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## ► To cite this version:

Florent Di Meglio, Pierre-Olivier Lamare, Ulf Jakob F. Aarsnes. Robust output feedback stabilization of an ODE–PDE–ODE interconnection. *Automatica*, 2020, 119, pp.109059 -. 10.1016/j.automatica.2020.109059 . hal-03490139

**HAL Id: hal-03490139**

**<https://hal.science/hal-03490139>**

Submitted on 21 Jun 2022

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# Robust output feedback stabilization of an ODE-PDE-ODE interconnection

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## Abstract

We propose a controller that robustly stabilizes an interconnection of Ordinary Differential Equations with two hyperbolic linear Partial Differential Equations. The proposed design guarantees closed-loop asymptotic stability as well as robustness to small delays. The result is obtained through a backstepping approach. A boundary observer is also derived, yielding an output feedback controller with several calibration parameters.

**Key words:** Hyperbolic Partial Differential Equations; stabilization; backstepping.

## 1 Introduction

In this paper, we consider the problem of stabilizing the interconnection of a system of two first-order linear hyperbolic Partial Differential Equations (PDE) with Ordinary Differential Equations (ODE). More precisely, the interconnection features a controlled, scalar ODE coupled at one boundary of the spatial domain with the PDE, which we call the proximal boundary. At the distal boundary the other ODE is uncontrolled and may have a vector state. This situation is schematically depicted on Figure 1.

Such systems arise in many practical applications involving delay or transport phenomena, due to lumped physical elements or actuation and measurement dynamics. If the lumped dynamics are not taken explicitly into account in the controller design, e.g. by assuming they can be inverted or canceled, it could easily lead to robustness issues. This is illustrated in this paper by a discussion on delay robustness.

Specific examples of this kind of systems include the: control of mining shaft elevators [20], hydraulic flow-lines with lumped elements [13], control of loaded heavy chains with actuation dynamics [19].

Of particular interest are the modeling and control of

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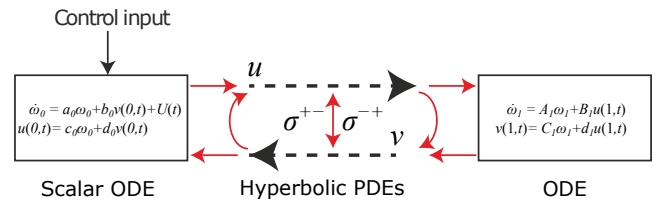


Fig. 1. Schematic description of the considered system. The black dotted arrows represent transport phenomena while the red arrows represent couplings.

torsional and axial vibrations in deep drilling systems. The hyperbolic PDE models the propagation of torsional and axial waves along the long (typically several thousand meters), slender drill string [1–3]. At the surface boundary, an electric motor applies a torque (the control input) to the so-called top drive, which has a large inertia and is responsible for the first-order dynamics at this boundary. At the bottom of the well, the drillstring is rigidly connected to the Bottom Hole Assembly (BHA), which consist of heavy collar sections that can be represented as a lumped element. Ignoring the lumped dynamics in this system leads to controllers which are not usable in practice. This is evident when considering the commercially available controllers [11,15,16].

Several contributions focus on ODE-PDE-ODE interconnections such as the one considered in this paper. In [5], a backstepping approach is used to design a stabilizing full-state feedback controller for actuator dynamics of arbitrary order. Using the design of [10], the PDE in-domain coupling terms are canceled and the distal ODE is stabilized. Then, using an additional step of backstepping, the proximal wave reflection is entirely

canceled and the influence of the PDE onto the state of the controlled ODE is entirely canceled. In an ideal setting, this control strategy efficiently stabilizes the plant regardless of the magnitude of coupling terms and of the natural time scales of the ODEs. However, as pointed out in [17,14,4] it is likely to have zero delay robustness margin, i.e. to result in an unstable system in the presence of arbitrarily small delays. In [20], a similar strategy is employed in the particular case where actuator dynamics are of second-order. In [9], these results are improved, in particular by alleviating some restrictions on the controlled ODE, and an observer is designed, relying on a measurement of the boundary value of one of the PDEs. Again, the proximal reflection of the PDE is entirely canceled, leading to a stable cascade, but to a non-robust design.

The main contribution of this paper is an output feedback law that:

- (1) stabilizes the plant depicted in Figure 1,
- (2) is robust to small actuator delays,
- (3) has tunable design parameters.

Our approach builds on the result from [5]. We propose small modifications of the design that “robustify” the controller: rather than canceling the entire proximal wave reflection, we assign it the dynamics of a high-pass filter. The resulting target system dynamics take the form of a cascade of the distal ODE into a first-order neutral system, which, when stable, is shown to be w-stable in the sense of [7,12]. Further, we propose an observer relying on collocated (proximal) measurements.

The paper is organized as follows: In Section 2 we define the problem and the notations used throughout the paper. Then, in Section 3, we design the controller and prove its robustness properties. In Section 4 we design the observer and formulate the main result, i.e. the output feedback control law. Finally, we illustrate the design with numerical simulations on a toy problem in Section 5.

## 2 Problem setup and notations

In this paper, we consider the problem of stabilizing the following interconnection

$$\dot{\omega}_0 = a_0 \omega_0 + b_0 v(t, 0) + U(t) \quad (1)$$

$$u(t, 0) = c_0 \omega_0(t) + d_0 v(t, 0) \quad (2)$$

$$u_t(t, x) = -\lambda(x) u_x(t, x) + \sigma^{++}(x) u(t, x) + \sigma^{+-}(x) v(t, x) \quad (3)$$

$$v_t(t, x) = \mu(x) v_x(t, x) + \sigma^{-+}(x) u(t, x) + \sigma^{--}(x) v(t, x) \quad (4)$$

$$v(t, 1) = C_1^\top \omega_1(t) + d_1 u(t, 1) \quad (5)$$

$$\dot{\omega}_1 = A_1 \omega_1 + B_1 u(t, 1) \quad (6)$$

with  $\lambda, \mu \in \mathcal{C}^1([0, 1], \mathbb{R}_+^*)$ ,  $\omega_0 \in \mathbb{R}$ ,  $(u, v) \in L^2((0, 1), \mathbb{R})$  and  $\omega_1 \in \mathbb{R}^n$ , using solely the measurement of

the (scalar) ODE state at  $x = 0$ , i.e.

$$y(t) = \omega_0(t) \quad (7)$$

under the following assumptions

- A1 The pair  $(A_1, B_1)$  (resp.  $(A_1, C_1)$ ) is stabilizable (resp. detectable). We denote  $\tilde{\kappa}_1$  (resp.  $\tilde{L}_1$ ) a set of control (resp. observer) gains such that  $A_1 + B_1 \tilde{\kappa}_1$  (resp.  $A_1 + \tilde{L}_1 C_1$ ) is Hurwitz.
- A2 The reflection coefficients  $d_0, d_1$  satisfy

$$R < 1 \quad (8)$$

where

$$R = \exp \left[ \int_0^1 \frac{\sigma^{++}(x)}{\lambda(x)} dx + \int_1^0 \frac{\sigma^{--}(x)}{\mu(x)} dx \right] |d_0 d_1| \quad (9)$$

- A3 The proximal ODE has an effect on the rest of the system, i.e.  $c_0 \neq 0$ . Conversely, the PDE states have an effect on the proximal ODE, i.e.  $b_0 \neq 0$ .

Assumption A2 is a necessary and sufficient condition for the existence of a stabilizing controller robust to at least small delays [17]. Indeed, when it is not satisfied, the open-loop system features an infinite number of poles in the Right-Half Plane. We first design a full-state feedback controller taking the form

$$U(t) = k_0 \omega_0(t) + k_1 \omega_1(t) + K_0 v(t, 0) + K_1 u(t, 1) + \int_0^1 [K(y) u(t, y) + L(y) v(t, y)] dy, \quad (10)$$

before designing the observer.

## 3 Controller: design and robustness

### 3.1 Control design

The controller is based on a backstepping design similar to [5]. More precisely, we map the system to the following target system

$$\dot{\eta}_0(t) = \check{p} \eta_0(t) + d_0 \frac{\check{p} - \check{z}}{c_0} \beta(t, 0) \quad (11)$$

$$\alpha(t, 0) = c_0 \eta_0(t) + d_0 \beta(t, 0) \quad (12)$$

$$\alpha_t(t, x) + \lambda(x) \alpha_x(t, x) = \sigma^{++}(x) \alpha(t, x) \quad (13)$$

$$\beta_t(t, x) - \mu(x) \beta_x(t, x) = \sigma^{--}(x) \beta(t, x) \quad (14)$$

$$\beta(t, 1) = d_1 \alpha(t, 1). \quad (15)$$

$$\dot{\omega}_1(t) = (A_1 - B_1 \tilde{\kappa}_1^\top) \omega_1(t) + B_1 \alpha(t, 1) \quad (16)$$

where  $\check{p}$ ,  $\check{z}$  and  $\tilde{\kappa}_1$  are control design parameters. The main difference with [5] is that we cancel here the ODE

term in (15) but not the PDE boundary reflection term in (12). Besides, we introduce the degrees of liberty  $\tilde{p}$  and  $\tilde{z}$ . To do so, we consider the following backstepping transformation.

$$\begin{aligned}\alpha(t, x) &= u(t, x) - \int_x^1 K^{uu}(x, y)u(t, y)dy \\ &\quad - \int_x^1 K^{uv}(x, y)v(t, y)dy - G_u(x)^\top \omega_1(t) \quad (17) \\ \beta(t, x) &= v(t, x) - \int_x^1 K^{vu}(x, y)u(t, y)dy \\ &\quad - \int_x^1 K^{vv}(x, y)v(t, y)dy - G_v(x)^\top \omega_1(t), \quad (18)\end{aligned}$$

where the transformation kernels are defined by the following set of well-posed [5] equations

$$\begin{aligned}\lambda(x)K_x^{uu}(x, y) + \lambda(y)K_y^{uu}(x, y) = \\ [\sigma^{++}(x) - \sigma^{++}(y) - \lambda'(y)]K^{uu}(x, y) - \sigma^{+-}(y)K^{uv}(x, y)\end{aligned} \quad (19)$$

$$\begin{aligned}\lambda(x)K_x^{uv}(x, y) - \mu(y)K_y^{uv}(x, y) = \\ [\sigma^{++}(x) - \sigma^{--}(y) + \mu'(y)]K^{uv}(x, y) - \sigma^{-+}(y)K^{uu}(x, y)\end{aligned} \quad (20)$$

$$K^{uv}(x, x) = -\frac{\sigma^{+-}(x)}{\lambda(x) + \mu(x)} \quad (21)$$

$$\lambda K^{uu}(x, 1) = \mu(1)d_1 K^{uv}(x, 1) + G_u(x)^\top B_1 \quad (22)$$

$$\begin{aligned}\lambda(x)G'_u(x) &= (\sigma^{++}(x)I_n - A_1^\top)G_u(x) \\ &\quad - \mu(1)K^{uv}(x, 1)C_1^\top\end{aligned} \quad (23)$$

$$G_u(1) = -\tilde{\kappa}_1 \quad (24)$$

where  $I_n$  is the identity matrix of size  $n$ , and

$$\begin{aligned}-\mu(x)K_x^{vu}(x, y) + \lambda(y)K_y^{vu}(x, y) = \\ [\sigma^{--}(x) - \sigma^{++}(y) - \lambda'(y)]K^{vv}(x, y) - \sigma^{-+}(y)K^{vv}(x, y)\end{aligned} \quad (25)$$

$$\begin{aligned}-\mu(x)K_x^{vv}(x, y) - \mu(y)K_y^{vv}(x, y) = \\ [\sigma^{--}(x) - \sigma^{--}(y) + \mu'(y)]K^{vv}(x, y) - \sigma^{+-}(y)K^{vu}(x, y)\end{aligned} \quad (26)$$

$$K^{vu}(x, x) = \frac{\sigma^{-+}(x)}{\lambda(x) + \mu(x)} \quad (27)$$

$$\mu d_1 K^{vv}(x, 1) = \lambda K^{vu}(x, 1) - G_v(x)^\top B_1 \quad (28)$$

$$\begin{aligned}\mu G'_v(x) &= -(\sigma^{--}(x)I_n - A_1^\top)G_v(x) \\ &\quad + \mu(1)K^{vv}(x, 1)C_1\end{aligned} \quad (29)$$

$$G_v(1) = -d_1 \tilde{\kappa}_1 + C_1. \quad (30)$$

With the following variable change for  $\omega_0$

$$\begin{aligned}\eta_0(t) &= \omega_0(t) \\ -c_0^{-1} \int_0^1 [\tilde{K}_\alpha(y)u(t, y) + \tilde{K}_\beta(y)v(t, y)] dy c_0^{-1} \tilde{G}^\top \omega_1(t)\end{aligned} \quad (31)$$

where

$$\tilde{K}_\alpha(y) = K^{uu}(0, y) - d_0 K^{vu}(0, y) \quad (32)$$

$$\tilde{K}_\beta(y) = K^{uv}(0, y) - d_0 K^{vv}(0, y) \quad (33)$$

$$\tilde{G} = G_u(0) - d_0 G_v(0). \quad (34)$$

In [5], the transformation  $\omega_0 \rightarrow \eta_0$  contains an additional term  $-d_0 v(t, 0)$  that is used to cancel the reflecting wave from the PDE at  $x = 0$ . Rather, here, Equation (31) maps (2) to the following boundary condition

$$\alpha(t, 0) = d_0 \beta(t, 0) + c_0 \eta_0(t). \quad (35)$$

This choice is made to ensure delay robustness of the controller, as proved in Section 3.2.2. Indeed, canceling entirely the reflecting wave of the PDE at the left boundary will cause the delay robustness margin to be exactly zero when  $R \geq 0.5$ , in (9), as detailed in [17, 4]. Moreover, the choices made in [5, 9] requires computing the time derivative of the PDE state at the boundary, or equivalently, its spatial derivative, resulting in both cases an acausal design necessitating approximations in the implementation. As will be made clear, this is not a requirement here. Plugging (17), (18), (31) into (1) yields

$$\begin{aligned}\dot{\eta}_0(t) &= U(t) + [a_0 - \lambda(0)\tilde{K}_\alpha(0)]\omega_0(t) \\ &\quad - c_0^{-1} [\tilde{G}^\top A_1 + C_1^\top \mu(1)\tilde{K}_\beta(1)] \omega_1(t) \\ &\quad + [b_0 + c_0^{-1} (\mu(0)\tilde{K}_\beta(0) - \lambda(0)d_0\tilde{K}_\alpha(0))] v(t, 0) \\ &\quad + c_0^{-1} [-\tilde{G}^\top b_1 + \lambda(1)\tilde{K}_\alpha(1) - \mu(1)d_1\tilde{K}_\beta(1)] u(t, 1) \\ &\quad - c_0^{-1} \int_0^1 [\lambda(y)\tilde{K}'_\alpha(y) + \lambda'(y)\tilde{K}_\alpha(y) + \sigma^{++}(y)\tilde{K}_\alpha(y) \\ &\quad \quad + \sigma^{-+}(y)\tilde{K}_\beta(y)] u(t, y) dy \\ &\quad - c_0^{-1} \int_0^1 [-\mu(y)\tilde{K}'_\beta(y) + \lambda'(y)\tilde{K}_\beta(y) + \sigma^{--}(y)\tilde{K}_\beta(y) \\ &\quad \quad + \sigma^{+-}(y)\tilde{K}_\alpha(y)] v(t, y) dy\end{aligned} \quad (36)$$

which, using the inverse backstepping transformations yields an expression of the form

$$\begin{aligned}\dot{\eta}_0(t) &= U(t) + A_0 \eta_0(t) + B_0 \beta(t, 0) + \tilde{D}_1 \omega_1(t) \\ &\quad + B_1 \alpha(t, 1) - \int_0^1 M(y) \alpha(t, y) dy + \int_0^1 N(y) \beta(t, y) dy.\end{aligned} \quad (37)$$

where  $A_0, B_0, B_1, \tilde{D}_1, M$  and  $N$  are given in Appendix A. Equation (37), which is the open-loop dynamics of  $\eta_0$  in the target system coordinates, will prove useful in the  $w$ -stability analysis in the next section while (36) is used to express the control law in the original coordinates. More precisely, we propose a control law of the form (10) where the control gains are given by

$$k_0 = \check{p} - A_0 \quad (38)$$

$$k_1 = -D_1 - \frac{\check{p}}{c_0} \tilde{G}^\top - d_0 \frac{\check{p} - \check{z}}{c_0} G_v(0)^\top \quad (39)$$

$$K_0 = d_0 \frac{\check{p} - \check{z}}{c_0} - B_0 \quad (40)$$

$$K_1 = -B_1 \quad (41)$$

$$K(x) = c_0^{-1} K_u^D(x) - \check{p} c_0^{-1} \tilde{K}_\alpha(x) - d_0 \frac{\check{p} - \check{z}}{c_0} K^{vu}(0, x) \quad (42)$$

$$L(x) = c_0^{-1} K_v^D(x) - \check{p} c_0^{-1} \tilde{K}_\beta(x) - d_0 \frac{\check{p} - \check{z}}{c_0} K^{vv}(0, x). \quad (43)$$

where  $\check{p}$  and  $\check{z}$  are design parameters. This choice maps, in closed-loop, (1)–(6) to (11)–(16). In the next section, we investigate the stability properties of (11)–(16).

### 3.2 Stability of the target system and controller robustness

In this Section, we give necessary and sufficient conditions on the control design parameters for the target system to be stable (Lemma 2). Then, we show that the closed-loop system is actually  $w$ -stable in the sense of [7]. Both results rely on the following Lemma.

**Lemma 1** Consider Equations (12)–(16). All the variables can be expressed, in the Laplace domain, as functions of  $\alpha(s, 0)$  using proper transfer functions.

We recall the definition of a (strictly) proper transfer function from [6]

**Definition 1** A transfer function  $G(s)$  is said to be proper if it satisfies

$$\limsup_{\Re\{s\} \geq 0, |s| > M, M \rightarrow \infty} |G(s)| < \infty \quad (44)$$

and strictly proper if it satisfies

$$\lim_{\Re\{s\} \geq 0, |s| \rightarrow \infty} |G(s)| = 0. \quad (45)$$

We are now ready to prove Lemma 1.

**Proof 1** Taking the Laplace transform of (12)–(16)

yields

$$\alpha(s, x) = \exp\left(\int_0^x \frac{\sigma^{++}(y) - s}{\lambda(y)} dy\right) \alpha(s, 0) \quad (46)$$

$$\begin{aligned} \omega_1(s) &= (sI - A_1 + B_1 \tilde{\kappa}_1)^{-1} B_1 \\ &\cdot \exp\left(\int_0^1 \frac{\sigma^{++}(y) - s}{\lambda(y)} dy\right) \alpha(s, 0) \end{aligned} \quad (47)$$

$$\begin{aligned} \beta(s, x) &= \\ d_1 \exp\left(\int_0^1 \frac{\sigma^{++}(y) - s}{\lambda(y)} dy + \int_x^1 \frac{\sigma^{--}(y) - s}{\mu(y)} dy\right) \alpha(s, 0) \end{aligned} \quad (48)$$

And denoting

$$\tau = \int_0^1 \left[ \frac{1}{\lambda(y)} + \frac{1}{\mu(y)} \right] dy, \quad (49)$$

$$k_d = \exp\left(\int_0^1 \frac{\sigma^{++}(y)}{\lambda(y)} dy + \int_0^1 \frac{\sigma^{--}(y)}{\mu(y)} dy\right), \quad (50)$$

yields

$$\beta(s, 0) = d_1 k_d e^{-\tau s} \alpha(s, 0) \quad (51)$$

$$c_0 \eta_0(s) = (1 - d_0 d_1 k_d e^{-\tau s}) \alpha(s, 0) \quad (52)$$

One can readily check that all the transfer functions appearing in (46)–(52) satisfy (44).

#### 3.2.1 Stability

The stability of the target system is assessed in the following Lemma.

**Lemma 2** Assume that  $\check{p}, \check{z}$  satisfy either the following set of conditions

$$\check{p} - d_0 d_1 k_d \check{z} < 0, \quad -\check{p} \geq |d_0 d_1 k_d \check{z}| \quad (53)$$

or

$$d_0 d_1 k_d \check{z} > |\check{p}|, \quad (54)$$

$$\tau < \quad (55)$$

$$\sqrt{\frac{1 - d_0^2 d_1^2 k_d^2}{d_0^2 d_1^2 k_d^2 \check{z}^2 - \check{p}^2}} \tan^{-1} \frac{d_0^2 d_1^2 k_d^2 \check{z} + \check{p}}{\sqrt{(d_0^2 d_1^2 k_d^2 \check{z}^2 - \check{p}^2)(1 - d_0^2 d_1^2 k_d^2)}} \quad (56)$$

Then (11)–(16) is asymptotically stable.

**Proof 2** Note that

$$\alpha(s, 0) = d_0 \frac{s - \check{z}}{s - \check{p}} \beta(s, 0) \quad (57)$$

Combining (51), (52) and (57) we find that the states  $\alpha(s, 0)$ ,  $\beta(s, 0)$  and  $\eta(s)$  all have the characteristic equation

$$s(1 - d_0 d_1 k_d e^{-\tau s}) - \check{p} + d_0 d_1 k_d \check{z} e^{-\tau s} = 0 \quad (58)$$

A system with the characteristic equation (58) has the conditions (53)–(56) for asymptotic stability as derived in [18, Proposition 3.16]. The distributed states  $\alpha(s, x)$  and  $\beta(s, x)$  and the distal ODE state  $\omega_1(s)$  can be expressed as transfer functions with no poles in the RHP and  $\alpha(s, 0)$  as input, and consequently these states are asymptotically stable as well.

### 3.2.2 Robustness

In this Section, we show that the closed-loop system is robust to approximate identities [7], that is, it is *w-stable*. A family of operators  $I_\delta$ ,  $\delta > 0$ , is said to be an approximate identity if, for any  $\delta$ ,  $I_\delta$  has transfer function  $I_\delta(s)$  satisfying  $\|I_\delta(s)\|_\infty = 1$  and  $\|(I_\delta(s) - I)\|_\infty \rightarrow 0$  as  $\delta \rightarrow 0$ , see [12]. This is important, as failure to be robust to approximate identities means that the system can be unstable for arbitrarily small uncertainty or delays.

To show this, we write the open-loop and feedback law transfer functions in the target system coordinates. More precisely, we write the full-state feedback law as a causal dynamic output feedback law (for a conveniently chosen output) by using the structure of the target system. First, we recall the definition of *w-stability*.

**Definition 2 (w-stability, [7] Definition 9.5.3)** A stable feedback system  $(G, K)$  is said to be *w-stable* if for any approximate identity  $I_\delta$  there exists a  $\delta_0$  such that  $(I_\delta G, K)$  is input-output stable for all  $\delta \in [0, \delta_0]$ .

We are now ready to state an important result of the paper: when stable, the closed-loop system is *w-stable*.

**Lemma 3** If system (37), (12)–(16) with a control law of the form (10) is stable, then it is *w-stable* in the sense of [12]. In particular, it is robust to small delays in the actuation path.

**Proof 3** The open-loop dynamics of  $\eta_0$  given by (37) rewrite as follows

$$\begin{aligned} s\eta_0 &= A_0\eta_0(s) + \tilde{D}_1^\top \omega_1(s) + B_0\beta(s, 0) + B_1\alpha(s, 1) \\ &\quad - \int_0^1 M(y)\alpha(s, y)dy - \int_0^1 N(y)\beta(s, y)dy + U(s). \end{aligned} \quad (59)$$

Therefore, using Lemma 1 and its proof, the open-loop transfer function reads

$$\alpha(s, 0) = \frac{c_0}{s[1 - Re^{-\tau s}] + H(s)} U(s) \quad (60)$$

$$:= G(s)U(s) \quad (61)$$

where  $R$  is defined by (9) and  $H$  can be computed from (46)–(59) and is proper according to Definition 1. Besides, since  $R$  satisfies (9), one has

$$\limsup_{|s| \rightarrow \infty, s \in \overline{\mathbb{C}^0}} 1 - Re^{-\tau s} > 0, \quad (62)$$

where  $\overline{\mathbb{C}^0}$  denotes the closed left half plane. This essentially means that there is no asymptotic chain of zeros converging towards the Right-Half-Plane, which, in turn, implies that  $G$  is strictly proper. Similarly, using the inverse backstepping transformation and relations (46)–(52), the control law (10) rewrites as

$$U(s) = K(s)\alpha(s, 0) \quad (63)$$

where  $K(s)$  is proper. Therefore, for some sufficiently large  $M > 0$ , the loop transfer function satisfies

$$\sup_{|s| > M} G(s)K(s) < 1 \quad (64)$$

which yields the result using [12, Proposition 16].

One should notice that a crucial element in the proof is that the feedback law is proper, which is not the case of, e.g., the feedback law in [5]. In the case of [9], the non-proper nature of the control law is avoided by replacing the derivative feedback term  $v_t(t, 0)$  with a term of the form  $v_x(t, 0)$  using (4). We have not investigated the robustness properties of such a control law. Lemma 3 guarantees a certain robustness of the design when the closed-loop system is stable.

## 4 Output feedback controller

In this Section, we design an observer using a transformation with a structure dual that is dual to (17), (18), before sketching the closed-loop stability proof for the corresponding output feedback controller.

### 4.1 Observer design

We consider the following Luenberger-like observer

$$\dot{\hat{\omega}}_0(t) = a_0\hat{\omega}_0(t) + b_0\hat{v}(t, 0) + U(t) - p_0(\hat{\omega}_0 - \omega_0) \quad (65)$$

$$\hat{u}(t, 0) = c_0\hat{\omega}_0(t) + d_0\hat{v}(t, 0) - P_0(\hat{\omega}_0 - \omega_0) \quad (66)$$

$$\begin{aligned} \hat{u}_t(t, x) &= -\lambda(x)\hat{u}_x(t, x) + \sigma^{++}(x)\hat{u}(t, x) \\ &\quad + \sigma^{+-}(x)\hat{v}(t, x) - p_u(x)(\hat{\omega}_0 - \omega_0) \end{aligned} \quad (67)$$

$$\begin{aligned} \hat{v}_t(t, x) &= \mu(x)\hat{v}_x(t, x) + \sigma^{-+}(x)\hat{u}(t, x) \\ &\quad + \sigma^{--}(x)\hat{v}(t, x) - p_v(x)(\hat{\omega}_0 - \omega_0) \end{aligned} \quad (68)$$

$$\hat{v}(t, 1) = C_1^\top \hat{\omega}_1(t) + d_1\hat{u}(t, 1) - P_1(\hat{\omega}_0 - \omega_0) \quad (69)$$

$$\dot{\hat{\omega}}_1(t) = A_1\hat{\omega}_1(t) + B_1\hat{u}(t, 1) - p_1(\hat{\omega}_0 - \omega_0). \quad (70)$$

We define the error variables  $\tilde{w} = \hat{w} - w$ . The following invertible mapping

$$\omega_0(t) = \tilde{\eta}_0(t) \quad (71)$$

$$\begin{aligned} \tilde{u}(t, x) = & \tilde{\alpha}(t, x) - \int_0^1 M^{uu}(x, y) \tilde{\alpha}(t, y) dy \\ & + \int_0^1 M^{uv}(x, y) \tilde{\beta}(t, y) dy - H_u(x) \tilde{\eta}_0(t) \end{aligned} \quad (72)$$

$$\begin{aligned} \tilde{v}(t, x) = & \tilde{\alpha}(t, x) - \int_0^1 M^{vu}(x, y) \tilde{\alpha}(t, y) dy \\ & + \int_0^1 M^{vv}(x, y) \tilde{\beta}(t, y) dy - H_v(x) \tilde{\eta}_0(t) \end{aligned} \quad (73)$$

$$\tilde{\omega}_1(t) = \tilde{\eta}_1(t) - \int_0^1 N^u(x, y) \tilde{\alpha}(t, y) dy \quad (74)$$

$$+ \int_0^1 N^v(x, y) \tilde{\beta}(t, y) dy - H_1 \tilde{\eta}_0(t) \quad (75)$$

is such that

$$\dot{\tilde{\eta}}_0(t) = \check{p}_{obs} \tilde{\eta}_0(t) + b_0 \tilde{\beta}(t, 0) \quad (76)$$

$$\tilde{\alpha}(t, 0) = d_0 \tilde{\beta}(t, 0) \quad (77)$$

$$\tilde{\alpha}_t(t, x) = -\lambda(x) \tilde{\alpha}_x(t, x)$$

$$\tilde{\beta}_t(t, x) = \mu(x) \tilde{\beta}_x(t, x)$$

$$\tilde{\beta}(t, 1) = C_1^\top \tilde{\eta}_1(t) + d_1 \tilde{\alpha}(t, 1) \quad (78)$$

$$\dot{\tilde{\eta}}_1(t) = (A_1 - \check{L}_1 C_1) \tilde{\eta}_1(t) \quad (79)$$

provided  $M^{uu}, M^{uv}, M^{vu}, M^{vv}, H^u, H^v, H_1$  satisfy an appropriate set of equation exactly equivalent to (19)–(30) with  $\check{z} = \check{p}$ , and  $\check{L}_1$  is such that  $A_1 - \check{L}_1 C_1$  is Hurwitz. Besides, the observer gains satisfy

$$p_0 = a_0 - b_0 H_v(0) - \check{p}_o b_s \quad (80)$$

$$P_0 = c_0 + d_0 H_v(0) - H_u(0) \quad (81)$$

$$p_u(x) = \sigma^{++} H_u(x) + \sigma^{+-} H_v(x) - \lambda H'_u(x) - \check{p} H_u(x) \quad (82)$$

$$p_v(x) = \sigma^{-+} H_u(x) + \sigma^{--} H_v(x) + \mu H'_v(x) - \check{p} H_v(x) \quad (83)$$

$$p_1 = B_1 H_u(1) + H_1(\check{p}_{obs} - a_1) \quad (84)$$

$$P_1 = d_1 H_u(1) - H_v(1) - C_1^\top H_1 \quad (85)$$

#### 4.2 Output feedback controller

The main result is stated in the following Theorem.

**Theorem 1** Consider System (1)–(6) with the following control law

$$\begin{aligned} U(t) = & k_0 \hat{\omega}_0(t) + k_1 \hat{\omega}_1(t) + K_0 \hat{v}(t, 0) + K_1 \hat{u}(t, 1) \\ & + \int_0^1 [K(y) \hat{u}(t, y) + L(y) \hat{v}(t, y)] dy, \end{aligned} \quad (86)$$

where  $\hat{\omega}_0, \hat{\omega}_1, \hat{v}, \hat{u}$  are defined by (65)–(70). The zero equilibrium is asymptotically stable.

**Proof 4** We only sketch here the proof for brevity's sake. Consider the observer error system in target system coordinates (76)–(79). It is a cascade of  $\tilde{\eta}_1$  into  $(\tilde{\alpha}, \tilde{\beta})$  into  $\tilde{\eta}_0$ , thus it is exponentially stable. Consider now (11)–(16) with the control law (86) expressed in target system coordinates. It rewrites

$$\dot{\eta}_0(t) = \check{p} \eta_0(t) + d_0 \frac{\check{p} - \check{z}}{c_0} \beta(t, 0) + \tilde{U}(t) \quad (87)$$

$$\alpha(t, 0) = c_0 \eta_0(t) + d_0 \beta(t, 0) \quad (88)$$

$$\alpha_t(t, x) + \lambda(x) \alpha_x(t, x) = \sigma^{++}(x) \alpha(t, x) \quad (89)$$

$$\beta_t(t, x) - \mu(x) \beta_x(t, x) = \sigma^{--}(x) \beta(t, x) \quad (90)$$

$$\beta(t, 1) = d_1 \alpha(t, 1). \quad (91)$$

$$\dot{\omega}_1(t) = (A_1 - B_1 \check{\kappa}_1^\top) \omega_1 + B_1 \alpha(t, 1) \quad (92)$$

where

$$\tilde{U}(t) = k_0 \tilde{\omega}_0(t) + k_1 \tilde{\omega}_1(t) + K_0 \tilde{v}(t, 0) + K_1 \tilde{u}(t, 1) \quad (93)$$

$$+ \int_0^1 [K(y) \tilde{u}(t, y) + L(y) \tilde{v}(t, y)] dy \quad (94)$$

vanishes to zero since the observer error converges asymptotically to zero. Besides, the target system (87)–(92) is linear, which implies by Lemma 1 and Proposition 3 in [8] that it is Input-to-State Stable (ISS) wrt to the input  $\tilde{U}$ . Therefore, target system (87)–(92) is asymptotically stable.

## 5 Numerical example

To illustrate the result we consider the following toy example

$$\dot{\omega}_0(t) = 0.2 \omega_0(t) + v(t, 0) + U(t) \quad (95)$$

$$u(t, 0) = \omega_0(t) + 0.9 v(t, 0) \quad (96)$$

$$u_t(t, x) + u_x(t, x) = -0.01 u(t, x) - 0.05 v(t, x) \quad (97)$$

$$v_t(t, x) - v_x(t, x) = 0.1 u(t, x) - 0.01(x) v(t, x) \quad (98)$$

$$v(t, 1) = \begin{bmatrix} 1 & 0 \end{bmatrix} \omega_1(t) - 0.9 u(t, 1) \quad (99)$$

$$\dot{\omega}_1(t) = \begin{bmatrix} 0.1 & 1 \\ -1 & 0 \end{bmatrix} \omega_1(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v(t, 1). \quad (100)$$

This plant has unstable ODEs at both boundaries as well as an instability caused by the in-domain source terms.

For this plant we design a stabilizing collocated output injection controller using an observer with  $\omega_0$  as the measurement. For the observer we use the design parameters  $\kappa_1 = \begin{bmatrix} 3 & 1 \end{bmatrix}$ ,  $\check{z} = 0$ ,  $\check{p} = -1000$ . For the controller

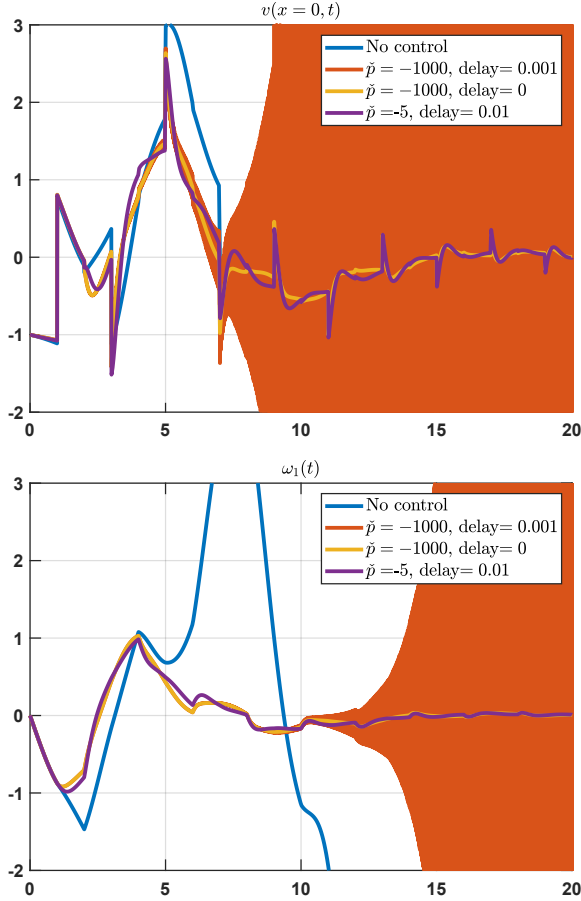


Fig. 2. Toy example trends.

we use  $\kappa_1 = [1.5 \ 0.6]$ ,  $\dot{z} = 0$ , while for the proximal pole placement we consider two cases:  $\tilde{p} = -1000$  and  $\tilde{p} = -5$ . The results are shown in Fig. 5. As is clearly illustrated in this Figure, increasing the magnitude of the  $\tilde{p}$  pole yields better performance, but at the cost of a more aggressive control effort and lower robustness to actuation delays. The limit where  $\tilde{p} \rightarrow -\infty$  corresponds to the approach of using a full cancellation of the reflection at the proximal boundary, but this also yields zero delay robustness.

## Acknowledgments

The work of the third author was supported by the Research Council of Norway, ConocoPhillips, AkerBP, Statoil and Wintershall through the research center DrillWell (203525/O30) at IRIS, and by the Marie Skłodowska-Curie (MSCA) - Individual Fellowships through the Robust Estimation and Control of Infinite Dimensional Systems (RECIDS) Project.

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## A Open-loop dynamics of the target system

We detail here the open-loop dynamics of the target system. First, one has to compute the inverse backstepping transformation  $(\eta_0, \alpha, \beta, \omega_1) \mapsto (\omega_0, u, v, \omega_1)$ . To do so, notice that (17),(18) are Volterra equations in  $u$  and  $v$ .

Denoting  $L = \begin{pmatrix} L^{uu} & L^{uv} \\ L^{vu} & L^{vv} \end{pmatrix}$  the inverse kernels and omitting the time argument, one has

$$\begin{aligned} u(x) = & \alpha(x) - \int_x^1 L^{uu}(x, y)\alpha(y) + L^{uv}(x, y)\beta(y)dy \\ & + \left[ G_u(x)^\top - \int_x^1 L^{uu}(x, y)G_u(y)^\top dy \right. \\ & \left. - \int_x^1 L^{uv}(x, y)G_v(y)^\top dy \right] \omega_1(t) \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} v(x) = & \beta(x) - \int_x^1 L^{vu}(x, y)\alpha(y) + L^{vv}(x, y)\beta(y)dy \\ & + \left[ G_v(x)^\top - \int_x^1 L^{vu}(x, y)G_u(y)^\top dy \right. \\ & \left. - \int_x^1 L^{vv}(x, y)G_v(y)^\top dy \right] \omega_1(t) \end{aligned} \quad (\text{A.2})$$

that we rewrite

$$u(x) = \alpha(x) - \int_x^1 L^{uu}(x, y)\alpha(y) + L^{uv}(x, y)\beta(y)dy + H_u(x)^\top \omega_1(t) \quad (\text{A.3})$$

$$v(x) = \beta(x) - \int_x^1 L^{vu}(x, y)\alpha(y) + L^{vv}(x, y)\beta(y)dy + H_v(x)^\top \omega_1(t) \quad (\text{A.4})$$

Plugging into (31) yields, similarly, an expression of the form

$$\begin{aligned} \omega_0(t) = & \eta_0(t) - \int_0^1 \tilde{L}^\alpha(y)\alpha(t, y) + \tilde{L}^\beta(y)\beta(t, y)dy \\ & - \tilde{H}^\top \omega_1 \end{aligned} \quad (\text{A.5})$$

Finally, plugging (A.3)–(A.5) into (36) yields (37) with

$$A_0 = a_0 - \lambda(0)\tilde{K}_\alpha(0) \quad (\text{A.6})$$

$$B_0 = b_0 + c_0^{-1} \left( \mu(0)\tilde{K}_\beta(0) - \lambda(0)d_0\tilde{K}_\alpha(0) \right) \quad (\text{A.7})$$

$$B_1 = c_0^{-1} \left[ -\tilde{G}^\top b_1 + \lambda(1)\tilde{K}_\alpha(1) - \mu(1)d_1\tilde{K}_\beta(1) \right] \quad (\text{A.8})$$

$$\begin{aligned} \tilde{D}_1^\top = & -c_0^{-1} \left[ \tilde{G}^\top A_1 + C_1^\top \mu(1)\tilde{K}_\beta(1) \right] \\ & - A_0 \tilde{H}^\top + B_0 H_v(0)^\top + B_1 H_u(1)^\top \\ & + \int_0^1 \tilde{M}(x)H_u(x)^\top + \tilde{N}(x)H_v(x)^\top dx \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} M(x) = & \tilde{M}(x) - A_0 L^\alpha(y) - B_0 L^{vu}(0, y) \\ & - \int_0^x \tilde{M}(y)L^{uu}(y, x) + N(y)L^{vu}(y, x)dy \end{aligned} \quad (\text{A.10})$$

$$\begin{aligned} N(x) = & \tilde{N}(x) - A_0 L^\beta(y) - B_0 L^{vv}(0, y) \\ & - \int_0^x \tilde{M}(y)L^{uv}(y, x) + \tilde{N}(y)L^{vv}(y, x)dy \end{aligned} \quad (\text{A.11})$$

$$(\text{A.12})$$

where

$$\begin{aligned} \tilde{M}(x) = & -c_0^{-1} \left[ \lambda(y)\tilde{K}'_\alpha(y) + \lambda'(y)\tilde{K}_\alpha(y) \right. \\ & \left. + \sigma^{++}(y)\tilde{K}_\alpha(y) + \sigma^{-+}(y)\tilde{K}_\beta(y) \right] \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned} \tilde{N}(x) = & -c_0^{-1} \left[ -\mu(y)\tilde{K}'_\beta(y) + \lambda'(y)\tilde{K}_\beta(y) \right. \\ & \left. + \sigma^{--}(y)\tilde{K}_\beta(y) + \sigma^{+-}(y)\tilde{K}_\alpha(y) \right] \end{aligned} \quad (\text{A.14})$$