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► To cite this version:

Ahmad Makki, Alain Miranville, Wafa Saoud. On a Cahn–Hilliard/Allen–Cahn system coupled with a type III heat equation and singular potentials. *Nonlinear Analysis: Theory, Methods and Applications*, 2020, 196, pp.111804 -. [10.1016/j.na.2020.111804](https://doi.org/10.1016/j.na.2020.111804). [hal-03489793](https://hal.science/hal-03489793)

HAL Id: hal-03489793

<https://hal.science/hal-03489793v1>

Submitted on 7 Mar 2022

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On A Cahn-Hilliard/Allen-Cahn System Coupled With A Type III Heat Equation And Singular Potentials

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Abstract

Our aim in this article is to study the asymptotic behavior of a Cahn-Hilliard/Allen-Cahn system coupled with a heat equation based on the type III heat conduction law with singular potentials. We also show further regularity results and we prove a strict separation property (from the pure states) in one space dimension.

Key words and phrases. Cahn-Hilliard/Allen-Cahn equations; type III heat conduction; logarithmic nonlinear terms; well-posedness; dissipativity.

AMS Subject Classification. 35B45, 35K55, 35L15.

1 Introduction

The Cahn-Hilliard/Allen-Cahn system

$$\begin{aligned}\frac{\partial u}{\partial t} &= h^2 \Delta (f(u+v) + f(u-v) - h^2 \Delta u), \\ \frac{\partial v}{\partial t} &= -f(u+v) + f(u-v) - \gamma v + h^2 \Delta v,\end{aligned}$$

was initially introduced by J. Cahn and A. Novick-Cohen in [3].

The function u represents the concentration of one of the components and it is a conserved quantity, the function v is an order parameter, the variable h is a (positive) parameter which represents the lattice spacing, the variable γ is a parameter that reflects the location of the system within the phase diagram (it may be either positive or negative), and the nonlinear term f is the derivative of a double-well potential F .

The system models simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point. We note that it also represents a gradient flow in $(H^1)' \times L^2$ for the free energy

$$J(u, v) = \int_{\Omega} \left\{ F(u + v) + F(u - v) + \frac{\gamma}{2} v^2 + \frac{1}{2} h^2 (|\nabla u|^2 + |\nabla v|^2) \right\} dx.$$

The mathematicians A. Novick-Cohen, D. Brochet, and D. Hilhorst studied the previous system with Neumann boundary conditions in [2]. They proved the well-posedness and the existence of maximal attractors and exponential attractors for the classic cubic nonlinear term $f(s) = s^3 - \beta s$ in three space dimensions. These results were recently improved in [22]. There, the authors took initial conditions in $H^2(\Omega)$ which allowed them to prove the existence of exponential attractors and of the finite-dimensional global attractor for a large class of nonlinear terms containing polynomials of arbitrary odd degree, with a strictly positive leading coefficient in three space dimensions. This model has been also studied in [23], where an exponential attractor for singular potentials was found, and by consequence a global attractor of finite dimension.

Likewise, in [6], the authors studied a similar system with a non-constant mobility. They proved the existence of weak solutions for the Neumann problem for a degenerate parabolic system. In addition, asymptotics for a similar system was studied in [24]. Then the partial wetting case was studied in [25].

Solving coupled AC/CH systems using numerical methods was also studied in, e.g. [15, 35], and [37]-[40].

In the Cahn-Hilliard theory, a thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model (see [16, 17, 18, 19]):

$$F(s) = \frac{\theta_c}{2} (1 - s^2) + \frac{\theta}{2} \left[(1 - s) \ln \left(\frac{1 - s}{2} \right) + (1 + s) \ln \left(\frac{1 + s}{2} \right) \right], \quad (1.1)$$

$$s \in (-1, 1), \quad 0 < \theta < \theta_c.$$

Furthermore,

$$f(s) = (F'(s)) = -\theta_c s + \frac{\theta}{2} \ln \left(\frac{1 + s}{1 - s} \right). \quad (1.2)$$

This logarithmic potential F is usually approximated by regular ones, typically,

$$F(s) = \frac{1}{4} (s^2 - 1)^2, \quad (1.3)$$

leading to the cubic nonlinear term $f(s) = s^3 - s$.

It should be noted that this approximation is acceptable when the absolute temperature θ is close to the critical one θ_c . Also note that the nonlinear term f given by (1.2) leads to essential difficulties, due to the fact that the order parameter has to remain in the physically relevant interval $(-1, 1)$ (see, e.g. [4, 19, 18]).

Now, a critical disadvantage of the Fourier law is that it predicts the propagation of thermal signals with an infinite speed, which violates the 'paradox of heat conduction', (see, e.g., [5]). Therefore, many efforts have been done to correct this unrealistic feature.

For example, in [20], the authors assumed the classical Fourier law $q = -\nabla\theta$ and considered

$$\left(1 + \eta \frac{\partial}{\partial t}\right) q = -\nabla\theta, \quad \eta > 0.$$

That led to

$$\eta \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial \theta}{\partial t} - \Delta \theta = -\eta \frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial t},$$

(see also [12], [13] the Maxwell-Cattaneo law).

Another example was proposed in [10]-[11] by Green and Naghdi. It is a different treatment for a thermomechanical theory of deformable media that replaces the usual entropy inequality by an entropy balance.

We recall that three different theories concerning the heat conduction were proposed. They were entitled: type I (the Fourier law linearized), type II and type III. The linearized versions of the types II and III are given by:

$$q = -k \nabla \alpha, \quad k > 0, \tag{1.4}$$

and

$$q = -k \nabla \alpha - k^* \nabla \theta, \quad k, k^* > 0, \tag{1.5}$$

respectively, where

$$\alpha(t) = \int_{t_0}^t \theta(\tau) d\tau + \alpha_0,$$

is the thermal displacement variable and $\theta = \frac{\partial \alpha}{\partial t}$.

The qualitative study of the solutions was also treated in [27]-[32]. Non-linear acceleration waves for types II and III and fluids without energy dissipation have been studied too, in [33] and [34].

Observe that if we sum equations (1.4) and (1.5) with

$$\frac{\partial H}{\partial t} + \operatorname{div} q = 0, \tag{1.6}$$

where H is the enthalpy defined by

$$H = v + \theta$$

and θ is the relative temperature, we obtain

$$\frac{\partial^2 \alpha}{\partial t^2} - k \Delta \alpha = -\frac{\partial v}{\partial t} \quad (\text{Type II}),$$

and

$$\frac{\partial^2 \alpha}{\partial t^2} - k^* \frac{\partial}{\partial t} \Delta \alpha - k \Delta \alpha = -\frac{\partial v}{\partial t} \quad (\text{Type III}).$$

Setting of the problem

We consider the following initial and boundary value problem, in a bounded and regular domain $\Omega \subset \mathbb{R}^n$, $n = 1, 2$ or 3 , with boundary Γ , (for simplicity, we take $k = k^* = 1$):

$$\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u+v) + f(u-v)) = 0, \quad (1.7)$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u+v) - f(u-v) = \frac{\partial \alpha}{\partial t}, \quad (1.8)$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial v}{\partial t}, \quad (1.9)$$

$$u = \Delta u = v = \alpha = 0 \quad \text{on } \Gamma, \quad (1.10)$$

$$u|_{t=0} = u_0, \quad v|_{t=0} = v_0, \quad \alpha|_{t=0} = \alpha_0, \quad \frac{\partial \alpha}{\partial t}|_{t=0} = \alpha_1. \quad (1.11)$$

As far as the nonlinear term f is concerned, we assume more generally that

$$f \in C^1(-1, 1), \quad f(0) = 0, \quad (1.12)$$

$$\lim_{s \rightarrow \pm 1} f(s) = \pm \infty \quad \text{and} \quad \lim_{s \rightarrow \pm 1} f'(s) = +\infty. \quad (1.13)$$

In particular, it follows from these assumptions that

$$f' \geq -c_0, \quad c_0 \geq 0, \quad (1.14)$$

and

$$-c_1 \leq F(s) \leq f(s)s + c_2, \quad c_1, c_2 \geq 0, \quad s \in (-1, 1), \quad (1.15)$$

where $F(s) = \int_0^s f(\tau) d\tau$ (in particular, in order to obtain the right-hand side of (1.15), we can study the variations of the function $s \mapsto f(s)s - F(s) + \frac{c_0}{2}s^2$, whose derivative has, owing to (1.15), the sign of s).

Remark 1.1. *In particular, the thermodynamically relevant logarithmic functions (1.2) satisfy the above assumptions.*

In this work, we study the Cahn-Hilliard and Allen-Cahn equations coupled with a heat equation based on the type III law of thermoelasticity that was first studied in [21], with regular potentials and exponential attractors were found. Here, we first prove the existence of weak solutions for (1.7)-(1.11) with a singular nonlinear term that satisfies (1.13). First, we approximate the singular nonlinear terms by regular ones and prove the convergence of the solution for the approximated problem to that of the limit singular one. Then, we prove the uniqueness of the solution, which allows us to define the corresponding dissipative semigroup. We finally prove some higher-order regularity results which lead to a strict separation property in one space dimension.

We note that, in [14], the authors also studied a coupled AC/CH system with temperature and long-time oscillating properties were found.

Notations

We denote by $((\cdot, \cdot))$ the usual L^2 -scalar product, with associated norm $\|\cdot\|$. We further set $((\cdot, \cdot))_{-1} = (((-\Delta)^{-\frac{1}{2}}\cdot, (-\Delta)^{-\frac{1}{2}}\cdot))$, with associated norm $\|\cdot\|_{-1}$, where $(-\Delta)^{-1}$ denotes the inverse minus Laplace operator associated with Dirichlet boundary conditions. Note that $\|\cdot\|_{-1}$ is equivalent to the usual H_{-1} -norm on $H^{-1}(\Omega) = H_0^1(\Omega)'$. More generally, $\|\cdot\|_X$ denotes the norm in the Banach space X .

Throughout the article, the same letter c, c', c'' (and, sometimes, C) denotes (generally positive) constants which may vary from line to line.

Following the idea of Debussche and Dettori (see [7] and [8]), we consider the following approximated function $f_N \in \mathcal{C}^1(\mathbb{R})$, for $N \in \mathbb{N}$, by

$$f_N(s) = \begin{cases} f(-1 + \frac{1}{N}) + f'(-1 + \frac{1}{N})(s + 1 - \frac{1}{N}), & s < -1 + \frac{1}{N}, \\ f(s), & |s| \leq 1 - \frac{1}{N}, \\ f(1 - \frac{1}{N}) + f'(1 - \frac{1}{N})(s - 1 + \frac{1}{N}), & s > 1 - \frac{1}{N}. \end{cases}$$

Then, we introduce the approximated problem:

$$\frac{\partial u_N}{\partial t} + \Delta^2 u_N - \Delta(f_N(u_N + v_N) + f_N(u_N - v_N)) = 0, \quad (1.16)$$

$$\frac{\partial v_N}{\partial t} - \Delta v_N + f_N(u_N + v_N) - f_N(u_N - v_N) = \frac{\partial \alpha_N}{\partial t}, \quad (1.17)$$

$$\frac{\partial^2 \alpha_N}{\partial t^2} - \Delta \frac{\partial \alpha_N}{\partial t} - \Delta \alpha_N = -\frac{\partial v_N}{\partial t}, \quad (1.18)$$

$$u_N = \Delta u_N = v_N = \alpha_N = 0 \text{ on } \Gamma, \quad (1.19)$$

$$u_N|_{t=0} = u_0, \quad v_N|_{t=0} = v_0, \quad \alpha_N|_{t=0} = \alpha_0, \quad \frac{\partial \alpha_N}{\partial t}|_{t=0} = \alpha_1, \quad (1.20)$$

As far as the approximated function f_N is concerned, we have

$$f'_N \geq -c_0 \quad (1.21)$$

and, setting $F_N(s) = \int_0^s f_N(\tau) d\tau$,

$$-c_3 \leq F_N(s) \leq c_4 f_N(s)s + c_5, \quad c_4 > 0, \quad c_3, c_5 \geq 0, \quad s \in \mathbb{R}, \quad (1.22)$$

$$f_N(s)s \geq c_6 |f_N(s)| - c_7, \quad c_6 > 0, \quad c_7 \geq 0, \quad s \in \mathbb{R}, \quad (1.23)$$

where the constants $c_i, i = 3, \dots, 7$, are independent of N (see [18]).

2 A Priori Estimates

We start by assuming that :

$$\|u_0 + v_0\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \delta \in (0, 1) \quad (2.1)$$

and

$$\|u_0 - v_0\|_{L^\infty(\Omega)} \leq 1 - \delta', \quad \delta' \in (0, 1), \quad (2.2)$$

where δ and δ' are fixed positive constants.

We rewrite (1.16) in the equivalent form

$$(-\Delta)^{-1} \frac{\partial u_N}{\partial t} - \Delta u_N + f_N(u_N + v_N) + f_N(u_N - v_N) = 0. \quad (2.3)$$

We multiply (2.3) by u_N , (1.17) by v_N and sum the resulting equalities to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u_N\|_{-1}^2 + \|v_N\|^2) + \|\nabla u_N\|^2 + \|\nabla v_N\|^2 \\ + ((f_N(u_N + v_N), u_N + v_N)) + ((f_N(u_N - v_N), u_N - v_N)) = \left(\left(\frac{\partial \alpha_N}{\partial t}, v_N \right) \right). \end{aligned} \quad (2.4)$$

Now, owing to (1.23), the previous equation gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u_N\|_{-1}^2 + \|v_N\|^2) + \|\nabla u_N\|^2 + \|\nabla v_N\|^2 \\ + \|f_N(u_N + v_N)\|_{L^1(\Omega)} + \|f_N(u_N - v_N)\|_{L^1(\Omega)} \leq c \left(\left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 + \|v_N\|^2 + 1 \right). \end{aligned} \quad (2.5)$$

Multipling then (1.18) by α_N we obtain

$$\frac{d}{dt} \left(\|\nabla \alpha_N\|^2 + 2 \left(\left(\frac{\partial \alpha_N}{\partial t}, \alpha_N \right) \right) \right) + c \|\nabla \alpha_N\|^2 \leq c' \left\| \frac{\partial v_N}{\partial t} \right\|^2 + c'' \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2. \quad (2.6)$$

Next, we multiply (2.3) by $\frac{\partial u_N}{\partial t}$, (1.17) by $\frac{\partial v_N}{\partial t}$ and have, summing the results,

$$\begin{aligned} \left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial v_N}{\partial t} \right\|^2 + \frac{1}{2} \frac{d}{dt} \left(\|\nabla u_N\|^2 + \|\nabla v_N\|^2 + 2 \int_{\Omega} [F_N(u_N + v_N) + F_N(u_N - v_N)] dx \right) \\ = \left(\left(\frac{\partial \alpha_N}{\partial t}, \frac{\partial v_N}{\partial t} \right) \right). \end{aligned} \quad (2.7)$$

We then multiply (1.18) by $\frac{\partial \alpha_N}{\partial t}$ to obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla \alpha_N\|^2 + \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) + \left\| \nabla \frac{\partial \alpha_N}{\partial t} \right\|^2 = - \left(\left(\frac{\partial v_N}{\partial t}, \frac{\partial \alpha_N}{\partial t} \right) \right). \quad (2.8)$$

Now, summing (2.7) and (2.8), we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|\nabla u_N\|^2 + \|\nabla v_N\|^2 + 2 \int_{\Omega} [F_N(u_N + v_N) + F_N(u_N - v_N)] dx + \|\nabla \alpha_N\|^2 + \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) \\ + \left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial v_N}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \alpha_N}{\partial t} \right\|^2 = 0 \end{aligned} \quad (2.9)$$

Summing finally (2.9), ϵ_1 times (2.5) and ϵ_2 times (2.6), where $\epsilon_1, \epsilon_2 > 0$ are chosen small enough so that

$$\left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 + \epsilon_2 \left(\|\nabla \alpha_N\|^2 + 2 \left(\left(\frac{\partial \alpha_N}{\partial t}, \alpha_N \right) \right) \right) \geq c \left(\left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 + \|\nabla \alpha_N\|^2 \right), \quad (2.10)$$

$$2 - \epsilon_1 c - \epsilon_2 c' > 0, \quad (2.11)$$

$$2 - \epsilon_2 c'' > 0, \quad (2.12)$$

we obtain an inequality of the form

$$\begin{aligned} \frac{d}{dt} E_1^N + c \left(E_1^N + \|f_N(u_N + v_N)\|_{L^1(\Omega)} + \|f_N(u_N - v_N)\|_{L^1(\Omega)} \right. \\ \left. + \left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial v_N}{\partial t} \right\|^2 + \|\nabla \alpha_N\|^2 + \left\| \nabla \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) \leq c, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} E_1^N = \epsilon_1 \|u_N\|_{-1}^2 + \epsilon_1 \|v_N\|^2 + \|\nabla u_N\|^2 + \|\nabla v_N\|^2 + 2 \int_{\Omega} [F_N(u_N + v_N) + F_N(u_N - v_N)] dx \\ + (1 + \epsilon_2) \|\nabla \alpha_N\|^2 + 2\epsilon_2 \left(\left(\frac{\partial \alpha_N}{\partial t}, \alpha_N \right) \right) + \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2, \end{aligned} \quad (2.14)$$

satisfies

$$\begin{aligned} E_1^N \geq c \left(\|\nabla u_N\|^2 + \|\nabla v_N\|^2 + 2 \int_{\Omega} [F_N(u_N + v_N) + F_N(u_N - v_N)] dx \right. \\ \left. + \|\nabla \alpha_N\|^2 + \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) - c', \quad c > 0. \end{aligned} \quad (2.15)$$

We note that (2.13) and Gronwall's lemma imply the dissipative estimate

$$E_1^N(t) \leq e^{-ct} E_1^N(0) + c', \quad c > 0, \quad t \geq 0. \quad (2.16)$$

Integrating now (2.13) with respect to time, we have, for $r > 0$ fixed,

$$\begin{aligned} \|f_N(u_N + v_N)\|_{L^1((t,t+r) \times \Omega)} + \|f_N(u_N - v_N)\|_{L^1((t,t+r) \times \Omega)} \\ \leq ce^{-c't} \left(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\alpha_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla \alpha_0\|^2 + \|\alpha_1\|^2 \right. \\ \left. + c \int_{\Omega} (F_N(u_0 + v_0) + F_N(u_0 - v_0)) dx \right) + c''(r). \end{aligned} \quad (2.17)$$

Furthermore,

$$\begin{aligned} \int_t^{t+r} \left(\left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial v_N}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) d\tau \\ \leq ce^{-c't} \left(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\alpha_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla \alpha_0\|^2 + \|\alpha_1\|^2 \right. \\ \left. + c \int_{\Omega} (F_N(u_0 + v_0) + F_N(u_0 - v_0)) dx \right) + c''(r), \quad c' > 0, \quad t \geq 0. \end{aligned} \quad (2.18)$$

Also, we note that, owing to (2.1)-(2.2), we have

$$\int_{\Omega} (F_N(u_0 + v_0) + F_N(u_0 - v_0)) dx \leq c.$$

Consequently, (2.17) yields

$$\begin{aligned} \|f_N(u_N + v_N)\|_{L^1((t,t+r) \times \Omega)} + \|f_N(u_N - v_N)\|_{L^1((t,t+r) \times \Omega)} \\ \leq ce^{-c't} \left(\|u_0\|_{-1}^2 + \|v_0\|^2 + \|\alpha_0\|^2 + \|\nabla u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla \alpha_0\|^2 + \|\alpha_1\|^2 + c''(r) \right). \end{aligned} \quad (2.19)$$

We have thus found an estimate on the L^1 -norm of $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$.

We finally multiply (2.3) by $-\Delta u_N$, (1.17) by $-\Delta v_N$ and (1.18) by $\frac{\partial \alpha_N}{\partial t}$. Summing the result and using (1.21), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|u_N\|^2 + \|\nabla v_N\|^2 + \|\nabla \alpha_N\|^2 + \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) + \|\Delta u_N\|^2 + c \|\Delta v_N\|^2 + c' \left\| \nabla \frac{\partial \alpha_N}{\partial t} \right\|^2 \\ \leq c'' (\|\nabla u_N\|^2 + \|\nabla v_N\|^2) + \left\| \frac{\partial v_N}{\partial t} \right\|^2. \end{aligned} \quad (2.20)$$

Using an interpolation inequality and Young's inequality, we have

$$\|u_N\|_{H^1(\Omega)}^2 \leq c \|u_N\| \|u_N\|_{H^2(\Omega)} \leq \frac{1}{2} \|\Delta u_N\|^2 + c \|u_N\|^2, \quad c \geq 0.$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \left(\|u_N\|^2 + \|\nabla v_N\|^2 + \|\nabla \alpha_N\|^2 + \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) + c \left(\|\Delta u_N\|^2 + \|\Delta v_N\|^2 + \left\| \nabla \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) \\ \leq c' (\|u_N\|^2 + \|\nabla v_N\|^2) + \left\| \frac{\partial v_N}{\partial t} \right\|^2. \end{aligned} \quad (2.21)$$

Using now Gronwall's lemma, we obtain

$$\|u_N\|^2 + \|\nabla v_N\|^2 + \|\nabla \alpha_N\|^2 + \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 \leq e^{c't} (\|u_0\|^2 + \|\nabla v_0\|^2 + \|\nabla \alpha_0\|^2 + \|\alpha_1\|^2) + c''. \quad (2.22)$$

Furthermore, multiplying (1.18) by $-\Delta \frac{\partial \alpha_N}{\partial t}$ yields

$$\frac{d}{dt} \left(\|\Delta \alpha_N\|^2 + \left\| \nabla \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) + \left\| \Delta \frac{\partial \alpha_N}{\partial t} \right\|^2 \leq \left\| \frac{\partial v_N}{\partial t} \right\|^2. \quad (2.23)$$

3 Existence of solutions

Theorem 3.1. *We assume that $(u_0, v_0, \alpha_0, \alpha_1) \in (H^2(\Omega) \cap H_0^1(\Omega))^3 \times H_0^1(\Omega)$ such that u_0, v_0 satisfies (2.1)-(2.2). Then, the problem (1.7)-(1.11) admits at least one solution $(u, v, \alpha, \frac{\partial \alpha}{\partial t})$ such that, $\forall T > 0$*

$$\begin{aligned} (u, v) &\in L^\infty(\mathbb{R}^+; H_0^1(\Omega)^2) \cap L^2(0, T; H^2(\Omega)^2), \\ \alpha &\in L^\infty(\mathbb{R}^+; H_0^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega)), \\ \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right) &\in L^2(0, T; H^{-1}(\Omega) \times L^2(\Omega)) \end{aligned}$$

and

$$\frac{\partial \alpha}{\partial t} \in L^\infty(\mathbb{R}^+; H_0^1(\Omega)) \cap L^2(0, T; H^2(\Omega)).$$

Furthermore, $\forall t > 0$,

$$\|u(t) + v(t)\|_{L^\infty(\Omega)} \leq 1 \quad \text{and} \quad \|u(t) - v(t)\|_{L^\infty(\Omega)} \leq 1,$$

and the sets

$$\{(t, x) \in (0, T) \times \Omega; |u(t, x) + v(t, x)| \geq 1\} \quad \text{and} \quad \{(t, x) \in (0, T) \times \Omega; |u(t, x) - v(t, x)| \geq 1\},$$

are of null measures.

Proof. We have, owing to the dissipative estimate (2.15), that

$$\sup_{[0,T]} \left\{ \|u_N(t)\|_{H^1(\Omega)}^2 + \|v_N(t)\|_{H^1(\Omega)}^2 + \|\alpha_N(t)\|_{H^1(\Omega)}^2 + \left\| \frac{\partial \alpha_N}{\partial t}(t) \right\|^2 \right\} \leq c', \quad (3.1)$$

where c' is independent of N . Letting N tends to $+\infty$ and considering a subsequence, we have by estimation (3.1)

$$u_N \rightharpoonup u \quad \text{weak star in } L^\infty(0, T; H^1(\Omega)), \quad (3.2)$$

$$v_N \rightarrow v \quad \text{weak star in } L^\infty(0, T; H^1(\Omega)) \quad (3.3)$$

$$\alpha_N \rightarrow \alpha \quad \text{weak star in } L^\infty(0, T; H^1(\Omega)) \quad (3.4)$$

$$\frac{\partial \alpha_N}{\partial t} \rightharpoonup \frac{\partial \alpha}{\partial t} \quad \text{weak star in } L^\infty(0, T; L^2(\Omega)) \quad (3.5)$$

Integrating now (2.13) between 0 and t , we have, in view of (3.1),

$$E_1^N(t) + c \int_0^t \left(\left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial v_N}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \alpha_N}{\partial t} \right\|^2 \right) ds \leq c', \quad \forall t \in [0, T], \quad c > 0, \quad (3.6)$$

where c' is independent of N . It thus follows that

$$\frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{weakly in } L^2(0, T; H^{-1}(\Omega)), \quad (3.7)$$

$$\frac{\partial v_N}{\partial t} \rightharpoonup \frac{\partial v}{\partial t} \quad \text{weakly in } L^2(0, T; L^2(\Omega)). \quad (3.8)$$

Also, integrating (2.23) between 0 and t , we have,

$$\|\Delta \alpha_N(t)\|^2 + \left\| \nabla \frac{\partial \alpha_N}{\partial t}(t) \right\|^2 + \int_0^t \left\| \Delta \frac{\partial \alpha_N}{\partial t}(s) \right\|^2 ds \leq c', \quad \forall t \in [0, T], \quad (3.9)$$

where c' independent of N . It thus follows that

$$\alpha_N \rightarrow \alpha \quad \text{weak star in } L^\infty(0, T; H^2(\Omega)), \quad (3.10)$$

$$\frac{\partial \alpha_N}{\partial t} \rightharpoonup \frac{\partial \alpha}{\partial t} \quad \text{weak star in } L^\infty(0, T; H_0^1(\Omega)), \quad \text{weakly in } L^2(0, T; H^2(\Omega)). \quad (3.11)$$

Integrating (2.21) between 0 and t , we deduce

$$\begin{aligned} & \|u_N(t)\|^2 + \|\nabla v_N(t)\|^2 + \|\nabla \alpha_N(t)\|^2 + \left\| \frac{\partial \alpha_N}{\partial t}(t) \right\|^2 \\ & + c \int_0^t \left\{ \|\Delta u_N(s)\|^2 + \|\Delta v_N(s)\|^2 \right\} ds \leq c', \quad \forall t \in [0, T], \end{aligned} \quad (3.12)$$

where c' is independent of N . This results that

$$\begin{aligned} \Delta u_N & \rightharpoonup \Delta u \quad \text{weakly in } L^2((0, T) \times \Omega), \\ \Delta v_N & \rightharpoonup \Delta v \quad \text{weakly in } L^2((0, T) \times \Omega). \end{aligned} \quad (3.13)$$

The only difficulty, when passing to the limit, is to pass to the limit in the nonlinear terms containing f_N . First, it follows from (2.19) that $f_N(u_N + v_N)$ and $f_N(u_N - v_N)$ are bounded, independently of N , in $L^1((0, T) \times \Omega)$. Then, it follows from the explicit expression of f_N that

$$\text{meas}(F_{N,M}) \leq \mu\left(\frac{1}{N}\right), \quad N \leq M,$$

where

$$F_{N,M} = \{(t, x) \in (0, T) \times \Omega; |u_M(t, x) + v_M(t, x)| > 1 - \frac{1}{N}\}$$

and

$$\mu(s) = \frac{c}{\min(|f(1-s)|, |f(s-1)|)},$$

where, here, the constant c is independent of N and M . Note that there holds

$$\int_0^T \int_{\Omega} |f_M(u_M + v_M)| dx dt \geq \int_{F_{N,M}} |f_M(u_M + v_M)| dx dt \geq c' \text{meas}(F_{N,M}) \frac{1}{\mu(\frac{1}{N})}, \quad (3.14)$$

where the constant c' is independent of N and M .

Passing now to the limit M tends to $+\infty$ (employing Fatou's lemma on (3.14)) and the N tends to $+\infty$ (noting that $\lim_{s \rightarrow 0} \mu(s) = 0$) to find

$$\text{meas}\{(t, x) \in (0, T) \times \Omega; |u(t, x) + v(t, x)| \geq 1\} = 0,$$

so that

$$-1 < u(t, x) + v(t, x) < 1 \quad \text{a.e. } (t, x). \quad (3.15)$$

In the same way, we can prove that

$$-1 < u(t, x) - v(t, x) < 1 \quad \text{a.e. } (t, x). \quad (3.16)$$

Next, it follows from the above almost everywhere convergence of u_N and v_N , (3.15)-(3.16) and the explicit expression of f_N that

$$f_N(u_N + v_N) \rightarrow f(u + v) \quad \text{a.e. } (t, x) \in (0, T) \times \Omega \quad (3.17)$$

and

$$f_N(u_N - v_N) \rightarrow f(u - v) \quad \text{a.e. } (t, x) \in (0, T) \times \Omega. \quad (3.18)$$

Finally, rewriting (1.17) and (2.3) as

$$f_N(u_N + v_N) + f_N(u_N - v_N) = -(-\Delta)^{-1} \frac{\partial u_N}{\partial t} + \Delta u_N, \quad (3.19)$$

$$f_N(u_N + v_N) - f_N(u_N - v_N) = \frac{\partial \alpha_N}{\partial t} - \frac{\partial v_N}{\partial t} + \Delta v_N. \quad (3.20)$$

We sum (3.19) and (3.20), then we subtract them, to have

$$2f_N(u_N + v_N) = -(-\Delta)^{-1} \frac{\partial u_N}{\partial t} + \Delta u_N + \frac{\partial \alpha_N}{\partial t} - \frac{\partial v_N}{\partial t} + \Delta v_N,$$

$$2f_N(u_N - v_N) = -(-\Delta)^{-1} \frac{\partial u_N}{\partial t} + \Delta u_N - \frac{\partial \alpha_N}{\partial t} + \frac{\partial v_N}{\partial t} - \Delta v_N.$$

Therefore,

$$\begin{aligned} & \|f_N(u_N + v_N)\|^2 + \|f_N(u_N - v_N)\|^2 \\ & \leq c \left(\left\| \frac{\partial u_N}{\partial t} \right\|_{-1}^2 + \|\Delta u_N\|^2 + \left\| \frac{\partial \alpha_N}{\partial t} \right\|^2 + \left\| \frac{\partial v_N}{\partial t} \right\|^2 + \|\Delta v_N\|^2 \right), \end{aligned} \quad (3.21)$$

integrating (3.21) between 0 and t , owing to (3.6) and (3.12), we have

$$\|f_N(u_N + v_N)\|_{L^2((0,T) \times \Omega)}^2 + \|f_N(u_N - v_N)\|_{L^2((0,T) \times \Omega)}^2 \leq c, \quad (3.22)$$

where c is independent of N . Thus, it follows from (3.17) and (3.18) that

$$f_N(u_N + v_N) \rightarrow f(u + v) \quad \text{weakly in } L^2((0, T) \times \Omega),$$

and

$$f_N(u_N - v_N) \rightarrow f(u - v) \quad \text{weakly in } L^2((0, T) \times \Omega),$$

which finishes the proof of the passage to the limit (the weak continuity property follows from Strauss's lemma, see, e.g., [36]).

□

The following result gives us the uniqueness of the solution of the problem (1.7)-(1.11).

Theorem 3.2. *Under the assumptions of Theorem 3.1, the problem (1.7)-(1.11) admits a unique solution with the above regularity.*

Proof. We actually prove a more general result, namely, the uniqueness of solutions such that $|u(t, x) + v(t, x)| < 1$ and $|u(t, x) - v(t, x)| < 1$ almost everywhere in $(0, T) \times \Omega$ and which does not necessarily satisfy the separation property (when this property is satisfied, the proof of uniqueness is straightforward).

Let $\left(u^{(1)}, v^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right)$ and $\left(u^{(2)}, v^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right)$ be two solutions of (1.7)-(1.11) with the same initial data.

We set

$$\left(u, v, \alpha, \frac{\partial \alpha}{\partial t}\right) = \left(u^{(1)}, v^{(1)}, \alpha^{(1)}, \frac{\partial \alpha^{(1)}}{\partial t}\right) - \left(u^{(2)}, v^{(2)}, \alpha^{(2)}, \frac{\partial \alpha^{(2)}}{\partial t}\right).$$

Then (u, v, α) satisfies

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}) + f(u^{(1)} - v^{(1)}) - f(u^{(2)} - v^{(2)}) = 0, \quad (3.23)$$

$$\frac{\partial v}{\partial t} - \Delta v + f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}) - f(u^{(1)} - v^{(1)}) + f(u^{(2)} - v^{(2)}) = \frac{\partial \alpha}{\partial t}, \quad (3.24)$$

$$\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha = -\frac{\partial v}{\partial t}, \quad (3.25)$$

$$u = \Delta u = v = \alpha = 0 \text{ on } \Gamma, \quad (3.26)$$

$$u|_{t=0} = v|_{t=0} = \alpha|_{t=0} = \frac{\partial \alpha}{\partial t}|_{t=0} = 0. \quad (3.27)$$

We multiply (3.23) by u , (3.24) by v and sum to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u\|_{-1}^2 + \|v\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 + ((f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}), u + v)) \\ + ((f(u^{(1)} - v^{(1)}) - f(u^{(2)} - v^{(2)}), u - v)) = \left(\left(\frac{\partial \alpha}{\partial t}, v \right) \right). \end{aligned} \quad (3.28)$$

Now, let $p = u^{(1)} + v^{(1)}$, $q = u^{(2)} + v^{(2)}$, $h = u^{(1)} - v^{(1)}$, $l = u^{(2)} - v^{(2)}$ and have, owing to (1.14)

$$((f(u^{(1)} + v^{(1)}) - f(u^{(2)} + v^{(2)}), u + v)) = ((f(p) - f(q), p - q)) \geq -c_0 \|u + v\|^2 \quad (3.29)$$

and

$$((f(u^{(1)} - v^{(1)}) - f(u^{(2)} - v^{(2)}), u - v)) = ((f(h) - f(l), h - l)) \geq -c_0 \|u - v\|^2. \quad (3.30)$$

Employing the interpolation inequality

$$\|u\|^2 \leq c \|u\|_{-1}^2 \|\nabla u\|^2 \leq \frac{1}{2} \|\nabla u\|^2 + c \|u\|_{-1}^2,$$

we obtain

$$\frac{d}{dt} (\|u\|_{-1}^2 + \|v\|^2) + \|\nabla u\|^2 + \|\nabla v\|^2 \leq c \left(\|u\|_{-1}^2 + \|v\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right). \quad (3.31)$$

We then integrate (3.25) between 0 and t and have, noting that $u(0) = v(0) = \alpha(0) = \frac{\partial \alpha}{\partial t}(0) = 0$,

$$\frac{\partial \alpha}{\partial t} - \Delta \alpha - \Delta \int_0^t \alpha \, ds = v. \quad (3.32)$$

Multiplying (3.32) by α , we find

$$\frac{d}{dt} \left(\|\alpha\|^2 + \left\| \nabla \int_0^t \alpha \, ds \right\|^2 \right) + \|\nabla \alpha\|^2 \leq c \|v\|^2 \quad (3.33)$$

and, multiplying (3.32) by $\frac{\partial \alpha}{\partial t}$ then noting that

$$\left(\left(\nabla \int_0^t \alpha \, ds, \nabla \frac{\partial \alpha}{\partial t} \right) \right) = \frac{d}{dt} \left(\left(\nabla \int_0^t \alpha \, ds, \nabla \alpha \right) \right) - \|\nabla \alpha\|^2,$$

we obtain

$$\frac{d}{dt} \left(\|\nabla \alpha\|^2 + 2 \left(\left(\nabla \int_0^t \alpha \, ds, \nabla \alpha \right) \right) \right) + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \leq 2 \|\nabla \alpha\|^2 + c \|v\|^2. \quad (3.34)$$

We finally add (3.33) and $\delta_1 \times (3.34)$, where $\delta_1 \in \left(0, \frac{1}{2}\right)$ is such that

$$\left\| \nabla \int_0^t \alpha \, ds \right\|^2 + 2\delta_1 \left(\left(\nabla \int_0^t \alpha \, ds, \nabla \alpha \right) \right) + \delta_1 \|\nabla \alpha\|^2 \geq c \left(\left\| \nabla \int_0^t \alpha \, ds \right\|^2 + \|\nabla \alpha\|^2 \right), \quad c > 0, \quad (3.35)$$

and have

$$\frac{dE}{dt} + c \left(\|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) \leq c \|v\|^2, \quad c > 0, \quad (3.36)$$

where

$$E = \|\alpha\|^2 + \delta_1 \|\nabla \alpha\|^2 + \left\| \nabla \int_0^t \alpha ds \right\|^2 + 2\delta_1 \left(\left(\nabla \int_0^t \alpha ds, \nabla \alpha \right) \right).$$

We now add $\delta_2 \times (3.31)$ and (3.36) and obtain, taking $\delta_2 > 0$ small enough

$$\frac{d}{dt} \left(\delta_2 (\|u\|_{-1}^2 + \|v\|^2) + E \right) + c \left(\|\nabla u\|^2 + \|\nabla v\|^2 + \|\nabla \alpha\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 \right) \leq c' (\|u\|_{-1}^2 + \|v\|^2), \quad c > 0. \quad (3.37)$$

The uniqueness follows from (3.35), (3.37) and Gronwall's lemma. We can note that this would not give a continuity result (with respect to the initial data) for $\frac{\partial \alpha}{\partial t}$, but such a continuity would then follow from (3.3). \square

Moreover, it follows from Theorems 3.1 and 3.2 that we can define the continuous family of operators

$$S(t) : \Phi_1 \rightarrow \Phi, (u_0, v_0, \alpha_0, \alpha_1) \rightarrow (u(t), v(t), \alpha(t), \frac{\partial \alpha(t)}{\partial t}), t \geq 0,$$

where

$$\Phi := \{(u, v, \alpha, \frac{\partial \alpha}{\partial t}) \in H^1(\Omega)^2 \times H^2(\Omega) \times L^2(\Omega); |u + v| < 1 \text{ and } |u - v| < 1 \text{ a.e.}\}.$$

and

$$\Phi_1 := \Phi \cap \{(u, v, \alpha, \frac{\partial \alpha}{\partial t}) \in L^\infty(\Omega)^2 \times H^1(\Omega) \times L^2(\Omega); \|u + v\|_{L^\infty(\Omega)} < 1 \text{ and } \|u - v\|_{L^\infty(\Omega)} < 1\}.$$

We then deduce from (2.16) the

Theorem 3.3. *The semigroup $S(t)$ is dissipative in $H^1(\Omega)^2 \times H^2(\Omega) \times L^2(\Omega)$, in the sense it possesses a bounded absorbing set $\beta_1 \subset H^1(\Omega)^2 \times H^2(\Omega) \times L^2(\Omega)$ (i.e., $\forall B \subset \Phi_1$ bounded, $\exists t_0 = t_0(B)$ such that $t \geq t_0 \Rightarrow S(t)B \subset \beta_1$).*

4 Further Regularity Results

In what follows, we set $V = H_0^1(\Omega)$ and $W = H^2(\Omega)$. We also denote by V' the dual space of V and by $\|\cdot\|_{V'}$ its norm.

We can decompose the singular potential F as

$$F(x) = S(x) + \frac{\theta_c}{2}(1 - x^2),$$

with

$$\lim_{x \rightarrow -1} S'(x) = -\infty, \quad \lim_{x \rightarrow 1} S'(x) = +\infty, \quad S''(x) \geq \theta > 0, \quad \forall x \in (-1, 1), \quad (4.1)$$

and we let

$$\theta_c - \theta = \kappa > 0.$$

We also require that S satisfies

$$|S''(x)| \leq e^{c|S'(x)|+c}, \quad \forall x \in (-1, 1), \quad (4.2)$$

for some positive constant c , and S is convex.

We mention below a Trudinger-Moser type inequality (see, e.g. [26]) which is needed later on.

Lemma 4.1. *Let Ω be a bounded smooth domain of \mathbb{R}^2 . Then, there exists a positive constant c such that*

$$\int_{\Omega} e^{|u|} dx \leq c e^{c\|u\|_V^2}, \quad \forall u \in V. \quad (4.3)$$

Let us now define the free energy functional

$$\mathcal{E}(u+v) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\nabla v\|^2 + 2 \int_{\Omega} F(u+v) dx + 2 \int_{\Omega} F(u-v) dx.$$

Theorem 4.1. *Under the assumptions of Theorem 3.1, the solution further fulfills the estimate*

$$\mathcal{E}_1(u, v, \alpha, \alpha_t)(t) + \int_t^{t+1} \left(\|\nabla \mu(s)\|^2 + \|v_t(s)\|^2 + \|\nabla \alpha_t(s)\|^2 \right) ds \leq \mathcal{E}_1(u_0, v_0, \alpha_0, \alpha_1), \quad \forall t \geq 0, \quad (4.4)$$

where

$$\mathcal{E}_1(u, v, \alpha, \alpha_t) = \mathcal{E}(u+v) + \|\alpha_t\|^2 + \|\nabla \alpha\|^2.$$

Proof. There remains to prove estimate (4.4).

We rewrite equation (1.7) in the form

$$(-\Delta)^{-1} \frac{\partial u}{\partial t} - \Delta u + f(u+v) + f(u-v) = 0. \quad (4.5)$$

We start by differentiating equations (4.5) and (1.8) with respect to time to find, using (1.9),

$$(-\Delta)^{-1} \frac{\partial}{\partial t} \frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t} + f'(u+v) \left(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} \right) + f'(u-v) \left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} \right) = 0, \quad (4.6)$$

$$\frac{\partial}{\partial t} \frac{\partial v}{\partial t} - \Delta \frac{\partial v}{\partial t} + f'(u+v) \left(\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} \right) - f'(u-v) \left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t} \right) = -\frac{\partial v}{\partial t} + \Delta \frac{\partial \alpha}{\partial t} + \Delta \alpha, \quad (4.7)$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0 \text{ on } \Gamma. \quad (4.8)$$

Summing (4.6) times $t \frac{\partial u}{\partial t}$, (4.7) times $t \frac{\partial v}{\partial t}$ and (1.9) times $t \frac{\partial \alpha}{\partial t}$, then using (1.14) and an interpolation inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(t \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + t \left\| \frac{\partial v}{\partial t} \right\|^2 + t \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + t \|\nabla \alpha\|^2 \right) + ct \left(\left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial v}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 \right) \\ & \leq \frac{1}{2} \left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \frac{1}{2} \left\| \frac{\partial v}{\partial t} \right\|^2 + \frac{1}{2} \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + \frac{1}{2} \|\nabla \alpha\|^2 \\ & \quad + c' t \left(\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + \|\nabla \alpha\|^2 \right). \end{aligned} \quad (4.9)$$

We deduce from (2.18)-(2.19) (which hold when $N \rightarrow +\infty$), (4.9) and Gronwall's lemma that

$$\begin{aligned} & \left\| \frac{\partial u}{\partial t}(t) \right\|_{-1}^2 + \left\| \frac{\partial v}{\partial t}(t) \right\|^2 + \left\| \frac{\partial \alpha}{\partial t}(t) \right\|^2 + \|\nabla \alpha(t)\|^2 \\ & \leq \frac{1}{t} Q(\|u_0\|_{H^1(\Omega)}, \|v_0\|_{H^1(\Omega)}, \|\alpha_0\|_{H^1(\Omega)}, \|\alpha_1\|), \quad t \in (0, 1]. \end{aligned} \quad (4.10)$$

Summing (4.6) times $\frac{\partial u}{\partial t}$, (4.7) times $\frac{\partial v}{\partial t}$, (1.9) times $\frac{\partial \alpha}{\partial t}$ and using (1.14) and an interpolation inequality, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + \|\nabla \alpha\|^2 \right) + c \left\| \nabla \frac{\partial u}{\partial t} \right\|^2 \\ & + \left\| \nabla \frac{\partial v}{\partial t} \right\|^2 + c' \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 \leq c'' \left(\left\| \frac{\partial u}{\partial t} \right\|_{-1}^2 + \left\| \frac{\partial v}{\partial t} \right\|^2 + \left\| \frac{\partial \alpha}{\partial t} \right\|^2 + \|\nabla \alpha\|^2 \right). \end{aligned} \quad (4.11)$$

Finally, we conclude from (4.10)-(4.11) and Gronwall's lemma that

$$\begin{aligned} & \left\| \frac{\partial u}{\partial t}(t) \right\|_{-1}^2 + \left\| \frac{\partial v}{\partial t}(t) \right\|^2 + \left\| \frac{\partial \alpha}{\partial t}(t) \right\|^2 + \|\nabla \alpha(t)\|^2 \\ & \leq e^{ct} Q(\|u_0\|_{H^1(\Omega)}, \|v_0\|_{H^1(\Omega)}, \|\alpha_0\|_{H^1(\Omega)}, \|\alpha_1\|), \quad t \geq 1, \end{aligned} \quad (4.12)$$

where Q denotes a monotone increasing function.

Integrating now (4.11) between t and $t+1$ and using (4.12), we find

$$\int_t^{t+1} \left(\left\| \nabla \frac{\partial u}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial v}{\partial t} \right\|^2 + \left\| \nabla \frac{\partial \alpha}{\partial t} \right\|^2 \right) ds \leq e^{ct} Q(\|u_0\|_{H^1(\Omega)}, \|v_0\|_{H^1(\Omega)}, \|\alpha_0\|_{H^1(\Omega)}, \|\alpha_1\|), \quad (4.13)$$

for $t \geq 1$. Therefore, $(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial \alpha}{\partial t}) \in L^2(t, t+1; H_0^1(\Omega)^3)$.

Then, we can rewrite equations (1.7)-(1.8) as

$$u_t = \Delta \mu, \quad (4.14)$$

where

$$\mu = -\Delta u + F'(u+v) + F'(u-v),$$

and

$$v_t + F'(u+v) - F'(u-v) = \Delta v + \alpha_t. \quad (4.15)$$

Equations (4.14) and (4.15) are equivalent to

$$((u_t, q)) + ((\nabla \mu, \nabla q)) = 0, \quad \forall q \in H_0^1(\Omega), \quad (4.16)$$

$$((v_t, q')) + ((F'(u+v), q')) - ((F'(u-v), q')) + ((\nabla v, \nabla q')) = ((\alpha_t, q')), \quad \forall q' \in H_0^1(\Omega). \quad (4.17)$$

Using equations (4.16)-(4.17) and the standard chain rule in $L^2(0, T; V) \cap H^1(0, T; V')$, we get

$$((u_t, \mu)) + ((\nabla \mu, \nabla \mu)) + ((v_t, v_t)) + ((F'(u+v), v_t)) - ((F'(u-v), v_t)) + ((\nabla v, \nabla v_t)) = ((\alpha_t, v_t)),$$

Multiplying also (1.9) by $\frac{\partial \alpha}{\partial t}$, we obtain

$$\frac{1}{2} \frac{d}{dt} (\|\nabla \alpha\|^2 + \|\alpha_t\|^2) + \|\nabla \alpha_t\|^2 = -((v_t, \alpha_t)).$$

Noting that

$$((u_t, \mu)) = \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + ((F'(u+v), u_t)) + ((F'(u-v), u_t)),$$

the above estimates give the energy equality

$$\frac{d}{dt}\mathcal{E}_1(u, v, \alpha, \alpha_t)(t) + \|\nabla\mu\|^2 + \|v_t\|^2 + \|\nabla\alpha_t\|^2 = 0.$$

It finally follows from the Gronwall's lemma that

$$\mathcal{E}_1(u, v, \alpha, \alpha_t)(t) + \int_t^{t+1} \left(\|\nabla\mu(s)\|^2 + \|v_t(s)\|^2 + \|\nabla\alpha_t(s)\|^2 \right) ds \leq \mathcal{E}_1(u_0, v_0, \alpha_0, \alpha_1), \quad \forall t \geq 0.$$

□

In what follows, according to (4.4), the generic positive constant c may also depend on the initial energy $\mathcal{E}_1(u_0, v_0, \alpha_0, \alpha_1)$. In particular, we will use

$$\mathcal{E}_1(u, v, \alpha, \alpha_t)(t) + \int_t^{t+1} \left(\|\nabla\mu(s)\|^2 + \|v_t(s)\|^2 + \|\nabla\alpha_t(s)\|^2 \right) ds \leq c, \quad \forall t \geq 0. \quad (4.18)$$

Theorem 4.2. *Let the assumptions of Theorem 3.1 hold. Then, there exists a positive constant c such that*

$$\|\mu\|_{L^\infty(1,t;V)} + \|v_t\|_{L^\infty(1,t;L^2(\Omega))} + \|\alpha_t\|_{L^\infty(1,t;L^2(\Omega))} \leq c, \quad \forall t \geq 1, \quad (4.19)$$

and

$$\begin{aligned} \|u_t + v_t\|_{L^\infty(1,t;V')} + \|u_t - v_t\|_{L^\infty(1,t;V')} + \|u_t\|_{L^2(t,t+1;V)} + \|v_t\|_{L^2(t,t+1;V)} \\ \leq c, \quad \forall t \geq 1. \end{aligned} \quad (4.20)$$

Proof. We start by differentiating equation (4.15) with respect to time, which yields, using (1.9)

$$\frac{\partial}{\partial t}v_t + \frac{\partial}{\partial t}[F'(u+v) - F'(u-v)] = \Delta v_t - v_t + \Delta\alpha_t + \Delta\alpha. \quad (4.21)$$

Testing (4.14) by μ_t , (4.21) by v_t and multiplying (1.9) by $\frac{\partial\alpha}{\partial t}$, then summing the resulting equations, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|\nabla\mu\|^2 + \|v_t\|^2 + \|\nabla\alpha\|^2 + \left\| \frac{\partial\alpha}{\partial t} \right\|^2) + \|v_t\|^2 + \|\nabla v_t\|^2 + \|\nabla\alpha_t\|^2 + ((u_t, -\Delta u_t)) \\ + ((F''(u+v)(u_t+v_t), u_t+v_t)) + ((F''(u-v)(u_t-v_t), u_t-v_t)) \\ = -((\nabla\alpha_t, \nabla v_t)) - ((\nabla\alpha, \nabla v_t)) - ((v_t, \alpha_t)). \end{aligned}$$

We observe that

$$\begin{aligned} ((u_t, -\Delta u_t)) + ((F''(u_t+v_t)(u_t+v_t), u_t+v_t)) + ((F''(u_t-v_t)(u_t-v_t), u_t-v_t)) \\ \geq \|\nabla u_t\|^2 - \kappa \|u_t + v_t\|^2 - \kappa \|u_t - v_t\|^2 \\ \geq \frac{1}{2} \|\nabla u_t\|^2 - \frac{1}{2} \|\nabla v_t\|^2 - c \|u_t + v_t\|_{V'}^2 - c \|u_t - v_t\|_{V'}^2. \end{aligned} \quad (4.22)$$

Accordingly, setting

$$\Psi(t) = \frac{1}{2} \|\nabla\mu(t)\|^2 + \frac{1}{2} \|v_t(t)\|^2 + \frac{1}{2} \|\nabla\alpha\|^2 + \frac{1}{2} \|\alpha_t\|^2,$$

we end up with the differential inequality

$$\frac{d}{dt}\Psi + \frac{1}{2} \|\nabla u_t\|^2 + c(\|v_t\|_{H^1(\Omega)}^2 + \|\nabla\alpha_t\|^2) \leq c\|u_t + v_t\|_{V'}^2 + c\|u_t - v_t\|_{V'}^2 + c(\|\nabla\alpha\|^2 + \|\alpha_t\|^2). \quad (4.23)$$

Using (4.18), the definition of \mathcal{E}_1 , and (1.15) we can deduce that

$$\int_t^{t+1} \Psi(s) ds \leq c, \quad c \geq 0. \quad (4.24)$$

Also, by comparison,

$$\int_t^{t+1} \|u_t + v_t\|_{V'}^2 ds \leq c \quad (4.25)$$

and

$$\int_t^{t+1} \|u_t - v_t\|_{V'}^2 ds \leq c. \quad (4.26)$$

Therefore, the uniform Gronwall's lemma leads to

$$\Psi(t) \leq c, \quad \forall t \geq 1.$$

In particular, we have the bound

$$\|\mu\|_{L^\infty(1,t;V)} + \|v_t\|_{L^\infty(1,t;L^2(\Omega))} + \|\alpha_t\|_{L^\infty(1,t;L^2(\Omega))} \leq c, \quad \forall t \geq 1,$$

which in turn gives

$$\|u_t\|_{L^\infty(1,t;V')} + \|v_t\|_{L^\infty(1,t;V')} + \|\alpha_t\|_{L^\infty(1,t;V')} \leq c, \quad \forall t \geq 1.$$

The desired conclusion (4.20) follows from an integration in time of (4.23) on $(t, t+1)$, $t \geq 1$, combined with the previous inequality. □

Remark 4.1. *The proof of Theorem 4.1 is formal, but it can be justified within a Galerkin scheme as in the proof of Theorem 3.3. More precisely, all the computations can be rigorously performed within the Galerkin scheme. Given that F'' is controlled from below, the estimates turn out to be independent of the approximation parameter and a final passage to the limit gives the result.*

Lemma 4.2. *Let $N=2$ and the assumptions of Theorem 3.1 hold. Then, for any $p \geq 1$, there exist two positive constants c and c' depending on p such that*

$$\|S''(u+v)\|_{L^p(t,t+1;L^p(\Omega))} \leq c(p), \quad \forall t \geq 1, \quad (4.27)$$

and

$$\|S''(u-v)\|_{L^p(t,t+1;L^p(\Omega))} \leq c'(p), \quad \forall t \geq 1. \quad (4.28)$$

Proof. Equations (1.7) and (1.8) can be written in the equivalent form

$$-\Delta u + S'(u+v) + S'(u-v) = \tilde{\mu}, \quad (4.29)$$

$$-\Delta v + S'(u+v) - S'(u-v) = \tilde{\mu}', \quad (4.30)$$

where

$$\tilde{\mu} = -(-\Delta)^{-1} \frac{\partial u}{\partial t} + 2\theta_c u$$

and

$$\tilde{\mu}' = -\frac{\partial v}{\partial t} + \frac{\partial \alpha}{\partial t} + 2\theta_c v.$$

Combining (4.29) and (4.30), we obtain

$$-\Delta(u+v) + 2S'(u+v) = \tilde{\mu} + \tilde{\mu}' \quad (4.31)$$

and

$$-\Delta(u-v) + 2S'(u-v) = \tilde{\mu} - \tilde{\mu}'. \quad (4.32)$$

For any $L > 0$, we consider

$$g_1 = S'(u+v)e^{L|S'(u+v)|}$$

and

$$g_2 = S'(u-v)e^{L|S'(u-v)|}.$$

We observe that

$$\nabla g_1 = S''(u+v)[1 + L|S'(u+v)|]e^{L|S'(u+v)|}\nabla(u+v).$$

Then we consider equation (4.31) and test it with g_1 , which yields

$$\begin{aligned} \int_{\Omega} \nabla(u+v)\nabla(u+v)S''(u+v)[1 + L|S'(u+v)|]e^{L|S'(u+v)|} dx \\ + 2 \int_{\Omega} S'(u+v)^2 e^{L|S'(u+v)|} dx = \int_{\Omega} \tilde{\mu}g_1 dx + \int_{\Omega} \tilde{\mu}'g_1 dx. \end{aligned}$$

The first term on the left-hand side is nonnegative (S'' is convex). Therefore, the previous relation is equivalent to

$$2 \int_{\Omega} S'(u+v)^2 e^{L|S'(u+v)|} dx \leq \int_{\Omega} \tilde{\mu}g_1 dx + \int_{\Omega} \tilde{\mu}'g_1 dx.$$

Now, the right-hand side can be controlled by means of a generalized Young's inequality (see [1], Section 8.2) as follows:

$$\begin{aligned} \int_{\Omega} \tilde{\mu}S'(u+v)e^{L|S'(u+v)|} dx &\leq \int_{\Omega} |\tilde{\mu}||S'(u+v)|e^{L|S'(u+v)|} dx \\ &\leq \frac{1}{2} \int_{\Omega} S'(u+v)^2 e^{L|S'(u+v)|} dx + \int_{\Omega} e^{c(L)\tilde{\mu}} dx + c. \end{aligned}$$

In the same way,

$$\int_{\Omega} \tilde{\mu}'g_1 dx \leq \frac{1}{2} \int_{\Omega} S'(u+v)^2 e^{L|S'(u+v)|} dx + \int_{\Omega} e^{c(L)\tilde{\mu}'} dx + c.$$

Using Lemma 4.1, we end up with

$$2 \int_{\Omega} S'(u+v)^2 e^{L|S'(u+v)|} dx \leq \int_{\Omega} S'(u+v)^2 e^{L|S'(u+v)|} dx + 2c,$$

whence

$$\int_{\Omega} S'(u+v)^2 e^{L|S'(u+v)|} dx \leq c, \quad (4.33)$$

where c depends on L . On account of (4.2), we observe that, for any $p \geq 1$,

$$|S''(x)|^p \leq e^{pc}(c + |S'(x)|^2 e^{pc|S'(x)|}), \quad \forall x \in (-1, 1). \quad (4.34)$$

Combining (4.33) and (4.34) and taking $L = pc$, we deduce that

$$\|S''(u+v)\|_{L^p(t, t+1; L^p(\Omega))} \leq c(p), \quad \forall t \geq 1,$$

Similarly, if we consider equation (4.32) and test it with g_2 , we can easily obtain

$$\|S''(u-v)\|_{L^p(t, t+1; L^p(\Omega))} \leq c(p), \quad \forall t \geq 1.$$

□

Lemma 4.3. *Let the assumptions of Lemma 4.2 hold. Then, there exists a constant c such that*

$$\begin{aligned} & \|u_t\|_{L^\infty(2,t;L^2(\Omega))} + \|u_t\|_{L^2(t,t+1,H^2(\Omega))} + \|v_t\|_{L^\infty(2,t;H^1(\Omega))} \\ & + \|v_t\|_{L^2(t,t+1;H^2(\Omega))} \leq c, \quad \forall t \geq 2. \end{aligned} \quad (4.35)$$

Proof. Differentiating equations (4.14) and (4.15) with respect to time, we obtain

$$\frac{\partial}{\partial t} u_t + \Delta^2 u_t - \Delta \left[f'(u+v) \frac{\partial(u+v)}{\partial t} + f'(u-v) \frac{\partial(u-v)}{\partial t} \right] = 0 \quad (4.36)$$

and

$$\frac{\partial}{\partial t} v_t - \Delta v_t + f'(u+v) \frac{\partial(u+v)}{\partial t} - f'(u-v) \frac{\partial(u-v)}{\partial t} = -v_t + \Delta \alpha_t + \Delta \alpha \equiv h. \quad (4.37)$$

Multiplying (4.36) by u_t and (4.37) by $-\Delta v_t$ and summing up, we find

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|u_t\|^2 + \|\nabla v_t\|^2) + \|\Delta u_t\|^2 + \|\Delta v_t\|^2 = ((f'(u+v)(u_t+v_t), \Delta(u_t+v_t))) \\ & + ((f'(u-v)(u_t-v_t), \Delta(u_t-v_t))) + ((h, -\Delta v_t)) \end{aligned} \quad (4.38)$$

We can note that, employing the Hölder, Ladyzhenskaya and Young inequalities, we have

$$\begin{aligned} & |((f'(u+v)(u_t+v_t), \Delta(u_t+v_t)))| \leq \|f'(u+v)\|_{L^4(\Omega)} \|u_t+v_t\|_{L^4(\Omega)} \|\Delta(u_t+v_t)\| \\ & \leq c \|f'(u+v)\|_{L^4(\Omega)} \|u_t+v_t\|^{\frac{1}{2}} \|\Delta(u_t+v_t)\|^{\frac{3}{2}} \\ & \leq \frac{1}{2} \|\Delta(u_t+v_t)\|^2 + c \|f'(u+v)\|_{L^4(\Omega)}^4 \|u_t+v_t\|^2 \end{aligned}$$

and

$$\begin{aligned} & |((f'(u-v)(u_t-v_t), \Delta(u_t-v_t)))| \leq \|f'(u-v)\|_{L^4(\Omega)} \|u_t-v_t\|_{L^4(\Omega)} \|\Delta(u_t-v_t)\| \\ & \leq c \|f'(u-v)\|_{L^4(\Omega)} \|u_t-v_t\|^{\frac{1}{2}} \|\Delta(u_t-v_t)\|^{\frac{3}{2}} \\ & \leq \frac{1}{2} \|\Delta(u_t-v_t)\|^2 + c \|f'(u-v)\|_{L^4(\Omega)}^4 \|u_t-v_t\|^2. \end{aligned}$$

We thus have the differential inequality

$$\begin{aligned} & \frac{d}{dt} (\|u_t\|^2 + \|\nabla v_t\|^2) + \|\Delta u_t\|^2 + \|\Delta v_t\|^2 \\ & \leq c' (\|f'(u+v)\|_{L^4(\Omega)}^4 \|u_t+v_t\|^2 + \|f'(u-v)\|_{L^4(\Omega)}^4 \|u_t-v_t\|^2) + \|h\|^2. \end{aligned} \quad (4.39)$$

Using (4.19), (4.20), and the uniform Gronwall lemma to (4.39), we obtain the desired result. \square

Remark 4.2. *The proof of Lemma 4.3 is carried out with the solution itself since (4.27) and (4.28) cannot be guaranteed within a Galerkin scheme.*

Theorem 4.3. *Let $N = 1$. Then, there exist two positive constants δ and δ' and $T > 0$ such that*

$$\|(u+v)(t)\|_{L^\infty(\Omega)} \leq 1 - \delta, \quad \forall 2 \leq t \leq T \quad (4.40)$$

and

$$\|(u-v)(t)\|_{L^\infty(\Omega)} \leq 1 - \delta', \quad \forall 2 \leq t \leq T. \quad (4.41)$$

Proof. We consider the elliptic equation (4.31).

Due to Lemma 4.3 and to the elliptic regularity, $\tilde{\mu}$ and $\tilde{\mu}'$ satisfy

$$\|\tilde{\mu} + \tilde{\mu}'\|_{L^\infty(\Omega \times (t, t+1))} \leq C, \quad \forall t \geq 2.$$

Testing (4.31) by $|S'(u+v)|^{p-2}S'(u+v)$, we get

$$\begin{aligned} (p-1) \int_{\Omega} |S'(u+v)|^{p-2} S''(u+v) |\nabla(u+v)|^2 dx + 2 \|S'(u+v)\|_{L^p(\Omega)}^p \\ = \int_{\Omega} (\tilde{\mu} + \tilde{\mu}') |S'(u+v)|^{p-2} S'(u+v) dx. \end{aligned}$$

The first term on the left-hand side is nonnegative, so that an application of the Hölder inequality yields

$$\|S'(u+v)\|_{L^p(\Omega)} \leq c \|\tilde{\mu} + \tilde{\mu}'\|_{L^p(\Omega)}. \quad (4.42)$$

Hence, integrating (4.42) in time from t to $t+1$, we have for all $t \geq 2$

$$\|S'(u+v)\|_{L^p(\Omega \times (t, t+1))} \leq \|\tilde{\mu} + \tilde{\mu}'\|_{L^p(\Omega \times (t, t+1))} \leq C \|\tilde{\mu} + \tilde{\mu}'\|_{L^\infty(\Omega \times (t, t+1))} \leq C,$$

where C is independent of p and t . Applying Theorem 2.14 in [1], we obtain

$$\|S'(u+v)\|_{L^\infty(\Omega \times (t, t+1))} \leq C, \quad \forall t \geq 2.$$

This implies that there exists $\delta > 0$ such that

$$\|u+v\|_{L^\infty(\Omega \times (t, t+1))} \leq 1 - \delta, \quad \forall t \geq 2.$$

Since $u+v \in L^\infty(0, t; C(\overline{\Omega}))$ for all $t \geq 0$, we also infer that

$$\|u+v\|_{L^\infty(\Omega \times (2, t))} \leq 1 - \delta, \quad \forall t \geq 2.$$

Finally, we deduce (4.40) from the continuity in time.

In the same way, we can obtain the second inequality (4.41) using equation (4.32) instead of (4.31). □

An immediate consequence of Theorem 4.3 is the following

Corollary 4.1. *Let the assumptions of Theorem 4.3 hold. Then, there exists a positive constant C such that*

$$\|u(t)\|_{H^4(\Omega)} \leq C, \quad \forall t \geq 3.$$

Corollary 4.2. *It also follows from Theorem 4.3 that u and v are strictly separated from the pure states, i.e. there exist two positive constants γ and γ' such that*

$$|u(x, t)| \leq 1 - \gamma, \quad \forall (x, t) \in \overline{\Omega} \times (2, T),$$

and

$$|v(x, t)| \leq 1 - \gamma', \quad \forall (x, t) \in \overline{\Omega} \times (2, T).$$

Remark 4.3. *We can note that the strict separation property in two space dimensions can be obtained from the estimates of section 4 provided that we can prove that $\|v\|_{L^\infty} \leq c$ and $\|v_t\|_{L^\infty} \leq c'$, where c and c' are two constants which are independent of p (see [9]; see also [16, 19]). Such estimates seem difficult to derive, due to the coupling, and will be addressed elsewhere.*

Acknowledgments

The authors wish to thank an anonymous referee for her/his careful reading of the manuscript and many useful comments which helped improve it.

References

- [1] A. Adams and J. Fournier, *Sobolev Spaces*, 2nd edition, Academic Press, 2003.
- [2] D. Brochet, D. Hilhorst and A. Novick-Cohen, *Finite-Dimensional exponential attractor for a model for order-disorder and phase separation*, Appl. Math. Lett. **7** (1994), 83-87.
- [3] J.W. Cahn and A. Novick-Cohen, *Evolution equations for phase separation and ordering in binary alloys*, Statistical Phys. **76** (1994), 877-909.
- [4] L. Cherfilis, A. Miranville and S. Zelik, *The Cahn-Hilliard equation with logarithmic nonlinear terms*, Milan J. Math. **79** (2011), 561-596.
- [5] C.I. Christov and P.M. Jordan, *Heat conduction paradox involving second-sound propagation in moving media*, Phys. Rev. Lett. **94** (2005), 154301.
- [6] R. Dal Passo, L. Giacomelli and A. Novick-Cohen, *Existence for an Allen-Cahn/Cahn-Hilliard system with degenerate mobility*, Interfaces free boundaries **1** (1999), 199-226.
- [7] A. Debussche and L. Dettori, *On the Cahn-Hilliard equation with a logarithmic free energy*, Nonlinear Anal. **24** (1995), 1491-1514.
- [8] C. Dupaix, *A singularly perturbed phase field model with a logarithmic nonlinearity*, Nonlinear Anal. **41** (2000), 725-744.
- [9] A. Giorgini, M. Grasselli and A. Miranville, *The Cahn-Hilliard-Oono equation with singular potential*, Math. Models Methods Appl. Sci. **27** (2017), 2485-2510.
- [10] A.E. Green and P.M. Naghdi, *A re-examination of the basic postulates of thermomechanics*, Proc. Roy. Soc. Lond. A **432** (1991), 171-194.
- [11] A.E. Green and P.M. Naghdi, *A new thermoviscous theory for fluids*, J. Non-Newtonian Fluid Mech. **56** (1995), 289-306.
- [12] J. Jiang, *Convergence to equilibrium for a parabolic-hyperbolic phase-field model with Cattaneo heat flux law*, J. Math. Anal. Appl. **341** (2008), 149-169.
- [13] J. Jiang, *Convergence to equilibrium for a fully hyperbolic phase field model with Cattaneo heat flux law*, Math. Methods Appl. Sci. **32** (2009), 1156-1182.
- [14] I.B. Krasnyuk, R.M. Taranets and M. Chugunova, *Long-time oscillating properties of confined disordered binary alloys*, Journal of Advanced Research in Applied Mathematics **7** (2015), 1-16.
- [15] P.C. Millett, S. Rokkam, A. El-Azab, M. Tonks and D. Wolf, *Void nucleation and growth in irradiated polycrystalline metals: A phase-field model*, Modelling Simul. Mater. Sci. Eng. **17** (2009), 0064003.
- [16] A. Miranville, *The Cahn-Hilliard equation and some of its variants*, AIMS Math. **2** (2017), 479-544.

- [17] A. Miranville, *The Cahn-Hilliard equation: recent advances and applications*, CBMS-NSF Regional Conference Series in Applied Mathematics **95**, SIAM, 2019.
- [18] A. Miranville and S. Zelik, *The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions*, Discrete cont. Dyn. Sys. **28** (2010), 275-310.
- [19] A. Miranville and S. Zelik, *Robust exponential attractors for Cahn-Hilliard type equations with singular potentials*, Math. Methods Appl. Sci. **27** (2004), 545-582.
- [20] A. Miranville and R. Quintanilla, *A generalization of the Caginalp phase-field system based on the Cattaneo heat flux law*, Nonlinear Anal. TMA **71** (2009), 2278-2290.
- [21] A. Miranville, R. Quintanilla and W. Saoud, *Asymptotic Behavior of a Cahn-Hilliard/Allen-Cahn System with Temperature*, to appear.
- [22] A. Miranville, W. Saoud and R. Talhouk, *Asymptotic behavior of a model for order-disorder and phase separation*, Asympt. Anal. **103** (2017), 57-76.
- [23] A. Miranville, W. Saoud and R. Talhouk, *On the Cahn-Hilliard/Allen-Cahn equations with singular potentials*, Discrete Cont. Dynam. Systems Ser. B **24** (2019), 2278-2290.
- [24] A. Novick-Cohen, *Triple-junction motion for an Allen-Cahn/Cahn-Hilliard system*, Phys. D **137** (2000), 1-24.
- [25] A. Novick-Cohen and L. Peres Hari, *Geometric motion for a degenerate Allen-Cahn/Cahn-Hilliard system: The partial wetting case*, Phys. D **209** (2005), 205-235.
- [26] T. Nagai, T. Senba and K. Yoshida, *Application of the Trudinger-Moser Inequality to a Parabolic System of Chemotaxis*, Funkcial. Ekvac. **40** (1997), 411-433.
- [27] R. Quintanilla, *Damping of end effects in a thermoelastic theory*, Appl. Math. Lett. **14** (2001), 137-141.
- [28] R. Quintanilla, *On existence in thermoelasticity without energy dissipation*, J. Therm. Stresses **25** (2002), 195-202.
- [29] R. Quintanilla, *Impossibility of localization in linear thermoelasticity*, Proc. Roy. Soci. Lond. A **463** (2007), 3311-3322.
- [30] R. Quintanilla and R. Racke, *Stability in thermoelasticity of type III*, Discr. Cont. Dyn. Sys. Ser. B **3** (2003), 383-400.
- [31] R. Quintanilla and B. Straughan, *Growth and uniqueness in thermoelasticity*, Proc. Roy. Soci. Lond. A **456** (2000), 1419-1429.
- [32] R. Quintanilla and B. Straughan, *Energy bounds for some non-standard problems in thermoelasticity*, Proc. Roy. Soci. Lond. A **461** (2005), 1147-1162.
- [33] R. Quintanilla and B. Straughan, *A note on discontinuity waves in type III thermoelasticity*, Proc. Roy. Soci. Lond. A **60** (2004), 1169-1175.
- [34] R. Quintanilla and B. Straughan, *Nonlinear waves in a Green-Naghdi dissipationless fluid*, J. Non-Newtonian Fluid Mech. **154** (2008), 207-210.
- [35] S. Rokkam, A. El-Azab, P. Millett and D. Wolf, *Phase field modeling of void nucleation and growth in irradiated metals*, Modelling Simul. Mater. Sci. Eng. **17** (2009), 0064002.

- [36] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, Providence (R1): AMS Chelsea Publishing, American Mathematical Society 2001.
- [37] M.R. Tonks, D. Gaston, P.C. Millett, D. Andrs and P. Talbot, *An object-oriented finite element framework for multiphysics phase field simulations*, Comput. Mater. Sci. **51** (2012), 20-29.
- [38] L. Wang, J. Lee, M. Anitescu, A.E. Azab, L.C. Mcinnes, T. Munson and B. Smith, *A differential variational inequality approach for the simulation of heterogeneous materials*, in: Proc. SciDAC, 2011.
- [39] Y. Xia, Y. Xu and C.W. Shu, *Application of the local discontinuous Galerkin method for the Allen-Cahn/Cahn-Hilliard system*, Commun. Comput. Phys. **5** (2009), 821-835.
- [40] C. Yang, X.C. Cai, D.E. Keyes and M. Pernice, *NKS Method for the Implicit Solution of a Coupled Allen-Cahn/Cahn-Hilliard System*, Proceedings of the 21th International Conference on Domain Decomposition Methods, 2012.