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# A filtered high gain observer for a class of non uniformly observable systems - Application to a phytoplanktonic growth model

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**Abstract:** The design of a high gain observer with filtering capabilities for a class of non uniformly observable systems is proposed. The filtering feature enables to perform a smooth estimation in the presence of noisy output measurements. Of fundamental interest, the Filtered High Gain Observer (FHGO) design is first carried out by assuming that the output measurements are available in a continuous manner before being extended to the case of sampled outputs leading thereby to a continuous-discrete time FHGO which inherits the main properties of the original FHGO. The performance of the FHGO and its main properties are highlighted and compared to those of a Standard High Gain Observer (SHGO) through a bioprocess dealing with the growth of a phytoplanktonic population.

**keywords** Nonlinear systems, Standard high gain observers, Filtered high gain observers, phytoplanktonic.

## 1 Introduction

Since the eighties, we have witnessed the emergence of nonlinear state observers performing an admissible estimation of the component concentrations inside bioreactors (see e.g. [1] and references therein). These observers have been designed using various approaches that have been progressively developed for specific classes of nonlinear systems using various design principles (See for instance [2, 3, 4, 5, 6]). The high gain principle has shown to provide an appealing approach to design observers for nonlinear systems as pointed out in [5]. Such an attraction is motivated by implementation simplicity and fundamental considerations. Indeed, the Standard High Gain Observer (SHGO) structure is very simple since it consists in adding a copy of the system dynamics with a corrective involving an observer gain which is essentially parameterized by a positive design parameter  $\theta$ . Of a fundamental interest, one can naturally recover the genuine separation principle for nonlinear systems when designing output feedback control systems incorporating a SHGO (see [7] and references therein). However, the design parameter  $\theta$  has to be taken high enough for observer convergence purposes and it intervenes with positive powers in the observer gain, leading thereby to three well known issues. The first deals with a numerical implementation issue occurring for systems with high dimensions when relatively high values of  $\theta$  are required. The second issue consists in the intrinsic sensitivity of SHGO to the non avoidable output measurement noise. The third issue is the peaking phenomena of the state variable estimates, the magnitude of which is proportional to an appropriate power of  $\theta$ , leading to poor transient behavior of SHGO.

An important research activity has been recently devoted to alleviate the SHGO sensitivity to measurement noise using an appropriate filtering. The seminal contribution has been proposed in [8] for a class of single output nonlinear systems of dimension  $n \geq 3$ . The filtering feature is mainly used to limit the observer gain power to 2, reducing thereby the amplification of noise measurements. Two design features are worth to be emphasized. Firstly, the dimension of the proposed observer is equal to  $2(n-1)$ . Secondly, the observer provides an estimate of the first and last components of the system state whereas the intermediary states are provided by a couple of estimates. This contribution has been subsequently reconsidered in [9] and [10] where redesigned versions involving nested saturations have been proposed to deal with the peaking phenomena. A nice contribution has been proposed in [7], it consists in a redesigned SHGO having the same dimension as the original system with two important design features, namely the power of the observer design parameter is limited to one and the peaking phenomena is well reduced thanks to nested saturation functions. Further developments related to the design of SHGO with filtering capabilities have been carried out in [11, 12] for a class of systems with dimension  $n \geq 2$ . The underlying observers consist in two cascaded sub-systems each one of which has the same dimension as the original system. The first subsystem is a copy of the original system with a simple correction term which (linearly) depends on the state of the second subsystem. The latter subsystem is a linear filter driven by the output observation error, namely the error between the output measurements and an output prediction provided by the first subsystem. As in [8], the power of the observer gain is limited to 2.

It is worth noticing that all the high gain observers incorporating filtering capabilities, cited above, were proposed for uniformly observable systems, i.e. systems which are observable for any input. Up to the authors best knowledge, no equivalent solution has been proposed for non uniformly observable systems, i.e. systems where the considered inputs have to satisfy an adequate persistent excitation condition to ensure their observability. In this paper, one aims at providing a new Filtered High Gain Observer (FHGO) for a class of Multi-Input/Multi-Output (MIMO) non uniformly observable nonlinear systems. The underlying design is quite different from those given in [11, 12] for uniformly observable systems, namely the observer gain is issued from the resolution of a Riccati or Lyapunov Ordinary Differential Equation (ODE).

The paper is organized as follows. A comprehensive presentation of the class of systems to be considered together with an appropriate SHGO are given in section 2, constituting thereby the preliminaries of the paper. Section 3 is devoted to the FHGO design assuming that the output measurements are continuously available with a particular emphasis on a certain persistent excitation condition required for the observer convergence. The redesigned version of the FHGO dealing with sampled outputs is given in section 4 with a particular emphasis on the admissible sampling process. Section 5 is dedicated to an application, namely an admissible estimation of the component concentrations within a bioreactor in a realistic simulation framework. A particular attention is paid on the FHGO performance with respect to the SHGO in the presence of noise measurements. For clarity purposes, the convergence analysis related to the FHGO with continuous output measurements as well as with sampled ones are given in Appendix.

Throughout the paper, for any positive integers  $k$  and  $m$ ,  $I_k$  and  $0_k$  denote the  $k$ -dimensional identity and zero matrices respectively,  $0_{k \times m}$  is the  $k \times m$  zero matrix,  $\|\cdot\|$  denotes the euclidian norm and for any Symmetric Positive Definite (SPD) time-varying matrix  $Q(t)$ ,  $\lambda_M(Q(t))$  (resp.  $\lambda_m(Q(t))$ ) will be used to denote the largest (resp. the smallest) eigenvalue of  $Q(t)$  and  $\bar{\lambda}_M(Q(t)) = \sup_{t \geq t_0} \lambda_M(Q(t))$ ,  $\underline{\lambda}_m(Q(t)) = \min_{t \geq t_0} \lambda_m(Q(t))$  where  $t_0$  is any fixed non negative real number. Moreover, the arguments will be omitted when clear from the context.

## 2 Preliminaries

Recall that the ultimate motivation of this paper consists in addressing the SHGO sensitivity with respect to the ubiquitous noise measurements for a class of non uniformly observable systems. More specifically, we will consider the following class of nonlinear MIMO systems

$$\begin{cases} \dot{x}(t) = F(u(t), x(t))x(t) + \varphi(u(t), x(t)) + B\varepsilon(t) \\ y(t) = Cx(t) + w(t) \end{cases} \quad (1)$$

with

$$x = \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(q)} \end{pmatrix} \quad \text{with } x^{(k)} \in \mathbb{R}^{n_k} \text{ and } \sum_{k=1}^q n_k = n, \quad (2)$$

$$F(u, x) = \begin{pmatrix} 0 & F_1(u, x) & 0 & 0 \\ \vdots & & F_2(u, x) & \\ \vdots & & & 0 \\ 0 & & 0 & F_{q-1}(u, x) \\ 0 & & & 0 \end{pmatrix} \quad \text{with } F_k(u, x) = F_k(u, x^{(1)}, \dots, x^{(k)}), \quad (3)$$

$$\varphi(u, x) = \begin{pmatrix} \varphi^{(1)}(u, x) \\ \varphi^{(2)}(u, x) \\ \vdots \\ \varphi^{(q-1)}(u, x) \\ \varphi^{(q)}(u, x) \end{pmatrix} \in \mathbb{R}^n \quad \text{with } \varphi^{(k)}(u, x) = \varphi^{(k)}(u, x^{(1)}, \dots, x^{(k)}), \quad (4)$$

$$B = \begin{pmatrix} 0_{n_1 \times n_q} \\ \vdots \\ 0_{n_{q-1} \times n_q} \\ I_{n_q} \end{pmatrix}, C = ( I_{n_1} \quad 0_{n_1 \times n_2} \quad \cdots \quad 0_{n_1 \times n_q} ), \quad (5)$$

where the state  $x(t) \in \mathbb{R}^n$ , each  $F_k(u, x)$  is a  $n_k \times n_{k+1}$  matrix which is triangular w.r.t.  $x$ , i.e.  $F_k(u, x) = F_k(u, x^{(1)}, \dots, x^{(k)})$ ,  $k = 1, \dots, q-1$ ,  $\varphi(x(t), u(t))$  is a nonlinear vector function that has a triangular structure w.r.t.  $x$ ;  $u \in \mathbb{R}^s$  denotes the system input,  $y \in \mathbb{R}^p$  is the system output;  $\varepsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_q}$ , is an unknown function which denotes the system uncertainties and  $w : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_q}$  is the output noise.

The observer design will be performed under the following assumptions.

- A1.** The state  $x(t)$  and the control  $u(t)$  are bounded, i.e.  $x(t) \in X$  and  $u(t) \in U$  where  $X \subset \mathbb{R}^n$  and  $U \subset \mathbb{R}^s$  are compact sets. One sets  $x_M = \sup_{x \in X} \|x\|$ .
- A2.** Each  $\varphi^k(u, x)$ ,  $k = 1, \dots, q$  and each  $F_k(u, x)$ ,  $k = 1, \dots, q-1$ , is Lipschitz on  $X$  with respect to  $x$  uniformly in  $u$ , i.e.

$$\exists L_\varphi > 0 / \forall u \in U; \forall (x, \bar{x}) \in X \times X, \text{ one has } \|\varphi^{(k)}(u, x) - \varphi^{(k)}(u, \bar{x})\| \leq L_\varphi \|x - \bar{x}\|,$$

$$\exists L_F > 0 / \forall u \in U; \forall (x, \bar{x}) \in X \times X, \text{ one has } \|F_k(u, x) - F_k(u, \bar{x})\| \leq L_F \|x - \bar{x}\|.$$

**A3.** The unknown uncertainty  $\varepsilon$  and the noise signal  $w$  are essentially bounded functions, i.e.

$$\exists (\delta_\varepsilon, \delta_w) \in \mathbb{R}^+ \times \mathbb{R}^+ / \|\varepsilon\|_\infty \stackrel{\Delta}{=} \text{ess sup}_{t \geq 0} \|\varepsilon(t)\| \leq \delta_\varepsilon \text{ and } \|w\|_\infty \stackrel{\Delta}{=} \text{ess sup}_{t \geq 0} \|w(t)\| \leq \delta_w. \quad (6)$$

**Remark 2.1** *Since the state is bounded, one can construct global Lipschitz prolongations of the nonlinearities using saturation functions ([13]). To avoid symbol redundancy, one shall assume as in [14] that such prolongations have been achieved and that the functions  $F$  and  $\varphi$  result from these prolongations.*

System (1) has been already considered in [15] in the free uncertainties and noise measurements case and a SHGO has been designed under a set of assumptions amongst which **A1** to **A3**. Similarly, this system belongs to the class of systems considered in [16] and [17] where a SHGO has been proposed in the absence of noise measurements. For comparison purposes and in order to highlight the main steps of the FHGO design which shall be detailed later, let us recall the equations of a SHGO proposed in the above references ([16, 17])

$$\dot{\hat{x}}(t) = F(u, \hat{x})\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta \Delta_\theta^{-1} P(t) C^T (\hat{x}(t) - y(t)), \quad (7)$$

with

$$\Delta_\theta = \text{diag} \left[ I_{n_1} \quad \frac{1}{\theta} I_{n_2} \quad \cdots \quad \frac{1}{\theta^{q-1}} I_{n_q} \right], \quad (8)$$

where  $\hat{x} = \begin{pmatrix} \hat{x}^{(1)} \\ \vdots \\ \hat{x}^{(q)} \end{pmatrix} \in \mathbb{R}^n$ , with  $x^{(k)} \in \mathbb{R}^{n_k}, k = 1, \dots, q$ , denotes the state estimate and  $P(t)$  is a  $n \times n$  symmetric matrix governed by the following Riccati ODE

$$\begin{cases} \dot{P}(t) &= \theta (P(t) + F(u, \hat{x})P(t) + P(t)F^T(u, \hat{x}) - P(t)C^T C P(t)), \\ P(0) &= P^T(0) > 0. \end{cases} \quad (9)$$

Recall that the convergence of the underlying observation error has been established under Assumptions **A1** to **A3**, together with an appropriate persistent excitation condition similar to the one which shall be considered later when designing the FHGO.

### 3 The FHGO design

Recall that one aims at alleviating the sensitivity of the SHGO (7) to the ubiquitous noise measurements. A suitable filtering process is used to this end by cascading two subsystems. The first subsystem corresponds to a copy of the original system, with a (time-varying) corrective term that linearly depends on the state of the second subsystem. The latter is a linear filter the input of which is the difference between the output predicted by the observer and the output measurement, namely the output observation error. The proposed observer is given by the following set of equations

$$\begin{cases} \dot{\hat{x}}(t) &= F(u, \hat{x})\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta K(t)\eta(t), \\ \dot{\eta}(t) &= \delta\theta \{-\eta(t) + \theta A^T \eta(t) + C_{qn_1}^T (C\hat{x}(t) - y(t))\}, \quad \eta(0) = 0, \end{cases} \quad (10)$$

where  $\hat{x} = \begin{pmatrix} \hat{x}^{(1)} \\ \vdots \\ \hat{x}^{(q)} \end{pmatrix} \in \mathbb{R}^n$  with  $x^{(k)} \in \mathbb{R}^{n_k}, k = 1, \dots, q$ ,  $\eta = \begin{pmatrix} \eta^{(1)} \\ \vdots \\ \eta^{(q)} \end{pmatrix} \in \mathbb{R}^{qn_1}$  with  $\eta^{(k)} \in \mathbb{R}^{n_1}$  and  $A$  is the  $qn_1 \times qn_1$  anti-shift matrix, i.e.

$$A = \begin{bmatrix} 0_{n_1} & I_{n_1} & 0_{n_1} & 0_{n_1} \\ \vdots & \ddots & \ddots & 0_{n_1} \\ 0_{n_1} & \cdots & \cdots & I_{n_1} \\ 0_{n_1} & \cdots & \cdots & 0_{n_1} \end{bmatrix}, \quad C_{qn_1}^T = \begin{bmatrix} I_{n_1} \\ 0_{n_1} \\ \vdots \\ 0_{n_1} \end{bmatrix}. \quad (11)$$

The terms  $\delta$  and  $\theta > 0$  are positive real design parameters and finally,  $K(t)$  is the following diagonal matrix

$$K(t) = \text{diag}(P(t)C^T) \triangleq \text{diag}\left(K^{(1)}(t) \dots K^{(q)}(t)\right), \quad (12)$$

where each  $K^{(i)}$  is  $n_i \times n_1$  matrix and  $P(t)$  is a  $n \times n$  symmetric matrix governed by the following Riccati ODE

$$\begin{cases} \dot{P}(t) &= \theta \left( P(t) + F(u, \hat{x})P(t) + P(t)F^T(u, \hat{x}) - P(t)C^T C P(t) \right), \\ P(0) &= P^T(0) > 0. \end{cases} \quad (13)$$

Besides Assumptions **A1** to **A3**, the following assumption is needed for the observer design:

**A4.** The Riccati ODE (13) has a positive definite solution that satisfies the following property

$$\exists \rho^* > 0; \exists \theta^* > 0; \quad \forall \theta > \theta^*; \exists t^* \geq 0; \forall t \geq t^*, (1/\bar{\lambda}_M(P(t))) \geq (\rho^*/\alpha(\theta))I_n, \quad (14)$$

where  $\alpha(\theta)$  is a positive function satisfying

$$\forall \theta > 0, \alpha(\theta) \geq 1 \quad \text{and} \quad \exists 0 < \chi \leq 1, \quad \lim_{\theta \rightarrow \infty} \frac{\alpha(\theta)}{\theta^{2\chi}} = 0. \quad (15)$$

Notice that the dynamics of the Riccati equation (13) involved in Assumption A4 depends on the state estimate  $\hat{x}$  as in [18, 19] and not on the state  $x$ . At a first glance, it would seem more natural to express Assumption A4 using the state rather than its estimate. But this would lead to a non checkable assumption since the state is not available. Considering the observer states instead of the system states, as formulated in Assumption A4, would allow to check it at least on-line, even though not *a priori*. Indeed, Assumption A4 can be checked on-line by simply computing the inverse of the largest instantaneous eigenvalue of  $P(t)$ .

On other aspects, Assumption A4, which is similar to that considered in [18, 19], is of a primary importance for the stability of the observer. Indeed, as noted in [18], this assumption is satisfied for uniformly observable systems, i.e. systems which are observable for any input. For non uniformly observable systems, the characterization of the class of inputs which satisfy Assumption A4 is still an open problem despite that some interesting results dealing with some subclasses of systems (1) have already been obtained. Indeed, the authors in [20] introduced the notion of local regular inputs which are defined as those satisfying Assumption A4 with  $\alpha(\theta) \equiv 1$  for all  $\theta$ . However, these inputs were defined for a class of systems included in (1), i.e. the matrices  $F_k$  do not depend on the state. In [21], the class of systems considered in [20] was revisited and the authors enlarged the class of locally regular inputs to the class of regular inputs. In the spirit of the persistent excitation conditions proposed in [20, 21], a similar condition which allows Assumption **A4** to be satisfied has been formulated in [16, 17] where some classes of systems similar to (1) have been considered with a view to observer design in

the free noise case. As in [20, 21], the underlying formulation used the transition matrix  $\Phi_{u,\hat{x}}(t, s)$  of the following state affine system  $\dot{\xi}_{u,\hat{x}}(t) = A(u, \hat{x})\xi_{u,\hat{x}}(t)$  where  $\xi_{u,\hat{x}} \in \mathbb{R}^n$  is the system state,  $u$  and  $\hat{x}$  are the inputs of the system and they respectively correspond to the input of system (1) and to the state of the dynamical system (10). Recall that the matrix  $\Phi_{u,\hat{x}}(t, s)$  is defined as follows

$$\begin{cases} \frac{d}{dt}(\Phi_{u,\hat{x}}(t, s)) &= A(u(t), \hat{x}(t))\Phi_{u,\hat{x}}(t, s), \\ \Phi_{u,\hat{x}}(s, s) &= I_n. \end{cases} \quad (16)$$

The persistent excitation condition was formulated as follows in [16, 17]

•  $\exists \theta_0 > 0; \exists \rho_0 > 0; \exists T^* > 0; \forall \theta \geq \theta_0; \forall t \geq \frac{T^*}{\theta}$ , one has

$$\int_{t-\frac{T^*}{\theta}}^t \Phi_{u,\hat{x}}^T(s, t) C^T C \Phi_{u,\hat{x}}(s, t) ds \geq \frac{\rho_0}{\theta \alpha(\theta)} \Delta_\theta^2, \quad (17)$$

where the function  $\alpha(\theta)$  satisfies (15) and  $\Delta_\theta$  is given by (8).

It has been shown in [16, 17] that Assumption **A4** is satisfied under the persistent excitation condition (16)-(17) and one has

$$\theta^* = \theta_0, \quad t^* = \frac{T^*}{\theta} \text{ and } \rho^* = e^{-T^*} \rho_0.$$

Before stating the theorem where the main properties of observer (10) are summarized, one shall compare the structure of this observer with that of the SHGO (7) in order to put forward the rational

behind the proposed design. Let us first focus on the observer sub-state  $\eta = \begin{pmatrix} \eta^{(1)} \\ \vdots \\ \eta^{(q)} \end{pmatrix}$ . The first

component  $\eta^{(1)}$  is a filtered version of the output observation error  $\hat{x}^{(1)} - y$  with a filter of order 1 and the static gain of which does not depend on  $\theta$ . For  $i \geq 2$ , the component  $\eta^{(i)}$  is also a filtered version of  $\eta^{(i-1)}$  with a filter of order 1 and the static gain of which is proportional to  $\theta$ . The component  $\eta^{(i)}$  is therefore a filtered version of  $\hat{x}^{(1)} - y$  with a filter of order  $i$  and the static gain of which is proportional to  $\theta^{i-1}$ . Now, the ODE associated to the state  $\hat{x}_i$  in the FHGO (10) is obtained from that given by the SHGO (7) by replacing the corrective term  $\theta^i K^{(i)}(\hat{x}_1 - y)$  by its filtered version, namely  $\theta K^{(i)} \eta_i$ . Two features are worth to be mentioned. The first one deals with the fact that the power of the design parameter  $\theta$  does not exceed 2 in the FHGO. The second feature deals with the fact the substitution of the output observation error by an appropriate filtered version in the state estimate equations will definitely improve the behavior of the FHGO with respect to the output measurements noise.

Now, the main properties of the proposed observer (10) are summarized in the following theorem the proof of which is detailed in Appendix.

**Theorem 3.1** *Consider system (1) subject to assumptions **A1** to **A3**. Then, for every bounded input satisfying **A4**, there exist  $\delta^*, \theta^*$ , such that for all  $\theta \geq \theta^*$  and for all  $\delta > \max(\delta^*, \theta^x)$ , one has for all  $t \geq t^*$ ,*

$$\begin{aligned} \|\tilde{x}(t)\| &\leq \beta_f \theta^{q-1} \sqrt{\alpha(\theta)} e^{-\nu \frac{\theta}{2} \mu(\theta)(t-t^*)} ((1 + \sqrt{n}) \|\tilde{x}(t^*)\| + \|\eta(t^*)\|) \\ &\quad + 2 \frac{\beta_f}{\nu \mu(\theta)} \left( \sqrt{\frac{\alpha(\theta)}{\theta^2}} \delta_\varepsilon + \sqrt{\alpha(\theta)} \delta \theta^{q-1} \delta_w \right). \end{aligned} \quad (18)$$

where  $\tilde{x} = \hat{x} - x$  with  $\hat{x}$  being any trajectory of system (10),  $\delta_w$  and  $\delta_\varepsilon$  are the ultimate bounds of the uncertainties given by Assumption **A3**,  $\beta_f$  and  $\nu$  are positive constants independent of  $\theta$  and  $\delta$ ; the

positive reals  $t^*$ ,  $\rho^*$ ,  $\chi$  and the function  $\alpha$  are given by Assumption **A4** and finally  $\mu(\theta) > 0$  is such that  $\lim_{\theta \rightarrow \infty} \mu(\theta) = 1$ .

**Remark 3.1** Notice that in the uncertainties free case and in the absence of measurement noise, the observation error exponentially converges to zero. Moreover, in the presence of uncertainties and in the free noise case, the asymptotic estimation error can be made as small as desired by choosing  $\theta$  high enough.

## 4 The FHGO redesigned version with sampled outputs

The observer designed above assumes that the outputs are available in a continuous manner. This is rarely the case in practice since these outputs are generally available only at sampling time instants  $0 \leq t_0 < \dots < t_k < t_{k+1} < \dots$  with (time-varying) sampling intervals  $\tau_k = t_{k+1} - t_k$  and  $\lim_{k \rightarrow \infty} t_k = +\infty$ . In the sequel, one shall propose a redesigned version of the above observer that accounts for the output sampling process. For this sake, one naturally assumes that the time intervals  $\tau_k$ 's are bounded away from zero by  $\tau_m$  and are upperly bounded by the upper bound of the sampling partition diameter  $\tau_M$ , i.e.

$$0 < \tau_m \leq \tau_k = t_{k+1} - t_k \leq \tau_M, \quad \forall k \geq 0. \quad (19)$$

The approach which shall be adopted to redesign the continuous time output FHGO is similar to that proposed in [14]. One needs the following additional hypothesis on the boundedness of the noise samples  $w(t_k)$ .

**A5.** For all  $t_k$ , the samples  $w(t_k)$  are bounded by  $\delta_w$  where  $\delta_w$  is the essential bound given by Assumption **A3**.

The redesigned observer takes the following form

$$\begin{cases} \dot{\hat{x}}(t) &= F(u, \hat{x})\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta K(t)\eta(t), \\ \dot{\eta}(t) &= \delta\theta(-\eta(t) + \theta A^T \eta(t) + C_{q_{n_1}}^T (C\hat{x}(t_k) - y(t_k) + \alpha(t))), \\ \dot{\alpha}(t) &= -\delta\theta C P(t) C^T C_{q_{n_1}} \eta(t) = -\delta\theta K^{(1)}(t)\eta^{(1)}(t), \quad t \in [t_k, t_{k+1}[, \\ \eta(0) &= 0 \text{ and } \alpha(t_k) = 0, k \geq 0. \end{cases} \quad (20)$$

Notice that, the above continuous-discrete time observer involves a new state  $\alpha$  governed by an ODE which is re-initialized at zero at each sampling instant  $t_k$ .

The main properties of the proposed observer (20) are summarized in the following theorem the proof of which is given in Appendix.

**Theorem 4.1** Consider system (1) subject to assumptions **A1**, **A2**, **A3** and **A5**. Then, for every bounded input satisfying **A4**, there exist  $\delta^*$ ,  $\theta^*$ , such that for all  $\theta \geq \theta^*$  and for all  $\delta > \max(\delta^*, \theta^*)$ , there exist positive constants  $\varrho_\theta > 0$ ,  $\eta_\theta > 0$  and  $N_\theta$  such that if the upper bound of the sampling partition diameter  $\tau_M$  is such that  $\tau_M < \varrho_\theta$ , then for every  $u \in U$  and every  $\hat{x}(0) \in X$ , one has for all  $t \geq t^*$ ,

$$\begin{aligned} \|\tilde{x}(t)\| &\leq \beta_f \sqrt{\alpha(\theta)} \theta^{q-1} e^{-\zeta_\theta(t-t^*)} \left( (1 + \sqrt{n}) \|\tilde{x}(t^*)\| + \|\eta(t^*)\| \right) \\ &\quad + \beta_f N_\theta(\tau_m, \tau_M) \left( \sqrt{\frac{\alpha(\theta)}{\theta^2}} \delta_\varepsilon + \sqrt{\alpha(\theta)} \delta \theta^{q-1} \delta_w \right), \end{aligned} \quad (21)$$



where  $\tilde{x} = \hat{x} - x$  with  $\hat{x}$  being any trajectory of system (20),  $\delta_w, \delta_\varepsilon, \beta_f, \nu, t^*, \rho^*, \chi, \alpha(\theta)$  and  $\mu(\theta) > 0$  are as given in Theorem 3.1; the expressions  $\varrho_\theta, \zeta_\theta$  and  $N_\theta$  are

$$\begin{aligned} \varrho_\theta &= \frac{\beta_f \sqrt{\alpha(\theta)} (\theta F_M + L_\varphi)}{2\nu\theta\mu(\theta)}, \quad \zeta_\theta = a_\theta \left(1 - \frac{\tau_M}{\varrho_\theta}\right) e^{-a_\theta \tau_m}, \\ a_\theta &= \frac{\nu\theta\mu(\theta)}{2}, \quad N_\theta(\tau_m, \tau_M) = \theta\tau_M \frac{2 - e^{-\zeta_\theta \tau_m}}{1 - e^{-\zeta_\theta \tau_m}}, \end{aligned} \quad (22)$$

where  $F_M = \sup_{x \in X, u \in U} F(u, x)$  and  $L_\varphi$  is the Lipschitz constant of  $\varphi$  given by Assumption A2.

The following remark provides insights about the observer dynamics and the sampling process specification according to the above fundamental result.

**Remark 4.1** *The bound of the estimation error related to the continuous-discrete time observer, i.e. the right side of inequation (21) is very similar to that derived in the continuous output case, i.e. the right side of inequality (18), with the following two differences:*

- (i) *The rate of the exponential decay is equal to  $a_\theta = \nu\theta\mu(\theta)/2$  in the continuous output case while it is equal to  $\zeta_\theta$  when the output is sampled.*
- (ii) *The term  $2/(\nu\mu(\theta))$  involved in the expression of the ultimate bound in the continuous output case is replaced by  $N_\theta(\tau_m, \tau_M)$  when the output is sampled.*

Proceeding as in [14], one can show that in the case of a constant sampling period, i.e.  $\tau_M = \tau_m = T_s$ ,  $\zeta_\theta(T_s)$  is a decreasing function of  $T_s$  and  $\lim_{T_s \rightarrow 0} \zeta_\theta(T_s) = a_\theta$ . One can also show that  $N_\theta(T_s)$  is a non decreasing function of  $T_s$  and that  $\lim_{T_s \rightarrow 0} N_\theta(T_s) = 2/(\nu\mu(\theta))$ . Hence, the results obtained in the sampled output case are in accordance with those derived in the continuous output one: the decreasing to zero of the observation error is inversely proportional to the magnitude of the sampling period while the value of the ultimate bound is proportional to this magnitude. More specifically, when the sampling period tends to zero, the expressions for the decay rate and ultimate bound become identical to those derived in the continuous output case.

## 5 Application to a phytoplanktonic growth model

This section is devoted to the use of the proposed FHGO in a realistic simulation framework involving a phytoplanktonic growth model that has been validated in [22] through two high gain observers one of which is similar to (7) that uses continuous-time output measurements.

### 5.1 The phytoplanktonic growth model

One considers a bioreactor where a biomass (phytoplanktonic cells) with a concentration  $N$  is growing by consuming a substrate with a concentration  $S$ . The bioreactor is functioning under a continuous mode with a dilution rate  $D$  and an input substrate concentration  $S_{in}$ . A mathematical model which accounts for the biomass and substrate concentrations as well as the cell quota of assimilated nutrient,  $Q$ , has been considered in [22] and it can be written as follows

$$\begin{cases} \dot{N}(t) &= \mu_m \left(1 - \frac{K_Q}{Q(t)}\right) N(t) - D(t)N(t) \\ \dot{Q}(t) &= \rho_m \frac{S(t)}{K_\rho + S(t)} - \mu_m(Q(t) - K_Q) \\ \dot{S}(t) &= -\rho_m \frac{S(t)N(t)}{K_\rho + S(t)} + D(t)(S_{in} - S(t)) \end{cases} \quad (23)$$

where  $\mu_m, K_Q, \rho_m, K_\rho$  are constant kinetic parameters.

The biomass concentration is measured at equally spaced sampling instants and the observation objective is to provide continuous estimates of the biomass and substrate concentrations  $X$  and  $S$  and in particular the cell quota  $Q$  which can only be estimated with manual sampling and high cost measurements.

As in [22] and for writing convenience, one first adimensionalizes the state variables by considering the following changes

$$x_1 = \frac{\rho_m N}{S_{in}}, \quad x_2 = \frac{Q}{K_Q}, \quad x_3 = \frac{S}{S_{in}}, \quad a_1 = \frac{K_\rho}{S_{in}}, \quad a_2 = \mu_m, \quad a_3 = \frac{\rho_m}{K_Q}. \quad (24)$$

The resulting model which shall be used for the observer design specifies as follows

$$\begin{cases} \dot{x}_1 &= -a_2 x_1 \frac{1}{x_2} + (a_2 - u)x_1 \\ \dot{x}_2 &= -a_1 a_3 \frac{1}{a_1 + x_3} + a_3 - a_2(x_2 - 1) \\ \dot{x}_3 &= -\frac{x_1 x_3}{a_1 + x_3} + u(1 - x_3) \\ y(t_k) &= x_1(t_k) \end{cases} \quad (25)$$

where  $u \triangleq D$  is the input of the system.

Now, one shall consider the following change of coordinates which puts system (25) under the normal

form (1),  $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$  with

$$z_1 = x_1, \quad z_2 = \frac{1}{x_2}, \quad z_3 = \frac{1}{a_1 + x_3}. \quad (26)$$

Indeed, one can show that the above change of coordinates puts system (25) under form (1) with

$$x^{(k)} = z_k, \quad k = 1, \dots, 3, \text{ i.e. } n_k = 1, \dots, 3, \\ F_1(u, z) = -a_2 z_1, \quad F_2(u, z) = a_1 a_3 z_2^2, \quad \varphi(u, z) = \begin{pmatrix} (a_2 - u)z_1 \\ a_2 z_2 - (a_2 + a_3)z_2^2 \\ -a_1 z_1 z_3^3 + z_3^2(z_1 - u(1 + a_1)) + u z_3 \end{pmatrix}.$$

Note that, the original coordinates can be deduced from the new ones as follows

$$x_1 = z_1, \quad x_2 = \frac{1}{z_2}, \quad x_3 = \frac{1}{z_3} - a_1. \quad (27)$$

A discrete-continuous time FHGO observer of the form (20) has been designed for the resulting system in the new coordinates  $z$  to estimate the relevant state variables of the process. Notice that the observer equations could be derived in the original coordinates by considering the inverse of the transformation jacobian. Nevertheless, since the original dynamical system has already been expressed in the new coordinates  $z$  and the original variables can be easily deduced from the new ones through equations (27), one has simulated the underlying discrete-continuous time observer in the new coordinates  $z$ .

## 5.2 Estimation results

Two sets of simulation results are given. In each one, the measurements of the biomass are assumed to be available at equally spaced sampling instants with a sampling period equal to  $T_s$  which has been set to  $0.02 \text{ Day}$  in the first experiment and to  $0.0625 \text{ Day}$  (i.e.  $1.5 \text{ h}$  as in [22]) in the second one. Moreover and before being used by the observer, the output  $z_1(t_k)$  has been corrupted by a noise signal  $\mathcal{N}(t_k)$  to generate a noisy output  $y(t_k) = z_1(t_k)(1 + \mathcal{N}(t_k))$ . The noise signal  $\mathcal{N}(t_k)$  has been generated by considering the sum of 26 high frequency sinusoidal signals where the underlying frequencies are equally spaced between  $0.95/(2T_s)$  and  $1/(2T_s)$  (see Fig. 1). In each set of simulation results, the estimates provided by the continuous-discrete version of the FHGO are compared with their counterpart estimates provided by a continuous-discrete version of the SHGO (see [15] for more details). The values of the model parameters used in simulation are [22]

$$a_1 = 0.02 \text{ g.g}^{-1}, \quad a_2 = 4.18 \text{ Day}^{-1}, \quad a_3 = 5.32 \text{ Day}^{-1}.$$

The dilution rate  $D = u$  was chosen as a sinusoidal signal, i.e.  $u = 1.71 \left(1 + 0.6 \sin\left(\frac{2\pi}{T_0}t\right)\right)$  with  $T_0 = 1/3 \text{ Day}$ .

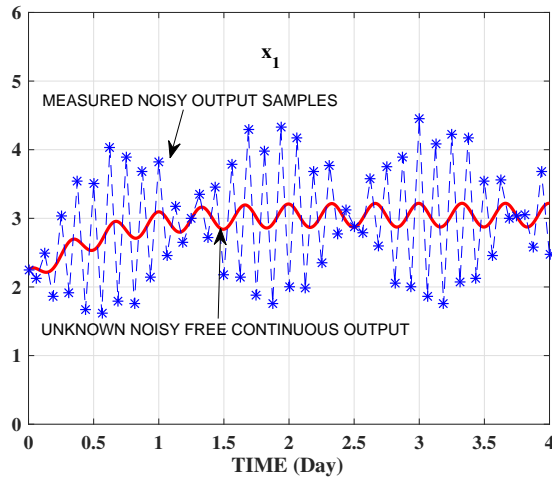


Figure 1: Noisy measurements of the output, with  $T_s = 0.0625 \text{ Day}$

Figure 2 shows estimates provided by the FHGO with  $T_s = 0.02 \text{ Day}$  together with their true values issued from the simulation model. The design parameters  $\theta$  and  $\delta$  have been set to 5 and 7, respectively. Figure 3 shows the estimates provided by the SHGO when the design parameter  $\theta$  has been set to 5 as for the SHGO. The fundamental results have been corroborated, namely the smooth behavior of the estimates provided by the FHGO is mainly due to its filtering capability. Figures 4 and 5 show the performance of the FHGO and SHGO, respectively during the second experiment with  $T_s = 0.0625 \text{ Day}$ . The underlying values of the design parameters are  $\theta = 3$  and  $\delta = 5$  for the FHGO and  $\theta = 3$  for the SHGO. Again, the FHGO outperforms the SHGO from the noise sensitivity point of view.

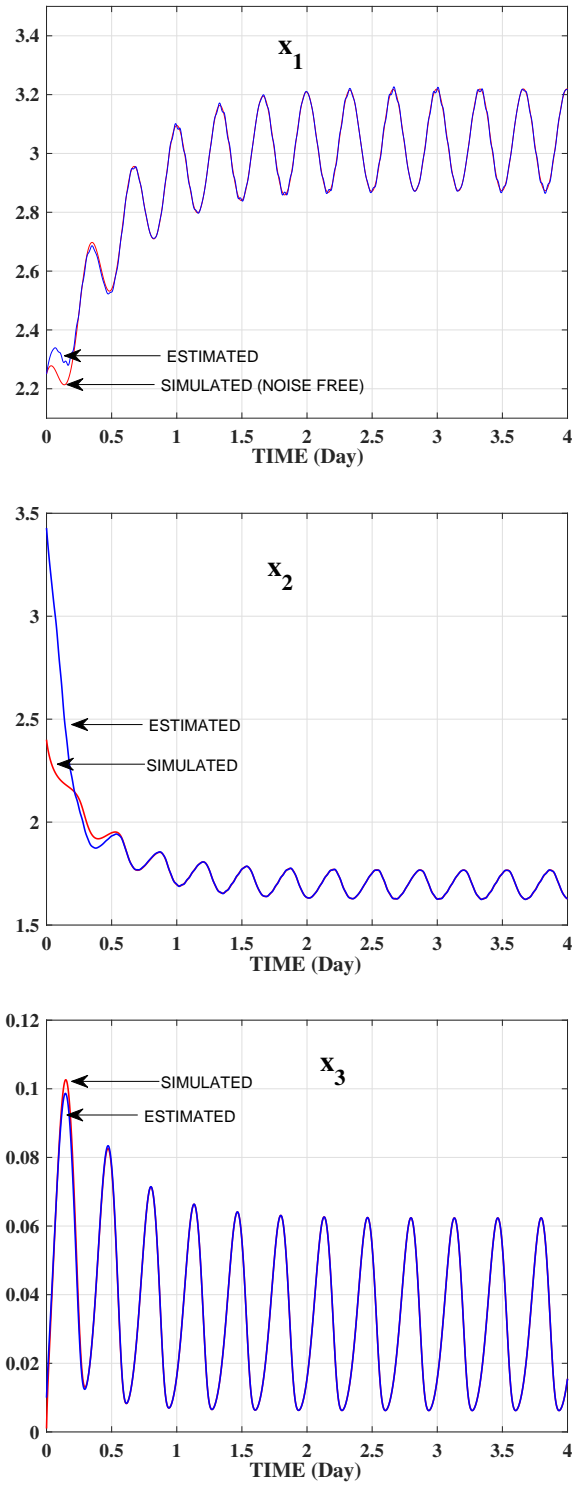


Figure 2: State estimation with FHGO, with  $T_s = 0.02$  Day ( $\theta = 5, \delta = 7$ )

## 6 Conclusion

A FHGO has been proposed for a class of non uniformly observable systems. This FHGO has been first designed assuming that the output measurements are continuously available and subsequently appropriately redesigned to account for the sampling process. A particular attention has been paid to

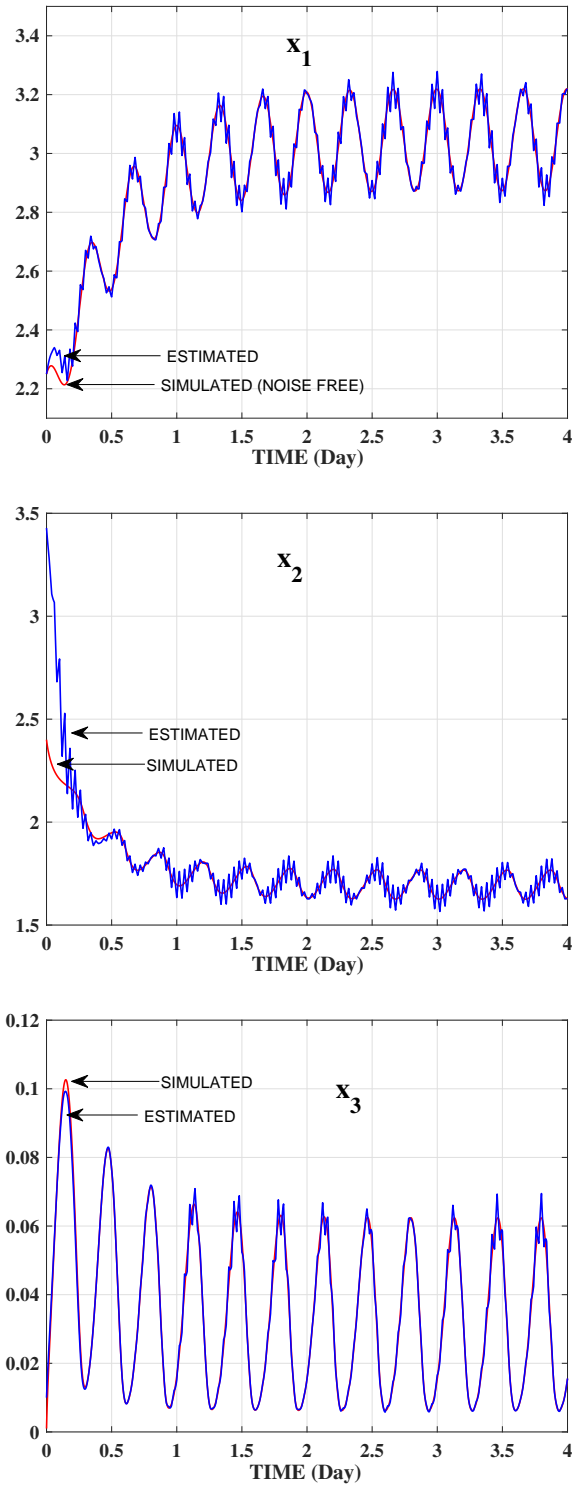


Figure 3: State estimation with SHGO, with  $T_s = 0.02$  Day ( $\theta = 5$ )

the observer convergence analysis while emphasizing the admissible sampling process. The feasibility and performance of the proposed FHGO have been highlighted through a bioreactor dealing with the growth of a population of phytoplanktonic cells. Smooth time-continuous estimates of the biomass and substrate concentrations as well as of the cell quota of assimilated nutrient have been provided

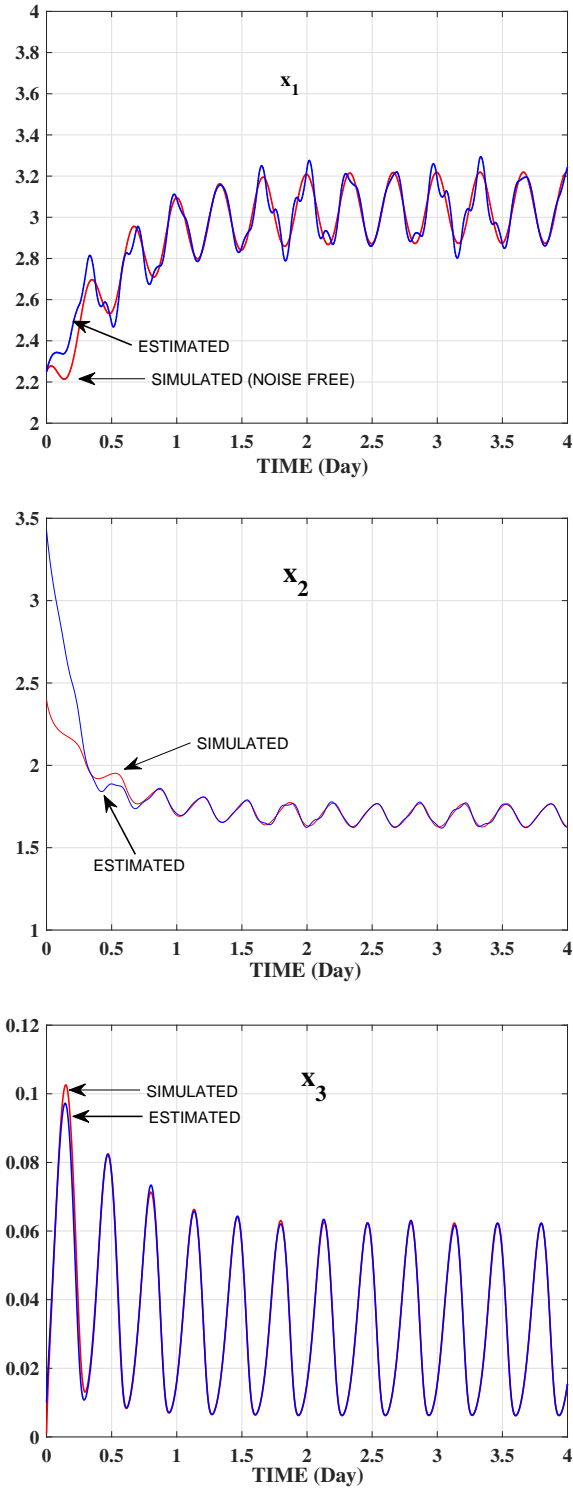


Figure 4: State estimation with FHGO, with  $T_s = 0.0625$  Day ( $\theta = 3, \delta = 5$ )

by the continuous-discrete time version of the FHGO from measured samples of the biomass. These samples were corrupted by a noise occurring at high frequencies. The so obtained estimates have been compared to their counterparts provided by a continuous-discrete time version of a SHGO which were rather noisy, in particular when the sampling periods are relatively high.

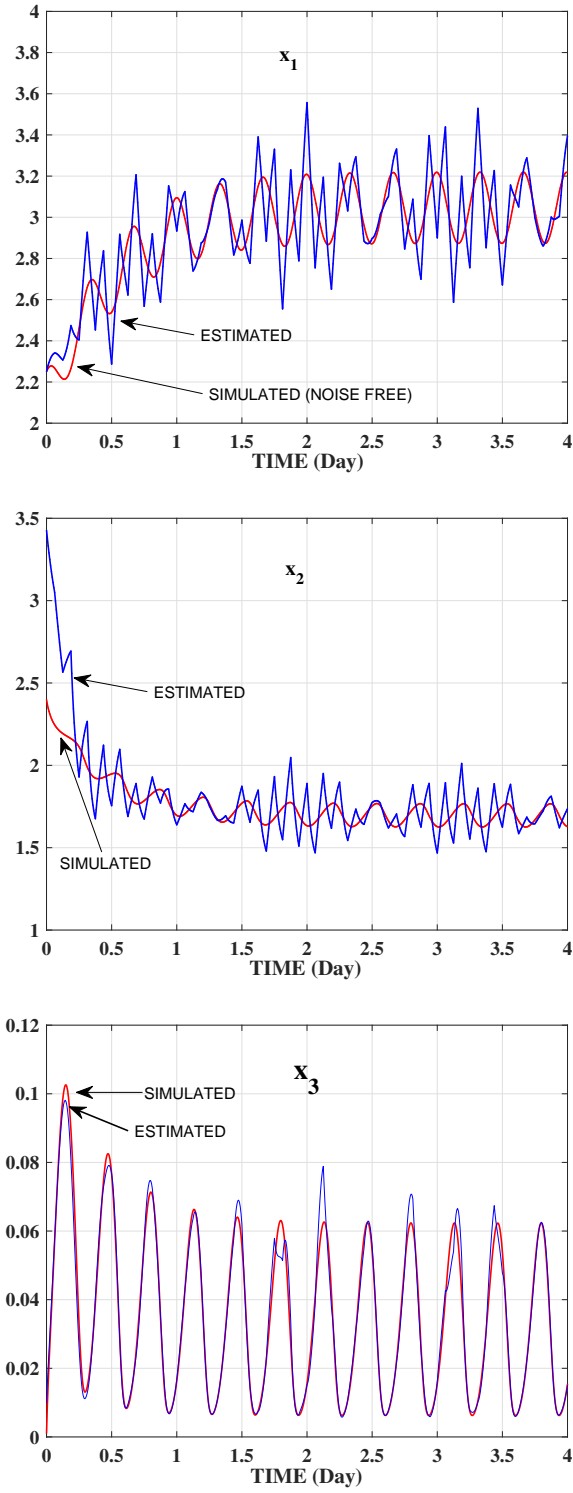


Figure 5: State estimation with SHGO, with  $T_s = 0.0625$  Day ( $\theta = 3$ )

Further studies are under consideration in the proposed FHGO design framework, namely how to redesign the proposed FHGO by including saturation functions in order to cope with the observer peaking phenomenon.

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## A Proof of Theorem 3.1

Set  $\tilde{x} = \hat{x} - x$ ,  $\bar{x} = \theta^{q-1}\Delta_\theta\tilde{x}$  and  $\bar{\eta} = \theta^{q-1}D_\theta\eta$  where  $\Delta_\theta$  is the diagonal matrix given by (8) and  $D_\theta$  is the following  $qn_1 \times qn_1$  diagonal matrix

$$D_\theta = \text{diag}(I_{n_1}, \frac{1}{\theta}I_{n_1}, \dots, \frac{1}{\theta^{q-1}}I_{n_1}). \quad (28)$$

Notice that one can easily check the following equalities

$$\Delta_\theta F(u, \hat{x})\Delta_\theta^{-1} = \theta F(u, \hat{x}); \quad \Delta_\theta K(t) = K(t)D_\theta; \quad D_\theta C_{qn_1}^T C = C_{qn_1}^T C \Delta_\theta. \quad (29)$$

From (1) and (10), one has

$$\dot{\tilde{x}} = F(u, \hat{x})\tilde{x} + \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) - \theta K(t)\eta(t) - B\varepsilon(t). \quad (30)$$

Now, using (29), one gets

$$\begin{aligned} \dot{\tilde{x}} &= \theta F(u, \hat{x})\bar{x} + \theta^{q-1}\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \Delta_\theta K(t)\eta(t) - \theta^{q-1}\Delta_\theta B\varepsilon(t) \\ &= \theta F(u, \hat{x})\bar{x} + \theta^{q-1}\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q K(t)D_\theta\eta(t) - B\varepsilon(t) \\ &= \theta F(u, \hat{x})\bar{x} + \theta^{q-1}\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta K(t)\bar{\eta}(t) - B\varepsilon(t). \end{aligned} \quad (31)$$

Similarly, from (1) and (10) and using (29), one gets

$$\begin{aligned} \dot{\bar{\eta}} &= -\theta\delta \left( (I_{qn_1} - A^T)\bar{\eta} + \theta^{q-1}D_\theta C_{qn_1}^T C(\tilde{x} - w(t)) \right) \\ &= -\theta\delta \left( (I_{qn_1} - A^T)\bar{\eta} + \theta^{q-1}C_{qn_1}^T C \Delta_\theta(\tilde{x} - w(t)) \right) \\ &= -\theta\delta \left( (I_{qn_1} - A^T)\bar{\eta} + C_{qn_1}^T C\bar{x} + \theta^{q-1}C_{qn_1}^T Cw(t) \right). \end{aligned} \quad (32)$$

Adding and subtracting the term  $\theta P(t)C^T C\bar{x}$  to the right side of equation (31), one gets

$$\begin{aligned}\dot{\hat{x}} &= \theta (F(u, \hat{x}) - P(t)C^T C) \bar{x} + \theta^{q-1} \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) \\ &\quad - B\varepsilon(t) - \theta (K(t)\bar{\eta} - P(t)C^T C\bar{x}).\end{aligned}\quad (33)$$

Now, since  $K(t) = \text{diag}(P(t)C^T)$ , one has  $K(t)U = P(t)C^T C$  where

$$U \triangleq \begin{pmatrix} I_{n_1} & 0_{n_1, n_2} & \cdots & 0_{n_1, n_q} \\ \vdots & \vdots & \vdots & \vdots \\ I_{n_1} & 0_{n_1, n_2} & \cdots & 0_{n_1, n_q} \end{pmatrix} \in \mathbb{R}^{q n_1 \times n}.$$
 (34)

Hence, equation (33) can be rewritten as follows

$$\dot{\hat{x}} = \theta (F(u, \hat{x}) - P(t)C^T C) \bar{x} + \theta^{q-1} \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - B\varepsilon(t) - \theta K(t)\bar{z}, \quad (35)$$

where

$$\bar{z} = \bar{\eta} - U\bar{x}. \quad (36)$$

Let now derive the time derivative of  $\bar{z}$ . Indeed, using (31), one gets

$$\begin{aligned}\dot{\bar{z}} &= \dot{\bar{\eta}} - U\dot{\hat{x}} \\ &= -\theta\delta \left( (I_{q n_1} - A^T)\bar{\eta} - C_{q n_1}^T C\bar{x} + \theta^{q-1} C_{q n_1}^T Cw(t) \right) - \theta U F(u, \hat{x})\bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) + \theta U K(t)\bar{\eta} + U B\varepsilon(t) \\ &= \theta \left( -\delta(I_{q n_1} - A^T) + U K(t) \right) \bar{\eta} + \theta \left( \delta C_{q n_1}^T C - U F(u, \hat{x}) \right) \bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \delta C_{q n_1}^T Cw(t) \\ &= \theta \left( -\delta(I_{q n_1} - A^T) + U K(t) \right) (\bar{z} + U\bar{x}) + \theta \left( \delta C_{q n_1}^T C - U F(u, \hat{x}) \right) \bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \delta C_{q n_1}^T Cw(t) \\ &= \theta \left( -\delta(I_{q n_1} - A^T) + U K(t) \right) \bar{z} + \theta \left( \delta C_{q n_1}^T C - U F(u, \hat{x}) - \delta(I_{q n_1} - A^T)U + U K(t)U \right) \bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \delta C_{q n_1}^T Cw(t).\end{aligned}$$

Now, one can check that  $(I_{q n_1} - A^T)U = C_{q n_1}^T C$  and the above last equation can therefore be written as follows

$$\begin{aligned}\dot{\bar{z}} &= \theta\delta \left( -(I_{q n_1} - A^T) + \frac{1}{\delta} U K(t) \right) \bar{z} + \theta \left( -U F(u, \hat{x}) + U K(t)U \right) \bar{x} \\ &\quad - \theta^{q-1} U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q \delta C_{q n_1}^T Cw(t).\end{aligned}\quad (37)$$

According to Assumption **A4**, the matrix  $P(t)$  governed by (13) is SPD. Let  $S(t) = P^{-1}(t)$ ; one can show that  $S(t)$  is governed by the following Lyapunov ODE

$$\begin{cases} \dot{S}(t) &= -\theta (S(t) + F^T(u, \hat{x})S(t) + S(t)F(u, \hat{x}) - C^T C), \\ S(0) &= S^T(0) > 0. \end{cases} \quad (38)$$

Again, according to Assumption **A4**, one has

$$\forall t \geq t^*, \lambda_m(S(t)) \geq \frac{\rho^*}{\alpha(\theta)}, \quad (39)$$

where  $\alpha(\theta)$  satisfies (15).

Now, proceeding as in [15, 16], one can show that the matrix  $S(t)$  is bounded and the underlying upper bound (or equivalently  $\bar{\lambda}_M(S(t))$ ) is independent of  $\theta$ . Let  $V_1(\bar{x}, t) = \bar{x}^T S(t) \bar{x}$ ; proceeding as in [15, 16], one can show that for  $\theta \geq 1$ ,

$$\begin{aligned} \dot{V}_1(\bar{x}, t) &\leq -\theta\mu(\theta)V_1(\bar{x}, t) + 2\sqrt{\bar{\lambda}_M(S(t))}\sqrt{V_1(\bar{x}, t)}\|\varepsilon(t)\| - 2\theta\bar{x}^T S(t)K(t)\bar{z}(t) \\ &\leq -\theta\mu(\theta)V_1(\bar{x}, t) + 2\sqrt{\bar{\lambda}_M(S(t))}\sqrt{V_1(\bar{x}, t)}\|\varepsilon(t)\| + 2\theta\sqrt{\bar{\lambda}_M(S(t))}\sqrt{V_1(\bar{x}, t)}\|K(t)\|\|\bar{z}(t)\|, \end{aligned} \quad (40)$$

where

$$\mu(\theta) = 1 - \gamma\sqrt{\frac{\alpha(\theta)}{\theta^2}} \text{ with } \gamma = 2\sqrt{n}(L_F x_M + L_\varphi)\sqrt{\frac{\bar{\lambda}_M(S(t))}{\rho^*}}, \quad (41)$$

where  $x_M$  is given by Assumption **A1**,  $L_F$  and  $L_\varphi$  by Assumption **A2** and finally  $\rho^*$  and  $\alpha(\theta)$  by Assumption **A4**.

Similarly, set  $V_2(\bar{z}) = \bar{z}^T \bar{z} = \|\bar{z}\|^2$ . Then, one has

$$\begin{aligned} \dot{V}_2 &= -\theta\delta\bar{z}^T(2I_{q_{n_1}} - (A^T + A))\bar{z} + 2\theta\bar{z}^T U K(t)\bar{z} + 2\theta\bar{z}^T (-UF(u, \hat{x}) + UK(t)U)\bar{x} \\ &\quad - 2\theta^{q-1}\bar{z}^T U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - 2\theta^q \delta \bar{z}^T C_{q_{n_1}}^T C w(t) \\ &\leq -\theta\delta\kappa\|\bar{z}\|^2 + 2\theta\|U\|\|K_1(t)\|\|\bar{z}\|^2 + 2\theta\|U\|(\|F_1(u, \hat{x})\| + \|K_1(t)\|)\|\bar{x}\|\|\bar{z}\| \\ &\quad - 2\theta^{q-1}\bar{z}^T U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - 2\theta^q \delta \bar{z}^T C_{q_{n_1}}^T C w(t), \end{aligned}$$

where  $\kappa$  is the smallest eigenvalue of the SPD matrix  $2I_{q_{n_1}} - (A + A^T)$ .

According to Assumption **A2**, one can show that

$$\begin{aligned} 2\theta^{q-1}\bar{z}^T U \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) &\leq 2\|\bar{z}\|\|U\|\sqrt{n}(L_F x_M + L_\varphi)\|\bar{x}\| \\ &\leq 2\|\bar{z}\|\|U\|\sqrt{n}(L_F x_M + L_\varphi)\frac{\sqrt{V_1(\bar{x}, t)}}{\sqrt{\lambda_m(S(t))}} \\ &\leq 2n(L_F x_M + L_\varphi)\sqrt{\frac{\alpha(\theta)}{\rho^*}}\|\bar{z}\|\sqrt{V_1(\bar{x}, t)} \\ &= 2n(L_F x_M + L_\varphi)\sqrt{\frac{\alpha(\theta)}{\rho^*}}\sqrt{V_1(\bar{x}, t)}\sqrt{V_2(\bar{z})}. \end{aligned} \quad (42)$$

Similarly, one has

$$\begin{aligned} 2\theta\|U\|(\|F_1(u, \hat{x})\| + \|K_1(t)\|)\|\bar{x}\|\|\bar{z}\| &\leq 2\theta\sqrt{n}(F_M + K_M)\frac{\sqrt{V_1(\bar{x}, t)}}{\sqrt{\lambda_m(S(t))}}\sqrt{V_2(\bar{z})} \\ &\leq 2\theta\sqrt{n}(F_M + K_M)\sqrt{\frac{\alpha(\theta)}{\rho^*}}\sqrt{V_1(\bar{x}, t)}\sqrt{V_2(\bar{z})}, \end{aligned} \quad (43)$$

where  $K_M = \sup_{t \geq 0} \|K_1(t)\|$ .

Hence,

$$\begin{aligned} \dot{V}_2 \leq & -\theta\delta\kappa \left(1 - 2\frac{K_M\sqrt{n}}{\kappa\delta}\right) V_2 + 2\theta^q\delta\sqrt{V_2}\|w(t)\| \\ & + 2\sqrt{n} \left\{ \sqrt{n}(L_F x_M + L_\varphi) + \theta(F_M + K_M) \right\} \sqrt{\frac{\alpha(\theta)}{\rho^*}} \sqrt{V_1(\bar{x}, t)} \sqrt{V_2(\bar{z})}. \end{aligned} \quad (44)$$

Now, choose  $\delta$  such that  $\left(1 - 2\frac{K_M\sqrt{n}}{\kappa\delta}\right) \geq \frac{1}{2}$ , i.e.  $\delta \geq 2\sqrt{n}\frac{K_M}{\kappa}$  and choose  $\theta$  such that  $\theta \geq 1$ . Then, inequality (44) leads to

$$\dot{V}_2 \leq -\frac{\theta\delta\kappa}{2}V_2 + c_2\theta\sqrt{\alpha(\theta)}\sqrt{V_1(\bar{x}, t)}\sqrt{V_2(\bar{z})} + 2\theta^q\delta\sqrt{V_2}\|w(t)\|,$$

where  $c_2 = 2\sqrt{\frac{n}{\rho^*}} \left\{ \sqrt{n}(L_F x_M + L_\varphi) + (F_M + K_M) \right\}$  is a constant independent of  $\theta$  and  $\delta$ . Using inequality (40), one gets

$$\dot{V}_1 \leq -\theta\mu(\theta)V_1 + c_1\theta\sqrt{V_1}\sqrt{V_2} + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_1}\|\varepsilon(t)\|, \quad (45)$$

where  $c_1 = 2K_M\sqrt{\bar{\lambda}_M(S(t))}$  is a positive constant independent of  $\theta$  and  $\delta$ . Set

$$V_1^* = \theta\mu(\theta)V_1, \quad V_2^* = \frac{\theta\delta\kappa}{2}V_2 \quad \text{and let } V^* = V_1^* + V_2^*. \quad (46)$$

Notice that one has

$$V_1^* \leq V^* \quad \text{and} \quad V_2^* \leq V^*. \quad (47)$$

Hence,

$$\begin{aligned} \dot{V}_1 & \leq -V_1^* + c_1\theta\sqrt{\frac{V_1^*}{\theta\mu(\theta)}}\sqrt{\frac{V_2^*}{\frac{\theta\delta\kappa}{2}}} + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_1}\|\varepsilon(t)\| \\ & = -V_1^* + \frac{c_1}{\sqrt{\delta}}\sqrt{\frac{2}{\kappa\mu(\theta)}}\sqrt{V_1^*}\sqrt{V_2^*} + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_1}\|\varepsilon(t)\|. \end{aligned}$$

Using (47), one gets

$$\dot{V}_1 \leq -V_1^* + \frac{c_1}{\sqrt{\delta}}\sqrt{\frac{2}{\kappa\mu(\theta)}}V^* + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_1}\|\varepsilon(t)\|. \quad (48)$$

Similarly, one has

$$\begin{aligned}
\dot{V}_2 &\leq -V_2^* + c_2\theta\sqrt{\alpha(\theta)}\sqrt{\frac{V_1^*}{\theta\mu(\theta)}}\sqrt{\frac{V_2^*}{\frac{\theta\delta\kappa}{2}}} + 2\theta^q\delta\sqrt{V_2}\|w(t)\| \\
&= -V_2^* + c_2\sqrt{\frac{\alpha(\theta)}{\delta}}\sqrt{\frac{2}{\kappa\mu(\theta)}}\sqrt{V_1^*}\sqrt{V_2^*} + 2\theta^q\delta\sqrt{V_2}\|w(t)\| \\
&\leq -V_2^* + c_2\sqrt{\frac{\alpha(\theta)}{\delta}}\sqrt{\frac{2}{\kappa\mu(\theta)}}V^* + 2\theta^q\delta\sqrt{V_2}\|w(t)\|. \tag{49}
\end{aligned}$$

Now, choose  $\delta$  such that  $\delta \geq \theta^\chi$ , i.e.  $\frac{1}{\delta} \leq \frac{1}{\theta^\chi}$  where  $\chi$  is given by (15). Then, inequality (49) becomes

$$\dot{V}_2 \leq -V_2^* + c_2\sqrt{\frac{\alpha(\theta)}{\theta^{2\chi}}}\sqrt{\frac{2}{\kappa\mu(\theta)}}V^* + 2\theta^q\delta\sqrt{V_2}\|w(t)\|.$$

Now, let  $V(\bar{x}, \bar{z}, t) = V_1(\bar{x}, t) + V_2(\bar{z})$  be the candidate Lyapunov function. Using (48) and (50) and from the facts that  $\sqrt{V_1} \leq V$  and  $\sqrt{V_2} \leq V$ , one gets

$$\dot{V} \leq -V^* + \sqrt{\frac{2}{\kappa\mu(\theta)}}\left(\frac{c_1}{\delta} + c_2\sqrt{\frac{\alpha(\theta)}{\theta^{2\chi}}}\right)V^* + 2\left(\sqrt{\bar{\lambda}_M(S)}\|\varepsilon(t)\| + \delta\theta^q\|w(t)\|\right)\sqrt{V}. \tag{50}$$

Now, for  $\theta$  and  $\delta$  sufficiently high and according to (15), one has

$$\lim_{\rho \rightarrow \infty} \frac{\alpha(\theta)}{\theta^{2\chi}} = \lim_{\delta \rightarrow \infty} \frac{1}{\delta} = 0, \tag{51}$$

and as a result, there exists  $0 < \nu < 1$  such that

$$1 - \sqrt{\frac{2}{\kappa\mu(\theta)}}\left(\frac{c_1}{\delta} + c_2\sqrt{\frac{\alpha(\theta)}{\theta^{2\chi}}}\right) > \nu. \tag{52}$$

Similarly, for such values of  $\theta$  and  $\delta$ , one has  $\theta\mu(\theta) \leq \frac{\theta\delta\kappa}{2}$  which implies that

$$\theta\mu(\theta)V \leq V^* \leq \frac{\theta\delta\kappa}{2}V. \tag{53}$$

Hence, combining (50), (52) and (53), one gets

$$\dot{V} \leq -\nu\theta\mu(\theta)V + 2\left(\sqrt{\bar{\lambda}_M(S)}\|\varepsilon(t)\| + \theta^q\delta\|w(t)\|\right)\sqrt{V}. \tag{54}$$

Using the comparison lemma, one gets for  $t \geq t^* \triangleq \frac{T^*}{\theta}$ ,

$$\sqrt{V(\bar{x}(t), \bar{z}(t), t)} \leq e^{-\nu\frac{\theta}{2}\mu(\theta)(t-t^*)}\sqrt{V(\bar{x}(t^*), \bar{z}(t^*), t^*)} + \frac{2}{\nu\theta\mu(\theta)}\left(\sqrt{\bar{\lambda}_M(S)}\delta_\varepsilon + \theta^q\delta\delta_w\right).$$

Coming back to  $\bar{x}$  and from the fact that  $V_1 \leq V$ , one gets

$$\|\bar{x}(t)\| \leq \sqrt{\frac{\lambda_M}{\underline{\lambda}_m(S(t))}} e^{-\nu \frac{\theta}{2} \mu(\theta)(t-t^*)} \|\bar{\xi}(t^*)\| + \sqrt{\frac{\lambda_M}{\underline{\lambda}_m(S(t))}} \frac{2}{\nu \theta \mu(\theta)} (\delta_\varepsilon + \theta^q \delta \delta_w),$$

where  $\bar{\xi}(t) = \begin{pmatrix} \bar{x}(t) \\ \bar{z}(t) \end{pmatrix}$  and  $\lambda_M = \max(\bar{\lambda}_M(S), 1)$ .

Now, using (39), the last inequality becomes

$$\begin{aligned} \|\bar{x}(t)\| &\leq \sqrt{\frac{\lambda_M}{\rho^*}} \sqrt{\alpha(\theta)} e^{-\nu \frac{\theta}{2} \mu(\theta)(t-t^*)} \|\bar{\xi}(t^*)\| + \sqrt{\frac{\lambda_M \alpha(\theta)}{\rho^*}} \frac{2}{\nu \theta \mu(\theta)} (\delta_\varepsilon + \theta^q \delta \delta_w) \\ &\leq \beta_f \sqrt{\alpha(\theta)} e^{-\nu \frac{\theta}{2} \mu(\theta)(t-t^*)} \|\bar{\xi}(t^*)\| + 2 \frac{\beta_f}{\nu \mu(\theta)} \left( \sqrt{\frac{\alpha(\theta)}{\theta^2}} \delta_\varepsilon + \sqrt{\alpha(\theta)} \theta^{q-1} \delta \delta_w \right), \end{aligned} \quad (55)$$

where

$$\beta_f = \sqrt{\frac{\lambda_M}{\rho^*}}. \quad (56)$$

Now, set  $\xi = \begin{pmatrix} \tilde{x}(t) \\ \tilde{z}(t) \end{pmatrix}$  where  $\tilde{z}$  is such that  $\bar{z} = \theta^{q-1} \Delta_\theta^{-1} \tilde{z}(t)$ . It is easy to check that

$$\|\tilde{\xi}(t)\| \leq \|\bar{\xi}(t)\| \leq \theta^{q-1} \|\tilde{\xi}(t)\|, \quad \forall t \geq t^* \text{ and } \forall \theta \geq 1. \quad (57)$$

Besides, one can show that  $\tilde{z} = \eta - U\tilde{x}$  and therefore, one has for all  $t \geq t^*$

$$\|\tilde{\xi}(t)\| \leq \|\tilde{x}(t)\| + (\|e(t)\| + \|U\| \|\tilde{x}(t)\|) \leq (1 + \sqrt{n}) \|\tilde{x}(t)\| + \|\eta(t)\|. \quad (58)$$

Using (57) and (58), inequality (55) leads to (18). The proof of Theorem 3.1 is ended.  $\square$ .

## B Proof of Theorem 4.1

Set  $\tilde{x} = \hat{x} - x$ ,  $\bar{x} = \theta^{q-1} \Delta_\theta \tilde{x}$  and  $\bar{\eta} = \theta^{q-1} D_\theta \eta$ . Proceeding as in the continuous output case, one gets (compare the equations below with (35) and (32), respectively),

$$\begin{cases} \dot{\tilde{x}}(t) &= \theta (F(u, \hat{x}) - P(t)C^T C) \bar{x} - B\varepsilon(t) - \theta K(t)\bar{z} \\ &\quad + \theta^{q-1} \Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) \\ \dot{\tilde{\eta}}(t) &= -\theta \delta ((I_{q n_1} - A^T)\bar{\eta}(t) + C_{q n_1}^T C \bar{x}(t) + \theta^{q-1} C_{q n_1}^T C w(t_k)) \\ &\quad - \theta \delta C_{q n_1}^T \bar{\gamma}(t), \end{cases}$$

where  $\bar{z}(t)$  is defined as in (36) and

$$\bar{\gamma} = C(\bar{x}(t) - \bar{x}(t_k)) - \theta^{q-1} \alpha(t) \quad (59)$$

The dynamics of  $\bar{z}$  is given by (compare with (37))

$$\begin{aligned}\dot{\bar{z}} &= \theta\delta \left( -(I_{q_{n_1}} - A^T) + \frac{1}{\delta}UK(t) \right) \bar{z} \\ &\quad + \theta (-UF(u, \hat{x}) + UK(t)U) \bar{x} \\ &\quad - \theta^{q-1}U\Delta_\theta \left( \tilde{F}(u, \hat{x}, x)x + \tilde{\varphi}(u, \hat{x}, x) \right) - \theta^q\delta C_{q_{n_1}}^T Cw(t) - \theta\delta C_{q_{n_1}}^T \bar{\gamma}(t).\end{aligned}\quad (60)$$

Notice that the first equation in (59) is similar to (35) while (60) differs from (37) by the presence of the term involving  $\bar{\gamma}$ . Hence, the forthcoming development will be very similar to those carried in the continuous out case. The sole difference consist in the accounting for the additional term  $\bar{\gamma}$ . Hence, let us derive a bound of  $\bar{\gamma}$ . One has

$$\begin{aligned}\dot{\bar{\gamma}} &= \theta F_1(u, \hat{x}^{(1)})\bar{x}^{(2)} + \theta^{q-1}\varphi^{(1)}(u, x^{(1)}) - \theta K^{(1)}(t)\bar{\eta}^{(1)}(t) - \theta^{q-1}\dot{\alpha}(t) \\ &= \theta F_1(u, \hat{x}^{(1)})\bar{x}^{(2)} + \theta^{q-1}\varphi^{(1)}(u, x^{(1)}) - \theta^{(q-1)}(\dot{\alpha}(t) + \theta\eta^{(1)}(t)) \\ &= \theta F_1(u, \hat{x}^{(1)})\bar{x}^{(2)} + \theta^{q-1}\varphi^{(1)}(u, x^{(1)}).\end{aligned}\quad (61)$$

Notice that the last inequality comes from the fact that  $\alpha(t)$  is governed by the ODE given by (59). Now, since  $\alpha(t_k) = 0$ , one can easily check that  $\bar{\gamma}(t_k) = 0$  and integrating the last equation from  $t^* \leq t$  to  $t$  gives

$$\bar{\gamma}(t) = \int_{t^*}^t \left( \theta F_1(u, \hat{x}^{(1)})\bar{x}^{(2)} + \theta^{q-1}\varphi^{(1)}(u, x^{(1)}) \right) ds,$$

and hence

$$\|\bar{\gamma}(t)\| \leq (\theta F_M + L_\varphi) \int_{t^*}^t \|\bar{x}(s)\| ds,$$

where  $F_M = \sup_{x \in X, u \in U} F(u, x)$  and  $L_\varphi$  is the Lipschitz constant of  $\varphi$  given by Assumption **A2**.

Now, as in the continuous output case, let  $V_1(\bar{x}, t) = \bar{x}^T S(t)\bar{x}$ ,  $V_2(\bar{z}) = \|\bar{z}\|^2$  and  $V = V_1 + V_2$ . Again, proceeding as in the continuous output case, one can show that (compare with (54)),

$$\begin{aligned}\dot{V} &\leq -\nu\theta\mu(\theta)V + 2\sqrt{\bar{\lambda}_M(S)}\sqrt{V_2}(\theta F_M + L_\varphi) \int_{t^*}^t \|\bar{x}(s)\| ds \\ &\quad + 2 \left( \sqrt{\bar{\lambda}_M(S)}\|\varepsilon(t)\| + \theta^q\delta\|w(t)\| \right) \sqrt{V}.\end{aligned}$$

Using (39), one gets

$$\begin{aligned}\dot{V} &\leq -\nu\theta\mu(\theta)V + 2\beta_f\sqrt{\alpha(\theta)}(\theta F_M + L_\varphi)\sqrt{V} \int_{t^*}^t \sqrt{V(s)} ds \\ &\quad + 2\lambda_M(\|\varepsilon(t)\| + \theta^q\delta\|w(t)\|)\sqrt{V},\end{aligned}$$

where  $\lambda_M = \max(\bar{\lambda}_M(S), 1)$  and  $\beta_f$  is as in (56).

The last inequality leads to

$$\begin{aligned} \frac{d}{dt}\sqrt{V} &\leq -\frac{1}{2}\nu\theta\mu(\theta)\sqrt{V} + \beta_f\sqrt{\alpha(\theta)}(\theta F_M + L_\varphi) \int_{t^*}^t \sqrt{V(s)}ds \\ &\quad + \lambda_M \left( \sqrt{\bar{\lambda}_M(S)}\|\varepsilon(t)\| + \theta^q\delta\|w(t)\| \right). \end{aligned}$$

Now, let  $a_\theta = \frac{1}{2}\nu\theta\mu(\theta)$ ,  $b_\theta = \beta_f\sqrt{\alpha(\theta)}(\theta F_M + L_\varphi)$  and  $p_\theta(t) = \sqrt{\bar{\lambda}_M}(\|\varepsilon(t)\| + \theta^q\delta\|w(t)\|)$  and assume that the upper diameter of the sampling partition  $\tau_M$  satisfies the following condition

$$\tau_M < \varrho_\theta \triangleq \frac{a_\theta}{b_\theta} = \frac{\beta_f\sqrt{\alpha(\theta)}(\theta F_M + L_\varphi)}{2\nu\theta\mu(\theta)}, \quad (62)$$

then, according to lemma 2.1 in [14], one has

$$\sqrt{V(\bar{x}(t), \bar{z}(t), t)} \leq e^{-\zeta_\theta(t-t^*)}\sqrt{V(\bar{x}(t^*), \bar{z}(t^*), t^*)} + c_\theta\tau_M \frac{2 - e^{-\zeta_\theta\tau_M}}{1 - e^{-\zeta_\theta\tau_M}},$$

where

$$ess \sup_{t \geq 0} p_\theta(t) \leq c_\theta = \sqrt{\bar{\lambda}_M}(\delta_\varepsilon + \delta\theta^q\delta_w), \quad \zeta_\theta(\tau_M) = a_\theta\left(1 - \frac{\tau_M}{\varrho_\theta}\right)e^{-a_\theta\tau_M}. \quad (63)$$

Coming back to  $\bar{x}$  and proceeding as in the continuous output case leads to (21). The proof of Theorem 4.1 is ended.