7)*, 2(3):541–554, 1988.
- [6] Alan J. Cain, Robert D. Gray, and António Malheiro. Finite Gröbner–Shirshov bases for Plactic algebras and biautomatic structures for Plactic monoids. *J. Algebra*, 423:37–53, 2015.
- [7] Alan J. Cain, Robert D. Gray, and António Malheiro. Crystal monoids & crystal bases: rewriting systems and biautomatic structures for plactic monoids of types  $A_n$ ,  $B_n$ ,  $C_n$ ,  $D_n$ , and  $G_2$ . *J. Combin. Theory Ser. A*, 162:406–466, 2019.
- [8] Alan J. Cain, António Malheiro, and Fábio M. Silva. The monoids of the patience sorting algorithm. *International Journal of Algebra and Computation*, 29(01):85–125, 2019.
- [9] Julien Cassaigne, Marc Espie, Daniel Krob, Jean-Christophe Novelli, and Florent Hivert. The Chinese monoid. *Internat. J. Algebra Comput.*, 11(3):301–334, 2001.
- [10] Neil O Connell. Conditioned random walks and the RSK correspondence. *Journal of Physics A: Mathematical and General*, 36(12):3049–3066, mar 2003.
- [11] William Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.
- [12] Stéphane Gaussent, Yves Guiraud, and Philippe Malbos. Coherent presentations of Artin monoids. *Compos. Math.*, 151(5):957–998, 2015.
- [13] Samuele Giraudo. Algebraic and combinatorial structures on pairs of twin binary trees. *J. Algebra*, 360:115–157, 2012.
- [14] Frank D Grosshans, Gian-Carlo Rota, and Joel A Stein. *Invariant theory and superalgebras*. Number 69. American Mathematical Soc., 1987.
- [15] Yves Guiraud and Philippe Malbos. Higher-dimensional normalisation strategies for acyclicity. *Adv. Math.*, 231(3-4):2294–2351, 2012.
- [16] Yves Guiraud and Philippe Malbos. Polygraphs of finite derivation type. *Math. Structures Comput. Sci.*, 28(2):155–201, 2018.
- [17] Nohra Hage. Finite convergent presentation of plactic monoid for type C. *Internat. J. Algebra Comput.*, 25(8):1239–1263, 2015.
- [18] Nohra Hage. Super jeu de taquin and combinatorics of super tableaux of type A. arxiv:2105.07819, submitted, May 2021.
- [19] Nohra Hage and Philippe Malbos. Knuth’s coherent presentations of plactic monoids of type A. *Algebr. Represent. Theory*, 20(5):1259–1288, 2017.
- [20] Nohra Hage and Philippe Malbos. Chinese syzygies by insertions. To appear in *Semigroup Forum*, arXiv:1901.09879, 2022.
- [21] F. Hivert, J.-C. Novelli, and J.-Y. Thibon. The algebra of binary search trees. *Theoret. Comput. Sci.*, 339(1):129–165, 2005.
- [22] Donald E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.

## REFERENCES

---

- [23] Łukasz Kubat and Jan Okniński. Gröbner-Shirshov bases for plactic algebras. *Algebra Colloq.*, 21(4):591–596, 2014.
- [24] Roberto La Scala, Vincenzo Nardoza, and Domenico Senato. Super RSK-algorithms and super plactic monoid. *Int. J. Algebra Comput.*, 16(2):377–396, 2006.
- [25] Alain Lascoux and Marcel-P. Schützenberger. Le monoïde plaxique. In *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*, volume 109 of *Quad. "Ricerca Sci."*, pages 129–156. CNR, Rome, 1981.
- [26] Cédric Lecouvey. Combinatorics of crystal graphs for the root systems of types  $A_n, B_n, C_n, D_n$  and  $G_2$ . In *Combinatorial aspect of integrable systems*, volume 17 of *MSJ Mem.*, pages 11–41. Math. Soc. Japan, Tokyo, 2007.
- [27] Jean-Louis Loday and Todor Popov. Parastatistics algebra, young tableaux and the super plactic monoid. *International Journal of Geometric Methods in Modern Physics*, 05(08):1295–1314, 2008.
- [28] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002.
- [29] Uran Meha. C-trees and a coherent presentation for the plactic monoid of type C, arXiv: 2006.03456, 2020.
- [30] Jean-Christophe Novelli. On the hypoplactic monoid. volume 217, pages 315–336. 2000. Formal power series and algebraic combinatorics (Vienna, 1997).
- [31] Jean-Baptiste Priez. Lattice of combinatorial hopf algebras: binary trees with multiplicities. *Discrete Mathematics & Theoretical Computer Science*, 2013.
- [32] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.
- [33] Craig C. Squier, Friedrich Otto, and Yuji Kobayashi. A finiteness condition for rewriting systems. *Theoret. Comput. Sci.*, 131(2):271–294, 1994.

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