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► **To cite this version:**

Florent Dewez. Stability of propagation features under time-asymptotic approximations for a class of dispersive equations. *Journal of Mathematical Analysis and Applications*, 2020, 491 (1), pp.124292. 10.1016/j.jmaa.2020.124292 . hal-03490225

HAL Id: hal-03490225

<https://hal.science/hal-03490225>

Submitted on 15 Jun 2022

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1 Stability of propagation features under time-asymptotic
2 approximations for a class of dispersive equations

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5 **Abstract**

We consider solutions of dispersive equations on the line defined by Fourier multipliers with initial data having compactly supported Fourier transforms. In this paper, a refinement of an existing method permitting to expand time-asymptotically the solution formulas is proposed. Here the first term of the expansion is supported in a space-time cone whose origin depends explicitly on the initial datum. As an important consequence of our refined method, the first term inherits the mean position of the solution together with a constant variance error and a shifted time-decay rate is obtained. Hence this refinement, which takes into account both spatial and frequency information of the initial datum, makes stable some propagation features under time-asymptotic approximations and permits a better description of the time-asymptotic behaviour of the solutions. The results are achieved firstly by making apparent the cone origin in the solution formula, secondly by applying precisely an adapted version of the stationary phase method with a new error bound, and finally by minimising the error bound with respect to the cone origin.

6 *Keywords:* wave packet, dispersive equation, oscillatory integral,
7 stationary phase method, frequency band
8 *2010 MSC:* 35B40, 35S10, 35B30, 35Q41, 35Q40

9 **1. Introduction**

10 In this paper, we are interested in the time-asymptotic behaviour of wave
11 packets of the form

$$u_f(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}u_0(p) e^{-itf(p)+ixp} dp, \quad (1)$$

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Preprint submitted to Journal of Mathematical Analysis and Applications May 28, 2020
(Florent Dewez)

12 where $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly convex symbol. We suppose
 13 that the Fourier transform $\mathcal{F}u_0$ of $u_0 \in \mathcal{S}(\mathbb{R})$ is supported in a bounded
 14 interval $[p_1, p_2]$, where $p_1 < p_2$ are finite real numbers. In terms of evolution
 15 equations, wave packets of the form (1) are solutions of the following type of
 16 dispersive equations:

$$\begin{cases} [i \partial_t - f(D)] u_f(t) = 0 \\ u_f(0) = u_0 \end{cases}, \quad (2)$$

17 for $t \in \mathbb{R}$, where $f(D)$ is the Fourier multiplier associated with f and
 18 $u_0 \in \mathcal{S}(\mathbb{R})$ the initial datum supposed to be in the frequency band $[p_1, p_2]$. For
 19 instance, the solutions of the free Schrödinger equation, of the Klein-Gordon
 20 equation or of certain higher-order evolution equations can be described by
 21 wave packets of the form (1); we refer to [9, Sec. 6] for further details. In the
 22 present setting, the frequency band hypothesis prevents the wave packet (1)
 23 from being too much localised in space according to the uncertainty principle
 24 and makes hence challenging the task of describing its spatial propagation.

25
 26 Some approaches solving this challenge time-asymptotically have been
 27 developed. In [4], the authors propose to approximate the solution of the
 28 Klein-Gordon equation on a star-shaped network by a spatially localised
 29 function, the latter tending to the true solution as the time tends to infinity.
 30 This has been achieved by applying precisely the version of the stationary
 31 phase method given in [23, Theorem 7.7.5] to an integral solution formula of
 32 the equation. The desired approximation is then given by the first term of
 33 the asymptotic expansion from the stationary phase method. The principle
 34 of the stationary phase method, which consists in evaluating the integrand
 35 of the oscillatory integral of interest at the stationary point of the phase
 36 function, combined with the bounded frequency band hypothesis leads to
 37 an approximation supported in a space-time cone: this cone describes both
 38 the motion and the dispersion of the solution for large times. In particular,
 39 the results exhibit in this setting the influence of the tunnel effect on the
 40 time-decay rate of the solution.

41 In [3], this approach has been adapted to the setting of the free Schrö-
 42 dinger equation on the line. In that paper, the initial data are assumed
 43 to have integrable singular frequencies in order to study the effect of such
 44 singularities on the time-asymptotic behaviour. The version of the stationary
 45 phase method proposed in [18, Sec. 2.9] (also proposed in [17]) has been used

46 since it covers the case of singular amplitudes; we mention that the authors
47 in [3] propose modern formulations and detailed proofs of the results from
48 [18]. The results show that a free particle with a singular frequency tends to
49 travel at the speed associated with this frequency. This is highlighted by the
50 existence of space-time cones, containing the direction given by the singular
51 frequency, in which the time-decay rates are below the rate of the classical
52 decay inherited from the classical dispersion.

53 However, the expansion provided in [3] is proved to blow up when ap-
54 proaching the space-time direction associated with the singular frequency:
55 this prevents the method from approximating uniformly the true solution in
56 regions containing this direction. This is due to the fact that the first term
57 of the expansion inherits the singularity of the initial datum; see [3, Sec. 3]
58 for more details. To tackle this issue, another approach has been proposed in
59 [9]: the precision of asymptotic expansions to one term is removed in favour
60 of less precise but more flexible uniform estimates. In particular, they cover
61 the above critical regions. Further this flexibility has permitted to consider
62 not only the free Schrödinger equation but also equations of type (2) with
63 initial data having singular frequencies. The uniform estimates for the so-
64 lutions have been achieved by applying a generalisation [9, Theorem 4.8]
65 of the classical van der Corput Lemma [28, Chap. VIII, Sec. 1, Proposition 2]
66 to the case of singular and integrable amplitudes.

67 The approach developed in [4] and the subsequent adaptations appearing
68 in [3, 9] permit then to describe both the motion and the dispersion of the
69 solutions via the inclination of space-time cones (which depends actually on
70 the frequency support of the initial datum). Nevertheless it is noteworthy
71 that the origin of the space-time cones resulting from the above methods is
72 always put at the space-time point $(0, 0)$, whatever the localisation of the
73 initial datum is: the approximations provided by the method do not have
74 the right spatial positions. As a consequence this method, although asymp-
75 totically correct, leads to inaccurate approximations of the solutions on large
76 but finite time intervals $[0, T]$ for initial data spatially far from the origin.

77
78 In view of this, we aim at proposing time-asymptotic approximations
79 whose mean positions and variances are close (even the same) to those of
80 the solutions of equation (2) for a better description of the spatial propaga-
81 tion. To this end, we proceed in four steps, each of them containing a new
82 argument:

1. The first step, which is actually independent from the setting of dispersive equations, consists in establishing new uniform and explicit remainder estimates for asymptotic expansions of oscillatory integrals of the form

$$\forall \omega > 0 \quad \int_{\mathbb{R}} U(p) e^{i\omega\psi(p)} dp ,$$

83 where the amplitude $U : \mathbb{R} \rightarrow \mathbb{C}$ is compactly supported and the phase
 84 $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is concave. The new remainder estimate we establish
 85 involves the L^2 -norm of the first derivative of the amplitude and not
 86 the L^∞ -norm as in the original proofs [3, 18]. This is achieved by
 87 applying Cauchy-Schwarz inequality to the integral representation of
 88 the remainder term. We refer to Theorems 3.3 and 3.4 for the precise
 89 statements of these results.

2. As a second step, we introduce a space-time shift parametrised by a
 90 two-dimensional parameter (t_0, x_0) in the integral formula (1). Broadly
 91 speaking, this new argument modifies the initial datum which is then
 92 given by the solution at time t_0 spatially translated by x_0 . Inspired by
 93 the main lines of the method developed in [4], we apply then carefully
 94 a stationary phase method to this shifted solution formula to obtain
 95 here a family of time-asymptotic expansions parametrised by (t_0, x_0) .
 96 This parameter turns out to be the origin of the cone in which lies the
 97 support of the first term. This result is given in Theorem 2.2.
 98

3. Here we use our new version of the stationary phase method described
 99 above to expand the formula (1). Since this version makes appear
 100 the L^2 -norm of the derivative of $\mathcal{F}u_0$, it offers the possibility to apply
 101 Plancherel's theorem. This leads to an error bound depending explicitly
 102 on the spatial part of the initial datum and on (t_0, x_0) . The third
 103 step consists then in computing explicitly the parameter (t^*, x^*) which
 104 minimises the (t_0, x_0) -dependent family of error bounds provided in
 105 Theorem 2.2. The expansion associated with this optimal parameter,
 106 given in Corollary 2.5, is then the one whose error bound is the smallest.
 107

4. In the last step, we prove that the mean position of the first term
 108 associated with (t^*, x^*) is the same as the one of the solution (1) and
 109 that the difference between the two variances is constant; this is stated
 110 in Theorem 2.6. It is noteworthy that the proof of this theorem shows
 111 that the first term associated with (t^*, x^*) is actually the only one in
 112 the family of first terms we provide which satisfies these two properties.
 113

114 Contrary to the original method developed in [4] and adapted in [3, 9], the
 115 present refined approach offers sufficient flexibility to provide time-asymptotic
 116 approximations for solutions of equations of type (2) which are more precise
 117 and which verify expected propagation properties.

118

119 Let us illustrate our main result in the case of the free Schrödinger equa-
 120 tion on the line with initial datum $u_0 \in \mathcal{S}(\mathbb{R})$, namely

$$\begin{cases} i \partial_t u_S(t) = -\frac{1}{2} \partial_{xx} u_S(t) \\ u_S(0) = u_0 \end{cases}, \quad (3)$$

121 for $t \in \mathbb{R}$. We note that equation (3) is actually of the form (2) with symbol
 122 $f(p) = \frac{1}{2} p^2$ and its solution is given by

$$\forall (t, x) \in \mathbb{R} \times \mathbb{R} \quad u_S(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}u_0(p) e^{-\frac{1}{2}itp^2 + ixp} dp. \quad (4)$$

123 In terms of quantum mechanics, the solution of the free Schrödinger equation
 124 is the wave function associated with the free particle being in the state u_0
 125 at the initial time. Further the frequency band hypothesis means that the
 126 particle has a momentum localised in the interval $[p_1, p_2]$. According to the
 127 physical principle of group velocity, the wave packet (4) will travel in space
 128 at different speeds between p_1 and p_2 over time. Hence a free wave packet in
 129 the frequency band $[p_1, p_2]$ is expected to be mainly spatially localised in an
 130 interval of the form $[p_1(t-t_0)+x_0, p_2(t-t_0)+x_0]$, where t_0 and x_0 have to be
 131 fixed, illustrating the propagation of the associated particle. For instance,
 132 a partial formalisation of this principle is given by Ehrenfest theorem [22,
 133 Proposition 3.19] which describes the evolution of the mean position of the
 134 particle (but not the dispersion).

135 In the following result, we apply Corollary 2.5 to the setting of the free
 136 Schrödinger equation (3) to provide a time-asymptotic expansion of the solu-
 137 tion (4) with explicit error estimates. According to Theorem 2.6, the resulting
 138 approximation has the right mean position and its variance is equal to the
 139 solution variance plus an explicit constant. This is actually the consequence
 140 of the fact that the origin of the space-time cone, in which lies the support of
 141 the approximation, is actually put at the mean spatial position of the solution
 142 at the time when the variance of the solution is minimal. Hence the resulting
 143 approximation describes both position and dispersion of the true solution

144 and the theorem provides a formalisation of the principle of group velocity
 145 which seems to be more precise than Ehrenfest theorem in the setting of a
 146 free particle.

Theorem 1.1. *Consider the free Schrödinger equation on the line (3) with initial datum $u_0 \in \mathcal{S}(\mathbb{R})$. Let p_1, p_2, \tilde{p}_1 and \tilde{p}_2 be finite real numbers such that $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$. Suppose $\|u_0\|_{L^2(\mathbb{R})} = 1$ and $\text{supp } \mathcal{F}u_0 \subseteq [p_1, p_2]$, and define*

- $t^* = \arg \min_{\tau \in \mathbb{R}} \left(\int_{\mathbb{R}} x^2 |u_S(\tau, x)|^2 dx - \left(\int_{\mathbb{R}} x |u_S(\tau, x)|^2 dx \right)^2 \right);$
- $x^* = \int_{\mathbb{R}} x |u_S(t^*, x)|^2 dx .$

Then for all $(t, x) \in \left\{ (t, x) \in (\mathbb{R} \setminus \{t^*\}) \times \mathbb{R} \mid p_1 \leq \frac{x - x^*}{t - t^*} \leq p_2 \right\}$, we have

$$\begin{aligned} & \left| u_S(t, x) - \frac{1}{\sqrt{2\pi}} e^{-\text{sgn}(t-t^*)i\frac{\pi}{4}} e^{-it\left(\frac{x-x^*}{t-t^*}\right)^2 + ix\frac{x-x^*}{t-t^*}} \mathcal{F}u_0\left(\frac{x-x^*}{t-t^*}\right) |t-t^*|^{-\frac{1}{2}} \right| \\ & \leq C_1(\delta, \tilde{p}_1, \tilde{p}_2) \sqrt{\int_{\mathbb{R}} x^2 |u_S(t^*, x)|^2 dx - \left(\int_{\mathbb{R}} x |u_S(t^*, x)|^2 dx \right)^2} |t-t^*|^{-\delta}, \end{aligned}$$

where the real number δ is arbitrarily chosen in $(\frac{1}{2}, \frac{3}{4})$. And for all $(t, x) \in \left\{ (t, x) \in (\mathbb{R} \setminus \{t^*\}) \times \mathbb{R} \mid \frac{x - x^*}{t - t^*} < p_1 \text{ or } p_2 < \frac{x - x^*}{t - t^*} \right\}$, we have

$$\begin{aligned} |u_S(t, x)| & \leq \left(C_2(p_1, p_2, \tilde{p}_1, \tilde{p}_2) \sqrt{\int_{\mathbb{R}} x^2 |u_S(t^*, x)|^2 dx - \left(\int_{\mathbb{R}} x |u_S(t^*, x)|^2 dx \right)^2} \right. \\ & \quad \left. + C_3(p_1, p_2, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) |t-t^*|^{-1}. \end{aligned}$$

147 All the above constants are defined in Theorem 2.2.

148 See Corollary 2.5 for the general result. Let us now make some comments on
 149 this result:

- 150 • On one hand, we observe that the first term is spatially well-localised
 151 for a solution in a narrow frequency band; on the other hand, the er-
 152 ror is bounded by the minimal value of the standard deviation of the

153 solution. Combined with the uncertainty principle, this exhibits a com-
 154 promise: a frequency well-localised solution (4) can be approximated
 155 by a function supported in a narrow space-time cone but a time suffi-
 156 ciently far from t^* is required to achieve a good precision; on the other
 157 hand, the approximation of the solution (4) with a small minimal stan-
 158 dard deviation lies in a larger cone but the error bound is smaller than
 159 in the preceding case.

- We remark that the time-decay rate is shifted by t^* , which is the time when the variance of the solution (4) is minimal; this corresponds to the fact that the origin of the cone belongs to the vertical space-time line $\{(t, x) \in \mathbb{R} \times \mathbb{R} \mid t = t^*\}$. Hence if we require an error smaller than a certain threshold $\varepsilon > 0$, then this precision is achieved for all $t \in \mathbb{R}$ satisfying

$$\begin{aligned}
 |t - t^*| &> C_1(\delta, \tilde{p}_1, \tilde{p}_2)^{\frac{1}{\delta}} \left(\int_{\mathbb{R}} x^2 |u_S(t^*, x)|^2 dx \right. \\
 &\quad \left. - \left(\int_{\mathbb{R}} x |u_S(t^*, x)|^2 dx \right)^2 \right)^{\frac{1}{2\delta}} \varepsilon^{-\frac{1}{\delta}} \\
 &=: \eta(\varepsilon) .
 \end{aligned}$$

160 In particular if we are interested in the evolution of the solution for
 161 positive times and if $t^* < -\eta(\varepsilon)$, then the error of the approximation
 162 is smaller than ε for all $t \geq 0$. This has to be compared with the
 163 results from the classical approach (as in [3, 4]) which always imply
 164 the existence of a small time-interval $[0, T]$ during which the error is
 165 larger than a given threshold. This is due to the lack of flexibility of
 166 the classical approach which enforces $t^* = 0$ (the decay rate is then
 167 $t^{-\frac{1}{2}}$) and puts automatically the origin of the cone at the origin of
 168 space-time.

169 Let us now comment on some possible improvements or applications of
 170 the present results. First of all, an interesting issue would be to apply the
 171 approach developed in this paper to more complicated settings. One may
 172 consider dispersive equations on certain networks where integral solution
 173 formulas are available, as for example the Schrödinger equation on a star-
 174 shaped network with infinite branches [1] or on a tadpole graph [2]. In
 175 both papers, the time-asymptotic behaviour is studied by a frequency band

176 hypothesis and one may hope a better description of physical phenomena by
 177 using our refined method.

178 We could also consider the Schrödinger equation with a potential. In [10],
 179 the time-asymptotic behaviour of the two first terms of the Dyson-Phillips
 180 series [16, Chap. III, Theorem 1.10] representing the perturbed solution is
 181 studied by means of asymptotic expansions. The results concerning the se-
 182 cond term of the series are interpreted as follows: if the initial state travels
 183 from left to right in space, then the positive frequencies of the potential tend
 184 to accelerate the motion of the second term while the negative frequencies
 185 tend to slow down or even reverse it, exhibiting advanced and retarded trans-
 186 missions as well as reflections. The application of the present results could
 187 bring more information on these phenomena, in particular precise spatial
 188 information on the transmitted and reflected wave packets.

189 As explained in this paper, the notion of frequency band is physically
 190 meaningful and permits to describe time-asymptotically the propagation of
 191 solutions of certain dispersive equations in a precise way. However it is a
 192 restrictive hypothesis: for example, a function in a finite frequency band is
 193 necessarily a \mathcal{C}^∞ -function. Hence it would be relevant to extend this notion
 194 to functions whose Fourier transform is not necessarily compactly supported
 195 but still localised in a weaker sense. In this setting, the first term of the
 196 expansion is no longer supported in a space-time cone and so one has to
 197 quantify the localisation by means of different tools. For instance, we can
 198 consider approaches based on weighted norms; such norms have been used
 199 in [19], [20] or in [21] to show that the continuous part of the perturbed
 200 Schrödinger evolution transports away from the origin with non-zero velocity.

201 Our approach makes appear naturally the shifted decay rate $|t - t^*|^{-\frac{1}{2}}$,
 202 where t^* minimises the variance of the solution. It would be also interesting to
 203 introduce this time-shift in other existing results to obtain greater precision.
 204 For instance, one may consider the important $L^p - L^{p'}$ estimates for which
 205 a simple argument makes apparent the shifted decay; this is proved in the
 206 following result:

Proposition 1.2. *Consider the free Schrödinger equation on the line (3)
 with $u_0 \in \mathcal{S}(\mathbb{R})$ and define $t^* \in \mathbb{R}$ as follows:*

$$t^* := \arg \min_{\tau \in \mathbb{R}} \left(\int_{\mathbb{R}} x^2 |u_S(\tau, x)|^2 dx - \left(\int_{\mathbb{R}} x |u_S(\tau, x)|^2 dx \right)^2 \right).$$

Then for all $p \in [2, \infty]$ and for all $t \in \mathbb{R} \setminus \{t^*\}$, we have

$$\|u_S(t, \cdot)\|_{L^p(\mathbb{R})} \leq \left(\frac{1}{4\pi}\right)^{-\frac{1}{2} + \frac{1}{p}} \|u_S(t^*, \cdot)\|_{L^{p'}(\mathbb{R})} |t - t^*|^{-\frac{1}{2} + \frac{1}{p}},$$

207 where p' is the conjugate of p .

Proof. For the sake of clarity, we use the one-parameter group $(e^{-it\partial_{xx}})_{t \in \mathbb{R}}$ which permits to describe the Schrödinger evolution as follows:

$$\forall t \in \mathbb{R} \quad u_S(t) = e^{-it\partial_{xx}} u_0.$$

Using the group property, we have for any $t \in \mathbb{R}$,

$$e^{-it\partial_{xx}} u_0 = e^{-i(t-t^*)\partial_{xx}} e^{-it^*\partial_{xx}} u_0,$$

and by applying the classical $L^p - L^{p'}$ estimate [6, Proposition 2.2.3] to the above right-hand side, we obtain for all $t \neq t^*$,

$$\begin{aligned} \|u_S(t, \cdot)\|_{L^p(\mathbb{R})} &= \left\| e^{-it\partial_{xx}} u_0 \right\|_{L^p(\mathbb{R})} \\ &\leq \left(\frac{1}{4\pi}\right)^{-\frac{1}{2} + \frac{1}{p}} \left\| e^{-it^*\partial_{xx}} u_0 \right\|_{L^{p'}(\mathbb{R})} |t - t^*|^{-\frac{1}{2} + \frac{1}{p}} \\ &= \left(\frac{1}{4\pi}\right)^{-\frac{1}{2} + \frac{1}{p}} \|u_S(t^*, \cdot)\|_{L^{p'}(\mathbb{R})} |t - t^*|^{-\frac{1}{2} + \frac{1}{p}}. \end{aligned}$$

208 Note that we are allowed to apply the classical $L^p - L^{p'}$ estimate since
209 $e^{-it^*\partial_{xx}} u_0 \in \mathcal{S}(\mathbb{R}) \subset L^{p'}(\mathbb{R})$ thanks to the hypothesis $u_0 \in \mathcal{S}(\mathbb{R})$. \square

210 Since the classical $L^p - L^{p'}$ estimates are exploited to establish Strichartz es-
211 timates which are themselves used to study non-linear dispersive phenomena,
212 it is necessary to extend the above shifted $L^p - L^{p'}$ estimates to spaces larger
213 than the Schwartz space in view of precise applications. In particular, one
214 may examine whether t^* defined above still satisfies some optimal conditions;
215 this could be linked with the results established in [7].

216 Regarding long-term perspectives of our work, one could consider the full
217 soliton resolution for non-linear dispersive equations [11, 12, 13, 14, 15], which
218 aims at classifying the asymptotic behaviour of the non-linear solutions. A
219 key argument for the results contained in this series of papers is the channel

220 energy method [26], which consists in estimating the associated free solution
 221 outside a space-time cone or channel; this estimate is then used to prove that
 222 a dispersive term appearing in the decomposition of the non-linear solution
 223 tends to 0 in the energy-space. In particular, we mention that the authors of
 224 [8] have to shift in time the cones and channels to derive the desired estimates.
 225 Hence one might hope that the ideas proposed in the present paper could
 226 help to understand the requirement for this shift and more generally to refine
 227 the channel energy method.

228 Finally we could also think about minimal escape velocities [25, 27] which
 229 aim at exhibiting propagation features for evolution operators of type e^{-itH} ,
 230 where H is a general Hamiltonian; for instance, one may consider the opera-
 231 tor $H = -\frac{1}{2}\partial_{xx} + V$ where V is real-valued potential. As explained in [24],
 232 the method to establish these estimates generalises the integration by parts
 233 which is actually crucial to describe the time-asymptotic behaviour of wave
 234 packets, as illustrated in the present paper. Our approach could bring more
 235 precision to the abstract setting and hence lead to estimates containing more
 236 information on the propagation of general wave packets.

237

238 The paper is organised as follows: in the following section, we state the
 239 main results on time-asymptotic expansions for dispersive equations of the
 240 form (2). Since the proofs of the main results are substantially based on a
 241 careful application of the stationary phase method, Section 3 is devoted to
 242 our new version of this method. Finally Section 4 contains the proofs of each
 243 result presented in Section 2.

244 **2. Main results on time-asymptotic approximations for dispersive** 245 **equations**

246 In this section, we present the main results of our paper. Prior to this,
 247 we introduce the mathematical setting as well as some notations.

248

The Fourier transform $\mathcal{F}u : \mathbb{R} \rightarrow \mathbb{C}$ of a function $u : \mathbb{R} \rightarrow \mathbb{C}$ belonging
 to the Schwartz space $\mathcal{S}(\mathbb{R})$ is defined by

$$\forall p \in \mathbb{R} \quad \mathcal{F}u(p) := \int_{\mathbb{R}} u(x) e^{-ixp} dx .$$

The Fourier transform defines an invertible operator from $\mathcal{S}(\mathbb{R})$ onto itself,
 and can be extended to the space of square-integrable functions $L^2(\mathbb{R})$ and

to the tempered distributions $\mathcal{S}'(\mathbb{R})$. Moreover, for $u \in L^2(\mathbb{R})$, Plancherel theorem assures the following equality:

$$\forall u \in L^2(\mathbb{R}) \quad \|u\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \|\mathcal{F}u\|_{L^2(\mathbb{R})} ;$$

249 see [23, Theorem 7.1.6].

Consider now a \mathcal{C}^∞ -function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that all its derivatives grow at most as a polynomial at infinity and consider the associated operator $f(D) : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ defined by

$$\forall x \in \mathbb{R} \quad f(D)u(x) := \frac{1}{2\pi} \int_{\mathbb{R}} f(p) \mathcal{F}u(p) e^{ixp} dp = \mathcal{F}^{-1}(f \mathcal{F}u)(x) ,$$

250 which can be extended to the tempered distributions $\mathcal{S}'(\mathbb{R})$. The operator
 251 $f(D) : \mathcal{S}'(\mathbb{R}) \rightarrow \mathcal{S}'(\mathbb{R})$ is called a *Fourier multiplier* associated with the
 252 *symbol* f .

253 Given such an operator, we introduce the following evolution equation on
 254 the line,

$$\begin{cases} [i \partial_t - f(D)] u_f(t) = 0 \\ u_f(0) = u_0 \end{cases} , \quad (5)$$

for $t \in \mathbb{R}$. If we suppose $u_0 \in \mathcal{S}'(\mathbb{R})$ then the equation (5) has a unique solution in $\mathcal{C}^1(\mathbb{R}, \mathcal{S}'(\mathbb{R}))$ given by the following solution formula,

$$u_f(t) = \mathcal{F}^{-1}(e^{-itf} \mathcal{F}u_0) .$$

255 We refer to [5] for a detailed study of this family of equations. In this paper,
 256 we suppose that the symbol f is strictly convex; an important example of
 257 such an equation is given by the free Schrödinger equation whose symbol is
 258 $f_S(p) = \frac{1}{2} p^2$.

259 For the sake of better presentation of the results, we consider initial data
 260 u_0 belonging only to the Schwartz space $\mathcal{S}(\mathbb{R})$ to focus on the approach we
 261 propose. We mention that it is possible to extend our results to the case of
 262 initial data in $L^2(\mathbb{R})$ with additional assumptions on regularity and decay;
 263 but this falls out of the scope of the paper.

264 Further the initial data are assumed to be in bounded frequency bands,
 265 meaning that their Fourier transforms are supported on bounded intervals
 266 $[p_1, p_2]$, where $p_1 < p_2$ are finite real numbers. Under such hypotheses, the

267 solution formula for the equation (5) defines a function $u_f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{C}$
 268 given by

$$u_f(t, x) = \frac{1}{2\pi} \int_{p_1}^{p_2} \mathcal{F}u_0(p) e^{-itf(p)+ixp} dp. \quad (6)$$

269 We define now the space-time cone related to the symbol f and to the
 270 frequency band $[\tilde{p}_1, \tilde{p}_2]$ with origin $(t_0, x_0) \in \mathbb{R}^2$.

271 **Definition 2.1.** Let t_0, x_0, \tilde{p}_1 and \tilde{p}_2 be finite real numbers such that $\tilde{p}_1 < \tilde{p}_2$
 272 and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a symbol.

1. We define the space-time cone $\mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t_0, x_0))$ as follows:

$$\begin{aligned} & \mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t_0, x_0)) \\ & := \left\{ (t, x) \in (\mathbb{R} \setminus \{t_0\}) \times \mathbb{R} \mid f'(\tilde{p}_1) \leq \frac{x - x_0}{t - t_0} \leq f'(\tilde{p}_2) \right\}. \end{aligned}$$

- 273 2. Let $\mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t_0, x_0))^c$ be the complement of the space-time cone
 274 $\mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t_0, x_0))$ in $(\mathbb{R} \setminus \{t_0\}) \times \mathbb{R}$.

275 We present now our first main result. Theorem 2.2 provides a time-
 276 asymptotic expansion to one term with explicit error estimate of the solution
 277 (6) in the space-time cone $\mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t_0, x_0))$, where $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$ and
 278 $(t_0, x_0) \in \mathbb{R}^2$ is arbitrarily chosen. A uniform estimate of the solution (6)
 279 outside the cone is also established. The result shows that the solution tends
 280 to be time-asymptotically localised in the cone $\mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t_0, x_0))$.

Theorem 2.2. Let p_1, p_2, \tilde{p}_1 and \tilde{p}_2 be finite real numbers such that
 $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$. Suppose that $u_0 \in \mathcal{S}(\mathbb{R})$ is a function whose Fourier
 transform satisfies

$$\text{supp } \mathcal{F}u_0 \subseteq [p_1, p_2],$$

281 and fix $(t_0, x_0) \in \mathbb{R}^2$. Then

1. for all $(t, x) \in \mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t_0, x_0))$, we have

$$\begin{aligned} & \left| u_f(t, x) - \frac{1}{\sqrt{2\pi}} e^{-\text{sgn}(t-t_0)i\frac{\pi}{4}} e^{-itf(p_0(t,x))+ixp_0(t,x)} \frac{\mathcal{F}u_0(p_0(t,x))}{\sqrt{f''(p_0(t,x))}} |t - t_0|^{-\frac{1}{2}} \right| \\ & \leq \left(C_1(f, \delta, \tilde{p}_1, \tilde{p}_2) \|(\cdot - x_0) u_f(t_0, \cdot)\|_{L^2(\mathbb{R})} \right. \\ & \quad \left. + C_2(f, \delta, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) |t - t_0|^{-\delta}, \quad (7) \end{aligned}$$

where the real number δ is arbitrarily chosen in $(\frac{1}{2}, \frac{3}{4})$ and

- $p_0(t, x) := (f')^{-1}\left(\frac{x - x_0}{t - t_0}\right)$;
- $C_1(f, \delta, \tilde{p}_1, \tilde{p}_2) := \frac{2^{\delta+\frac{1}{2}} L(\delta)}{\sqrt{\pi}\sqrt{3-4\delta}} (\tilde{p}_2 - \tilde{p}_1)^{\frac{3-4\delta}{2}} c_1(f, \delta, \tilde{p}_1, \tilde{p}_2)$;
- $C_2(f, \delta, \tilde{p}_1, \tilde{p}_2) := \frac{2^{\delta-2} L(\delta)}{\pi(1-\delta)} (\tilde{p}_2 - \tilde{p}_1)^{2-2\delta} c_2(f, \delta, \tilde{p}_1, \tilde{p}_2)$;
- $c_1(f, \delta, \tilde{p}_1, \tilde{p}_2) := \|f''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{2}-\delta} \min_{[\tilde{p}_1, \tilde{p}_2]} \{f''\}^{-\frac{3}{2}}$;
- $c_2(f, \delta, \tilde{p}_1, \tilde{p}_2) := \|f''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{5}{2}-\delta} \|f^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{f''\}^{-\frac{7}{2}}$
 $+ \frac{1}{3} \|f''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{7}{2}-\delta} \|f^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{f''\}^{-\frac{9}{2}}$.

282

The constant $L(\delta) > 0$ is defined in (13);

2. for all $(t, x) \in \mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t_0, x_0))^c$, we have

$$|u_f(t, x)| \leq \left(C_3(f, p_1, p_2, \tilde{p}_1, \tilde{p}_2) \|(\cdot - x_0) u_f(t_0, \cdot)\|_{L^2(\mathbb{R})} + C_4(f, p_1, p_2, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) |t - t_0|^{-1}, \quad (8)$$

where

- $C_3(f, p_1, p_2, \tilde{p}_1, \tilde{p}_2) := \frac{1}{\sqrt{2\pi}} (p_2 - p_1)^{\frac{1}{2}} \min\{f'(p_1) - f'(\tilde{p}_1), f'(\tilde{p}_2) - f'(p_2)\}^{-1}$;
- $C_4(f, p_1, p_2, \tilde{p}_1, \tilde{p}_2) := \frac{1}{2\pi} \min\{f'(p_1) - f'(\tilde{p}_1), f'(\tilde{p}_2) - f'(p_2)\}^{-1}$.

283

To state the two other main results, we define the following moment-type and variance-type quantities for normalized $u \in L^2(\mathbb{R})$.

284

Definition 2.3. Let $u \in L^2(\mathbb{R})$ such that $\|u\|_{L^2(\mathbb{R})} = 1$ and let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a symbol. If they exist, we define the real numbers $\mathcal{M}_f(u)$ and $\mathcal{V}_f(u)$ as follows,

$$\mathcal{M}_f(u) := \int_{\mathbb{R}} f(x) |u(x)|^2 dx \quad , \quad \mathcal{V}_f(u) := \mathcal{M}_{f^2}(u) - \mathcal{M}_f(u)^2 .$$

If $f(x) = x$, then we note for simplicity

$$\mathcal{M}_1(u) := \mathcal{M}_f(u) \quad , \quad \mathcal{M}_2(u) := \mathcal{M}_{f^2}(u) \quad , \quad \mathcal{V}(u) := \mathcal{V}_f(u) .$$

285 **Remark 2.4.** The above quantities $\mathcal{M}_1(u)$, $\mathcal{M}_2(u)$ and $\mathcal{V}(u)$ are respectively
286 the mean, the second moment and the variance of $|u|^2$.

287 The explicitness of the family of error bounds given in Theorem 2.2 allows
288 to compute the parameter (t^*, x^*) giving the smallest bound; this computa-
289 tion is carried out in Lemma 4.2.1. In the following result, we provide the
290 values of t^* and x^* and we provide the time-asymptotic expansion for the
291 solution of equation (5) associated with this optimal parameter.

Corollary 2.5. *Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then the (t_0, x_0) -dependent right-hand sides of inequalities (7) and (8) in Theorem 2.2 have a global minimum at the point $(t^*, x^*) \in \mathbb{R}^2$ with*

- $t^* = \arg \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot))$;
- $x^* = \mathcal{M}_1(u_f(t^*, \cdot))$.

In this case, for all $(t, x) \in \mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t^*, x^*))$, we have

$$\begin{aligned} & \left| u_f(t, x) - \frac{1}{\sqrt{2\pi}} e^{-\text{sgn}(t-t^*)i\frac{\pi}{4}} e^{-itf(p_*(t,x))+ixp_*(t,x)} \frac{\mathcal{F}u_0(p_*(t, x))}{\sqrt{f''(p_*(t, x))}} |t - t^*|^{-\frac{1}{2}} \right| \\ & \leq \left(C_1(f, \delta, \tilde{p}_1, \tilde{p}_2) \min_{\tau \in \mathbb{R}} \left(\sqrt{\mathcal{V}(u_f(\tau, \cdot))} \right) \right. \\ & \quad \left. + C_2(f, \delta, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) |t - t^*|^{-\delta} , \end{aligned}$$

where the real number δ is arbitrarily chosen in $(\frac{1}{2}, \frac{3}{4})$ and

$$p_*(t, x) := (f')^{-1} \left(\frac{x - x^*}{t - t^*} \right) .$$

And for all $(t, x) \in \mathfrak{C}_f([\tilde{p}_1, \tilde{p}_2], (t^*, x^*))^c$, we have

$$\begin{aligned} |u_f(t, x)| & \leq \left(C_3(f, p_1, p_2, \tilde{p}_1, \tilde{p}_2) \min_{\tau \in \mathbb{R}} \left(\sqrt{\mathcal{V}(u_f(\tau, \cdot))} \right) \right. \\ & \quad \left. + C_4(f, p_1, p_2, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) |t - t^*|^{-1} . \end{aligned}$$

292 All the constants are defined in Theorem 2.2.

293 In the last result of this section, we consider mean position and variance
 294 of the time-asymptotic approximation of the solution (6) associated with the
 295 optimal parameter (t^*, x^*) . Theorem 2.6 claims that the mean position of
 296 this approximation is equal to the one of the true solution and the difference
 297 between the two variances is constant; the value for this constant is provided.

Theorem 2.6. *Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. For all $t \in (\mathbb{R} \setminus \{t^*\})$ and $x \in \mathbb{R}$, we define*

$$H_f(t, x, u_0, t^*, x^*) := \frac{1}{\sqrt{2\pi}} e^{-\text{sgn}(t-t^*)i\frac{\pi}{4}} e^{-itf(p_*(t,x))+ixp_*(t,x)} \frac{\mathcal{F}u_0(p_*(t,x))}{\sqrt{f''(p_*(t,x))}} |t-t^*|^{-\frac{1}{2}},$$

where t^* and x^* are introduced in Corollary 2.5 and $p_*(t, x) := (f')^{-1}\left(\frac{x-x^*}{t-t^*}\right)$. Then for all $t \in \mathbb{R} \setminus \{t^*\}$, we have

$$\begin{cases} \mathcal{M}_1(u_f(t, \cdot)) = \mathcal{M}_1(H_f(t, \cdot, u_0, t^*, x^*)) \\ \mathcal{V}(u_f(t, \cdot)) - \mathcal{V}(H_f(t, \cdot, u_0, t^*, x^*)) = \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot)) \end{cases} .$$

298 3. Explicit error estimates for a stationary phase method via Cau- 299 chy-Schwarz inequality

300 The proof of Theorem 2.2 is substantially based on a careful application
 301 of a new version of the stationary phase method for oscillatory integrals of
 302 the form

$$\forall \omega > 0 \quad \int_{\mathbb{R}} U(p) e^{i\omega\psi(p)} dp, \quad (9)$$

303 where the amplitude $U : \mathbb{R} \rightarrow \mathbb{C}$ is a continuously differentiable function
 304 supported on a bounded interval and the phase $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly
 305 concave \mathcal{C}^3 -function having a unique stationary point p_0 . The remainder
 306 estimates we provide here are explicit, uniform with respect to p_0 and involve
 307 the L^2 -norm of the first derivative of the amplitude. This plays actually a
 308 key role in the proof of Theorem 2.2.

309 This section is fully devoted to this new version since it is generic and
 310 independent from the setting of Section 2. The asymptotic expansions to-
 311 gether with the uniform and explicit remainder estimates are established in

312 Theorems 3.3 and 3.4.

313

314 We start by stating two technical lemmas which will be substantially used
315 in the proof of Theorem 3.3.

316 The first step to expand ω -asymptotically integrals of type (9) consists
317 in making simpler the phase function in order to integrate then by parts. To
318 do so, we use the diffeomorphisms φ_j ($j = 1, 2$) defined and studied in the
319 following lemma. The values of these diffeomorphisms at the stationary point
320 p_0 are provided in order to compute explicitly the first term of the expansions
321 and two inequalities for φ_j are established to bound the remainders of these
322 expansions.

323 The proof of the following result lies mainly on an integral representation of
324 φ_j .

Lemma 3.1. *Let p_0 , \tilde{p}_1 and \tilde{p}_2 be finite real numbers such that $p_0 \in (\tilde{p}_1, \tilde{p}_2)$. Suppose that $\psi \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ is a strictly concave function which has a unique stationary point at p_0 . Then, for $j = 1, 2$, the function*

$$\begin{aligned} \varphi_j : I_j &\longrightarrow [0, s_j] \\ p &\longmapsto (\psi(p_0) - \psi(p))^{\frac{1}{2}} \end{aligned}$$

325 where $I_1 := [\tilde{p}_1, p_0]$, $I_2 := [p_0, \tilde{p}_2]$ and $s_j := \varphi_j(\tilde{p}_j)$, satisfies the following
326 properties:

- 327
1. the function φ_j is a \mathcal{C}^2 -diffeomorphism between I_j and $[0, s_j]$;
 2. we have

$$\varphi'_j(p_0) = (-1)^j \sqrt{-\frac{\psi''(p_0)}{2}} ;$$

3. for all $p \in I_j$, the absolute value of $\varphi'_j(p)$ is lower bounded as follows:

$$\left| \varphi'_j(p) \right| \geq \frac{1}{\sqrt{2}} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{-\frac{1}{2}} ;$$

4. we have the following L^∞ -norm estimate for $(\varphi_j^{-1})''$:

$$\begin{aligned} \left\| (\varphi_j^{-1})'' \right\|_{L^\infty(0, s_j)} &\leq \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{2}} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{7}{2}} \\ &\quad + \frac{1}{3} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{5}{2}} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{9}{2}} . \end{aligned}$$

328 *Proof.* Let $j \in \{1, 2\}$ and fix $p_0 \in (\tilde{p}_1, \tilde{p}_2)$. The proof of the present lemma
 329 is mainly based on the following integral representation of the function φ_j :

$$\varphi_j(p) = (-1)^j (p - p_0) \left(\int_0^1 \int_0^1 \tilde{\psi}(p, \tau, \nu) d\nu d\tau \right)^{\frac{1}{2}}, \quad (10)$$

for all $p \in I_j$, where

$$\tilde{\psi}(p, \tau, \nu) := -\psi''((1 - \tau)(1 - \nu)p + (\nu - \nu\tau + \tau)p_0) (1 - \tau).$$

This representation can be derived by noting firstly that

$$\psi(p_0) - \psi(p) = - \int_p^{p_0} \psi'(p_0) - \psi'(t) dt = \int_p^{p_0} \int_t^{p_0} -\psi''(v) dv dt,$$

where we have used the hypothesis $\psi'(p_0) = 0$ to obtain the first equality. Then we make the change of variable $(\nu, \tau) = \left(\frac{v-t}{p_0-t}, \frac{t-p}{p_0-p}\right)$, leading to

$$\psi(p_0) - \psi(p) = (p - p_0)^2 \int_0^1 \int_0^1 \tilde{\psi}(p, \tau, \nu) d\nu d\tau,$$

330 and we take finally the square root of the preceding equality to obtain the
 331 desired representation (10).

1. Since ψ is a strictly concave function on \mathbb{R} , the function φ_j is actually the square root of the non-negative \mathcal{C}^3 -function $p \mapsto \psi(p_0) - \psi(p)$, showing that φ_j is twice continuously differentiable on $I_j \setminus \{p_0\}$ (φ_j is actually a \mathcal{C}^3 -function on this domain). Let us prove that it is also twice differentiable on the whole I_j . To do so, note that we have for $p \in I_j \setminus \{p_0\}$,

$$\begin{aligned} \varphi_j'(p) &= -\frac{1}{2} \psi'(p) (\psi(p_0) - \psi(p))^{-\frac{1}{2}} \\ &= -\frac{1}{2} \left(\int_p^{p_0} -\psi''(q) dq \right) \varphi_j(p)^{-1} \\ &= \frac{1}{2} \left((p - p_0) \int_0^1 -\psi''((1 - t)p + tp_0) dt \right) \varphi_j(p)^{-1} \\ &= \frac{(-1)^j}{2} \left(\int_0^1 -\psi''((1 - t)p + tp_0) dt \right) \\ &\quad \times \left(\int_0^1 \int_0^1 \tilde{\psi}(p, \tau, \nu) d\nu d\tau \right)^{-\frac{1}{2}}. \end{aligned} \quad (11)$$

The preceding equality combined with the positivity of the \mathcal{C}^1 -function $-\psi''$ shows that φ'_j is continuously differentiable on I_j whose derivative is given by

$$\begin{aligned} \varphi''_j(p) &= \frac{(-1)^j}{2} \left(\int_0^1 -\psi^{(3)}((p-p_0)t+p_0)(1-t) dt \right) \\ &\quad \times \left(\int_0^1 \int_0^1 \tilde{\psi}(p, \tau, \nu) d\nu d\tau \right)^{-\frac{1}{2}} \\ &\quad + \frac{(-1)^j}{2} \left(\int_0^1 -\psi''((1-t)p+tp_0) dt \right) \\ &\quad \times \left(-\frac{1}{2} \right) \frac{\int_0^1 \int_0^1 \check{\psi}(p, \tau, \nu) d\nu d\tau}{\left(\int_0^1 \int_0^1 \tilde{\psi}(p, \tau, \nu) d\nu d\tau \right)^{\frac{3}{2}}}, \end{aligned}$$

for all $p \in I_j$, where

$$\check{\psi}(p, \tau, \nu) := -\psi^{(3)}((1-\tau)(1-\nu)p + (\nu - \nu\tau + \tau)p_0)(1-\tau)^2(1-\nu).$$

332 Now, according to equality (11), we observe that φ'_j is negative for
 333 $j = 1$ and positive for $j = 2$ since $-\psi'' > 0$. By the inverse function
 334 theorem, we deduce that φ_j is a \mathcal{C}^2 -diffeomorphism.

2. Thanks to the integral representation (10), we have

$$\begin{aligned} \varphi'_j(p_0) &= \lim_{p \rightarrow p_0} \frac{\varphi_j(p) - \varphi_j(p_0)}{p - p_0} \\ &= (-1)^j \lim_{p \rightarrow p_0} \left(\int_0^1 \int_0^1 \tilde{\psi}(p, \tau, \nu) d\nu d\tau \right)^{\frac{1}{2}} \\ &= (-1)^j \sqrt{-\frac{\psi'''(p_0)}{2}}. \end{aligned}$$

335 3. From equality (11) (which holds actually for all $p \in I_j$), we deduce the
 336 following lower estimate for φ'_j :

$$\forall p \in I_j \quad \left| \varphi'_j(p) \right| \geq \frac{1}{\sqrt{2}} \min_{[\bar{p}_1, \bar{p}_2]} \{ -\psi'' \} \left\| \psi'' \right\|_{L^\infty(\bar{p}_1, \bar{p}_2)}^{-\frac{1}{2}}. \quad (12)$$

4. From the expression of φ''_j computed above, we obtain the following

upper estimate:

$$\begin{aligned} \left| \varphi_j''(p) \right| &\leq \frac{1}{2\sqrt{2}} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{1}{2}} \\ &\quad + \frac{1}{6\sqrt{2}} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{3}{2}}, \end{aligned}$$

for all $p \in I_j$. By combining the preceding inequality with estimate (12) and the following relation,

$$\forall s \in [0, s_j] \quad (\varphi_j^{-1})''(s) = -\frac{\varphi_j''(\varphi_j^{-1}(s))}{\varphi_j'(\varphi_j^{-1}(s))^3},$$

we obtain finally for all $s \in [0, s_j]$,

$$\begin{aligned} \left| (\varphi_j^{-1})''(s) \right| &\leq \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{2}} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{7}{2}} \\ &\quad + \frac{1}{3} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{5}{2}} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{9}{2}}. \end{aligned}$$

337

□

338 After having applied the above diffeomorphism to the integral (9) (pre-
 339 viously split at p_0), the phase becomes the quadratic function $s \mapsto -s^2$.
 340 In order to make an integration by parts, creating then the first and the
 341 remainder terms of the integral, one needs an expression for a primitive of
 342 the function $s \in [0, s_0] \mapsto e^{-i\omega s^2} \in \mathbb{C}$, for fixed $s_0, \omega > 0$. In the following
 343 lemma, a useful integral representation of such a primitive is given. As in the
 344 preceding result, its value at the origin and an inequality are also provided
 345 to compute respectively the first term of the expansion and an upper bound
 346 for the remainder term.

347 To prove Lemma 3.2, we refer to the paper [3] which gives actually the
 348 successive primitives of more general functions by using essentially complex
 349 analysis; see [3, Theorems 6.4, 6.5 and Corollary 6.6].

Lemma 3.2. *Let $\omega, s_0 > 0$ be real numbers and let $\phi(\cdot, \omega) : [0, s_0] \rightarrow \mathbb{C}$ be the function defined by*

$$\phi(s, \omega) := - \int_{\Lambda(s)} e^{-i\omega z^2} dz,$$

where $\Lambda(s)$ is the half-line in the complex plane given by

$$\Lambda(s) := \left\{ s + t e^{-i\frac{\pi}{4}} \in \mathbb{C} \mid t \geq 0 \right\} .$$

350 Then

- 351 1. the function $\phi(\cdot, \omega)$ is a primitive of $s \in [0, s_0] \mapsto e^{-i\omega s^2} \in \mathbb{C}$;
 2. we have

$$\phi(0, \omega) = -\frac{1}{2} \sqrt{\pi} e^{-i\frac{\pi}{4}} \omega^{-\frac{1}{2}} ;$$

3. the function $\phi(\cdot, \omega)$ satisfies

$$\forall s \in (0, s_0] \quad |\phi(s, \omega)| \leq L(\delta) s^{1-2\delta} \omega^{-\delta} ,$$

352 where the real number δ is arbitrarily chosen in $(\frac{1}{2}, 1)$ and the constant
 353 $L(\delta) > 0$ is defined by

$$L(\delta) := \frac{\sqrt{\pi}}{2} \left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}} \right)^{2\delta-1} . \quad (13)$$

354 *Proof.* The function $\phi(\cdot, \omega)$ of the present paper corresponds actually to the
 355 function $\phi_1^{(2)}(\cdot, \omega, 2, 1)$ defined in [3, Theorem 2.3]. Hence we apply the results
 356 established in [3] to the present situation:

- 357 1. One proves this first point by applying [3, Corollary 6.6], which is a
 358 consequence of Theorems 6.4 and 6.5 of [3], in the case $n = 1$, $j = 2$,
 359 $\rho_j = 2$ and $\mu_j = 1$.
 360 2. The proof of this point lies only on basic computations which are carried
 361 out in the fourth step of the proof of [3, Theorem 2.3].
 362 3. The combination of Lemmas 2.4 and 2.6 of [3] assures this last point.

363 □

364 Thanks to the two preceding lemmas, we are now in position to establish
 365 the desired asymptotic expansions with respect to the parameter ω for oscil-
 366 latory integrals of type (9). In the following theorem, we are interested in the
 367 case where the stationary point p_0 of the phase belongs to a neighbourhood
 368 of the support of the amplitude. We emphasise that the remainder estimate
 369 we provide is different from those appearing in the original paper [18] and in
 370 [3].

371 Technically speaking, we split the integral at the stationary point p_0 and we
 372 study separately the two resulting integrals. In each situation, the method
 373 consists firstly in using the diffeomorphism introduced in Lemma 3.1 to make
 374 the phase function simpler, secondly in integrating by parts to create the ex-
 375 pansion by using Lemma 3.1 point 2, Lemma 3.2 point 1 and point 2, and
 376 finally in bounding the remainder term by combining Lemma 3.1 point 3,
 377 point 4 and Lemma 3.2 point 3 with Cauchy-Schwarz inequality.

Theorem 3.3. *Let p_1, p_2, \tilde{p}_1 and \tilde{p}_2 be finite real numbers such that $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$. Suppose that $\psi \in \mathcal{C}^3(\mathbb{R}, \mathbb{R}) : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly concave function which has a unique stationary point at $p_0 \in (\tilde{p}_1, \tilde{p}_2)$. And assume that $U \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ is a function satisfying*

$$\text{supp } U \subseteq [p_1, p_2] .$$

Then we have for all $\omega > 0$,

$$\left| \int_{\mathbb{R}} U(p) e^{i\omega\psi(p)} dp - \sqrt{2\pi} e^{-i\frac{\pi}{4}} e^{i\omega\psi(p_0)} \frac{U(p_0)}{\sqrt{-\psi''(p_0)}} \omega^{-\frac{1}{2}} \right| \\ \leq \left(C_5(\psi, \delta, \tilde{p}_1, \tilde{p}_2) \|U'\|_{L^2(\mathbb{R})} + C_6(\psi, \delta, \tilde{p}_1, \tilde{p}_2) \|U\|_{L^\infty(\mathbb{R})} \right) \omega^{-\delta} ,$$

where the real number δ is arbitrarily chosen in $(\frac{1}{2}, \frac{3}{4})$ and

- $C_5(\psi, \delta, \tilde{p}_1, \tilde{p}_2) := \frac{2^{\delta+1} L(\delta)}{\sqrt{3-4\delta}} (\tilde{p}_2 - \tilde{p}_1)^{\frac{3-4\delta}{2}} c_5(\psi, \delta, \tilde{p}_1, \tilde{p}_2) ;$
- $C_6(\psi, \delta, \tilde{p}_1, \tilde{p}_2) := \frac{2^{\delta-1} L(\delta)}{1-\delta} (\tilde{p}_2 - \tilde{p}_1)^{2-2\delta} c_6(\psi, \delta, \tilde{p}_1, \tilde{p}_2) ;$
- $c_5(\psi, \delta, \tilde{p}_1, \tilde{p}_2) := \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{2}-\delta} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{3}{2}} ;$
- $c_6(\psi, \delta, \tilde{p}_1, \tilde{p}_2) := \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{5}{2}-\delta} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{7}{2}}$
 $+ \frac{1}{3} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{7}{2}-\delta} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{9}{2}} .$

378 The constant $L(\delta) > 0$ is defined in (13).

Proof. Let $\omega > 0$ and choose $p_0 \in (\tilde{p}_1, \tilde{p}_2)$. First of all, since the support of the amplitude is included in $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$, we have clearly

$$\int_{\mathbb{R}} U(p) e^{i\omega\psi(p)} dp = \int_{\tilde{p}_1}^{\tilde{p}_2} U(p) e^{i\omega\psi(p)} dp =: I(\omega) .$$

Splitting the above integral at the point p_0 and using the two \mathcal{C}^2 -diffeomorphisms defined in Lemma 3.1, we obtain

$$\begin{aligned} I(\omega) &= - \int_0^{s_1} (U \circ \varphi_1^{-1})(p) (\varphi_1^{-1})'(p) e^{-i\omega s^2} ds e^{i\omega\psi(p_0)} \\ &\quad + \int_0^{s_2} (U \circ \varphi_2^{-1})(p) (\varphi_2^{-1})'(p) e^{-i\omega s^2} ds e^{i\omega\psi(p_0)} ; \end{aligned}$$

note that we have used the fact that φ_1 and φ_2 are respectively decreasing and increasing. We integrate now by parts by using the primitive $s \mapsto \phi(s, \omega)$ given in Lemma 3.2 and the regularity of φ_j :

$$\begin{aligned} &(-1)^j \int_0^{s_j} (U \circ \varphi_j^{-1})(s) (\varphi_j^{-1})'(s) e^{-i\omega s^2} ds \\ &= (-1)^j \left[(U \circ \varphi_j^{-1})(s) (\varphi_j^{-1})'(s) \phi(s, \omega) \right]_0^{s_j} \\ &\quad + (-1)^{j+1} \int_0^{s_j} \left((U \circ \varphi_j^{-1}) (\varphi_j^{-1})' \right)'(s) \phi(s, \omega) ds \\ &= (-1)^{j+1} (U \circ \varphi_j^{-1})(0) (\varphi_j^{-1})'(0) \phi(0, \omega) \\ &\quad + (-1)^{j+1} \int_0^{s_j} \left((U \circ \varphi_j^{-1}) (\varphi_j^{-1})' \right)'(s) \phi(s, \omega) ds \\ &= \frac{1}{2} \sqrt{2\pi} e^{-i\frac{\pi}{4}} \frac{U(p_0)}{\sqrt{-\psi''(p_0)}} \omega^{-\frac{1}{2}} \\ &\quad + (-1)^{j+1} \int_0^{s_j} \left((U \circ \varphi_j^{-1}) (\varphi_j^{-1})' \right)'(s) \phi(s, \omega) ds ; \end{aligned}$$

the second equality has been obtained by using the fact that $U(\tilde{p}_j) = 0$ and the last one by applying Lemma 3.1 point 2 and Lemma 3.2 point 2. Hence it follows

$$\begin{aligned} I(\omega) &= \sqrt{2\pi} e^{-i\frac{\pi}{4}} e^{i\omega\psi(p_0)} \frac{U(p_0)}{\sqrt{-\psi''(p_0)}} \omega^{-\frac{1}{2}} \\ &\quad + \sum_{j=1}^2 (-1)^{j+1} \int_0^{s_j} \left((U \circ \varphi_j^{-1}) (\varphi_j^{-1})' \right)'(s) \phi(s, \omega) ds e^{i\omega\psi(p_0)} . \end{aligned}$$

To estimate each term of the remainder, we proceed as follows:

$$\begin{aligned}
& \left| (-1)^{j+1} \int_0^{s_j} \left((U \circ \varphi_j^{-1}) (\varphi_j^{-1})' \right)'(s) \phi(s, \omega) ds \right| \\
& \leq \left| \int_0^{s_j} (U' \circ \varphi_j^{-1})(s) \left((\varphi_j^{-1})'(s) \right)^2 \phi(s, \omega) ds \right| \\
& \quad + \left| \int_0^{s_j} (U \circ \varphi_j^{-1})(s) (\varphi_j^{-1})''(s) \phi(s, \omega) ds \right| \\
& \leq \left(\int_0^{s_j} \left| (U' \circ \varphi_j^{-1})(s) \left((\varphi_j^{-1})'(s) \right)^2 \right|^2 ds \right)^{\frac{1}{2}} \left(\int_0^{s_j} |\phi(s, \omega)|^2 ds \right)^{\frac{1}{2}} \\
& \quad + \int_0^{s_j} |\phi(s, \omega)| ds \|U\|_{L^\infty(\mathbb{R})} \left\| (\varphi_j^{-1})'' \right\|_{L^\infty(0, s_j)} ;
\end{aligned}$$

let us remark that we have applied Cauchy-Schwarz inequality to the first integral. We continue the proof by estimating each resulting term; first of all, by making the change of variable $p = \varphi_j^{-1}(s)$ and by using Lemma 3.1 point 3, we obtain

$$\begin{aligned}
& \left(\int_0^{s_j} \left| (U' \circ \varphi_j^{-1})(s) \left((\varphi_j^{-1})'(s) \right)^2 \right|^2 ds \right)^{\frac{1}{2}} \\
& \leq 2^{\frac{3}{4}} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{4}} \min_{[\tilde{p}_1, \tilde{p}_2]} \{ -\psi'' \}^{-\frac{3}{2}} \|U'\|_{L^2(\mathbb{R})} .
\end{aligned}$$

Then we use the point 3 of Lemma 3.2 to derive the two following inequalities:

- $\int_0^{s_j} |\phi(s, \omega)| ds \leq L(\delta) \int_0^{s_j} s^{1-2\delta} ds \omega^{-\delta} \leq \frac{L(\delta)}{2-2\delta} \varphi_j(\tilde{p}_j)^{2-2\delta} \omega^{-\delta} ;$
- $\left(\int_0^{s_j} |\phi(s, \omega)|^2 ds \right)^{\frac{1}{2}} \leq \frac{L(\delta)}{\sqrt{3-4\delta}} \varphi_j(\tilde{p}_j)^{\frac{3-4\delta}{2}} \omega^{-\delta} .$

By using the integral representation (10) of φ_j , we obtain

$$\varphi_j(\tilde{p}_j) \leq \frac{1}{\sqrt{2}} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{1}{2}} (\tilde{p}_2 - \tilde{p}_1) ,$$

which permits to deduce

- $\int_0^{s_j} |\phi(s, \omega)| ds \leq \frac{1}{2^{1-\delta}} \frac{L(\delta)}{2-2\delta} (\tilde{p}_2 - \tilde{p}_1)^{2-2\delta} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{1-\delta} \omega^{-\delta}$;
- $\left(\int_0^{s_j} |\phi(s, \omega)|^2 ds \right)^{\frac{1}{2}} \leq \frac{1}{2^{\frac{3}{4}-\delta}} \frac{L(\delta)}{\sqrt{3-4\delta}} (\tilde{p}_2 - \tilde{p}_1)^{\frac{3-4\delta}{2}} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{4}-\delta} \omega^{-\delta}$.

And, from Lemma 3.1 point 4, we recall that

$$\begin{aligned} \left\| (\varphi_j^{-1})'' \right\|_{L^\infty(0, s_j)} &\leq \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{2}} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{-\psi''\}^{-\frac{7}{2}} \\ &\quad + \frac{1}{3} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{5}{2}} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{-\psi''\}^{-\frac{9}{2}} . \end{aligned}$$

Putting everything together provides the desired estimate, namely,

$$\begin{aligned} &\left| I(\omega) - \sqrt{2\pi} e^{-i\frac{\pi}{4}} e^{i\omega\psi(p_0)} \frac{U(p_0)}{\sqrt{-\psi''(p_0)}} \omega^{-\frac{1}{2}} \right| \\ &\leq \sum_{j=1}^2 \left| (-1)^{j+1} \int_0^{s_j} \left((U \circ \varphi_j^{-1}) (\varphi_j^{-1})' \right)'(s) \phi(s, \omega) ds e^{i\omega\psi(p_0)} \right| \\ &\leq \frac{2^{\delta+1} L(\delta)}{\sqrt{3-4\delta}} (\tilde{p}_2 - \tilde{p}_1)^{\frac{3-4\delta}{2}} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{2}-\delta} \min_{[\tilde{p}_1, \tilde{p}_2]} \{-\psi''\}^{-\frac{3}{2}} \|U'\|_{L^2(\mathbb{R})} \omega^{-\delta} \\ &\quad + \frac{2^{\delta-1} L(\delta)}{1-\delta} (\tilde{p}_2 - \tilde{p}_1)^{2-2\delta} \|U\|_{L^\infty(\mathbb{R})} \\ &\quad \times \left(\|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{5}{2}-\delta} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{-\psi''\}^{-\frac{7}{2}} \right. \\ &\quad \left. + \frac{1}{3} \|\psi''\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{7}{2}-\delta} \|\psi^{(3)}\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \{-\psi''\}^{-\frac{9}{2}} \right) \omega^{-\delta} . \end{aligned}$$

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□

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We end this section by providing an explicit and uniform bound for oscillatory integrals of type (9) in the case where there is no stationary point inside the support of the amplitude, making the decay with respect to ω faster. As above, the estimate involves the L^2 -norm of the first derivative of the amplitude.

382

The proof of the following result lies on classical arguments (as those in [28, Chap. VIII, Sec. 1, Proposition 2]) combined with Cauchy-Schwarz inequality.

385

386

387

Theorem 3.4. *Let p_1, p_2, \tilde{p}_1 and \tilde{p}_2 be finite real numbers such that $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$. Suppose that $\psi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}) : \mathbb{R} \rightarrow \mathbb{R}$ is a concave function such that $|\psi'| > 0$ on $[p_1, p_2]$. And assume that $U \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ is a function satisfying*

$$\text{supp } U \subseteq [p_1, p_2] .$$

Then we have for all $\omega > 0$,

$$\left| \int_{\mathbb{R}} U(p) e^{i\omega\psi(p)} dp \right| \leq \left(C_7(\psi, p_1, p_2) \|U'\|_{L^2(\mathbb{R})} + C_8(\psi, p_1, p_2) \|U\|_{L^\infty(\mathbb{R})} \right) \omega^{-1} ,$$

where

- $C_7(\psi, p_1, p_2) := (p_2 - p_1)^{\frac{1}{2}} \min \left\{ |\psi'(p_1)|, |\psi'(p_2)| \right\}^{-1} ;$
- $C_8(\psi, p_1, p_2) := \min \left\{ |\psi'(p_1)|, |\psi'(p_2)| \right\}^{-1} .$

Proof. Let $\omega > 0$. Since ψ' is monotonic and has a constant sign on $[p_1, p_2]$, we have

$$\forall p \in [p_1, p_2] \quad |\psi'(p)| \geq \min \left\{ |\psi'(p_1)|, |\psi'(p_2)| \right\} =: m_{p_1, p_2}(\psi') > 0 .$$

Hence we are allowed to integrate by parts as follows:

$$\int_{\mathbb{R}} U(p) e^{i\omega\psi(p)} dp = \int_{p_1}^{p_2} U(p) e^{i\omega\psi(p)} dp = -i \int_{p_1}^{p_2} \left(\frac{U}{\psi'} \right)'(p) e^{i\omega\psi(p)} dp \omega^{-1} .$$

Moreover we have

$$\begin{aligned} & \left| -i \int_{p_1}^{p_2} \left(\frac{U}{\psi'} \right)'(p) e^{i\omega\psi(p)} dp \right| \\ & \leq \left| \int_{p_1}^{p_2} U'(p) \psi'(p)^{-1} e^{i\omega\psi(p)} dp \right| + \int_{p_1}^{p_2} \left| U(p) \psi''(p) \psi'(p)^{-2} \right| dp \\ & \leq \|U'\|_{L^2(\mathbb{R})} \|(\psi')^{-1}\|_{L^2(p_1, p_2)} + \|U\|_{L^\infty(\mathbb{R})} \int_{p_1}^{p_2} \left| \psi''(p) \psi'(p)^{-2} \right| dp ; \end{aligned}$$

as in the preceding proof, we have applied Cauchy-Schwarz inequality to the first integral. Now the hypotheses $\psi'' \leq 0$ and ψ' is monotonic with a

constant sign allow to carry the following computations out:

$$\begin{aligned} \int_{p_1}^{p_2} \left| \psi''(p) \psi'(p)^{-2} \right| dp &= \left| - \int_{p_1}^{p_2} \psi''(p) \psi'(p)^{-2} dp \right| \\ &= \left| \psi'(p_2)^{-1} - \psi'(p_1)^{-1} \right| \\ &\leq m_{p_1, p_2} (\psi')^{-1} . \end{aligned}$$

Furthermore, we have

$$\|(\psi')^{-1}\|_{L^2(p_1, p_2)} \leq m_{p_1, p_2} (\psi')^{-1} (p_2 - p_1)^{\frac{1}{2}} .$$

Consequently we obtain

$$\begin{aligned} \left| \int_{\mathbb{R}} U(p) e^{i\omega\psi(p)} dp \right| &\leq \left(m_{p_1, p_2} (\psi')^{-1} (p_2 - p_1)^{\frac{1}{2}} \|U'\|_{L^2(\mathbb{R})} \right. \\ &\quad \left. + m_{p_1, p_2} (\psi')^{-1} \|U\|_{L^\infty(\mathbb{R})} \right) \omega^{-1} . \end{aligned}$$

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□

389 4. Proofs of the main results

390 This section is devoted to the proofs of the results stated in Section 2.

391 4.1. Proof of Theorem 2.2

392 The proof of Theorem 2.2 is inspired by the one of [3, Theorem 5.2]: it
393 consists mainly in rewriting wisely the solution formula (6) as an oscillatory
394 integral with respect to time and in applying then a version of the stationary
395 phase method.

396 In the present paper, the expansions in cones with arbitrary origin are
397 obtained thanks to a space-time shift in the integral defining (6). And the
398 explicitness of the error bounds with respect to the origin of the cones is
399 achieved thanks to the new generic remainder estimates given in Theorems
400 3.3 and 3.4; this allows the application of Plancherel theorem in the present
401 setting.

402 *Proof of Theorem 2.2.* The cases $t > t_0$ and $t < t_0$ are distinguished for the
403 sake of readability.

404

Case 1: $t > t_0$

We rewrite the solution formula as an oscillatory integral by proceeding as follows¹

$$\begin{aligned}
u_f(t, x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}u_0(p) e^{-itf(p)+ixp} dp \\
&= \int_{\mathbb{R}} \frac{1}{2\pi} \mathcal{F}u_0(p) e^{-it_0f(p)+ix_0p} e^{i(t-t_0)\left(\frac{x-x_0}{t-t_0}p - f(p)\right)} dp \\
&= \int_{\mathbb{R}} \mathbf{U}_f(p, t_0, x_0) e^{i(t-t_0)\Psi_f(p, t, x, t_0, x_0)} dp \\
&=: I_f(t, x, u_0, t_0, x_0) .
\end{aligned}$$

We note that the amplitude

$$\mathbf{U}_f(p, t_0, x_0) := \frac{1}{2\pi} \mathcal{F}u_0(p) e^{-it_0f(p)+ix_0p} ,$$

which is actually the Fourier transform of $\frac{1}{2\pi}u_f(t_0, \cdot + x_0)$, is a \mathcal{C}^∞ -function (with respect to the variable p) whose support is included in $[p_1, p_2]$. The phase function

$$\Psi_f(p, t, x, t_0, x_0) := \frac{x - x_0}{t - t_0}p - f(p)$$

is a \mathcal{C}^∞ -function on \mathbb{R} which is strictly concave since we have supposed $f'' > 0$ in this section.

Now we remark that the existence of a stationary point for the phase inside the interval $\tilde{I} := (\tilde{p}_1, \tilde{p}_2)$ depends on the value of $\frac{x-x_0}{t-t_0}$: it exists and is unique if and only if $\frac{x-x_0}{t-t_0} \in f'(\tilde{I})$. In this case, the stationary point $p_0(t, x)$ is given by

$$p_0(t, x) = (f')^{-1}\left(\frac{x - x_0}{t - t_0}\right) .$$

405 Let us now distinguish two sub-cases to apply Theorem 3.3 and Theorem 3.4.

- *Case $\frac{x-x_0}{t-t_0} \in f'(\tilde{I})$.* In this case, the stationary point belongs to \tilde{I} . Hence we are allowed to apply Theorem 3.3 to the oscillatory integral

¹In [3, 9], the parameters t_0 and x_0 are implicitly equal to 0. Allowing these parameters to be arbitrary produces a space-time shift in the solution formula and permits to consider cones with arbitrary origin.

$I_f(t, x, u_0, t_0, x_0)$ with $\omega = t - t_0$:

$$\begin{aligned} & \left| I_f(t, x, u_0, t_0, x_0) \right. \\ & \quad \left. - \frac{1}{\sqrt{2\pi}} e^{-i\frac{\pi}{4}} e^{-itf(p_0(t,x))+ixp_0(t,x)} \frac{\mathcal{F}u_0(p_0(t,x))}{\sqrt{f''(p_0(t,x))}} (t-t_0)^{-\frac{1}{2}} \right| \\ & \leq \frac{1}{2\pi} \left(C_5(\Psi_f, \delta, \tilde{p}_1, \tilde{p}_2) \left\| \partial_p [\mathcal{F}u_0(\cdot) e^{-it_0f(\cdot)+ix_0\cdot}] \right\|_{L^2(\mathbb{R})} \right. \\ & \quad \left. + C_6(\Psi_f, \delta, \tilde{p}_1, \tilde{p}_2) \|\mathcal{F}u_0\|_{L^\infty(\mathbb{R})} \right) (t-t_0)^{-\delta}, \end{aligned}$$

with $\delta \in (\frac{1}{2}, \frac{3}{4})$ and the constants $C_5(\Psi_f, \delta, \tilde{p}_1, \tilde{p}_2)$, $C_6(\Psi_f, \delta, \tilde{p}_1, \tilde{p}_2) > 0$ are defined in Theorem 3.3. Since we have

$$\partial_p^2 \Psi_f(p, t, x, t_0, x_0) = -f''(p) \quad , \quad \partial_p^3 \Psi_f(p, t, x, t_0, x_0) = -f^{(3)}(p) \quad ,$$

and since the constants $C_5(\Psi_f, \delta, \tilde{p}_1, \tilde{p}_2)$ and $C_6(\Psi_f, \delta, \tilde{p}_1, \tilde{p}_2)$ depend only on the second and third derivatives (with respect to p) of the phase, we can claim that these constants depend on f rather than Ψ_f . Furthermore, Plancherel theorem and standard properties of the Fourier transform provide

$$\begin{aligned} & \left\| \partial_p [\mathcal{F}u_0(\cdot) e^{-itf(\cdot)+ix_0\cdot}] \right\|_{L^2(\mathbb{R})} \\ & = \sqrt{2\pi} \left\| x \mapsto x \mathcal{F}^{-1} [\mathcal{F}u_0(\cdot) e^{-it_0f(\cdot)+ix_0\cdot}](x) \right\|_{L^2(\mathbb{R})} \\ & = \sqrt{2\pi} \left\| x \mapsto x \mathcal{F}^{-1} [\mathcal{F}u_0(\cdot) e^{-it_0f(\cdot)}](x+x_0) \right\|_{L^2(\mathbb{R})} \\ & = \sqrt{2\pi} \left\| x \mapsto (x-x_0) \mathcal{F}^{-1} [\mathcal{F}u_0(\cdot) e^{-it_0f(\cdot)}](x) \right\|_{L^2(\mathbb{R})} \\ & = \sqrt{2\pi} \left\| x \mapsto (x-x_0) u_f(t_0, x) \right\|_{L^2(\mathbb{R})} . \end{aligned}$$

Hence we obtain finally

$$\begin{aligned} & \left| u_f(t, x) - \frac{1}{\sqrt{2\pi}} e^{-i\frac{\pi}{4}} e^{-itf(p_0(t,x))+ixp_0(t,x)} \frac{\mathcal{F}u_0(p_0(t, x))}{\sqrt{f''(p_0(t, x))}} (t - t_0)^{-\frac{1}{2}} \right| \\ & \leq \left(\frac{1}{\sqrt{2\pi}} C_5(-f, \delta, \tilde{p}_1, \tilde{p}_2) \|(\cdot - x_0) u_f(t_0, \cdot)\|_{L^2(\mathbb{R})} \right. \\ & \quad \left. + \frac{1}{2\pi} C_6(-f, \delta, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) (t - t_0)^{-\delta}, \end{aligned}$$

406 where we have used the classical estimate $\|\mathcal{F}u_0\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^1(\mathbb{R})}$.

- *Case $\frac{x-x_0}{t-t_0} \notin f'(\tilde{I})$.* As previously, we rewrite the solution formula as the oscillatory integral $I_f(t, x, u_0, t_0, x_0)$. Here the phase $\Psi_f(\cdot, t, x, t_0, x_0)$ has no stationary point inside the interval $\tilde{I} = (\tilde{p}_1, \tilde{p}_2)$ and one has

$$\forall p \in [p_1, p_2] \quad \left| \partial_p \Psi_f(p, t, x, t_0, x_0) \right| = \left| \frac{x - x_0}{t - t_0} - f'(p) \right| \geq m_{\tilde{I}}(f) > 0,$$

where $m_{\tilde{I}}(f) := \min \{f'(p_1) - f'(\tilde{p}_1), f'(\tilde{p}_2) - f'(p_2)\}$. Consequently we can apply Theorem 3.4 which provides

$$\begin{aligned} |u_f(t, x)| & \leq \left(\frac{1}{\sqrt{2\pi}} C_7(-f, \tilde{p}_1, \tilde{p}_2) \|(\cdot - x_0) u_f(t_0, \cdot)\|_{L^2(\mathbb{R})} \right. \\ & \quad \left. + \frac{1}{2\pi} C_8(-f, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) (t - t_0)^{-1}, \end{aligned}$$

407 where the constants $C_7(-f, \tilde{p}_1, \tilde{p}_2)$, $C_8(-f, \tilde{p}_1, \tilde{p}_2) > 0$ are defined in
408 Theorem 3.4.

Case 2: $t < t_0$

Here we have

$$\begin{aligned} u_f(t, x) & = \int_{\mathbb{R}} \frac{1}{2\pi} \mathcal{F}u_0(p) e^{-it_0f(p)+ix_0p} e^{i(t-t_0)\left(\frac{x-x_0}{t-t_0}p - f(p)\right)} dp \\ & = \int_{\mathbb{R}} \frac{1}{2\pi} \overline{\mathcal{F}u_0(p)} e^{-it_0f(p)+ix_0p} e^{i(t_0-t)\left(\frac{x-x_0}{t-t_0}p - f(p)\right)} dp \\ & = \int_{\mathbb{R}} \overline{\mathbf{U}_f(p, t_0, x_0)} e^{i(t_0-t)\Psi_f(p, t, x, t_0, x_0)} dp. \end{aligned}$$

Following the arguments and computations of the preceding case $t > t_0$, we obtain

$$\begin{aligned} & \left| u_f(t, x) - \frac{1}{\sqrt{2\pi}} e^{+i\frac{\pi}{4}} e^{-itf(p_0(t,x))+ixp_0(t,x)} \frac{\mathcal{F}u_0(p_0(t, x))}{\sqrt{f''(p_0(t, x))}} (t_0 - t)^{-\frac{1}{2}} \right| \\ & \leq \left(\frac{1}{\sqrt{2\pi}} C_5(-f, \delta, \tilde{p}_1, \tilde{p}_2) \|(\cdot - x_0) u_f(t_0, \cdot)\|_{L^2(\mathbb{R})} \right. \\ & \quad \left. + \frac{1}{2\pi} C_6(-f, \delta, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) (t_0 - t)^{-\delta}, \end{aligned}$$

for all $(t, x) \in (-\infty, t_0) \times \mathbb{R}$ such that $\frac{x-x_0}{t-t_0} \in f'(\tilde{I})$, and

$$\begin{aligned} |u_f(t, x)| & \leq \left(\frac{1}{\sqrt{2\pi}} C_7(-f, \tilde{p}_1, \tilde{p}_2) \|(\cdot - x_0) u_f(t_0, \cdot)\|_{L^2(\mathbb{R})} \right. \\ & \quad \left. + \frac{1}{2\pi} C_8(-f, \tilde{p}_1, \tilde{p}_2) \|u_0\|_{L^1(\mathbb{R})} \right) (t_0 - t)^{-1}, \end{aligned}$$

409 for all $(t, x) \in (-\infty, t_0) \times \mathbb{R}$ such that $\frac{x-x_0}{t-t_0} \notin f'(\tilde{I})$, leading to the desired
410 estimates. \square

411 4.2. Proof of Corollary 2.5

412 This subsection is devoted to the proof of Corollary 2.5. It is based on
413 a combination between Theorem 2.2 and the following Lemma 4.2.1. This
414 lemma gives the minimum of the function $(t_0, x_0) \mapsto \|(\cdot - x_0)u_f(t_0, \cdot)\|_{L^2(\mathbb{R})}^2$
415 which is involved in the error estimates in Theorem 2.2. This function is
416 actually the moment of order 2 of the function $x \mapsto |u_f(t_0, x + x_0)|^2$ for
417 fixed $(t_0, x_0) \in \mathbb{R}^2$.

Lemma 4.2.1. *Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then the function $g : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ defined by*

$$g(t_0, x_0) = \|(\cdot - x_0) u_f(t_0, \cdot)\|_{L^2(\mathbb{R})}^2$$

has a global minimum at $(t^*, x^*) \in \mathbb{R}^2$ with

- $t^* = \arg \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot))$;
- $x^* = \mathcal{M}_1(u_f(t^*, \cdot))$.

Proof. For fixed $t_0 \in \mathbb{R}$, differentiating twice the function $g(t_0, \cdot)$ with respect to its second argument shows that

$$\partial_{x_0}^2 g(t_0, x_0) = 2 > 0 .$$

Hence $g(t_0, \cdot)$ is a polynomial function of degree 2 whose unique global minimum is

$$\tilde{x}(t_0) = \int_{\mathbb{R}} x |u_f(t_0, x)|^2 dx = \mathcal{M}_1(u_f(t_0, \cdot)) .$$

It follows that

$$g(t_0, \tilde{x}(t_0)) = \int_{\mathbb{R}} \left(x - \mathcal{M}_1(u_f(t_0, \cdot)) \right)^2 |u_f(t_0, x)|^2 dx = \mathcal{V}(u_f(t_0, \cdot)) .$$

Lemma Appendix A.3 assures that $t_0 \in \mathbb{R} \mapsto \mathcal{V}(u_f(t_0, \cdot)) \in \mathbb{R}_+$ is a polynomial of degree 2 whose leading coefficient is $\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)$. Since we have by simple calculations,

$$\begin{aligned} \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right) &= \mathcal{M}_{f'^2}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right) - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)^2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(f'(p) - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right) \right)^2 |\mathcal{F}u_0(p)|^2 dp , \end{aligned}$$

and since f is supposed to be strictly convex in this paper, the leading coefficient $\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)$ is necessarily positive. Thus $t_0 \in \mathbb{R} \mapsto g(t_0, \tilde{x}(t_0)) \in \mathbb{R}_+$ has a global minimum at a certain $t^* \in \mathbb{R}$, *i.e.*,

$$t^* = \arg \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot)) .$$

Finally we define

$$x^* := \tilde{x}(t^*) = \mathcal{M}_1(u_f(t^*, \cdot)) .$$

418

□

Remark 4.2.2. The polynomial nature of $t_0 \in \mathbb{R} \mapsto g(t_0, \tilde{x}(t_0)) \in \mathbb{R}_+$ permits to derive the following formula for t^* :

$$\begin{aligned} t^* &= \frac{1}{\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)} \left(-\frac{1}{2\pi} \Im \left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp \right) \right. \\ &\quad \left. + \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right) \mathcal{M}_1(u_0) \right) . \end{aligned} \tag{14}$$

Furthermore, from Lemma Appendix A.1, we have

$$\mathcal{M}_1(u_f(t^*, \cdot)) = \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) t^* + \mathcal{M}_1(u_0) ;$$

inserting formula (14) into the preceding equality provides

$$\begin{aligned} x^* &= \frac{1}{\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right)} \left(-\frac{1}{2\pi} \Im \left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp \right) \right. \\ &\quad \left. \times \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) + \mathcal{M}_{f'^2}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) \mathcal{M}_1(u_0) \right) . \end{aligned}$$

419 We are now in position to prove Corollary 2.5.

Proof of Corollary 2.5. We apply Theorem 2.2 with $t_0 = t^*$ and $x_0 = x^*$, where t^* and x^* are defined in Lemma 4.2.1, to the solution formula (6). This provides the time-asymptotic expansions appearing in (7) and (8) together with the error bounds in which the following term appears:

$$\|(\cdot - x^*) u_f(t^*, \cdot)\|_{L^2(\mathbb{R})} = \sqrt{g(t^*, x^*)} ,$$

where $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined in Lemma 4.2.1. By the definitions of t^* and x^* , we obtain

$$g(t^*, x^*) = \int_{\mathbb{R}} (x - x^*)^2 |u_f(t^*, x)|^2 dx = \mathcal{V}(u_f(t^*, \cdot)) = \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot)) .$$

420 Further it is clear that minimising the two families of (t_0, x_0) -dependent error
421 bounds given in inequalities (7) and (8) is equivalent to minimising the func-
422 tion g . By Lemma 4.2.1, the parameter (t^*, x^*) minimises these two families,
423 which ends the proof. \square

424 4.3. Proof of Theorem 2.6

425 We prove Theorem 2.6 in this last subsection. This will be done in two
426 main steps: for arbitrary $(t_0, x_0) \in \mathbb{R}^2$, we first compute the mean position
427 of the term $H_f(t, \cdot, u_0, t_0, x_0)$ defined in the statement of Theorem 2.6 and
428 compare it to the mean position of the solution (6); a similar study for the
429 variances is carried out. Theorem 2.6 is finally a direct consequence of these
430 two studies.

431

432 In the first proposition, we compute the mean position of $H_f(t, \cdot, u_0, t_0, x_0)$
433 for all $t \neq t_0$.

Proposition 4.3.1. *Let $(t_0, x_0) \in \mathbb{R}^2$. Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then for all $t \in \mathbb{R} \setminus \{t_0\}$, we have*

$$\mathcal{M}_1\left(H_f(t, \cdot, u_0, t_0, x_0)\right) = x_0 + \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right)(t - t_0).$$

Proof. Let $t \in \mathbb{R} \setminus \{t_0\}$. First of all, we note that

$$x_0 + f'(p_1)(t - t_0) < x_0 + f'(p_2)(t - t_0) \iff t > t_0.$$

Hence using the definition of $H_f(t, x, u_0, t_0, x_0)$ given in Theorem 2.6, we have

$$\begin{aligned} & \int_{\mathbb{R}} x \left| H_f(t, x, u_0, t_0, x_0) \right|^2 dx \\ &= \frac{\operatorname{sgn}(t - t_0)}{2\pi} \int_{x_0 + f'(p_1)(t - t_0)}^{x_0 + f'(p_2)(t - t_0)} x \frac{|\mathcal{F}u_0(p_0(t, x))|^2}{f''(p_0(t, x))} dx |t - t_0|^{-1} \\ &= \frac{1}{2\pi} \int_{x_0 + f'(p_1)(t - t_0)}^{x_0 + f'(p_2)(t - t_0)} x \frac{|\mathcal{F}u_0(p_0(t, x))|^2}{f''(p_0(t, x))} dx (t - t_0)^{-1}. \end{aligned}$$

We make now the change of variable $x = x_0 + f'(p)(t - t_0)$ to obtain the desired result:

$$\begin{aligned} & \int_{\mathbb{R}} x \left| H_f(t, x, u_0, t_0, x_0) \right|^2 dx \\ &= \frac{1}{2\pi} \int_{p_1}^{p_2} (x_0 + f'(p)(t - t_0)) |\mathcal{F}u_0(p)|^2 dp \\ &= x_0 + \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) t - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) t_0; \end{aligned}$$

434 note that we have used the fact that $\frac{1}{2\pi} \|\mathcal{F}u_0\|_{L^2(p_1, p_2)}^2 = 1$, which is a direct
435 consequence of the assumption $\|u_0\|_{L^2(\mathbb{R})} = 1$. \square

436 In the following result, we prove that the mean positions of the solution
437 of equation (5) and of the term $H_f(t, \cdot, u_0, t_0, x_0)$ are equal if and only if x_0
438 is equal to the mean position of $u_f(t_0, \cdot)$.

Proposition 4.3.2. *Let $(t_0, x_0) \in \mathbb{R}^2$. Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then for all $t \in \mathbb{R} \setminus \{t_0\}$, we have*

$$\mathcal{M}_1(u_f(t, \cdot)) = \mathcal{M}_1(H_f(t, \cdot, u_0, t_0, x_0)) \iff x_0 = \mathcal{M}_1(u_f(t_0, \cdot)).$$

Proof. Let $t \in \mathbb{R} \setminus \{t_0\}$. According to Propositions 4.3.1 and Appendix A.1, we have

- $\mathcal{M}_1(u_f(t, \cdot)) = \mathcal{M}_1(u_0) + \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)t$;
- $\mathcal{M}_1(H_f(t, \cdot, u_0, t_0, x_0)) = x_0 + \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)(t - t_0)$.

Hence these two mean positions are equal if and only if

$$\mathcal{M}_1(u_0) = x_0 - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)t_0 ,$$

439 which is equivalent to $x_0 = \mathcal{M}_1(u_f(t_0, \cdot))$. □

440 **Remark 4.3.3.** The definitions of t^* and x^* from Corollary 2.5 and the
441 preceding proposition assure that the mean positions of $u_f(t, \cdot)$ and the term
442 $H_f(t, \cdot, u_0, t^*, x^*)$ are equal for all $t \neq t^*$.

443 In the two following results, we focus on the variances of the solution
444 $u_f(t, \cdot)$ and of the term $H_f(t, \cdot, u_0, t_0, x_0)$ for all $t \neq t_0$. We give firstly a
445 formula for the difference between the two variances for arbitrary (t_0, x_0) .

Proposition 4.3.4. *Let $(t_0, x_0) \in \mathbb{R}^2$. Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then for all $t \in \mathbb{R} \setminus \{t_0\}$, we have*

$$\begin{aligned} & \mathcal{V}(u_f(t, \cdot)) - \mathcal{V}(H_f(t, \cdot, u_0, t_0, x_0)) \\ &= 2\left(\frac{1}{2\pi} \Im\left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp\right) - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right) \mathcal{M}_1(u_0)\right. \\ & \quad \left. + \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)t_0\right)t + \mathcal{V}(u_0) - \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)t_0^2 . \end{aligned} \quad (15)$$

Proof. Let $t \in \mathbb{R} \setminus \{t_0\}$. Similarly to the arguments employed in the proof of

Proposition 4.3.1, we have

$$\begin{aligned}
& \mathcal{M}_2(H_f(t, \cdot, u_0, t_0, x_0)) \\
&= \frac{1}{2\pi} \int_{x_0+f'(p_1)(t-t_0)}^{x_0+f'(p_2)(t-t_0)} x^2 \frac{|\mathcal{F}u_0(p_0(t, x))|^2}{f''(p_0(t, x))} dx (t-t_0)^{-1} \\
&= \frac{1}{2\pi} \int_{p_1}^{p_2} (x_0 + f'(p)(t-t_0))^2 |\mathcal{F}u_0(p)|^2 dp \\
&= x_0^2 + \mathcal{M}_{f^2} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) (t-t_0)^2 + 2x_0 \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) (t-t_0).
\end{aligned}$$

Moreover, applying Proposition 4.3.1 gives

$$\begin{aligned}
& \mathcal{M}_1(H_f(t, \cdot, u_0, t_0, x_0))^2 \\
&= x_0^2 + \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right)^2 (t-t_0)^2 + 2x_0 \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) (t-t_0).
\end{aligned}$$

It follows:

$$\begin{aligned}
\mathcal{V}(H_f(t, \cdot, u_0, t_0, x_0)) &= \left(\mathcal{M}_{f^2} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) - \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right)^2 \right) (t-t_0)^2 \\
&= \mathcal{V}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) (t-t_0)^2.
\end{aligned}$$

Using the formula for $\mathcal{V}(u_f(t, \cdot))$ from Proposition Appendix A.3, we obtain finally

$$\begin{aligned}
& \mathcal{V}(u_f(t, \cdot)) - \mathcal{V}(H_f(t, \cdot, u_0, t_0, x_0)) \\
&= \mathcal{V}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) t^2 + 2 \left(\frac{1}{2\pi} \Im \left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp \right) \right. \\
&\quad \left. - \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) \mathcal{M}_1(u_0) \right) t + \mathcal{V}(u_0) \\
&\quad - \mathcal{V}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) (t-t_0)^2 \\
&= 2 \left(\frac{1}{2\pi} \Im \left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp \right) - \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) \mathcal{M}_1(u_0) \right. \\
&\quad \left. + \mathcal{V}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) t_0 \right) t + \mathcal{V}(u_0) - \mathcal{V}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) t_0^2.
\end{aligned}$$

447 In view of the preceding result, the difference between the variances of
448 $u_f(t, \cdot)$ and $H_f(t, \cdot, u_0, t_0, x_0)$ is an affine function with respect to t . Conse-
449 quently the unique way to make this difference constant is to choose t_0 in
450 such way that the leading coefficient is equal to 0. It turns out that the
451 unique t_0 satisfying this property is the one minimising the variance of the
452 solution (6), namely t^* introduced in Corollary 2.5.

Proposition 4.3.5. *Let $(t_0, x_0) \in \mathbb{R}^2$. Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then we have the following equivalence:*

$$\begin{aligned} \exists C \in \mathbb{R} \quad \forall t \in \mathbb{R} \setminus \{t_0\} \quad \mathcal{V}(u_f(t, \cdot)) - \mathcal{V}(H_f(t, \cdot, u_0, t_0, x_0)) &= C \\ \iff t_0 = t^* := \arg \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot)) . \end{aligned}$$

In particular, we have

$$C = \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot)) .$$

Proof. According to Proposition 4.3.4, the difference between the variances of $u_f(t, \cdot)$ and $H_f(t, \cdot, u_0, t_0, x_0)$ is constant if and only if

$$\begin{aligned} t_0 = \frac{1}{\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right)} &\left(-\frac{1}{2\pi} \Im \left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp \right) \right. \\ &\left. + \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) \mathcal{M}_1(u_0) \right) , \end{aligned}$$

453 which is equal to t^* according to Remark 4.2.2. By evaluating equality (15)
454 at t^* , we obtain for all $t \in \mathbb{R} \setminus \{t^*\}$,

$$\mathcal{V}(u_f(t, \cdot)) - \mathcal{V}(H_f(t, \cdot, u_0, t^*, x_0)) = \mathcal{V}(u_0) - \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) (t^*)^2 . \quad (16)$$

We note now that

$$\begin{aligned}
& \mathcal{V}(u_f(t^*, \cdot)) \\
&= \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) (t^*)^2 + 2\left(\frac{1}{2\pi} \Im\left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp\right)\right) \\
&\quad - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) \mathcal{M}_1(u_0) t^* + \mathcal{V}(u_0) \\
&= \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) (t^*)^2 - 2\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) (t^*)^2 + \mathcal{V}(u_0) \\
&= -\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) (t^*)^2 + \mathcal{V}(u_0) . \tag{17}
\end{aligned}$$

From equalities (16) and (17), it follows finally

$$\mathcal{V}(u_f(t, \cdot)) - \mathcal{V}(H_f(t, \cdot, u_0, t^*, x_0)) = \mathcal{V}(u_f(t^*, \cdot)) = \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot)) ,$$

455 the last equality being obtained by the definition $t^* = \arg \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot))$. \square

456 Theorem 2.6 can be now proved in a straightforward way.

Proof of Theorem 2.6. Let $t \in \mathbb{R} \setminus \{t^*\}$. According to Proposition 4.3.2 (and Remark 4.3.3), we have

$$\mathcal{M}_1(u_f(t, \cdot)) = \mathcal{M}_1(H_f(t, \cdot, u_0, t^*, x^*)) ,$$

and Proposition 4.3.5 assures that

$$\mathcal{V}(u_f(t, \cdot)) - \mathcal{V}(H_f(t, \cdot, u_0, t^*, x^*)) = \min_{\tau \in \mathbb{R}} \mathcal{V}(u_f(\tau, \cdot)) ,$$

457 which ends the proof. \square

458 **Appendix A. Mean position and variance of the free wave packet**

459 In this appendix, we give the formulas for the mean position and the
460 variance of the wave packet defined in (1). The proofs we propose here are
461 substantially based on the fact that the wave packet is defined via the Fourier
462 transform, permitting to apply some properties of this transform.

463

464 We begin with the formula for the mean position.

Proposition Appendix A.1. *Suppose that $u_0 \in \mathcal{S}(\mathbb{R})$. Then for all $t \in \mathbb{R}$, we have*

$$\mathcal{M}_1(u_f(t, \cdot)) = \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) t + \mathcal{M}_1(u_0) .$$

Proof. For $t \in \mathbb{R}$, we have

$$\begin{aligned} \int_{\mathbb{R}} x |u_f(t, x)|^2 dx &= \int_{\mathbb{R}} x u_f(t, x) \overline{u_f(t, x)} dx \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[x \mapsto x u_f(t, x)](p) \overline{\mathcal{F}[x \mapsto u_f(t, x)](p)} dp \\ &= \frac{i}{2\pi} \int_{\mathbb{R}} \partial_p \mathcal{F}[x \mapsto u_f(t, x)](p) \overline{\mathcal{F}[x \mapsto u_f(t, x)](p)} dp ; \end{aligned} \tag{A.1}$$

the second and third equalities have been obtained by applying Plancherel theorem and basic properties of the Fourier transform. Using now the formula (1), we obtain for all $p \in \mathbb{R}$,

- $\partial_p \mathcal{F}[x \mapsto u_f(t, x)](p) = e^{-itf(p)} \left(-itf'(p) \mathcal{F}u_0(p) + (\mathcal{F}u_0)'(p) \right) ;$
- $\overline{\mathcal{F}[x \mapsto u_f(t, x)](p)} = e^{itf(p)} \overline{\mathcal{F}u_0(p)} .$

By combing the two last equalities with (A.1) and by using again basic properties of the Fourier transform, it follows

$$\begin{aligned} \int_{\mathbb{R}} x |u_f(t, x)|^2 dx &= \frac{i}{2\pi} \int_{\mathbb{R}} \left(-itf'(p) \mathcal{F}u_0(p) + (\mathcal{F}u_0)'(p) \right) \overline{\mathcal{F}u_0(p)} dp \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} f'(p) |\mathcal{F}u_0(p)|^2 dp t + \frac{i}{2\pi} \int_{\mathbb{R}} (\mathcal{F}u_0)'(p) \overline{\mathcal{F}u_0(p)} dp \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} f'(p) |\mathcal{F}u_0(p)|^2 dp t + \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[x \mapsto x u_0(x)](p) \overline{\mathcal{F}u_0(p)} dp \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} f'(p) |\mathcal{F}u_0(p)|^2 dp t + \int_{\mathbb{R}} x |u_0(x)|^2 dx , \end{aligned}$$

465 leading finally to the desired equality. □

Remark Appendix A.2. The preceding formula is actually an extension of the well-known Ehrenfest theorem [22, Proposition 3.19] to the family of dispersive equations of type (5). We recall that Ehrenfest theorem in the setting of the free Schrödinger equation (3) gives the following formula for the mean position of a free particle:

$$\mathcal{M}_1(u_S(t, \cdot)) = \mathcal{M}_1\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) t + \mathcal{M}_1(u_0) .$$

466 The formula for the variance is provided in the following result.

Proposition Appendix A.3. *Suppose that $u_0 \in \mathcal{S}(\mathbb{R})$. Then for all $t \in \mathbb{R}$, we have*

$$\begin{aligned} \mathcal{V}(u_f(t, \cdot)) &= \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) t^2 + 2\left(\frac{1}{2\pi} \Im\left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp\right)\right) \\ &\quad - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0\right) \mathcal{M}_1(u_0) t + \mathcal{V}(u_0) . \end{aligned}$$

Proof. Following the computational arguments of the proof of Proposition Appendix A.1, we have for all $t \in \mathbb{R}$,

$$\begin{aligned} \int_{\mathbb{R}} x^2 |u_f(t, x)|^2 dx &= \int_{\mathbb{R}} |x u_f(t, x)|^2 dx \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \mathcal{F}[x \mapsto x u_f(t, x)](p) \right|^2 dp \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \partial_p \mathcal{F}[x \mapsto u_f(t, x)](p) \right|^2 dp \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| -it f'(p) \mathcal{F}u_0(p) + (\mathcal{F}u_0)'(p) \right|^2 dp . \quad (\text{A.2}) \end{aligned}$$

Inserting the following relation

$$\forall p \in \mathbb{R} \quad (\mathcal{F}u_0)'(p) = -i \mathcal{F}[x \mapsto x u_0(x)](p)$$

into (A.2) and expanding then the square of the absolute value provides

$$\begin{aligned}
\int_{\mathbb{R}} x^2 |u_f(t, x)|^2 dx &= \frac{1}{2\pi} \int_{\mathbb{R}} f'(p)^2 |\mathcal{F}u_0(p)|^2 dp t^2 + \int_{\mathbb{R}} x^2 |u_0(x)|^2 dx \\
&\quad - \frac{1}{\pi} \int_{\mathbb{R}} \Re\left(i f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)}\right) dp t \\
&= \mathcal{M}_{f'^2} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) t^2 + \mathcal{M}_2(u_0) \\
&\quad + \frac{1}{\pi} \Im \left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp \right) t.
\end{aligned}$$

Now by using Proposition Appendix A.1, we have

$$\begin{aligned}
&\mathcal{M}_1(u_f(t, \cdot))^2 \\
&= \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right)^2 t^2 + \mathcal{M}_1(u_0)^2 + 2 \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) \mathcal{M}_1(u_0) t,
\end{aligned}$$

which leads finally to

$$\begin{aligned}
&\mathcal{V}(u_f(t, \cdot)) \\
&= \mathcal{M}_2(u_f(t, \cdot)) - \mathcal{M}_1(u_f(t, \cdot))^2 \\
&= \mathcal{M}_{f'^2} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) t^2 + \mathcal{M}_2(u_0) \\
&\quad + \frac{1}{\pi} \Im \left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp \right) t \\
&\quad - \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right)^2 t^2 - \mathcal{M}_1(u_0)^2 - 2 \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) \mathcal{M}_1(u_0) t \\
&= \mathcal{V}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) t^2 + \mathcal{V}(u_0) \\
&\quad + 2 \left(\frac{1}{2\pi} \Im \left(\int_{\mathbb{R}} f'(p) \mathcal{F}u_0(p) \overline{(\mathcal{F}u_0)'(p)} dp \right) \right. \\
&\quad \left. - \mathcal{M}_{f'} \left(\frac{1}{\sqrt{2\pi}} \mathcal{F}u_0 \right) \mathcal{M}_1(u_0) \right) t.
\end{aligned}$$

468 **Appendix B. Acknowledgements**

469 The author gratefully thanks Prof. Felix Ali Mehmeti whose numerous
470 and insightful comments helped to improve the present paper. The author
471 is also grateful to the referee for valuable remarks which helped to improve
472 significantly the presentation of the paper.

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