Stability of propagation features under time-asymptotic approximations for a class of dispersive equations

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Abstract

We consider solutions of dispersive equations on the line defined by Fourier multipliers with initial data having compactly supported Fourier transforms. In this paper, a refinement of an existing method permitting to expand time-asymptotically the solution formulas is proposed. Here the first term of the expansion is supported in a space-time cone whose origin depends explicitly on the initial datum. As an important consequence of our refined method, the first term inherits the mean position of the solution together with a constant variance error and a shifted time-decay rate is obtained. Hence this refinement, which takes into account both spatial and frequency information of the initial datum, makes stable some propagation features under time-asymptotic approximations and permits a better description of the time-asymptotic behaviour of the solutions. The results are achieved firstly by making apparent the cone origin in the solution formula, secondly by applying precisely an adapted version of the stationary phase method with a new error bound, and finally by minimising the error bound with respect to the cone origin.

Keywords: wave packet, dispersive equation, oscillatory integral, stationary phase method, frequency band
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1. Introduction

In this paper, we are interested in the time-asymptotic behaviour of wave packets of the form

$$u_f(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}u_0(p)\, e^{-itf(p)+ixp}\, dp \,, \tag{1}$$

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where $t \in \mathbb{R}$, $x \in \mathbb{R}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly convex symbol. We suppose that the Fourier transform $\mathcal{F}u_0$ of $u_0 \in \mathcal{S}(\mathbb{R})$ is supported in a bounded interval $[p_1, p_2]$, where $p_1 < p_2$ are finite real numbers. In terms of evolution equations, wave packets of the form (1) are solutions of the following type of dispersive equations:

$$\begin{cases} \left[i\,\partial_t - f\big(D\big)\right] u_f(t) = 0 \\ u_f(0) = u_0 \end{cases}, \tag{2}$$

for $t \in \mathbb{R}$, where $f(D)$ is the Fourier multiplier associated with f and $u_0 \in \mathcal{S}(\mathbb{R})$ the initial datum supposed to be in the frequency band $[p_1, p_2]$. For instance, the solutions of the free Schrödinger equation, of the Klein-Gordon equation or of certain higher-order evolution equations can be described by wave packets of the form (1); we refer to [9, Sec. 6] for further details. In the present setting, the frequency band hypothesis prevents the wave packet (1) from being too much localised in space according to the uncertainty principle and makes hence challenging the task of describing its spatial propagation.

Some approaches solving this challenge time-asymptotically have been developed. In [4], the authors propose to approximate the solution of the Klein-Gordon equation on a star-shaped network by a spatially localised function, the latter tending to the true solution as the time tends to infinity. This has been achieved by applying precisely the version of the stationary phase method given in [23, Theorem 7.7.5] to an integral solution formula of the equation. The desired approximation is then given by the first term of the asymptotic expansion from the stationary phase method. The principle of the stationary phase method, which consists in evaluating the integrand of the oscillatory integral of interest at the stationary point of the phase function, combined with the bounded frequency band hypothesis leads to an approximation supported in a space-time cone: this cone describes both the motion and the dispersion of the solution for large times. In particular, the results exhibit in this setting the influence of the tunnel effect on the time-decay rate of the solution.

In [3], this approach has been adapted to the setting of the free Schrödinger equation on the line. In that paper, the initial data are assumed to have integrable singular frequencies in order to study the effect of such singularities on the time-asymptotic behaviour. The version of the stationary phase method proposed in [18, Sec. 2.9] (also proposed in [17]) has been used since it covers the case of singular amplitudes; we mention that the authors in [3] propose modern formulations and detailed proofs of the results from [18]. The results show that a free particle with a singular frequency tends to travel at the speed associated with this frequency. This is highlighted by the existence of space-time cones, containing the direction given by the singular frequency, in which the time-decay rates are below the rate of the classical decay inherited from the classical dispersion.

However, the expansion provided in [3] is proved to blow up when approaching the space-time direction associated with the singular frequency: this prevents the method from approximating uniformly the true solution in regions containing this direction. This is due to the fact that the first term of the expansion inherits the singularity of the initial datum; see [3, Sec. 3] for more details. To tackle this issue, another approach has been proposed in [9]: the precision of asymptotic expansions to one term is removed in favour of less precise but more flexible uniform estimates. In particular, they cover the above critical regions. Further this flexibility has permitted to consider not only the free Schrödinger equation but also equations of type (2) with initial data having singular frequencies. The uniform estimates for the solutions have been achieved by applying a generalisation [9, Theorem 4.8] of the classical van der Corput Lemma [28, Chap. VIII, Sec. 1, Proposition 2] to the case of singular and integrable amplitudes.

The approach developed in [4] and the subsequent adaptations appearing in [3, 9] permit then to describe both the motion and the dispersion of the solutions via the inclination of space-time cones (which depends actually on the frequency support of the initial datum). Nevertheless it is noteworthy that the origin of the space-time cones resulting from the above methods is always put at the space-time point (0, 0), whatever the localisation of the initial datum is: the approximations provided by the method do not have the right spatial positions. As a consequence this method, although asymptotically correct, leads to inaccurate approximations of the solutions on large but finite time intervals $[0, T]$ for initial data spatially far from the origin.

In view of this, we aim at proposing time-asymptotic approximations whose mean positions and variances are close (even the same) to those of the solutions of equation (2) for a better description of the spatial propagation. To this end, we proceed in four steps, each of them containing a new argument:

1. The first step, which is actually independent from the setting of dispersive equations, consists in establishing new uniform and explicit remainder estimates for asymptotic expansions of oscillatory integrals of the form
$$\forall\,\omega > 0 \qquad \int_{\mathbb{R}} U(p)\, e^{i\omega\psi(p)}\, dp\ ,$$
where the amplitude $U : \mathbb{R} \longrightarrow \mathbb{C}$ is compactly supported and the phase $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is concave. The new remainder estimate we establish involves the L^2-norm of the first derivative of the amplitude and not the L^∞-norm as in the original proofs [3, 18]. This is achieved by applying Cauchy-Schwarz inequality to the integral representation of the remainder term. We refer to Theorems 3.3 and 3.4 for the precise statements of these results.
2. As a second step, we introduce a space-time shift parametrised by a two-dimensional parameter (t_0, x_0) in the integral formula (1). Broadly speaking, this new argument modifies the initial datum which is then given by the solution at time t_0 spatially translated by x_0. Inspired by the main lines of the method developed in [4], we apply then carefully a stationary phase method to this shifted solution formula to obtain here a family of time-asymptotic expansions parametrised by (t_0, x_0). This parameter turns out to be the origin of the cone in which lies the support of the first term. This result is given in Theorem 2.2.
3. Here we use our new version of the stationary phase method described above to expand the formula (1). Since this version makes appear the L^2-norm of the derivative of $\mathcal{F}u_0$, it offers the possibility to apply Plancherel's theorem. This leads to an error bound depending explicitly on the spatial part of the initial datum and on (t_0, x_0). The third step consists then in computing explicitly the parameter (t^*, x^*) which minimises the (t_0, x_0)-dependent family of error bounds provided in Theorem 2.2. The expansion associated with this optimal parameter, given in Corollary 2.5, is then the one whose error bound is the smallest.
4. In the last step, we prove that the mean position of the first term associated with (t^*, x^*) is the same as the one of the solution (1) and that the difference between the two variances is constant; this is stated in Theorem 2.6. It is noteworthy that the proof of this theorem shows that the first term associated with (t^*, x^*) is actually the only one in the family of first terms we provide which satisfies these two properties.

Contrary to the original method developed in [4] and adapted in [3, 9], the present refined approach offers sufficient flexibility to provide time-asymptotic approximations for solutions of equations of type (2) which are more precise and which verify expected propagation properties.

Let us illustrate our main result in the case of the free Schrödinger equation on the line with initial datum $u_0 \in \mathcal{S}(\mathbb{R})$, namely

$$\begin{cases} i\,\partial_t u_S(t) = -\dfrac{1}{2}\,\partial_{xx} u_S(t) \\ u_S(0) = u_0 \end{cases}, \tag{3}$$

for $t \in \mathbb{R}$. We note that equation (3) is actually of the form (2) with symbol $f(p) = \frac{1}{2}\,p^2$ and its solution is given by

$$\forall\,(t,x) \in \mathbb{R}\times\mathbb{R} \qquad u_S(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}u_0(p)\,e^{-\frac{1}{2}itp^2+ixp}\,dp\,. \tag{4}$$

In terms of quantum mechanics, the solution of the free Schrödinger equation is the wave function associated with the free particle being in the state u_0 at the initial time. Further the frequency band hypothesis means that the particle has a momentum localised in the interval $[p_1, p_2]$. According to the physical principle of group velocity, the wave packet (4) will travel in space at different speeds between p_1 and p_2 over time. Hence a free wave packet in the frequency band $[p_1, p_2]$ is expected to be mainly spatially localised in an interval of the form $\big[p_1\,(t-t_0)+x_0, p_2\,(t-t_0)+x_0\big]$, where t_0 and x_0 have to be fixed, illustrating the propagation of the associated particle. For instance, a partial formalisation of this principle is given by Ehrenfest theorem [22, Proposition 3.19] which describes the evolution of the mean position of the particle (but not the dispersion).

In the following result, we apply Corollary 2.5 to the setting of the free Schrödinger equation (3) to provide a time-asymptotic expansion of the solution (4) with explicit error estimates. According to Theorem 2.6, the resulting approximation has the right mean position and its variance is equal to the solution variance plus an explicit constant. This is actually the consequence of the fact that the origin of the space-time cone, in which lies the support of the approximation, is actually put at the mean spatial position of the solution at the time when the variance of the solution is minimal. Hence the resulting approximation describes both position and dispersion of the true solution and the theorem provides a formalisation of the principle of group velocity which seems to be more precise than Ehrenfest theorem in the setting of a free particle.

Theorem 1.1. *Consider the free Schrödinger equation on the line* (3) *with initial datum* $u_0 \in \mathcal{S}(\mathbb{R})$*. Let* p_1*,* p_2*,* $\tilde{p}_1$ *and* $\tilde{p}_2$ *be finite real numbers such that* $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$*. Suppose* $\|u_0\|_{L^2(\mathbb{R})} = 1$ *and* $\operatorname{supp} \mathcal{F}u_0 \subseteq [p_1, p_2]$*, and define*

- $t^* = \operatorname*{arg\,min}_{\tau \in \mathbb{R}} \left(\int_{\mathbb{R}} x^2 \left| u_S(\tau, x) \right|^2 dx - \left(\int_{\mathbb{R}} x \left| u_S(\tau, x) \right|^2 dx \right)^2 \right)$;
- $x^* = \int_{\mathbb{R}} x \left| u_S(t^*, x) \right|^2 dx$.

Then for all $(t, x) \in \left\{ (t, x) \in \big(\mathbb{R} \backslash \{t^*\}\big) \times \mathbb{R} \;\middle|\; p_1 \leqslant \frac{x - x^*}{t - t^*} \leqslant p_2 \right\}$*, we have*

$$\left| u_S(t, x) - \frac{1}{\sqrt{2\pi}}\, e^{-sgn(t-t^*) i \frac{\pi}{4}}\, e^{-it\left(\frac{x-x^*}{t-t^*}\right)^2 + ix\frac{x-x^*}{t-t^*}}\, \mathcal{F}u_0\left(\frac{x - x^*}{t - t^*} \right) |t - t^*|^{-\frac{1}{2}} \right|$$
$$\leqslant C_1(\delta, \tilde{p}_1, \tilde{p}_2) \sqrt{\int_{\mathbb{R}} x^2 \left| u_S(t^*, x) \right|^2 dx - \left(\int_{\mathbb{R}} x \left| u_S(t^*, x) \right|^2 dx \right)^2}\, |t - t^*|^{-\delta} ,$$

where the real number δ *is arbitrarily chosen in* $\left(\frac{1}{2}, \frac{3}{4}\right)$*. And for all* $(t, x) \in \left\{ (t, x) \in \big(\mathbb{R} \backslash \{t^*\}\big) \times \mathbb{R} \;\middle|\; \frac{x - x^*}{t - t^*} < p_1 \text{ or } p_2 < \frac{x - x^*}{t - t^*} \right\}$*, we have*

$$\left| u_S(t, x) \right| \leqslant \Bigg(C_2(p_1, p_2, \tilde{p}_1, \tilde{p}_2) \sqrt{\int_{\mathbb{R}} x^2 \left| u_S(t^*, x) \right|^2 dx - \left(\int_{\mathbb{R}} x \left| u_S(t^*, x) \right|^2 dx \right)^2}$$
$$+ C_3(p_1, p_2, \tilde{p}_1, \tilde{p}_2) \left\| u_0 \right\|_{L^1(\mathbb{R})} \Bigg) |t - t^*|^{-1} .$$

All the above constants are defined in Theorem 2.2.

See Corollary 2.5 for the general result. Let us now make some comments on this result:

- On one hand, we observe that the first term is spatially well-localised for a solution in a narrow frequency band; on the other hand, the error is bounded by the minimal value of the standard deviation of the solution. Combined with the uncertainty principle, this exhibits a compromise: a frequency well-localised solution (4) can be approximated by a function supported in a narrow space-time cone but a time sufficiently far from t^* is required to achieve a good precision; on the other hand, the approximation of the solution (4) with a small minimal standard deviation lies in a larger cone but the error bound is smaller than in the preceding case.

- We remark that the time-decay rate is shifted by t^*, which is the time when the variance of the solution (4) is minimal; this corresponds to the fact that the origin of the cone belongs to the vertical space-time line $\left\{(t,x) \in \mathbb{R} \times \mathbb{R} \,\middle|\, t = t^*\right\}$. Hence if we require an error smaller than a certain threshold $\varepsilon > 0$, then this precision is achieved for all $t \in \mathbb{R}$ satisfying

$$
\begin{aligned}
|t - t^*| > C_1(\delta, \tilde{p}_1, \tilde{p}_2)^{\frac{1}{\delta}} \Bigg(\int_{\mathbb{R}} x^2 \left| u_S(t^*, x) \right|^2 dx \\
- \Big(\int_{\mathbb{R}} x \left| u_S(t^*, x) \right|^2 dx \Big)^2 \Bigg)^{\frac{1}{2\delta}} \varepsilon^{-\frac{1}{\delta}} \\
=: \eta(\varepsilon) \ .
\end{aligned}
$$

  In particular if we are interested in the evolution of the solution for positive times and if $t^* < -\eta(\varepsilon)$, then the error of the approximation is smaller than ε for all $t \geqslant 0$. This has to be compared with the results from the classical approach (as in [3, 4]) which always imply the existence of a small time-interval $[0, T]$ during which the error is larger than a given threshold. This is due to the lack of flexibility of the classical approach which enforces $t^* = 0$ (the decay rate is then $t^{-\frac{1}{2}}$) and puts automatically the origin of the cone at the origin of space-time.

Let us now comment on some possible improvements or applications of the present results. First of all, an interesting issue would be to apply the approach developed in this paper to more complicated settings. One may consider dispersive equations on certain networks where integral solution formulas are available, as for example the Schrödinger equation on a star-shaped network with infinite branches [1] or on a tadpole graph [2]. In both papers, the time-asymptotic behaviour is studied by a frequency band hypothesis and one may hope a better description of physical phenomena by using our refined method.

We could also consider the Schrödinger equation with a potential. In [10], the time-asymptotic behaviour of the two first terms of the Dyson-Phillips series [16, Chap. III, Theorem 1.10] representing the perturbed solution is studied by means of asymptotic expansions. The results concerning the second term of the series are interpreted as follows: if the initial state travels from left to right in space, then the positive frequencies of the potential tend to accelerate the motion of the second term while the negative frequencies tend to slow down or even reverse it, exhibiting advanced and retarded transmissions as well as reflections. The application of the present results could bring more information on these phenomena, in particular precise spatial information on the transmitted and reflected wave packets.

As explained in this paper, the notion of frequency band is physically meaningful and permits to describe time-asymptotically the propagation of solutions of certain dispersive equations in a precise way. However it is a restrictive hypothesis: for example, a function in a finite frequency band is necessarily a $\mathcal{C}^\infty$-function. Hence it would be relevant to extend this notion to functions whose Fourier transform is not necessarily compactly supported but still localised in a weaker sense. In this setting, the first term of the expansion is no longer supported in a space-time cone and so one has to quantify the localisation by means of different tools. For instance, we can consider approaches based on weighted norms; such norms have been used in [19], [20] or in [21] to show that the continuous part of the perturbed Schrödinger evolution transports away from the origin with non-zero velocity.

Our approach makes appear naturally the shifted decay rate $|t - t^*|^{-\frac{1}{2}}$, where t^* minimises the variance of the solution. It would be also interesting to introduce this time-shift in other existing results to obtain greater precision. For instance, one may consider the important $L^p - L^{p'}$ estimates for which a simple argument makes apparent the shifted decay; this is proved in the following result:

Proposition 1.2. *Consider the free Schrödinger equation on the line* (3) *with $u_0 \in \mathcal{S}(\mathbb{R})$ and define $t^* \in \mathbb{R}$ as follows:*

$$t^* := \underset{\tau\in\mathbb{R}}{\arg\min} \left(\int_{\mathbb{R}} x^2 \left|u_S(\tau,x)\right|^2 dx - \left(\int_{\mathbb{R}} x \left|u_S(\tau,x)\right|^2 dx \right)^2 \right).$$

Then for all $p \in [2, \infty]$ *and for all* $t \in \mathbb{R}\backslash\{t^*\}$*, we have*

$$\left\| u_S(t,.) \right\|_{L^p(\mathbb{R})} \leqslant \left(\frac{1}{4\pi} \right)^{-\frac{1}{2}+\frac{1}{p}} \left\| u_S(t^*,.) \right\|_{L^{p'}(\mathbb{R})} |t - t^*|^{-\frac{1}{2}+\frac{1}{p}} \ ,$$

where p' *is the conjugate of* p*.*

Proof. For the sake of clarity, we use the one-parameter group $\left(e^{-it\partial_{xx}}\right)_{t\in\mathbb{R}}$ which permits to describe the Schrödinger evolution as follows:

$$\forall\, t \in \mathbb{R} \qquad u_S(t) = e^{-it\partial_{xx}} u_0 \ .$$

Using the group property, we have for any $t \in \mathbb{R}$,

$$e^{-it\partial_{xx}} u_0 = e^{-i(t-t^*)\partial_{xx}} \, e^{-it^*\partial_{xx}} u_0 \ ,$$

and by applying the classical $L^p - L^{p'}$ estimate [6, Proposition 2.2.3] to the above right-hand side, we obtain for all $t \neq t^*$,

$$\begin{aligned} \left\| u_S(t,.) \right\|_{L^p(\mathbb{R})} &= \left\| e^{-it\partial_{xx}} u_0 \right\|_{L^p(\mathbb{R})} \\ &\leqslant \left(\frac{1}{4\pi} \right)^{-\frac{1}{2}+\frac{1}{p}} \left\| e^{-it^*\partial_{xx}} u_0 \right\|_{L^{p'}(\mathbb{R})} |t - t^*|^{-\frac{1}{2}+\frac{1}{p}} \\ &= \left(\frac{1}{4\pi} \right)^{-\frac{1}{2}+\frac{1}{p}} \left\| u_S(t^*,.) \right\|_{L^{p'}(\mathbb{R})} |t - t^*|^{-\frac{1}{2}+\frac{1}{p}} \ . \end{aligned}$$

Note that we are allowed to apply the classical $L^p - L^{p'}$ estimate since $e^{-it^*\partial_{xx}} u_0 \in \mathcal{S}(\mathbb{R}) \subset L^{p'}(\mathbb{R})$ thanks to the hypothesis $u_0 \in \mathcal{S}(\mathbb{R})$. □

Since the classical $L^p - L^{p'}$ estimates are exploited to establish Strichartz estimates which are themselves used to study non-linear dispersive phenomena, it is necessary to extend the above shifted $L^p - L^{p'}$ estimates to spaces larger than the Schwartz space in view of precise applications. In particular, one may examine whether t^* defined above still satisfies some optimal conditions; this could be linked with the results established in [7].

Regarding long-term perspectives of our work, one could consider the full soliton resolution for non-linear dispersive equations [11, 12, 13, 14, 15], which aims at classifying the asymptotic behaviour of the non-linear solutions. A key argument for the results contained in this series of papers is the channel energy method [26], which consists in estimating the associated free solution outside a space-time cone or channel; this estimate is then used to prove that a dispersive term appearing in the decomposition of the non-linear solution tends to 0 in the energy-space. In particular, we mention that the authors of [8] have to shift in time the cones and channels to derive the desired estimates. Hence one might hope that the ideas proposed in the present paper could help to understand the requirement for this shift and more generally to refine the channel energy method.

Finally we could also think about minimal escape velocities [25, 27] which aim at exhibiting propagation features for evolution operators of type e^{-itH}, where H is a general Hamiltonian; for instance, one may consider the operator $H = -\frac{1}{2}\,\partial_{xx} + V$ where V is real-valued potential. As explained in [24], the method to establish these estimates generalises the integration by parts which is actually crucial to describe the time-asymptotic behaviour of wave packets, as illustrated in the present paper. Our approach could bring more precision to the abstract setting and hence lead to estimates containing more information on the propagation of general wave packets.

The paper is organised as follows: in the following section, we state the main results on time-asymptotic expansions for dispersive equations of the form (2). Since the proofs of the main results are substantially based on a careful application of the stationary phase method, Section 3 is devoted to our new version of this method. Finally Section 4 contains the proofs of each result presented in Section 2.

2. Main results on time-asymptotic approximations for dispersive equations

In this section, we present the main results of our paper. Prior to this, we introduce the mathematical setting as well as some notations.

The Fourier transform $\mathcal{F}u : \mathbb{R} \longrightarrow \mathbb{C}$ of a function $u : \mathbb{R} \longrightarrow \mathbb{C}$ belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$ is defined by

$$\forall\, p \in \mathbb{R} \qquad \mathcal{F}u(p) := \int_{\mathbb{R}} u(x)\, e^{-ixp}\, dx\ .$$

The Fourier transform defines an invertible operator from $\mathcal{S}(\mathbb{R})$ onto itself, and can be extended to the space of square-integrable functions $L^2(\mathbb{R})$ and to the tempered distributions $\mathcal{S}'(\mathbb{R})$. Moreover, for $u \in L^2(\mathbb{R})$, Plancherel theorem assures the following equality:

$$\forall\, u \in L^2(\mathbb{R}) \qquad \|u\|_{L^2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}} \big\|\mathcal{F}u\big\|_{L^2(\mathbb{R})} \;;$$

see [23, Theorem 7.1.6].

Consider now a $\mathcal{C}^\infty$-function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that all its derivatives grow at most as a polynomial at infinity and consider the associated operator $f(D) : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ defined by

$$\forall\, x \in \mathbb{R} \qquad f(D)u(x) := \frac{1}{2\pi} \int_{\mathbb{R}} f(p)\, \mathcal{F}u(p)\, e^{ixp}\, dp = \mathcal{F}^{-1}\Big(f\, \mathcal{F}u\Big)(x) \;,$$

which can be extended to the tempered distributions $\mathcal{S}'(\mathbb{R})$. The operator $f(D) : \mathcal{S}'(\mathbb{R}) \longrightarrow \mathcal{S}'(\mathbb{R})$ is called a *Fourier multiplier* associated with the *symbol* f.

Given such an operator, we introduce the following evolution equation on the line,

$$\begin{cases} \big[i\, \partial_t - f\big(D\big)\big]\, u_f(t) = 0 \\ u_f(0) = u_0 \end{cases} \;, \tag{5}$$

for $t \in \mathbb{R}$. If we suppose $u_0 \in \mathcal{S}'(\mathbb{R})$ then the equation (5) has a unique solution in $\mathcal{C}^1\big(\mathbb{R}, \mathcal{S}'(\mathbb{R})\big)$ given by the following solution formula,

$$u_f(t) = \mathcal{F}^{-1}\Big(e^{-itf}\, \mathcal{F}u_0\Big) \;.$$

We refer to [5] for a detailed study of this family of equations. In this paper, we suppose that the symbol f is strictly convex; an important example of such an equation is given by the free Schrödinger equation whose symbol is $f_S(p) = \frac{1}{2}\, p^2$.

For the sake of better presentation of the results, we consider initial data u_0 belonging only to the Schwartz space $\mathcal{S}(\mathbb{R})$ to focus on the approach we propose. We mention that it is possible to extend our results to the case of initial data in $L^2(\mathbb{R})$ with additional assumptions on regularity and decay; but this falls out of the scope of the paper.

Further the initial data are assumed to be in bounded frequency bands, meaning that their Fourier transforms are supported on bounded intervals $[p_1, p_2]$, where $p_1 < p_2$ are finite real numbers. Under such hypotheses, the solution formula for the equation (5) defines a function $u_f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$ given by

$$u_f(t,x) = \frac{1}{2\pi} \int_{p_1}^{p_2} \mathcal{F}u_0(p)\, e^{-itf(p)+ixp}\, dp\ . \tag{6}$$

We define now the space-time cone related to the symbol f and to the frequency band $[\tilde{p}_1, \tilde{p}_2]$ with origin $(t_0, x_0) \in \mathbb{R}^2$.

Definition 2.1. *Let t_0, x_0, $\tilde{p}_1$ and $\tilde{p}_2$ be finite real numbers such that $\tilde{p}_1 < \tilde{p}_2$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a symbol.*

1. *We define the space-time cone $\mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t_0, x_0)\big)$ as follows:*

$$\mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t_0, x_0)\big) := \left\{ (t,x) \in \big(\mathbb{R}\backslash\{t_0\}\big) \times \mathbb{R} \;\middle|\; f'(\tilde{p}_1) \leqslant \frac{x - x_0}{t - t_0} \leqslant f'(\tilde{p}_2) \right\}\ .$$

2. *Let $\mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t_0, x_0)\big)^c$ be the complement of the space-time cone $\mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t_0, x_0)\big)$ in $\big(\mathbb{R}\backslash\{t_0\}\big) \times \mathbb{R}$.*

We present now our first main result. Theorem 2.2 provides a time-asymptotic expansion to one term with explicit error estimate of the solution (6) in the space-time cone $\mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t_0, x_0)\big)$, where $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$ and $(t_0, x_0) \in \mathbb{R}^2$ is arbitrarily chosen. A uniform estimate of the solution (6) outside the cone is also established. The result shows that the solution tends to be time-asymptotically localised in the cone $\mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t_0, x_0)\big)$.

Theorem 2.2. *Let p_1, p_2, $\tilde{p}_1$ and $\tilde{p}_2$ be finite real numbers such that $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$. Suppose that $u_0 \in \mathcal{S}(\mathbb{R})$ is a function whose Fourier transform satisfies*

$$supp\, \mathcal{F}u_0 \subseteq [p_1, p_2]\ ,$$

and fix $(t_0, x_0) \in \mathbb{R}^2$. Then

1. *for all $(t,x) \in \mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t_0, x_0)\big)$, we have*

$$\left| u_f(t,x) - \frac{1}{\sqrt{2\pi}}\, e^{-sgn(t-t_0) i \frac{\pi}{4}}\, e^{-itf(p_0(t,x)) + ixp_0(t,x)}\, \frac{\mathcal{F}u_0\big(p_0(t,x)\big)}{\sqrt{f''\big(p_0(t,x)\big)}}\, |t - t_0|^{-\frac{1}{2}} \right|$$
$$\leqslant \Big(C_1(f, \delta, \tilde{p}_1, \tilde{p}_2)\, \big\| (. - x_0)\, u_f(t_0, .) \big\|_{L^2(\mathbb{R})} + C_2(f, \delta, \tilde{p}_1, \tilde{p}_2)\, \big\| u_0 \big\|_{L^1(\mathbb{R})} \Big)\, |t - t_0|^{-\delta}\ , \tag{7}$$

where the real number δ is arbitrarily chosen in $\left(\frac{1}{2}, \frac{3}{4}\right)$ and

- $p_0(t,x) := (f')^{-1}\left(\dfrac{x-x_0}{t-t_0}\right)$;
- $C_1(f,\delta,\tilde{p}_1,\tilde{p}_2) := \dfrac{2^{\delta+\frac{1}{2}}\, L(\delta)}{\sqrt{\pi}\sqrt{3-4\delta}} \left(\tilde{p}_2 - \tilde{p}_1\right)^{\frac{3-4\delta}{2}} c_1(f,\delta,\tilde{p}_1,\tilde{p}_2)$;
- $C_2(f,\delta,\tilde{p}_1,\tilde{p}_2) := \dfrac{2^{\delta-2} L(\delta)}{\pi(1-\delta)} \left(\tilde{p}_2 - \tilde{p}_1\right)^{2-2\delta} c_2(f,\delta,\tilde{p}_1,\tilde{p}_2)$;
- $c_1(f,\delta,\tilde{p}_1,\tilde{p}_2) := \left\|f''\right\|^{\frac{3}{2}-\delta}_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \min\limits_{[\tilde{p}_1,\tilde{p}_2]} \left\{f''\right\}^{-\frac{3}{2}}$;
- $$c_2(f,\delta,\tilde{p}_1,\tilde{p}_2) := \left\|f''\right\|^{\frac{5}{2}-\delta}_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \left\|f^{(3)}\right\|_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \min_{[\tilde{p}_1,\tilde{p}_2]} \left\{f''\right\}^{-\frac{7}{2}} \\ + \frac{1}{3} \left\|f''\right\|^{\frac{7}{2}-\delta}_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \left\|f^{(3)}\right\|_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \min_{[\tilde{p}_1,\tilde{p}_2]} \left\{f''\right\}^{-\frac{9}{2}} .$$

The constant $L(\delta) > 0$ is defined in (13)*;*

2. *for all $(t,x) \in \mathfrak{C}_f\big([\tilde{p}_1,\tilde{p}_2],(t_0,x_0)\big)^c$, we have*

$$\left|u_f(t,x)\right| \leqslant \Big(C_3(f,p_1,p_2,\tilde{p}_1,\tilde{p}_2) \left\|(.-x_0)\, u_f(t_0,.)\right\|_{L^2(\mathbb{R})} \\ + C_4(f,p_1,p_2,\tilde{p}_1,\tilde{p}_2) \left\|u_0\right\|_{L^1(\mathbb{R})}\Big) \left|t-t_0\right|^{-1} , \qquad (8)$$

where

- $C_3(f,p_1,p_2,\tilde{p}_1,\tilde{p}_2)$
  $:= \dfrac{1}{\sqrt{2\pi}} (p_2-p_1)^{\frac{1}{2}} \min\left\{f'(p_1) - f'(\tilde{p}_1), f'(\tilde{p}_2) - f'(p_2)\right\}^{-1}$;
- $C_4(f,p_1,p_2,\tilde{p}_1,\tilde{p}_2) := \dfrac{1}{2\pi} \min\left\{f'(p_1) - f'(\tilde{p}_1), f'(\tilde{p}_2) - f'(p_2)\right\}^{-1}$.

To state the two other main results, we define the following moment-type and variance-type quantities for normalized $u \in L^2(\mathbb{R})$.

Definition 2.3. *Let $u \in L^2(\mathbb{R})$ such that $\|u\|_{L^2(\mathbb{R})} = 1$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a symbol. If they exist, we define the real numbers $\mathcal{M}_f(u)$ and $\mathcal{V}_f(u)$ as follows,*

$$\mathcal{M}_f(u) := \int_{\mathbb{R}} f(x) \left|u(x)\right|^2 dx \quad , \quad \mathcal{V}_f(u) := \mathcal{M}_{f^2}(u) - \mathcal{M}_f(u)^2 .$$

If $f(x)=x$, then we note for simplicity

$$\mathcal{M}_1(u) := \mathcal{M}_f(u) \quad , \quad \mathcal{M}_2(u) := \mathcal{M}_{f^2}(u) \quad , \quad \mathcal{V}(u) := \mathcal{V}_f(u) \ .$$

Remark 2.4. The above quantities $\mathcal{M}_1(u)$, $\mathcal{M}_2(u)$ and $\mathcal{V}(u)$ are respectively the mean, the second moment and the variance of $|u|^2$.

The explicitness of the family of error bounds given in Theorem 2.2 allows to compute the parameter (t^*, x^*) giving the smallest bound; this computation is carried out in Lemma 4.2.1. In the following result, we provide the values of t^* and x^* and we provide the time-asymptotic expansion for the solution of equation (5) associated with this optimal parameter.

Corollary 2.5. *Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then the (t_0, x_0)-dependent right-hand sides of inequalities (7) and (8) in Theorem 2.2 have a global minimum at the point $\big(t^*, x^*\big) \in \mathbb{R}^2$ with*

- $t^* = \arg\min\limits_{\tau \in \mathbb{R}} \mathcal{V}\big(u_f(\tau, .)\big)$;
- $x^* = \mathcal{M}_1\big(u_f(t^*, .)\big)$.

In this case, for all $(t,x) \in \mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t^*, x^*)\big)$, we have

$$\left| u_f(t,x) - \frac{1}{\sqrt{2\pi}}\, e^{-sgn(t-t^*) i \frac{\pi}{4}}\, e^{-itf(p_*(t,x)) + ixp_*(t,x)}\, \frac{\mathcal{F}u_0\big(p_*(t,x)\big)}{\sqrt{f''\big(p_*(t,x)\big)}}\, |t-t^*|^{-\frac{1}{2}} \right|$$
$$\leqslant \Bigg(C_1(f, \delta, \tilde{p}_1, \tilde{p}_2) \min_{\tau \in \mathbb{R}} \left(\sqrt{\mathcal{V}\big(u_f(\tau, .)\big)} \right) + C_2(f, \delta, \tilde{p}_1, \tilde{p}_2) \left\| u_0 \right\|_{L^1(\mathbb{R})} \Bigg) |t-t^*|^{-\delta} \ ,$$

where the real number δ is arbitrarily chosen in $\left(\frac{1}{2}, \frac{3}{4}\right)$ and

$$p_*(t,x) := (f')^{-1} \left(\frac{x-x^*}{t-t^*} \right) \ .$$

And for all $(t,x) \in \mathfrak{C}_f\big([\tilde{p}_1, \tilde{p}_2], (t^*, x^*)\big)^c$, we have

$$\big|u_f(t,x)\big| \leqslant \Bigg(C_3(f, p_1, p_2, \tilde{p}_1, \tilde{p}_2) \min_{\tau \in \mathbb{R}} \left(\sqrt{\mathcal{V}\big(u_f(\tau, .)\big)} \right) + C_4(f, p_1, p_2, \tilde{p}_1, \tilde{p}_2) \left\| u_0 \right\|_{L^1(\mathbb{R})} \Bigg) |t-t^*|^{-1} \ .$$

All the constants are defined in Theorem 2.2.

In the last result of this section, we consider mean position and variance of the time-asymptotic approximation of the solution (6) associated with the optimal parameter (t^*, x^*). Theorem 2.6 claims that the mean position of this approximation is equal to the one of the true solution and the difference between the two variances is constant; the value for this constant is provided.

Theorem 2.6. *Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that* $\|u_0\|_{L^2(\mathbb{R})} = 1$*. For all* $t \in \big(\mathbb{R}\backslash\{t^*\}\big)$ *and* $x \in \mathbb{R}$*, we define*

$$H_f(t, x, u_0, t^*, x^*) \\ := \frac{1}{\sqrt{2\pi}}\, e^{-sgn(t-t^*) i \frac{\pi}{4}}\, e^{-itf(p_*(t,x)) + ixp_*(t,x)}\, \frac{\mathcal{F}u_0\big(p_*(t,x)\big)}{\sqrt{f''\big(p_*(t,x)\big)}}\, |t - t^*|^{-\frac{1}{2}} \ ,$$

where t^* *and* x^* *are introduced in Corollary 2.5 and* $p_*(t,x) := (f')^{-1}\big(\frac{x-x^*}{t-t^*}\big)$*. Then for all* $t \in \mathbb{R}\backslash\{t^*\}$*, we have*

$$\left\{ \begin{array}{l} \mathcal{M}_1\big(u_f(t,.)\big) = \mathcal{M}_1\big(H_f(t,.,u_0,t^*,x^*)\big) \\ \mathcal{V}\big(u_f(t,.)\big) - \mathcal{V}\big(H_f(t,.,u_0,t^*,x^*)\big) = \min\limits_{\tau\in\mathbb{R}} \mathcal{V}\big(u_f(\tau,.)\big) \end{array} \right. .$$

3. Explicit error estimates for a stationary phase method via Cauchy-Schwarz inequality

The proof of Theorem 2.2 is substantially based on a careful application of a new version of the stationary phase method for oscillatory integrals of the form

$$\forall\, \omega > 0 \qquad \int_{\mathbb{R}} U(p)\, e^{i\omega\psi(p)}\, dp \ , \tag{9}$$

where the amplitude $U : \mathbb{R} \longrightarrow \mathbb{C}$ is a continuously differentiable function supported on a bounded interval and the phase $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly concave $\mathcal{C}^3$-function having a unique stationary point p_0. The remainder estimates we provide here are explicit, uniform with respect to p_0 and involve the L^2-norm of the first derivative of the amplitude. This plays actually a key role in the proof of Theorem 2.2.

This section is fully devoted to this new version since it is generic and independent from the setting of Section 2. The asymptotic expansions together with the uniform and explicit remainder estimates are established in

Theorems 3.3 and 3.4.

We start by stating two technical lemmas which will be substantially used in the proof of Theorem 3.3.

The first step to expand ω-asymptotically integrals of type (9) consists in making simpler the phase function in order to integrate then by parts. To do so, we use the diffeomorphisms φ_j ($j = 1, 2$) defined and studied in the following lemma. The values of these diffeomorphisms at the stationary point p_0 are provided in order to compute explicitly the first term of the expansions and two inequalities for φ_j are established to bound the remainders of these expansions.

The proof of the following result lies mainly on an integral representation of φ_j.

Lemma 3.1. *Let p_0, $\tilde{p}_1$ and $\tilde{p}_2$ be finite real numbers such that $p_0 \in (\tilde{p}_1, \tilde{p}_2)$. Suppose that $\psi \in \mathcal{C}^3(\mathbb{R}, \mathbb{R})$ is a strictly concave function which has a unique stationary point at p_0. Then, for $j = 1, 2$, the function*

$$\begin{array}{rcrcl} \varphi_j & : & I_j & \longrightarrow & [0, s_j] \\ & & p & \longmapsto & \big(\psi(p_0) - \psi(p)\big)^{\frac{1}{2}} \end{array}$$

where $I_1 := [\tilde{p}_1, p_0]$, $I_2 := [p_0, \tilde{p}_2]$ and $s_j := \varphi_j(\tilde{p}_j)$, satisfies the following properties:

1. *the function φ_j is a $\mathcal{C}^2$-diffeomorphism between I_j and $[0, s_j]$;*
2. *we have*
$$\varphi_j'(p_0) = (-1)^j \sqrt{-\frac{\psi''(p_0)}{2}} \ ;$$
3. *for all $p \in I_j$, the absolute value of $\varphi_j'(p)$ is lower bounded as follows:*
$$\left|\varphi_j'(p)\right| \geqslant \frac{1}{\sqrt{2}} \min_{[\tilde{p}_1, \tilde{p}_2]} \big\{ -\psi'' \big\} \left\|\psi''\right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{-\frac{1}{2}} \ ;$$
4. *we have the following L^∞-norm estimate for $\big(\varphi_j^{-1}\big)''$:*
$$\begin{aligned} \left\|\big(\varphi_j^{-1}\big)''\right\|_{L^\infty(0, s_j)} \leqslant & \left\|\psi''\right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{2}} \left\|\psi^{(3)}\right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \big\{ -\psi'' \big\}^{-\frac{7}{2}} \\ & + \frac{1}{3} \left\|\psi''\right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{5}{2}} \left\|\psi^{(3)}\right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \big\{ -\psi'' \big\}^{-\frac{9}{2}} . \end{aligned}$$

Proof. Let $j \in \{1,2\}$ and fix $p_0 \in (\tilde{p}_1, \tilde{p}_2)$. The proof of the present lemma is mainly based on the following integral representation of the function φ_j:

$$\varphi_j(p) = (-1)^j \,(p - p_0) \left(\int_0^1 \int_0^1 \widetilde{\psi}(p,\tau,\nu)\, d\nu d\tau \right)^{\frac{1}{2}} , \tag{10}$$

for all $p \in I_j$, where

$$\widetilde{\psi}(p,\tau,\nu) := -\psi''\big((1-\tau)(1-\nu)p + (\nu - \nu\tau + \tau)p_0\big)\,(1-\tau) \;.$$

This representation can be derived by noting firstly that

$$\psi(p_0) - \psi(p) = -\int_p^{p_0} \psi'(p_0) - \psi'(t)\, dt = \int_p^{p_0} \int_t^{p_0} -\psi''(v)\, dv\, dt \;,$$

where we have used the hypothesis $\psi'(p_0) = 0$ to obtain the first equality. Then we make the change of variable $(\nu,\tau) = \left(\frac{v-t}{p_0-t}, \frac{t-p}{p_0-p}\right)$, leading to

$$\psi(p_0) - \psi(p) = (p - p_0)^2 \int_0^1 \int_0^1 \widetilde{\psi}(p,\tau,\nu)\, d\nu d\tau \;,$$

and we take finally the square root of the preceding equality to obtain the desired representation (10).

1. Since ψ is a strictly concave function on $\mathbb{R}$, the function φ_j is actually the square root of the non-negative $\mathcal{C}^3$-function $p \longmapsto \psi(p_0) - \psi(p)$, showing that φ_j is twice continuously differentiable on $I_j \backslash \{p_0\}$ (φ_j is actually a $\mathcal{C}^3$-function on this domain). Let us prove that it is also twice differentiable on the whole I_j. To do so, note that we have for $p \in I_j \backslash \{p_0\}$,

$$\begin{aligned}
\varphi_j'(p) &= -\frac{1}{2}\, \psi'(p) \left(\psi(p_0) - \psi(p)\right)^{-\frac{1}{2}} \\
&= -\frac{1}{2} \left(\int_p^{p_0} -\psi''(q)\, dq \right) \varphi_j(p)^{-1} \\
&= \frac{1}{2} \left((p - p_0) \int_0^1 -\psi''\big((1-t)p + tp_0\big)\, dt \right) \varphi_j(p)^{-1} \\
&= \frac{(-1)^j}{2} \left(\int_0^1 -\psi''\big((1-t)p + tp_0\big)\, dt \right) \\
&\qquad \times \left(\int_0^1 \int_0^1 \widetilde{\psi}(p,\tau,\nu)\, d\nu d\tau \right)^{-\frac{1}{2}} .
\end{aligned} \tag{11}$$

The preceding equality combined with the positivity of the $\mathcal{C}^1$-function $-\psi''$ shows that φ_j' is continuously differentiable on I_j whose derivative is given by

$$\begin{aligned}\varphi_j''(p) = \frac{(-1)^j}{2} &\left(\int_0^1 -\psi^{(3)}\big((p-p_0)t+p_0\big)\,(1-t)\,dt \right) \\ &\times \left(\int_0^1 \int_0^1 \widetilde{\psi}(p,\tau,\nu)\,d\nu d\tau \right)^{-\frac{1}{2}} \\ &+ \frac{(-1)^j}{2} \left(\int_0^1 -\psi''\big((1-t)p+tp_0\big)\,dt \right) \\ &\times \left(-\frac{1}{2}\right) \frac{\int_0^1\int_0^1 \check{\psi}(p,\tau,\nu)\,d\nu d\tau}{\left(\int_0^1\int_0^1 \widetilde{\psi}(p,\tau,\nu)\,d\nu d\tau\right)^{\frac{3}{2}}}\,,\end{aligned}$$

for all $p \in I_j$, where

$$\check{\psi}(p,\tau,\nu) := -\psi^{(3)}\big((1-\tau)(1-\nu)p + (\nu-\nu\tau+\tau)p_0\big)\,(1-\tau)^2\,(1-\nu)\,.$$

Now, according to equality (11), we observe that φ_j' is negative for $j=1$ and positive for $j=2$ since $-\psi''>0$. By the inverse function theorem, we deduce that φ_j is a $\mathcal{C}^2$-diffeomorphism.

2. Thanks to the integral representation (10), we have

$$\begin{aligned}\varphi_j'(p_0) &= \lim_{p\to p_0} \frac{\varphi_j(p)-\varphi_j(p_0)}{p-p_0} \\ &= (-1)^j \lim_{p\to p_0} \left(\int_0^1 \int_0^1 \widetilde{\psi}(p,\tau,\nu)\,d\nu d\tau \right)^{\frac{1}{2}} \\ &= (-1)^j \sqrt{-\frac{\psi''(p_0)}{2}}\,.\end{aligned}$$

3. From equality (11) (which holds actually for all $p\in I_j$), we deduce the following lower estimate for φ_j':

$$\forall\, p \in I_j \qquad \Big|\varphi_j'(p)\Big| \geqslant \frac{1}{\sqrt{2}} \min_{[\tilde{p}_1,\tilde{p}_2]} \big\{-\psi''\big\}\, \big\|\psi''\big\|_{L^\infty(\tilde{p}_1,\tilde{p}_2)}^{-\frac{1}{2}}\,. \tag{12}$$

4. From the expression of φ_j'' computed above, we obtain the following

upper estimate:

$$\left|\varphi_j''(p)\right| \leqslant \frac{1}{2\sqrt{2}} \left\|\psi^{(3)}\right\|_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \min_{[\tilde{p}_1,\tilde{p}_2]} \left\{-\psi''\right\}^{-\frac{1}{2}} + \frac{1}{6\sqrt{2}} \left\|\psi''\right\|_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \left\|\psi^{(3)}\right\|_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \min_{[\tilde{p}_1,\tilde{p}_2]} \left\{-\psi''\right\}^{-\frac{3}{2}} ,$$

for all $p \in I_j$. By combining the preceding inequality with estimate (12) and the following relation,

$$\forall\, s \in [0, s_j] \qquad \left(\varphi_j^{-1}\right)''(s) = -\frac{\varphi_j''\left(\varphi_j^{-1}(s)\right)}{\varphi_j'\left(\varphi_j^{-1}(s)\right)^3} ,$$

we obtain finally for all $s \in [0, s_j]$,

$$\left|\left(\varphi_j^{-1}\right)''(s)\right| \leqslant \left\|\psi''\right\|^{\frac{3}{2}}_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \left\|\psi^{(3)}\right\|_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \min_{[\tilde{p}_1,\tilde{p}_2]} \left\{-\psi''\right\}^{-\frac{7}{2}} + \frac{1}{3} \left\|\psi''\right\|^{\frac{5}{2}}_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \left\|\psi^{(3)}\right\|_{L^\infty(\tilde{p}_1,\tilde{p}_2)} \min_{[\tilde{p}_1,\tilde{p}_2]} \left\{-\psi''\right\}^{-\frac{9}{2}} .$$

□

After having applied the above diffeomorphism to the integral (9) (previously split at p_0), the phase becomes the quadratic function $s \longmapsto -s^2$. In order to make an integration by parts, creating then the first and the remainder terms of the integral, one needs an expression for a primitive of the function $s \in [0, s_0] \longmapsto e^{-i\omega s^2} \in \mathbb{C}$, for fixed s_0, $\omega > 0$. In the following lemma, a useful integral representation of such a primitive is given. As in the preceding result, its value at the origin and an inequality are also provided to compute respectively the first term of the expansion and an upper bound for the remainder term.
To prove Lemma 3.2, we refer to the paper [3] which gives actually the successive primitives of more general functions by using essentially complex analysis; see [3, Theorems 6.4, 6.5 and Corollary 6.6].

Lemma 3.2. *Let $\omega, s_0 > 0$ be real numbers and let $\phi(.,\omega) : [0, s_0] \longrightarrow \mathbb{C}$ be the function defined by*

$$\phi(s,\omega) := -\int_{\Lambda(s)} e^{-i\omega z^2}\, dz ,$$

where $\Lambda(s)$ is the half-line in the complex plane given by

$$\Lambda(s) := \left\{ s + t\, e^{-i\frac{\pi}{4}} \in \mathbb{C} \,\middle|\, t \geqslant 0 \right\} .$$

Then

1. *the function $\phi(.,\omega)$ is a primitive of $s \in [0, s_0] \longmapsto e^{-i\omega s^2} \in \mathbb{C}$;*
2. *we have*
$$\phi(0,\omega) = -\frac{1}{2}\sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, \omega^{-\frac{1}{2}} \ ;$$
3. *the function $\phi(.,\omega)$ satisfies*
$$\forall\, s \in (0, s_0] \qquad \big|\phi(s,\omega)\big| \leqslant L(\delta)\, s^{1-2\delta}\, \omega^{-\delta} \ ,$$
where the real number δ is arbitrarily chosen in $\left(\frac{1}{2}, 1\right)$ and the constant $L(\delta) > 0$ is defined by
$$L(\delta) := \frac{\sqrt{\pi}}{2} \left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}} \right)^{2\delta - 1} . \tag{13}$$

Proof. The function $\phi(.,\omega)$ of the present paper corresponds actually to the function $\phi_1^{(2)}(.,\omega,2,1)$ defined in [3, Theorem 2.3]. Hence we apply the results established in [3] to the present situation:

1. One proves this first point by applying [3, Corollary 6.6], which is a consequence of Theorems 6.4 and 6.5 of [3], in the case $n = 1$, $j = 2$, $\rho_j = 2$ and $\mu_j = 1$.
2. The proof of this point lies only on basic computations which are carried out in the fourth step of the proof of [3, Theorem 2.3].
3. The combination of Lemmas 2.4 and 2.6 of [3] assures this last point.

□

Thanks to the two preceding lemmas, we are now in position to establish the desired asymptotic expansions with respect to the parameter ω for oscillatory integrals of type (9). In the following theorem, we are interested in the case where the stationary point p_0 of the phase belongs to a neighbourhood of the support of the amplitude. We emphasise that the remainder estimate we provide is different from those appearing in the original paper [18] and in [3].

Technically speaking, we split the integral at the stationary point p_0 and we study separately the two resulting integrals. In each situation, the method consists firstly in using the diffeomorphism introduced in Lemma 3.1 to make the phase function simpler, secondly in integrating by parts to create the expansion by using Lemma 3.1 point 2, Lemma 3.2 point 1 and point 2, and finally in bounding the remainder term by combining Lemma 3.1 point 3, point 4 and Lemma 3.2 point 3 with Cauchy-Schwarz inequality.

Theorem 3.3. *Let p_1, p_2, $\tilde{p}_1$ and $\tilde{p}_2$ be finite real numbers such that $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$. Suppose that $\psi \in \mathcal{C}^3(\mathbb{R}, \mathbb{R}) : \mathbb{R} \longrightarrow \mathbb{R}$ is a strictly concave function which has a unique stationary point at $p_0 \in (\tilde{p}_1, \tilde{p}_2)$. And assume that $U \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ is a function satisfying*

$$\operatorname{supp} U \subseteq [p_1, p_2] \, .$$

Then we have for all $\omega > 0$,

$$\left| \int_{\mathbb{R}} U(p)\, e^{i\omega\psi(p)}\, dp - \sqrt{2\pi}\, e^{-i\frac{\pi}{4}}\, e^{i\omega\psi(p_0)} \frac{U(p_0)}{\sqrt{-\psi''(p_0)}}\, \omega^{-\frac{1}{2}} \right| \leqslant \Big(C_5(\psi, \delta, \tilde{p}_1, \tilde{p}_2) \left\| U' \right\|_{L^2(\mathbb{R})} + C_6(\psi, \delta, \tilde{p}_1, \tilde{p}_2) \left\| U \right\|_{L^\infty(\mathbb{R})} \Big)\, \omega^{-\delta} \, ,$$

where the real number δ is arbitrarily chosen in $\left(\frac{1}{2}, \frac{3}{4}\right)$ and

- $C_5(\psi, \delta, \tilde{p}_1, \tilde{p}_2) := \dfrac{2^{\delta+1} L(\delta)}{\sqrt{3-4\delta}} \big(\tilde{p}_2 - \tilde{p}_1\big)^{\frac{3-4\delta}{2}}\, c_5(\psi, \delta, \tilde{p}_1, \tilde{p}_2)$;
- $C_6(\psi, \delta, \tilde{p}_1, \tilde{p}_2) := \dfrac{2^{\delta-1} L(\delta)}{1-\delta} \big(\tilde{p}_2 - \tilde{p}_1\big)^{2-2\delta}\, c_6(\psi, \delta, \tilde{p}_1, \tilde{p}_2)$;
- $c_5(\psi, \delta, \tilde{p}_1, \tilde{p}_2) := \left\| \psi'' \right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{3}{2}-\delta} \min\limits_{[\tilde{p}_1, \tilde{p}_2]} \big\{ -\psi'' \big\}^{-\frac{3}{2}}$;
- $$c_6(\psi, \delta, \tilde{p}_1, \tilde{p}_2) := \left\| \psi'' \right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{5}{2}-\delta} \left\| \psi^{(3)} \right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \big\{ -\psi'' \big\}^{-\frac{7}{2}} + \frac{1}{3} \left\| \psi'' \right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)}^{\frac{7}{2}-\delta} \left\| \psi^{(3)} \right\|_{L^\infty(\tilde{p}_1, \tilde{p}_2)} \min_{[\tilde{p}_1, \tilde{p}_2]} \big\{ -\psi'' \big\}^{-\frac{9}{2}} .$$

The constant $L(\delta) > 0$ is defined in (13).

Proof. Let $\omega > 0$ and choose $p_0 \in (\tilde{p}_1, \tilde{p}_2)$. First of all, since the support of the amplitude is included in $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$, we have clearly

$$\int_{\mathbb{R}} U(p)\, e^{i\omega\psi(p)}\, dp \, = \int_{\tilde{p}_1}^{\tilde{p}_2} U(p)\, e^{i\omega\psi(p)}\, dp \, =: I(\omega) \, .$$

Splitting the above integral at the point p_0 and using the two $\mathcal{C}^2$-diffeomorphisms defined in Lemma 3.1, we obtain

$$\begin{aligned} I(\omega) = & -\int_0^{s_1} \big(U \circ \varphi_1^{-1}\big)(p)\, \big(\varphi_1^{-1}\big)'(p)\, e^{-i\omega s^2}\, ds\, e^{i\omega\psi(p_0)} \\ & + \int_0^{s_2} \big(U \circ \varphi_2^{-1}\big)(p)\, \big(\varphi_2^{-1}\big)'(p)\, e^{-i\omega s^2}\, ds\, e^{i\omega\psi(p_0)}\ ; \end{aligned}$$

note that we have used the fact that φ_1 and φ_2 are respectively decreasing and increasing. We integrate now by parts by using the primitive $s \longmapsto \phi(s,\omega)$ given in Lemma 3.2 and the regularity of φ_j:

$$\begin{aligned} (-1)^j & \int_0^{s_j} \big(U \circ \varphi_j^{-1}\big)(s)\, \big(\varphi_j^{-1}\big)'(s)\, e^{-i\omega s^2}\, ds \\ & = (-1)^j \Big[\big(U \circ \varphi_j^{-1}\big)(s)\, \big(\varphi_j^{-1}\big)'(s)\, \phi(s,\omega)\Big]_0^{s_j} \\ & \qquad + (-1)^{j+1} \int_0^{s_j} \Big(\big(U \circ \varphi_j^{-1}\big)\, \big(\varphi_j^{-1}\big)'\Big)'(s)\, \phi(s,\omega)\, ds \\ & = (-1)^{j+1} \big(U \circ \varphi_j^{-1}\big)(0)\, \big(\varphi_j^{-1}\big)'(0)\, \phi(0,\omega) \\ & \qquad + (-1)^{j+1} \int_0^{s_j} \Big(\big(U \circ \varphi_j^{-1}\big)\, \big(\varphi_j^{-1}\big)'\Big)'(s)\, \phi(s,\omega)\, ds \\ & = \frac{1}{2}\sqrt{2\pi}\, e^{-i\frac{\pi}{4}}\, \frac{U(p_0)}{\sqrt{-\psi''(p_0)}}\, \omega^{-\frac{1}{2}} \\ & \qquad + (-1)^{j+1} \int_0^{s_j} \Big(\big(U \circ \varphi_j^{-1}\big)\, \big(\varphi_j^{-1}\big)'\Big)'(s)\, \phi(s,\omega)\, ds\ ; \end{aligned}$$

the second equality has been obtained by using the fact that $U(\tilde{p}_j) = 0$ and the last one by applying Lemma 3.1 point 2 and Lemma 3.2 point 2. Hence it follows

$$\begin{aligned} I(\omega) = & \sqrt{2\pi}\, e^{-i\frac{\pi}{4}}\, e^{i\omega\psi(p_0)}\, \frac{U(p_0)}{\sqrt{-\psi''(p_0)}}\, \omega^{-\frac{1}{2}} \\ & + \sum_{j=1}^{2} (-1)^{j+1} \int_0^{s_j} \Big(\big(U \circ \varphi_j^{-1}\big)\, \big(\varphi_j^{-1}\big)'\Big)'(s)\, \phi(s,\omega)\, ds\, e^{i\omega\psi(p_0)}\ . \end{aligned}$$

To estimate each term of the remainder, we proceed as follows:

$$\begin{aligned}
&\left|(-1)^{j+1}\int_0^{s_j}\Big(\big(U\circ\varphi_j^{-1}\big)\,\big(\varphi_j^{-1}\big)'\Big)'(s)\,\phi(s,\omega)\,ds\right| \\
&\qquad\leqslant \left|\int_0^{s_j}\big(U'\circ\varphi_j^{-1}\big)(s)\,\Big(\big(\varphi_j^{-1}\big)'(s)\Big)^2\,\phi(s,\omega)\,ds\right| \\
&\qquad\qquad + \left|\int_0^{s_j}\big(U\circ\varphi_j^{-1}\big)(s)\,\big(\varphi_j^{-1}\big)''(s)\,\phi(s,\omega)\,ds\right| \\
&\qquad\leqslant \left(\int_0^{s_j}\left|\big(U'\circ\varphi_j^{-1}\big)(s)\,\Big(\big(\varphi_j^{-1}\big)'(s)\Big)^2\right|^2 ds\right)^{\frac12}\left(\int_0^{s_j}\big|\phi(s,\omega)\big|^2\,ds\right)^{\frac12} \\
&\qquad\qquad + \int_0^{s_j}\big|\phi(s,\omega)\big|\,ds\,\big\|U\big\|_{L^\infty(\mathbb{R})}\,\Big\|\big(\varphi_j^{-1}\big)''\Big\|_{L^\infty(0,s_j)}\ ;
\end{aligned}$$

let us remark that we have applied Cauchy-Schwarz inequality to the first integral. We continue the proof by estimating each resulting term; first of all, by making the change of variable $p = \varphi_j^{-1}(s)$ and by using Lemma 3.1 point 3, we obtain

$$\begin{aligned}
&\left(\int_0^{s_j}\left|\big(U'\circ\varphi_j^{-1}\big)(s)\,\Big(\big(\varphi_j^{-1}\big)'(s)\Big)^2\right|^2 ds\right)^{\frac12} \\
&\qquad\qquad \leqslant 2^{\frac34}\,\big\|\psi''\big\|_{L^\infty(\tilde p_1,\tilde p_2)}^{\frac34}\,\min_{[\tilde p_1,\tilde p_2]}\big\{-\psi''\big\}^{-\frac32}\,\big\|U'\big\|_{L^2(\mathbb{R})}\ .
\end{aligned}$$

Then we use the point 3 of Lemma 3.2 to derive the two following inequalities:

- $\displaystyle\int_0^{s_j}\big|\phi(s,\omega)\big|\,ds \leqslant L(\delta)\int_0^{s_j}s^{1-2\delta}\,ds\,\omega^{-\delta} \leqslant \frac{L(\delta)}{2-2\delta}\,\varphi_j(\tilde p_j)^{2-2\delta}\,\omega^{-\delta}\ ;$
- $\displaystyle\left(\int_0^{s_j}\big|\phi(s,\omega)\big|^2\,ds\right)^{\frac12} \leqslant \frac{L(\delta)}{\sqrt{3-4\delta}}\,\varphi_j(\tilde p_j)^{\frac{3-4\delta}{2}}\,\omega^{-\delta}\ .$

By using the integral representation (10) of φ_j, we obtain

$$\varphi_j(\tilde p_j) \leqslant \frac{1}{\sqrt2}\,\big\|\psi''\big\|_{L^\infty(\tilde p_1,\tilde p_2)}^{\frac12}\,(\tilde p_2-\tilde p_1)\ ,$$

which permits to deduce

- $\displaystyle \int_0^{s_j} \big|\phi(s,\omega)\big|\, ds \leqslant \frac{1}{2^{1-\delta}}\,\frac{L(\delta)}{2-2\delta}\big(\tilde p_2 - \tilde p_1\big)^{2-2\delta}\,\big\|\psi''\big\|^{1-\delta}_{L^\infty(\tilde p_1,\tilde p_2)}\,\omega^{-\delta}\ ;$
- $\displaystyle \left(\int_0^{s_j} \big|\phi(s,\omega)\big|^2\, ds\right)^{\frac12} \leqslant \frac{1}{2^{\frac34-\delta}}\,\frac{L(\delta)}{\sqrt{3-4\delta}}\,\big(\tilde p_2 - \tilde p_1\big)^{\frac{3-4\delta}{2}}\,\big\|\psi''\big\|^{\frac34-\delta}_{L^\infty(\tilde p_1,\tilde p_2)}\,\omega^{-\delta}\ .$

And, from Lemma 3.1 point 4, we recall that

$$\begin{aligned}\Big\|\big(\varphi_j^{-1}\big)''\Big\|_{L^\infty(0,s_j)} \leqslant{}& \big\|\psi''\big\|^{\frac32}_{L^\infty(\tilde p_1,\tilde p_2)}\,\big\|\psi^{(3)}\big\|_{L^\infty(\tilde p_1,\tilde p_2)}\,\min_{[\tilde p_1,\tilde p_2]}\big\{-\psi''\big\}^{-\frac72}\\ &+\frac13\,\big\|\psi''\big\|^{\frac52}_{L^\infty(\tilde p_1,\tilde p_2)}\,\big\|\psi^{(3)}\big\|_{L^\infty(\tilde p_1,\tilde p_2)}\,\min_{[\tilde p_1,\tilde p_2]}\big\{-\psi''\big\}^{-\frac92}\ .\end{aligned}$$

Putting everything together provides the desired estimate, namely,

$$\begin{aligned}&\left|I(\omega) - \sqrt{2\pi}\,e^{-i\frac{\pi}{4}}\,e^{i\omega\psi(p_0)}\,\frac{U(p_0)}{\sqrt{-\psi''(p_0)}}\,\omega^{-\frac12}\right|\\ &\quad\leqslant \sum_{j=1}^{2}\left|(-1)^{j+1}\int_0^{s_j}\Big(\big(U\circ\varphi_j^{-1}\big)\,\big(\varphi_j^{-1}\big)'\Big)'(s)\,\phi(s,\omega)\,ds\,e^{i\omega\psi(p_0)}\right|\\ &\quad\leqslant \frac{2^{\delta+1}\,L(\delta)}{\sqrt{3-4\delta}}\,\big(\tilde p_2-\tilde p_1\big)^{\frac{3-4\delta}{2}}\,\big\|\psi''\big\|^{\frac32-\delta}_{L^\infty(\tilde p_1,\tilde p_2)}\,\min_{[\tilde p_1,\tilde p_2]}\big\{-\psi''\big\}^{-\frac32}\,\big\|U'\big\|_{L^2(\mathbb{R})}\,\omega^{-\delta}\\ &\qquad+\frac{2^{\delta-1}L(\delta)}{1-\delta}\,\big(\tilde p_2-\tilde p_1\big)^{2-2\delta}\,\|U\|_{L^\infty(\mathbb{R})}\\ &\qquad\quad\times\Bigg(\big\|\psi''\big\|^{\frac52-\delta}_{L^\infty(\tilde p_1,\tilde p_2)}\,\big\|\psi^{(3)}\big\|_{L^\infty(\tilde p_1,\tilde p_2)}\,\min_{[\tilde p_1,\tilde p_2]}\big\{-\psi''\big\}^{-\frac72}\\ &\qquad\qquad+\frac13\,\big\|\psi''\big\|^{\frac72-\delta}_{L^\infty(\tilde p_1,\tilde p_2)}\,\big\|\psi^{(3)}\big\|_{L^\infty(\tilde p_1,\tilde p_2)}\,\min_{[\tilde p_1,\tilde p_2]}\big\{-\psi''\big\}^{-\frac92}\Bigg)\,\omega^{-\delta}\ .\end{aligned}$$

□

We end this section by providing an explicit and uniform bound for oscillatory integrals of type (9) in the case where there is no stationary point inside the support of the amplitude, making the decay with respect to ω faster. As above, the estimate involves the L^2-norm of the first derivative of the amplitude.
The proof of the following result lies on classical arguments (as those in [28, Chap. VIII, Sec. 1, Proposition 2]) combined with Cauchy-Schwarz inequality.

Theorem 3.4. *Let p_1, p_2, $\tilde{p}_1$ and $\tilde{p}_2$ be finite real numbers such that $[p_1, p_2] \subset (\tilde{p}_1, \tilde{p}_2)$. Suppose that $\psi \in \mathcal{C}^2(\mathbb{R}, \mathbb{R}) : \mathbb{R} \longrightarrow \mathbb{R}$ is a concave function such that $|\psi'| > 0$ on $[p_1, p_2]$. And assume that $U \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ is a function satisfying*

$$\mathit{supp}\, U \subseteq [p_1, p_2] \, .$$

Then we have for all $\omega > 0$,

$$\left| \int_{\mathbb{R}} U(p)\, e^{i\omega\psi(p)}\, dp \right| \leqslant \Big(C_7(\psi, p_1, p_2) \left\| U' \right\|_{L^2(\mathbb{R})} + C_8(\psi, p_1, p_2) \left\| U \right\|_{L^\infty(\mathbb{R})} \Big) \, \omega^{-1} \, ,$$

where

- $C_7(\psi, p_1, p_2) := (p_2 - p_1)^{\frac{1}{2}} \, \min \Big\{ \big|\psi'(p_1)\big|, \big|\psi'(p_2)\big| \Big\}^{-1}$;
- $C_8(\psi, p_1, p_2) := \min \Big\{ \big|\psi'(p_1)\big|, \big|\psi'(p_2)\big| \Big\}^{-1}$.

Proof. Let $\omega > 0$. Since ψ' is monotonic and has a constant sign on $[p_1, p_2]$, we have

$$\forall\, p \in [p_1, p_2] \qquad \big|\psi'(p)\big| \geqslant \min \Big\{ \big|\psi'(p_1)\big|, \big|\psi'(p_2)\big| \Big\} =: m_{p_1,p_2}(\psi') > 0 \, .$$

Hence we are allowed to integrate by parts as follows:

$$\int_{\mathbb{R}} U(p)\, e^{i\omega\psi(p)}\, dp = \int_{p_1}^{p_2} U(p)\, e^{i\omega\psi(p)}\, dp = -i \int_{p_1}^{p_2} \left(\frac{U}{\psi'} \right)'(p)\, e^{i\omega\psi(p)}\, dp\, \omega^{-1} \, .$$

Moreover we have

$$\begin{aligned}
&\left| -i \int_{p_1}^{p_2} \left(\frac{U}{\psi'} \right)'(p)\, e^{i\omega\psi(p)}\, dp \right| \\
&\qquad \leqslant \left| \int_{p_1}^{p_2} U'(p)\, \psi'(p)^{-1}\, e^{i\omega\psi(p)}\, dp \right| + \int_{p_1}^{p_2} \Big| U(p)\, \psi''(p)\, \psi'(p)^{-2} \Big|\, dp \\
&\qquad \leqslant \left\| U' \right\|_{L^2(\mathbb{R})} \left\| (\psi')^{-1} \right\|_{L^2(p_1,p_2)} + \left\| U \right\|_{L^\infty(\mathbb{R})} \int_{p_1}^{p_2} \Big| \psi''(p)\, \psi'(p)^{-2} \Big|\, dp \, ;
\end{aligned}$$

as in the preceding proof, we have applied Cauchy-Schwarz inequality to the first integral. Now the hypotheses $\psi'' \leqslant 0$ and ψ' is monotonic with a constant sign allow to carry the following computations out:

$$\begin{aligned}\int_{p_1}^{p_2} \left|\psi''(p)\,\psi'(p)^{-2}\right| dp &= \left|-\int_{p_1}^{p_2} \psi''(p)\,\psi'(p)^{-2}\,dp\right| \\ &= \left|\psi'(p_2)^{-1} - \psi'(p_1)^{-1}\right| \\ &\leqslant m_{p_1,p_2}(\psi')^{-1}\,.\end{aligned}$$

Furthermore, we have

$$\left\|(\psi')^{-1}\right\|_{L^2(p_1,p_2)} \leqslant m_{p_1,p_2}(\psi')^{-1}\,(p_2-p_1)^{\frac{1}{2}}\,.$$

Consequently we obtain

$$\begin{aligned}\left|\int_{\mathbb{R}} U(p)\,e^{i\omega\psi(p)}\,dp\right| \leqslant \Big(& m_{p_1,p_2}(\psi')^{-1}\,(p_2-p_1)^{\frac{1}{2}}\left\|U'\right\|_{L^2(\mathbb{R})} \\ &+ m_{p_1,p_2}(\psi')^{-1}\left\|U\right\|_{L^\infty(\mathbb{R})}\Big)\,\omega^{-1}\,.\end{aligned}$$

□

4. Proofs of the main results

This section is devoted to the proofs of the results stated in Section 2.

4.1. Proof of Theorem 2.2

The proof of Theorem 2.2 is inspired by the one of [3, Theorem 5.2]: it consists mainly in rewriting wisely the solution formula (6) as an oscillatory integral with respect to time and in applying then a version of the stationary phase method.

In the present paper, the expansions in cones with arbitrary origin are obtained thanks to a space-time shift in the integral defining (6). And the explicitness of the error bounds with respect to the origin of the cones is achieved thanks to the new generic remainder estimates given in Theorems 3.3 and 3.4; this allows the application of Plancherel theorem in the present setting.

Proof of Theorem 2.2. The cases $t > t_0$ and $t < t_0$ are distinguished for the sake of readability.

Case 1: $t > t_0$
We rewrite the solution formula as an oscillatory integral by proceeding as follows[1]

$$\begin{aligned}
u_f(t,x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}u_0(p)\, e^{-itf(p)+ixp}\, dp \\
&= \int_{\mathbb{R}} \frac{1}{2\pi}\, \mathcal{F}u_0(p)\, e^{-it_0f(p)+ix_0p}\, e^{i(t-t_0)\left(\frac{x-x_0}{t-t_0}p\,-\,f(p)\right)}\, dp \\
&= \int_{\mathbb{R}} \mathbf{U}_f(p,t_0,x_0)\, e^{i(t-t_0)\Psi_f(p,t,x,t_0,x_0)}\, dp \\
&=: I_f(t,x,u_0,t_0,x_0)\ .
\end{aligned}$$

We note that the amplitude

$$\mathbf{U}_f(p,t_0,x_0) := \frac{1}{2\pi}\, \mathcal{F}u_0(p)\, e^{-it_0f(p)+ix_0p}\ ,$$

which is actually the Fourier transform of $\frac{1}{2\pi}u_f(t_0,\,.+x_0)$, is a $\mathcal{C}^\infty$-function (with respect to the variable p) whose support is included in $[p_1,p_2]$. The phase function

$$\Psi_f(p,t,x,t_0,x_0) := \frac{x-x_0}{t-t_0}p\ -\ f(p)$$

is a $\mathcal{C}^\infty$-function on $\mathbb{R}$ which is strictly concave since we have supposed $f'' > 0$ in this section.
Now we remark that the existence of a stationary point for the phase inside the interval $\tilde{I} := (\tilde{p}_1,\tilde{p}_2)$ depends on the value of $\frac{x-x_0}{t-t_0}$: it exists and is unique if and only if $\frac{x-x_0}{t-t_0} \in f'\big(\tilde{I}\big)$. In this case, the stationary point $p_0(t,x)$ is given by

$$p_0(t,x) = (f')^{-1}\left(\frac{x-x_0}{t-t_0}\right)\ .$$

Let us now distinguish two sub-cases to apply Theorem 3.3 and Theorem 3.4.

- *Case* $\frac{x-x_0}{t-t_0} \in f'\big(\tilde{I}\big)$. In this case, the stationary point belongs to $\tilde{I}$. Hence we are allowed to apply Theorem 3.3 to the oscillatory integral

[1] In [3, 9], the parameters t_0 and x_0 are implicitly equal to 0. Allowing these parameters to be arbitrary produces a space-time shift in the solution formula and permits to consider cones with arbitrary origin.

$I_f(t,x,u_0,t_0,x_0)$ with $\omega = t-t_0$:

$$\begin{aligned}
&\Bigg| I_f(t,x,u_0,t_0,x_0) \\
&\qquad - \frac{1}{\sqrt{2\pi}}\, e^{-i\frac{\pi}{4}}\, e^{-itf(p_0(t,x))+ixp_0(t,x)}\, \frac{\mathcal{F}u_0\big(p_0(t,x)\big)}{\sqrt{f''\big(p_0(t,x)\big)}}\,(t-t_0)^{-\frac{1}{2}}\Bigg| \\
&\qquad\qquad \leqslant \frac{1}{2\pi}\Bigg(C_5(\Psi_f,\delta,\tilde{p}_1,\tilde{p}_2)\left\| \partial_p\left[\mathcal{F}u_0(\cdot)\, e^{-it_0 f(\cdot)+ix_0\cdot}\right]\right\|_{L^2(\mathbb{R})} \\
&\qquad\qquad\qquad + C_6(\Psi_f,\delta,\tilde{p}_1,\tilde{p}_2)\left\|\mathcal{F}u_0\right\|_{L^\infty(\mathbb{R})}\Bigg)\,(t-t_0)^{-\delta}\ ,
\end{aligned}$$

with $\delta \in \left(\frac{1}{2},\frac{3}{4}\right)$ and the constants $C_5(\Psi_f,\delta,\tilde{p}_1,\tilde{p}_2)$, $C_6(\Psi_f,\delta,\tilde{p}_1,\tilde{p}_2) > 0$ are defined in Theorem 3.3. Since we have

$$\partial_p^2\Psi_f(p,t,x,t_0,x_0) = -f''(p) \quad , \quad \partial_p^3\Psi_f(p,t,x,t_0,x_0) = -f^{(3)}(p)\ ,$$

and since the constants $C_5(\Psi_f,\delta,\tilde{p}_1,\tilde{p}_2)$ and $C_6(\Psi_f,\delta,\tilde{p}_1,\tilde{p}_2)$ depend only on the second and third derivatives (with respect to p) of the phase, we can claim that these constants depend on f rather than Ψ_f. Furthermore, Plancherel theorem and standard properties of the Fourier transform provide

$$\begin{aligned}
&\left\|\partial_p\left[\mathcal{F}u_0(.)e^{-itf(\cdot)+ix_0\cdot}\right]\right\|_{L^2(\mathbb{R})} \\
&\qquad = \sqrt{2\pi}\left\| x\longmapsto x\,\mathcal{F}^{-1}\big[\mathcal{F}u_0(.)e^{-it_0f(\cdot)+ix_0\cdot}\big](x)\right\|_{L^2(\mathbb{R})} \\
&\qquad = \sqrt{2\pi}\left\| x\longmapsto x\,\mathcal{F}^{-1}\big[\mathcal{F}u_0(.)\,e^{-it_0f(\cdot)}\big](x+x_0)\right\|_{L^2(\mathbb{R})} \\
&\qquad = \sqrt{2\pi}\left\| x\longmapsto (x-x_0)\,\mathcal{F}^{-1}\big[\mathcal{F}u_0(.)\,e^{-it_0f(\cdot)}\big](x)\right\|_{L^2(\mathbb{R})} \\
&\qquad = \sqrt{2\pi}\left\| x\longmapsto (x-x_0)\,u_f(t_0,x)\right\|_{L^2(\mathbb{R})}\ .
\end{aligned}$$

Hence we obtain finally

$$\left| u_f(t,x) - \frac{1}{\sqrt{2\pi}}\, e^{-i\frac{\pi}{4}}\, e^{-itf(p_0(t,x))+ixp_0(t,x)}\, \frac{\mathcal{F}u_0\big(p_0(t,x)\big)}{\sqrt{f''\big(p_0(t,x)\big)}}\,(t-t_0)^{-\frac{1}{2}} \right|$$
$$\leqslant \Bigg(\frac{1}{\sqrt{2\pi}}\, C_5(-f,\delta,\tilde{p}_1,\tilde{p}_2)\, \big\|(.-x_0)\, u_f(t_0,.)\big\|_{L^2(\mathbb{R})}$$
$$+ \frac{1}{2\pi}\, C_6(-f,\delta,\tilde{p}_1,\tilde{p}_2)\, \big\|u_0\big\|_{L^1(\mathbb{R})} \Bigg)\,(t-t_0)^{-\delta}\ ,$$

where we have used the classical estimate $\big\|\mathcal{F}u_0\big\|_{L^\infty(\mathbb{R})} \leqslant \|u_0\|_{L^1(\mathbb{R})}$.

- *Case* $\frac{x-x_0}{t-t_0} \notin f'\big(\tilde{I}\big)$. As previously, we rewrite the solution formula as the oscillatory integral $I_f(t,x,u_0,t_0,x_0)$. Here the phase $\Psi_f(.,t,x,t_0,x_0)$ has no stationary point inside the interval $\tilde{I} = (\tilde{p}_1,\tilde{p}_2)$ and one has

$$\forall\, p \in [p_1,p_2] \qquad \Big|\partial_p \Psi_f(p,t,x,t_0,x_0)\Big| = \left| \frac{x-x_0}{t-t_0} - f'(p) \right| \geqslant m_{\tilde{I}}(f) > 0\,,$$

where $m_{\tilde{I}}(f) := \min\big\{f'(p_1)-f'(\tilde{p}_1), f'(\tilde{p}_2)-f'(p_2)\big\}$. Consequently we can apply Theorem 3.4 which provides

$$\big|u_f(t,x)\big| \leqslant \Bigg(\frac{1}{\sqrt{2\pi}}\, C_7(-f,\tilde{p}_1,\tilde{p}_2)\, \big\|(.-x_0)\, u_f(t_0,.)\big\|_{L^2(\mathbb{R})}$$
$$+ \frac{1}{2\pi}\, C_8(-f,\tilde{p}_1,\tilde{p}_2)\, \big\|u_0\big\|_{L^1(\mathbb{R})} \Bigg)(t-t_0)^{-1}\ ,$$

where the constants $C_7(-f,\tilde{p}_1,\tilde{p}_2)$, $C_8(-f,\tilde{p}_1,\tilde{p}_2) > 0$ are defined in Theorem 3.4.

<u>Case 2:</u> $t < t_0$

Here we have

$$\begin{aligned} u_f(t,x) &= \int_{\mathbb{R}} \frac{1}{2\pi}\, \mathcal{F}u_0(p)\, e^{-it_0 f(p)+ix_0 p}\, e^{i(t-t_0)\left(\frac{x-x_0}{t-t_0}p - f(p)\right)}\, dp \\ &= \overline{\int_{\mathbb{R}} \frac{1}{2\pi}\, \overline{\mathcal{F}u_0(p)\, e^{-it_0 f(p)+ix_0 p}}\, e^{i(t_0-t)\left(\frac{x-x_0}{t-t_0}p - f(p)\right)}\, dp} \\ &= \overline{\int_{\mathbb{R}} \overline{\mathbf{U}_f(p,t_0,x_0)}\, e^{i(t_0-t)\Psi_f(p,t,x,t_0,x_0)}\, dp}\ . \end{aligned}$$

Following the arguments and computations of the preceding case $t > t_0$, we obtain

$$\left| u_f(t,x) - \frac{1}{\sqrt{2\pi}}\, e^{+i\frac{\pi}{4}}\, e^{-itf(p_0(t,x))+ixp_0(t,x)}\, \frac{\mathcal{F}u_0\big(p_0(t,x)\big)}{\sqrt{f''\big(p_0(t,x)\big)}}\, (t_0-t)^{-\frac{1}{2}} \right|$$
$$\leqslant \bigg(\frac{1}{\sqrt{2\pi}}\, C_5(-f,\delta,\tilde{p}_1,\tilde{p}_2) \big\| (.-x_0)\, u_f(t_0,.) \big\|_{L^2(\mathbb{R})} + \frac{1}{2\pi}\, C_6(-f,\delta,\tilde{p}_1,\tilde{p}_2) \big\| u_0 \big\|_{L^1(\mathbb{R})} \bigg)\, (t_0-t)^{-\delta} \ ,$$

for all $(t,x) \in (-\infty, t_0) \times \mathbb{R}$ such that $\frac{x-x_0}{t-t_0} \in f'\big(\tilde{I}\big)$, and

$$\big| u_f(t,x) \big| \leqslant \Bigg(\frac{1}{\sqrt{2\pi}}\, C_7(-f,\tilde{p}_1,\tilde{p}_2)\, \big\| (.-x_0)\, u_f(t_0,.) \big\|_{L^2(\mathbb{R})} + \frac{1}{2\pi}\, C_8(-f,\tilde{p}_1,\tilde{p}_2)\, \big\| u_0 \big\|_{L^1(\mathbb{R})} \Bigg) (t_0-t)^{-1} \ ,$$

for all $(t,x) \in (-\infty, t_0) \times \mathbb{R}$ such that $\frac{x-x_0}{t-t_0} \notin f'\big(\tilde{I}\big)$, leading to the desired estimates. □

4.2. Proof of Corollary 2.5

This subsection is devoted to the proof of Corollary 2.5. It is based on a combination between Theorem 2.2 and the following Lemma 4.2.1. This lemma gives the minimum of the function $(t_0, x_0) \longmapsto \big\| (.-x_0) u_f(t_0,.) \big\|^2_{L^2(\mathbb{R})}$ which is involved in the error estimates in Theorem 2.2. This function is actually the moment of order 2 of the function $x \longmapsto \big| u_f(t_0, x+x_0) \big|^2$ for fixed $(t_0, x_0) \in \mathbb{R}^2$.

Lemma 4.2.1. *Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that* $\|u_0\|_{L^2(\mathbb{R})} = 1$. *Then the function* $g : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ *defined by*

$$g(t_0, x_0) = \big\| (.-x_0)\, u_f(t_0,.) \big\|^2_{L^2(\mathbb{R})}$$

has a global minimum at $(t^*, x^*) \in \mathbb{R}^2$ *with*

- $t^* = \underset{\tau \in \mathbb{R}}{\arg\min}\, \mathcal{V}\big(u_f(\tau,.)\big)$;
- $x^* = \mathcal{M}_1\big(u_f(t^*,.)\big)$.

Proof. For fixed $t_0 \in \mathbb{R}$, differentiating twice the function $g(t_0, .)$ with respect to its second argument shows that

$$\partial_{x_0}^2 g(t_0, x_0) = 2 > 0 \ .$$

Hence $g(t_0, .)$ is a polynomial function of degree 2 whose unique global minimum is

$$\tilde{x}(t_0) = \int_{\mathbb{R}} x \big|u_f(t_0, x)\big|^2 \, dx = \mathcal{M}_1\big(u_f(t_0, .)\big) \ .$$

It follows that

$$g\big(t_0, \tilde{x}(t_0)\big) = \int_{\mathbb{R}} \Big(x - \mathcal{M}_1\big(u_f(t_0, .)\big)\Big)^2 \, \big|u_f(t_0, x)\big|^2 \, dx = \mathcal{V}\big(u_f(t_0, .)\big) \ .$$

Lemma Appendix A.3 assures that $t_0 \in \mathbb{R} \longmapsto \mathcal{V}\big(u_f(t_0, .)\big) \in \mathbb{R}_+$ is a polynomial of degree 2 whose leading coefficient is $\mathcal{V}_{f'}\big(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\big)$. Since we have by simple calculations,

$$\begin{aligned}
\mathcal{V}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\bigg) &= \mathcal{M}_{f'^2}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg) - \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)^2 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \bigg(f'(p) - \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\bigg)^2 \, \big|\mathcal{F}u_0(p)\big|^2 \, dp \ ,
\end{aligned}$$

and since f is supposed to be strictly convex in this paper, the leading coefficient $\mathcal{V}_{f'}\big(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\big)$ is necessarily positive. Thus $t_0 \in \mathbb{R} \longmapsto g\big(t_0, \tilde{x}(t_0)\big) \in \mathbb{R}_+$ has a global minimum at a certain $t^* \in \mathbb{R}$, *i.e.*,

$$t^* = \underset{\tau \in \mathbb{R}}{\arg\min}\, \mathcal{V}\big(u_f(\tau, .)\big) \ .$$

Finally we define

$$x^* := \tilde{x}(t^*) = \mathcal{M}_1\big(u_f(t^*, .)\big) \ .$$

□

Remark 4.2.2. The polynomial nature of $t_0 \in \mathbb{R} \longmapsto g\big(t_0, \tilde{x}(t_0)\big) \in \mathbb{R}_+$ permits to derive the following formula for t^*:

$$\begin{aligned}
t^* = \frac{1}{\mathcal{V}_{f'}\big(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\big)} &\bigg(-\frac{1}{2\pi}\Im\bigg(\int_{\mathbb{R}} f'(p)\,\mathcal{F}u_0(p)\,\overline{\big(\mathcal{F}u_0\big)'(p)}\,dp\bigg) \\
&\quad + \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,\mathcal{M}_1(u_0)\bigg) \ . \qquad (14)
\end{aligned}$$

Furthermore, from Lemma Appendix A.1, we have

$$\mathcal{M}_1\big(u_f(t^*,.)\big) = \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,t^* + \mathcal{M}_1(u_0)\ ;$$

inserting formula (14) into the preceding equality provides

$$x^* = \frac{1}{\mathcal{V}_{f'}\big(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\big)}\Bigg(-\frac{1}{2\pi}\Im\bigg(\int_{\mathbb{R}} f'(p)\,\mathcal{F}u_0(p)\,\overline{\big(\mathcal{F}u_0\big)'(p)}\,dp\bigg) \\ \times \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg) + \mathcal{M}_{f'^2}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,\mathcal{M}_1(u_0)\Bigg)\ .$$

We are now in position to prove Corollary 2.5.

Proof of Corollary 2.5. We apply Theorem 2.2 with $t_0 = t^*$ and $x_0 = x^*$, where t^* and x^* are defined in Lemma 4.2.1, to the solution formula (6). This provides the time-asymptotic expansions appearing in (7) and (8) together with the error bounds in which the following term appears:

$$\big\|(.-x^*)\,u_f(t^*,.)\big\|_{L^2(\mathbb{R})} = \sqrt{g(t^*,x^*)}\ ,$$

where $g:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is defined in Lemma 4.2.1. By the definitions of t^* and x^*, we obtain

$$g(t^*,x^*) = \int_{\mathbb{R}} (x-x^*)^2\,\big|u_f(t^*,x)\big|^2\,dx = \mathcal{V}\big(u_f(t^*,.)\big) = \min_{\tau\in\mathbb{R}} \mathcal{V}\big(u_f(\tau,.)\big)\ .$$

Further it is clear that minimising the two families of (t_0,x_0)-dependent error bounds given in inequalities (7) and (8) is equivalent to minimising the function g. By Lemma 4.2.1, the parameter (t^*,x^*) minimises these two families, which ends the proof. □

4.3. Proof of Theorem 2.6

We prove Theorem 2.6 in this last subsection. This will be done in two main steps: for arbitrary $(t_0,x_0)\in\mathbb{R}^2$, we first compute the mean position of the term $H_f(t,.,u_0,t_0,x_0)$ defined in the statement of Theorem 2.6 and compare it to the mean position of the solution (6); a similar study for the variances is carried out. Theorem 2.6 is finally a direct consequence of these two studies.

In the first proposition, we compute the mean position of $H_f(t,.,u_0,t_0,x_0)$ for all $t\neq t_0$.

Proposition 4.3.1. *Let* $(t_0, x_0) \in \mathbb{R}^2$*. Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that* $\|u_0\|_{L^2(\mathbb{R})} = 1$*. Then for all* $t \in \mathbb{R}\backslash\{t_0\}$*, we have*

$$\mathcal{M}_1\Big(H_f(t,.,u_0,t_0,x_0)\Big) = x_0 + \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,(t-t_0)\ .$$

Proof. Let $t \in \mathbb{R}\backslash\{t_0\}$. First of all, we note that

$$x_0 + f'(p_1)(t-t_0) < x_0 + f'(p_2)(t-t_0) \quad \Longleftrightarrow \quad t > t_0\ .$$

Hence using the definition of $H_f(t,x,u_0,t_0,x_0)$ given in Theorem 2.6, we have

$$\begin{aligned}
&\int_{\mathbb{R}} x\,\Big|H_f(t,x,u_0,t_0,x_0)\Big|^2\,dx \\
&\qquad\qquad = \frac{sgn(t-t_0)}{2\pi}\int_{x_0+f'(p_1)(t-t_0)}^{x_0+f'(p_2)(t-t_0)} x\,\frac{\big|\mathcal{F}u_0\big(p_0(t,x)\big)\big|^2}{f''\big(p_0(t,x)\big)}\,dx\,|t-t_0|^{-1} \\
&\qquad\qquad = \frac{1}{2\pi}\int_{x_0+f'(p_1)(t-t_0)}^{x_0+f'(p_2)(t-t_0)} x\,\frac{\big|\mathcal{F}u_0\big(p_0(t,x)\big)\big|^2}{f''\big(p_0(t,x)\big)}\,dx\,(t-t_0)^{-1}\ .
\end{aligned}$$

We make now the change of variable $x = x_0 + f'(p)(t-t_0)$ to obtain the desired result:

$$\begin{aligned}
&\int_{\mathbb{R}} x\,\Big|H_f\big(t,x,u_0,t_0,x_0\big)\Big|^2\,dx \\
&\qquad\qquad = \frac{1}{2\pi}\int_{p_1}^{p_2}\big(x_0 + f'(p)(t-t_0)\big)\big|\mathcal{F}u_0(p)\big|^2\,dp \\
&\qquad\qquad = x_0 + \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\bigg)\,t - \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\bigg)\,t_0\ ;
\end{aligned}$$

note that we have used the fact that $\frac{1}{2\pi}\big\|\mathcal{F}u_0\big\|^2_{L^2(p_1,p_2)} = 1$, which is a direct consequence of the assumption $\|u_0\|_{L^2(\mathbb{R})} = 1$. $\square$

In the following result, we prove that the mean positions of the solution of equation (5) and of the term $H_f(t,.,u_0,t_0,x_0)$ are equal if and only if x_0 is equal to the mean position of $u_f(t_0,.)$.

Proposition 4.3.2. *Let* $(t_0, x_0) \in \mathbb{R}^2$*. Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that* $\|u_0\|_{L^2(\mathbb{R})} = 1$*. Then for all* $t \in \mathbb{R}\backslash\{t_0\}$*, we have*

$$\mathcal{M}_1\big(u_f(t,.)\big) = \mathcal{M}_1\big(H_f(t,.,u_0,t_0,x_0)\big) \quad \Longleftrightarrow \quad x_0 = \mathcal{M}_1\big(u_f(t_0,.)\big)\ .$$

Proof. Let $t \in \mathbb{R}\backslash\{t_0\}$. According to Propositions 4.3.1 and Appendix A.1, we have

- $\mathcal{M}_1\big(u_f(t,.)\big) = \mathcal{M}_1(u_0) + \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right) t$;
- $\mathcal{M}_1\big(H_f(t,.,u_0,t_0,x_0)\big) = x_0 + \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)(t-t_0)$.

Hence these two mean positions are equal if and only if

$$\mathcal{M}_1(u_0) = x_0 - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right) t_0 \ ,$$

which is equivalent to $x_0 = \mathcal{M}_1\big(u_f(t_0,.)\big)$. $\square$

Remark 4.3.3. The definitions of t^* and x^* from Corollary 2.5 and the preceding proposition assure that the mean positions of $u_f(t,.)$ and the term $H_f(t,.,u_0,t^*,x^*)$ are equal for all $t \neq t^*$.

In the two following results, we focus on the variances of the solution $u_f(t,.)$ and of the term $H_f(t,.,u_0,t_0,x_0)$ for all $t \neq t_0$. We give firstly a formula for the difference between the two variances for arbitrary (t_0,x_0).

Proposition 4.3.4. *Let $(t_0,x_0) \in \mathbb{R}^2$. Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then for all $t \in \mathbb{R}\backslash\{t_0\}$, we have*

$$\begin{aligned}
&\mathcal{V}\big(u_f(t,.)\big) - \mathcal{V}\big(H_f(t,.,u_0,t_0,x_0)\big) \\
&\qquad = 2\left(\frac{1}{2\pi}\,\Im\left(\int_{\mathbb{R}} f'(p)\,\mathcal{F}u_0(p)\,\overline{\big(\mathcal{F}u_0\big)'(p)}\,dp\right) - \mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)\mathcal{M}_1(u_0)\right. \\
&\qquad\qquad \left. + \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right) t_0\right) t + \mathcal{V}(u_0) - \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right) t_0^2 \ . \qquad (15)
\end{aligned}$$

Proof. Let $t \in \mathbb{R}\backslash\{t_0\}$. Similarly to the arguments employed in the proof of

Proposition 4.3.1, we have

$$
\begin{aligned}
&\mathcal{M}_2\big(H_f(t,.,u_0,t_0,x_0)\big)\\
&\quad = \frac{1}{2\pi}\int_{x_0+f'(p_1)(t-t_0)}^{x_0+f'(p_2)(t-t_0)} x^2\,\frac{\big|\mathcal{F}u_0\big(p_0(t,x)\big)\big|^2}{f''\big(p_0(t,x)\big)}\,dx\,(t-t_0)^{-1}\\
&\quad = \frac{1}{2\pi}\int_{p_1}^{p_2}\big(x_0+f'(p)(t-t_0)\big)^2\big|\mathcal{F}u_0(p)\big|^2\,dp\\
&\quad = x_0^2+\mathcal{M}_{f'^2}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)(t-t_0)^2+2\,x_0\,\mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)(t-t_0)\ .
\end{aligned}
$$

Moreover, applying Proposition 4.3.1 gives

$$
\begin{aligned}
&\mathcal{M}_1\big(H_f(t,.,u_0,t_0,x_0)\big)^2\\
&\quad = x_0^2+\mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)^2(t-t_0)^2+2\,x_0\,\mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)(t-t_0)\ .
\end{aligned}
$$

It follows:

$$
\begin{aligned}
\mathcal{V}\big(H_f(t,.,u_0,t_0,x_0)\big) &= \left(\mathcal{M}_{f'^2}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)-\mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)^2\right)(t-t_0)^2\\
&= \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)(t-t_0)^2\ .
\end{aligned}
$$

Using the formula for $\mathcal{V}\big(u_f(t,.)\big)$ from Proposition Appendix A.3, we obtain finally

$$
\begin{aligned}
&\mathcal{V}\big(u_f(t,.)\big)-\mathcal{V}\Big(H_f(t,.,u_0,t_0,x_0)\Big)\\
&\quad = \mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)t^2+2\left(\frac{1}{2\pi}\,\Im\left(\int_{\mathbb{R}} f'(p)\,\mathcal{F}u_0(p)\,\overline{\big(\mathcal{F}u_0\big)'(p)}\,dp\right)\right.\\
&\qquad\quad \left.-\,\mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)\mathcal{M}_1(u_0)\right)t+\mathcal{V}(u_0)\\
&\qquad\qquad -\,\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)(t-t_0)^2\\
&\quad = 2\left(\frac{1}{2\pi}\,\Im\left(\int_{\mathbb{R}} f'(p)\,\mathcal{F}u_0(p)\,\overline{\big(\mathcal{F}u_0\big)'(p)}\,dp\right)-\mathcal{M}_{f'}\left(\frac{1}{\sqrt{2\pi}}\mathcal{F}u_0\right)\mathcal{M}_1(u_0)\right.\\
&\qquad\quad \left.+\,\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)t_0\right)t+\mathcal{V}(u_0)-\mathcal{V}_{f'}\left(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\right)t_0^2\ .
\end{aligned}
$$

$\square$

In view of the preceding result, the difference between the variances of $u_f(t,.)$ and $H_f(t,.,u_0,t_0,x_0)$ is an affine function with respect to t. Consequently the unique way to make this difference constant is to choose t_0 in such way that the leading coefficient is equal to 0. It turns out that the unique t_0 satisfying this property is the one minimising the variance of the solution (6), namely t^* introduced in Corollary 2.5.

Proposition 4.3.5. *Let $(t_0,x_0) \in \mathbb{R}^2$. Assume that the hypotheses of Theorem 2.2 are satisfied and assume in addition that $\|u_0\|_{L^2(\mathbb{R})} = 1$. Then we have the following equivalence:*

$$\begin{aligned} \exists\, C \in \mathbb{R} \quad \forall\, t \in \mathbb{R}\backslash\{t_0\} \qquad & \mathcal{V}\big(u_f(t,.)\big) - \mathcal{V}\big(H_f(t,.,u_0,t_0,x_0)\big) = C \\ \iff \qquad & t_0 = t^* := \arg\min_{\tau\in\mathbb{R}} \mathcal{V}\big(u_f(\tau,.)\big)\ . \end{aligned}$$

In particular, we have

$$C = \min_{\tau\in\mathbb{R}} \mathcal{V}\big(u_f(\tau,.)\big)\ .$$

Proof. According to Proposition 4.3.4, the difference between the variances of $u_f(t,.)$ and $H_f(t,.,u_0,t_0,x_0)$ is constant if and only if

$$\begin{aligned} t_0 = \frac{1}{\mathcal{V}_{f'}\big(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\big)} \Bigg(& -\frac{1}{2\pi}\Im\bigg(\int_{\mathbb{R}} f'(p)\,\mathcal{F}u_0(p)\,\overline{\big(\mathcal{F}u_0\big)'(p)}\,dp\bigg) \\ & + \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,\mathcal{M}_1(u_0)\Bigg)\ , \end{aligned}$$

which is equal to t^* according to Remark 4.2.2. By evaluating equality (15) at t^*, we obtain for all $t \in \mathbb{R}\backslash\{t^*\}$,

$$\mathcal{V}\big(u_f(t,.)\big) - \mathcal{V}\big(H_f(t,.,u_0,t^*,x_0)\big) = \mathcal{V}(u_0) - \mathcal{V}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,(t^*)^2\ . \qquad (16)$$

We note now that

$$\begin{aligned}
&\mathcal{V}\big(u_f(t^*,.)\big)\\
&\quad = \mathcal{V}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,(t^*)^2 + 2\bigg(\frac{1}{2\pi}\,\Im\bigg(\int_{\mathbb{R}} f'(p)\,\mathcal{F}u_0(p)\,\overline{\big(\mathcal{F}u_0\big)'(p)}\,dp\bigg)\\
&\qquad\qquad - \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,\mathcal{M}_1(u_0)\bigg)t^* + \mathcal{V}(u_0)\\
&\quad = \mathcal{V}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,(t^*)^2 - 2\mathcal{V}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,(t^*)^2 + \mathcal{V}(u_0)\\
&\quad = -\mathcal{V}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,(t^*)^2 + \mathcal{V}(u_0)\ .
\end{aligned} \tag{17}$$

From equalities (16) and (17), it follows finally

$$\mathcal{V}\big(u_f(t,.)\big) - \mathcal{V}\big(H_f(t,.,u_0,t^*,x_0\big) = \mathcal{V}\big(u_f(t^*,.)\big) = \min_{\tau\in\mathbb{R}} \mathcal{V}\big(u_f(\tau,.)\big)\ ,$$

the last equality being obtained by the definition $t^* = \arg\min_{\tau\in\mathbb{R}} \mathcal{V}\big(u_f(\tau,.)\big)$. □

Theorem 2.6 can be now proved in a straightforward way.

Proof of Theorem 2.6. Let $t \in \mathbb{R}\backslash\{t^*\}$. According to Proposition 4.3.2 (and Remark 4.3.3), we have

$$\mathcal{M}_1\big(u_f(t,.)\big) = \mathcal{M}_1\big(H_f(t,.,u_0,t^*,x^*)\big)\ ,$$

and Proposition 4.3.5 assures that

$$\mathcal{V}\big(u_f(t,.)\big) - \mathcal{V}\big(H_f(t,.,u_0,t^*,x^*)\big) = \min_{\tau\in\mathbb{R}} \mathcal{V}\big(u_f(\tau,.)\big)\ ,$$

which ends the proof. □

Appendix A. Mean position and variance of the free wave packet

In this appendix, we give the formulas for the mean position and the variance of the wave packet defined in (1). The proofs we propose here are substantially based on the fact that the wave packet is defined via the Fourier transform, permitting to apply some properties of this transform.

We begin with the formula for the mean position.

Proposition Appendix A.1. *Suppose that $u_0 \in \mathcal{S}(\mathbb{R})$. Then for all $t \in \mathbb{R}$, we have*

$$\mathcal{M}_1\big(u_f(t,.)\big) = \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,t + \mathcal{M}_1(u_0)\ .$$

Proof. For $t \in \mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{R}} x\,\big|u_f(t,x)\big|^2\,dx &= \int_{\mathbb{R}} x\,u_f(t,x)\,\overline{u_f(t,x)}\,dx\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}\big[x\mapsto x\,u_f(t,x)\big](p)\,\overline{\mathcal{F}\big[x\mapsto u_f(t,x)\big](p)}\,dp\\ &= \frac{i}{2\pi}\int_{\mathbb{R}} \partial_p\mathcal{F}\big[x\mapsto u_f(t,x)\big](p)\,\overline{\mathcal{F}\big[x\mapsto u_f(t,x)\big](p)}\,dp\ ;\end{aligned} \tag{A.1}$$

the second and third equalities have been obtained by applying Plancherel theorem and basic properties of the Fourier transform. Using now the formula (1), we obtain for all $p \in \mathbb{R}$,

- $\partial_p\mathcal{F}\big[x\mapsto u_f(t,x)\big](p) = e^{-itf(p)}\Big(-itf'(p)\,\mathcal{F}u_0(p) + (\mathcal{F}u_0)'(p)\Big)\ ;$
- $\overline{\mathcal{F}\big[x\mapsto u_f(t,x)\big](p)} = e^{itf(p)}\,\overline{\mathcal{F}u_0(p)}\ .$

By combing the two last equalities with (A.1) and by using again basic properties of the Fourier transform, it follows

$$\begin{aligned}&\int_{\mathbb{R}} x\,\big|u_f(t,x)\big|^2\,dx\\ &\quad= \frac{i}{2\pi}\int_{\mathbb{R}}\Big(-itf'(p)\,\mathcal{F}u_0(p)\ +\ (\mathcal{F}u_0)'(p)\Big)\,\overline{\mathcal{F}u_0(p)}\,dp\\ &\quad= \frac{1}{2\pi}\int_{\mathbb{R}} f'(p)\,\big|\mathcal{F}u_0(p)\big|^2\,dp\,t\ +\ \frac{i}{2\pi}\int_{\mathbb{R}}(\mathcal{F}u_0)'(p)\,\overline{\mathcal{F}u_0(p)}\,dp\\ &\quad= \frac{1}{2\pi}\int_{\mathbb{R}} f'(p)\,\big|\mathcal{F}u_0(p)\big|^2\,dp\,t\ +\ \frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{F}\big[x\mapsto x\,u_0(x)\big](p)\,\overline{\mathcal{F}u_0(p)}\,dp\\ &\quad= \frac{1}{2\pi}\int_{\mathbb{R}} f'(p)\,\big|\mathcal{F}u_0(p)\big|^2\,dp\,t\ +\ \int_{\mathbb{R}} x\,\big|u_0(x)\big|^2\,dx\ ,\end{aligned}$$

leading finally to the desired equality. $\square$

Remark Appendix A.2. The preceding formula is actually an extension of the well-known Ehrenfest theorem [22, Proposition 3.19] to the family of dispersive equations of type (5). We recall that Ehrenfest theorem in the setting of the free Schrödinger equation (3) gives the following formula for the mean position of a free particle:

$$\mathcal{M}_1\big(u_S(t,.)\big) = \mathcal{M}_1\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,t + \mathcal{M}_1(u_0)\ .$$

The formula for the variance is provided in the following result.

Proposition Appendix A.3. *Suppose that $u_0 \in \mathcal{S}(\mathbb{R})$. Then for all $t \in \mathbb{R}$, we have*

$$\begin{aligned}\mathcal{V}\big(u_f(t,.)\big) = \mathcal{V}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,t^2 + 2\bigg(\frac{1}{2\pi}\,\Im\bigg(\int_{\mathbb{R}} f'(p)\,\mathcal{F}u_0(p)\,\overline{\big(\mathcal{F}u_0\big)'(p)}\,dp\bigg)\\ - \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,\mathcal{M}_1(u_0)\bigg)t + \mathcal{V}(u_0)\ .\end{aligned}$$

Proof. Following the computational arguments of the proof of Proposition Appendix A.1, we have for all $t \in \mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}} x^2\,\big|u_f(t,x)\big|^2\,dx &= \int_{\mathbb{R}} \big|x\,u_f(t,x)\big|^2\,dx\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \Big|\mathcal{F}\big[x \mapsto x\,u_f(t,x)\big](p)\Big|^2\,dp\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \Big|\partial_p\mathcal{F}\big[x \mapsto u_f(t,x)\big](p)\Big|^2\,dp\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \Big| - itf'(p)\,\mathcal{F}u_0(p) + (\mathcal{F}u_0)'(p)\Big|^2\,dp\ . \qquad \text{(A.2)}\end{aligned}$$

Inserting the following relation

$$\forall\, p \in \mathbb{R} \qquad (\mathcal{F}u_0)'(p) = -i\,\mathcal{F}\big[x \mapsto x\,u_0(x)\big](p)$$

into (A.2) and expanding then the square of the absolute value provides

$$
\begin{aligned}
\int_{\mathbb{R}} x^2 \left|u_f(t,x)\right|^2 dx &= \frac{1}{2\pi}\int_{\mathbb{R}} f'(p)^2 \left|\mathcal{F}u_0(p)\right|^2 dp\, t^2 \;+\; \int_{\mathbb{R}} x^2 \left|u_0(x)\right|^2 dx \\
&\qquad\qquad -\;\frac{1}{\pi}\int_{\mathbb{R}} \Re\Big(i\, f'(p)\, \mathcal{F}u_0(p)\, \overline{\big(\mathcal{F}u_0\big)'(p)}\Big)\, dp\, t \\
&= \mathcal{M}_{f'^2}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\, t^2 \;+\; \mathcal{M}_2(u_0) \\
&\qquad\qquad +\;\frac{1}{\pi}\,\Im\bigg(\int_{\mathbb{R}} f'(p)\, \mathcal{F}u_0(p)\, \overline{\big(\mathcal{F}u_0\big)'(p)}\, dp\bigg)\, t\;.
\end{aligned}
$$

Now by using Proposition Appendix A.1, we have

$$
\begin{aligned}
&\mathcal{M}_1\big(u_f(t,.)\big)^2 \\
&\qquad = \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)^2 t^2 + \mathcal{M}_1(u_0)^2 + 2\,\mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,\mathcal{M}_1(u_0)\, t\;,
\end{aligned}
$$

which leads finally to

$$
\begin{aligned}
&\mathcal{V}\big(u_f(t,.)\big) \\
&\quad = \mathcal{M}_2\big(u_f(t,.)\big) - \mathcal{M}_1\big(u_f(t,.)\big)^2 \\
&\quad = \mathcal{M}_{f'^2}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\, t^2 \;+\; \mathcal{M}_2(u_0) \\
&\qquad\quad +\;\frac{1}{\pi}\,\Im\bigg(\int_{\mathbb{R}} f'(p)\, \mathcal{F}u_0(p)\, \overline{\big(\mathcal{F}u_0\big)'(p)}\, dp\bigg)\, t \\
&\qquad\qquad - \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)^2 t^2 - \mathcal{M}_1(u_0)^2 \;-\; 2\,\mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,\mathcal{M}_1(u_0)\, t \\
&\quad = \mathcal{V}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\, t^2 + \mathcal{V}(u_0) \\
&\qquad\quad + 2\bigg(\frac{1}{2\pi}\,\Im\bigg(\int_{\mathbb{R}} f'(p)\, \mathcal{F}u_0(p)\, \overline{\big(\mathcal{F}u_0\big)'(p)}\, dp\bigg) \\
&\qquad\qquad - \mathcal{M}_{f'}\bigg(\frac{1}{\sqrt{2\pi}}\,\mathcal{F}u_0\bigg)\,\mathcal{M}_1(u_0)\bigg)\, t\;.
\end{aligned}
$$

□

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