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On some open questions concerning determinantal inequalities

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Abstract

In 2017, M. Lin formulated two conjectures concerning determinantal inequalities for positive semi-definite matrices A and B, and which can be stated as follows

$$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p) \quad \text{for } p \geq 0$$

and

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p) \qquad \text{for } 0 \leq p \leq 2.$$

The main goal of this paper is to confirm the first conjecture in a slightly more general setting namely in the case when A and B are Hermitian, and also to prove the second conjecture when $0 \leq p \leq \frac{4}{3}$. Various related inequalities are then presented and we conclude with an open log-majorization question.

Keywords: Determinantal inequalities; Hermitian matrix; Positive semi-definite matrix; Log-majorization; Eigenvalues; Furuta inequality.
2010 MSC: 15A45, 15A60, 47A64

1. Introduction

Audenaert [2] proved the following determinantal inequality, for $n \times n$ positive semi-definite matrices A and B,

$$\det(A^2 + |BA|) \leq \det(A^2 + AB) \tag{1}$$

that answers a question arising in the study of interpolation methods for image processing in diffusion tensor imaging. Recently, Lin [6] generalized Audenaert's result by proving

$$\det(A^2 + |BA|^p) \leq \det(A^2 + A^p B^p), \qquad 0 \leq p \leq 2. \tag{2}$$

In the same paper, he complemented (1) by proving that

$$\det(A^2 + |AB|) \geq \det(A^2 + AB). \tag{3}$$

Clearly, Inequalities (1) and (3) imply that

$$\det(A^2 + |AB|) \geq \det(A^2 + |BA|). \tag{4}$$

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In the same paper, it was asked whether it is possible to find a generalization of (4) and put forward the following conjecture.

Conjecture 1.1. *Let A and B be positive semi-definite matrices. Then, for all $p \geq 0$, we have*

$$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p).$$

In addition, he introduced the following conjecture which is a generalization of (3) and a complementing result for (2).

Conjecture 1.2. *Let A and B be $n \times n$ positive semi-definite matrices. Then, for $0 \leq p \leq 2$ it holds that*

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p).$$

Lin [6] was able to prove Conjecture 1.1 for $p = 1$ and for all p positive even integers, and Conjecture 1.2 for $p = 0, 1$ and 2. All other cases for both conjectures remain unsolved.

The main purpose of this paper is to confirm Conjecture 1.1 in a slightly more general setting; namely in the case where A and B are Hermitian matrices, and also to show that Conjecture 1.2 is valid for $0 \leq p \leq \frac{4}{3}$. In addition, we shall prove that the determinantal inequality of Conjecture 1.2 is also true for all $2 \leq p \leq 4$, however it remains open for $\frac{4}{3} < p < 2$.

To proceed, we first fix some notation. Let M_n be the space of $n \times n$ complex matrices where its identity matrix is denoted by I_n. The modulus of a complex matrix X is defined as $|X| = (X^*X)^{1/2}$. As usual, we shall write $X \geq 0$ to indicate that X is positive semi-definite. Also, for Hermitian matrices $X, Y \in M_n$, we shall say that $X \geq Y$ if $X - Y$ is positive semi-definite matrix. Moreover, if the eigenvalues $\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X)$ of a matrix X are real, then we shall always assume that they are arranged in decreasing order, that is

$$\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X).$$

For a Hermitian matrix $X \in M_n$, we shall denote

$$\lambda(X) = (\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X))^t$$

which is clearly a real vector of order n.

Majorization relations are great tools for deriving determinantal inequalities, see for example [9, Chapter 10] for more details on this subject. If $\lambda(A), \lambda(B) \in \mathbb{R}^n_+$, then by $\lambda(A) \prec_{wlog} \lambda(B)$, we mean that $\lambda(A)$ is *weakly log-majorized* by $\lambda(B)$, that is

$$\prod_{i=1}^{k} \lambda_i(A) \leq \prod_{i=1}^{k} \lambda_i(B) \qquad \text{for all } k = 1, 2, ..., n. \tag{5}$$

In addition, we shall write $\lambda(A) \prec_{log} \lambda(B)$ and we will say that $\lambda(A)$ is *log-majorized* by $\lambda(B)$ if (5) is true and equality holds for $k = n$.

2. Comparing $\det(A^2 + |BA|^p)$ with $\det(A^2 + A^pB^p)$ when $A, B \geq 0$.

The main purpose here is to study the relation between the two determinantal quantities

$$\det(A^2 + |BA|^p) \text{ and } \det(A^2 + A^pB^p) \text{ for } p \in [0, +\infty) \text{ and with } A, B \geq 0.$$

We shall start with the following lemma which is obtained from a result proved by A. Matsumoto, R. Nakamoto and M. Fujii [7] by using anti-symmetric tensor product argument.

Lemma 2.1. *Let X and Y be two positive semi-definite matrices. Then,*

1. $\lambda(X^{\frac{k+t}{2}}Y^tX^{\frac{k+t}{2}}) \prec_{wlog} \lambda(X^{\frac{k}{2}}(X^{\frac{1}{2}}YX^{\frac{1}{2}})^tX^{\frac{k}{2}})$ *for all* $0 \leq t \leq 1$ *and* $k \geq 0$.
2. $\lambda(X^{\frac{k+t}{2}}Y^tX^{\frac{k+t}{2}}) \succ_{wlog} \lambda(X^{\frac{k}{2}}(X^{\frac{1}{2}}YX^{\frac{1}{2}})^tX^{\frac{k}{2}})$ *for all* $t \geq 1$ *and* $0 \leq k \leq 1$.

Next, we prove the following elementary lemma which constitutes one of the basis of our main results. In fact, it is a slight generalization[1] of Lemma A on page 129 of [5].

Lemma 2.2. *Let X and Y be two invertible matrices. Then, for all $t \in \mathbb{R}$,*

$$(X^*Y^*YX)^t = X^*Y^*(YXX^*Y^*)^{t-1}YX.$$

Proof. Let $X^*Y^* = U|X^*Y^*|$ be the polar decomposition of the matrix X^*Y^*, where U is unitary. Then, clearly we obtain $U = X^*Y^*|X^*Y^*|^{-1}$, $U^* = |X^*Y^*|^{-1}YX$, and $YX = |X^*Y^*|U^*$.

$$\begin{aligned}
(X^*Y^*YX)^t &= (U|X^*Y^*||X^*Y^*|U^*)^t \\
&= \left(U|X^*Y^*|^2U^*\right)^t \\
&= U|X^*Y^*|^{2t}U^* \\
&= X^*Y^*|X^*Y^*|^{-1}|X^*Y^*|^{2t}|X^*Y^*|^{-1}YX \\
&= X^*Y^*(YXX^*Y^*)^{t-1}YX.
\end{aligned}$$

□

The next lemma is also needed for our purposes and it shows a close connection between log-majorization and determinantal inequalities and can be found in [6, (P2)].

Lemma 2.3. *Let X and Y be two matrices in M_n. If $\lambda(X), \lambda(Y)$ are in $\mathbb{R}^n_+$ such that $\lambda(X) \prec_{wlog} \lambda(Y)$, then*

$$\det(I_n + X) \leq \det(I_n + Y).$$

As a consequence, we have the following result.

Theorem 2.1. *Let A and B be two positive semi-definite matrices. Then*

1. $\det(A^2 + |BA|^p) \geq \det(A^2 + A^pB^p)$ *for* $2 \leq p \leq 4$.
2. *If* $0 \leq p \leq 2$ *or* $p \geq 4$, *then* $\det(A^2 + |BA|^p) \leq \det(A^2 + A^pB^p)$.

[1] Thanks to the reviewer for pointing this out.

Proof. 1. Without loss of generality, we shall assume that A and B are positive definite matrices as the general case can be then obtained by a continuity argument. For $2 \leq p \leq 4$, then we can write

$$\begin{aligned}
\lambda(A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) &= \lambda\left(A^{-1}(AB(BA^2B)^{\frac{p}{2}-1}BA)A^{-1}\right) \qquad \text{(by Lemma 2.2)} \\
&= \lambda\left(B(BA^2B)^{\frac{p}{2}-1}B\right) \\
&= \lambda\left((B^2)^{\frac{1}{2}}\left((B^2)^{\frac{1}{2}}(A^2)(B^2)^{\frac{1}{2}}\right)^{\frac{p}{2}-1}(B^2)^{\frac{1}{2}}\right) \\
&\succ_{wlog} \lambda\left((B^2)^{1+\frac{p}{2}-1}(A^2)^{\frac{p}{2}-1}\right) \qquad \text{(by Lemma 2.1, Part 1)} \\
&= \lambda(A^{p-2}B^p).
\end{aligned}$$

So,

$$\lambda(A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \succ_{wlog} \lambda(A^{p-2}B^p). \tag{6}$$

Next, applying Lemma 2.3 on (6) gives

$$\det(I_n + A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \geq \det(I_n + A^{p-2}B^p). \tag{7}$$

Multiplying both sides of (7) by $\det(A^2) > 0$ completes the proof of the first part.

2. The proof for the case where $0 \leq p \leq 2$, is due to [6], while the case where $p \geq 4$ can be done in a similar fashion as in the first case by making use this time of Part 2 of Lemma 2.1. □

3. Proof of Conjecture 1.1

The starting point here is a lemma dealing with a majorization inequality which appears in [8]. For the sake of completeness, we reproduce the proof here.

Lemma 3.1. *Let Y be a positive semi-definite matrix and X be any Hermitian matrix. Then for all $p, q \in [0, +\infty)$, it holds that*

$$\lambda(XY^pXY^q) \prec_{wlog} \lambda(X^2Y^{p+q}).$$

Proof. By appealing to a standard argument (anti-symmetric product), then it suffices to prove that

$$\lambda_1(XY^pXY^q) \leq \lambda_1(X^2Y^{p+q}).$$

Without loss of generality, we shall assume that X is invertible as the general case can be done by continuity argument. In addition, we shall assume that $q \leq p$ and $\lambda_1(X^2Y^{p+q}) = 1$. Now, obviously proving our claim is equivalent to showing that

$$\lambda_1(XY^pXY^q) \leq 1.$$

The fact that the largest eigenvalue of the matrix X^2Y^{p+q} is equal to 1, clearly implies that

$$\lambda_j(X^2Y^{p+q}) \leq 1 \text{ for all } 1 \leq j \leq n.$$

But this is equivalent to $XY^{p+q}X \leq I_n$ which in turn gives

$$Y^{p+q} \leq (X^{-1})^2. \tag{8}$$

Next, applying Lowner-Heinz on (8) for a power $0 \leq \frac{p}{p+q} \leq 1$, yields

$$Y^p \leq ((X^{-1})^2)^{\frac{p}{p+q}}. \tag{9}$$

Again, taking a power $0 \leq \frac{q}{p} \leq 1$ in both sides of (9), we get

$$Y^q \leq \left((X^{-1})^2\right)^{\frac{q}{p+q}}.$$

Hence,

$$(X^2)^{\frac{q}{p+q}} \leq Y^{-q}. \tag{10}$$

Therefore,

$$\begin{aligned}
\lambda_1(XY^pXY^q) &= \lambda_1\left(Y^{q/2}XY^pXY^{q/2}\right) \\
&\leq \lambda_1\left(Y^{q/2}X((X^{-1})^2)^{\frac{p}{p+q}}XY^{q/2}\right) && \text{(by using (9))} \\
&= \lambda_1\left((Y^{q/2}(X^2)^{\frac{q}{p+q}}Y^{q/2}\right) \\
&\leq \lambda_1\left(Y^{q/2}Y^{-q}Y^{q/2}\right) && \text{(by using (10))} \\
&= \lambda_1(I_n) \\
&= 1.
\end{aligned}$$

□

As a result, we have the following theorem.

Theorem 3.1. *Let Y be a positive definite matrix and X be a Hermitian matrix. Then for all $p, q \in [0, \infty)$*

$$\lambda(XY^pXY^{-q}) \succ_{wlog} \lambda(X^2Y^{p-q}). \tag{11}$$

Proof. The proof will be divided into three cases.

<u>Case 1:</u> Let $q \geq 2p$. By Schur's complement we have

$$M = \begin{bmatrix} Y^{-\frac{q}{2}}XY^pXY^{-\frac{q}{2}} & Y^{-\frac{q}{2}}X^2Y^{p-\frac{q}{2}} \\ Y^{p-\frac{q}{2}}X^2Y^{-\frac{q}{2}} & Y^{p-\frac{q}{2}}XY^{-p}XY^{p-\frac{q}{2}} \end{bmatrix} \geq 0.$$

Applying Theorem 10.20 in [9, p. 352] gives

$$\lambda_1\left(Y^{-\frac{q}{2}}XY^pXY^{-\frac{q}{2}}\right) \cdot \lambda_1\left(Y^{p-\frac{q}{2}}XY^{-p}XY^{p-\frac{p}{2}}\right) \geq \lambda_1\left(Y^{-\frac{q}{2}}X^2Y^{p-\frac{q}{2}}\right)^2.$$

That is,

$$\lambda_1\left(XY^pXY^{-q}\right) \cdot \lambda_1\left(XY^{2p-q}XY^{-p}\right) \geq \lambda_1\left(X^2Y^{p-q}\right)^2. \tag{12}$$

In view of Lemma 3.1, it is worthy to observe that

$$\begin{aligned}
\lambda\left(X(Y^{-1})^{q-2p}X(Y^{-1})^p\right) &\prec_{wlog} \lambda\left(X^2(Y^{-1})^{q-p}\right) \\
&= \lambda\left(X^2Y^{p-q}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda_1\left(Y^{2p-q}XY^{-p}X\right) &= \lambda_1\left(X(Y^{-1})^{q-2p}X(Y^{-1})^{p}\right)\\ &\leq \lambda_1\left(X^2Y^{p-q}\right).\end{aligned}$$

Thus, from (12) we obtain

$$\lambda_1\left(XY^{p}XY^{-q}\right) \geq \lambda_1\left(X^2Y^{p-q}\right) \quad \text{for } q \geq 2p \geq 0.$$

By a standard anti-symmetric tensor product argument, inequality (11) is true for all $q \geq 2p$.

<u>Case 2:</u> Let $p \leq q \leq 2p$. The idea of the proof here depends on writing the interval $[p, 2p] = \bigcup_{k=2}^{\infty} [\frac{k+1}{k}p, 2p]$ and then proving (11) for each subinterval. We start with with case $k = 2$ i.e. for $2p \geq q \geq \frac{3p}{2}$. Then,

$$\begin{aligned}\lambda(Y^pXY^{-q}X) &= \lambda\left(Y^{\frac{3p}{2}}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{p-q}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{\frac{p}{2}}\right)\\ &= \lambda\left(Y^{2p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{-(q-p)}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})\right)\\ &= \lambda\left((Y^{-1})^{-2p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})(Y^{-1})^{q-p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})\right).\end{aligned}$$

Now considering this last expression and noticing that $2p \geq 2(q-p) \geq 0$, then in view of Case 1; replacing Y with Y^{-1}, X with $Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}$, $2p$ with q and lastly $q-p$ with p, we obtain

$$\begin{aligned}\lambda(Y^pXY^{-q}X) &\succ_{wlog} \lambda\left(Y^{3p-q}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})^2\right)\\ &= \lambda\left(Y^{2p-q}XY^{-p}X\right)\\ &\succ_{wlog} \lambda(Y^{p-q}X^2) \quad \text{(again using Case 1 as } p \geq 2(2p-q) \geq 0).\end{aligned}$$

Next, using a similar argument, we prove (11) is true for $k = 3$ i.e. for $2p \geq q \geq \frac{4p}{3}$. As earlier, we can write

$$\begin{aligned}\lambda(Y^pXY^{-q}X) &= \lambda\left(Y^{\frac{3p}{2}}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{p-q}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{\frac{p}{2}}\right)\\ &= \lambda\left(Y^{2p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{-(q-p)}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})\right)\\ &= \lambda\left((Y^{-1})^{-2p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})(Y^{-1})^{q-p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})\right)\\ &\succ_{wlog} \lambda\left(Y^{3p-q}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})^2\right) \qquad \text{(in view of of Case 1 as } 2p \geq 2(q-p) \geq 0)\\ &= \lambda\left(Y^{2p-q}XY^{-p}X\right)\\ &\succ_{wlog} \lambda(Y^{p-q}X^2) \qquad \text{(similarly in view of Case } k=2 \text{ as } p \geq \frac{3}{2}(2p-q) \geq 0).\end{aligned}$$

Continuing this way, one can easily see (using induction) that for any positive integer k, inequality (11) is true for all q with $2p \geq q \geq \frac{(k+1)p}{k}$. Finally, the proof for this case can be achieved by letting k tends to infinity.

Case 3: Let $q \leq p$. To complete the proof of this case, it suffices to apply the preceding two cases on Y^{-1}.

$\square$

As applications, we have the following result.

Corollary 3.1. *Let A and B be two Hermitian matrices. Then, for all $p \in [0, \infty)$ we have*

$$\det(A^2 + |BA|^p) \leq \det(A^2 + |AB|^p).$$

Proof. As usual, we shall assume that A and B are invertible, the general case is by continuity argument. Then, for all $p \in [0, +\infty)$ we have

$$\begin{aligned}
\lambda\left(A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}\right) &= \lambda\left(B(BA^2B)^{\frac{p}{2}-1}B\right) && \text{(Using Lemma 2.2)}\\
&= \lambda\left(B^2(BA^2B)^{\frac{p}{2}-1}\right)\\
&\prec_{wlog} \lambda\left(B(BA^2B)^{\frac{p}{2}}B(BA^2B)^{-1}\right) && \text{(Using Theorem 3.1)}\\
&= \lambda\left(A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}\right).
\end{aligned}$$

Next, applying Lemma 2.3 gives

$$\det(I_n + A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \leq \det(I_n + A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}).$$

Finally, multiplying both sides with $\det(A^2) > 0$ yields

$$\det(A^2 + |BA|^p) \leq \det(A^2 + |AB|^p).$$

$\square$

4. Conjecture 1.2

In this section, our purpose is to find for what values of k and t the following majorization inequality

$$\lambda(A^{\frac{k}{2}-t}B^t) \prec_{log} \lambda(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}}) \tag{13}$$

is valid, where A and B are positive definite matrices.

We shall start here with the following lemma which is needed for our purposes and is well known as Furuta's inequality [4].

Lemma 4.1. *Let A, B be two positive semi-definite matrices such that $A \geq B$. Then, for all $p \geq 1$, $r \geq 0$,*

$$A^{(p+2r)/p} \geq (A^rB^pA^r)^{1/p}.$$

Now we are in a position to prove the next theorem which shows that (13) is valid for all $0 \leq t \leq 1$ and $k \geq 4t$.

Theorem 4.1. *Let A and B be two positive definite matrices. Then for all $0 \leq t \leq 1$ and $k \geq 4t$*

$$\lambda(A^{\frac{k}{2}-t}B^t) \prec_{log} \lambda(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}}).$$

Proof. Let $0 \leq t \leq 1$ and $k \geq 4t$. Using Schur's complement, we know that

$$M = \begin{bmatrix} A^{\frac{k}{4}-t}B^t(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^tB^tA^{\frac{k}{4}-t} & A^{\frac{k}{4}-t}B^tA^{\frac{k}{4}} \\ A^{\frac{k}{4}}B^tA^{\frac{k}{4}-t} & A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}} \end{bmatrix} \geq 0.$$

Then

$$\left(\lambda_1(A^{\frac{k}{2}-t}B^t)\right)^2 \leq \lambda_1(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}}) \cdot \lambda_1(A^{\frac{k}{4}-t}B^t(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^tB^tA^{\frac{k}{4}-t}). \tag{14}$$

As mentioned earlier, in order to prove our claim, then it is enough to prove that

$$\lambda_1(A^{\frac{k}{2}-t}B^t) \geq \lambda_1(A^{\frac{k}{4}-t}B^t(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^tB^tA^{\frac{k}{4}-t}), \tag{15}$$

which is in turn equivalent to showing that

$$B^{\frac{t}{2}}A^{\frac{k}{2}-t}B^{\frac{t}{2}} \leq I_n \Rightarrow A^{\frac{k}{4}-t}B^t(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^tB^tA^{\frac{k}{4}-t} \leq I_n.$$

For this, let $B^{\frac{t}{2}}A^{\frac{k}{2}-t}B^{\frac{t}{2}} \leq I_n$, then $A^{\frac{k}{2}-t} \leq B^{-t}$. First, making use of Löwner-Heinz inequality for $0 \leq \frac{t}{\frac{k}{2}-t} \leq 1$ gives

$$A^t \leq B^{-\frac{t^2}{\frac{k}{2}-t}}.$$

Next, applying Lemma 4.1 yields

$$\left(B^{-\frac{t^2}{\frac{k}{2}-t}}\right)^{\frac{p+2r}{p}} \geq \left[\left(B^{-\frac{t^2}{\frac{k}{2}-t}}\right)^r (A^t)^p \left(B^{-\frac{t^2}{\frac{k}{2}-t}}\right)^r\right]^{\frac{1}{p}}.$$

Now, replacing p with $\frac{1}{t} \geq 1$ and r with $\frac{\frac{k}{2}-t}{2t^2} \geq 0$ gives

$$B^{-\frac{\frac{kt}{2}}{\frac{k}{2}-t}} \geq (B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^t.$$

Pre-post multiplying both sides by $B^t > 0$ implies that

$$(B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}} = B^tB^{-\frac{\frac{kt}{2}}{\frac{k}{2}-t}}B^t \geq B^t(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^tB^t. \tag{16}$$

However, $A^{\frac{k}{2}-t} \leq B^{-t}$, so that $A^{-(\frac{k}{2}-t)} \geq B^t$. By again appealing to Löwner-Heinz inequality for $0 \leq \frac{\frac{k}{2}-2t}{\frac{k}{2}-t} \leq 1$ we obtain

$$(A^{-(\frac{k}{2}-t)})^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}} \geq (B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}},$$

which gives

$$A^{-(\frac{k}{2}-2t)} \geq (B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}}. \tag{17}$$

Now it is worthy to observe that inequalities (16) and (17) yield

$$A^{-(\frac{k}{2}-2t)} \geq B^t(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^tB^t.$$

Hence, $A^{\frac{k}{4}-t}B^t(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^tB^tA^{\frac{k}{4}-t} \leq I_n$, and therefore, (15) is true for all $0 \leq t \leq 1$ and $k \geq 4t$. On the other hand, using (14) and (15) gives

$$\lambda_1(A^{\frac{k}{2}-t}B^t) \leq \lambda_1(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}}), \qquad 0 \leq t \leq 1, \;\; k \geq 4t.$$

Thus, by a standard anti-symmetric tensor product argument, we get

$$\lambda(A^{\frac{k}{2}-t}B^t) \prec_{wlog} \lambda(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}}), \qquad 0 \leq t \leq 1, \;\; k \geq 4t.$$

Finally, the proof is complete by making use of the fact that

$$\det(A^{\frac{k}{2}-t}B^t) = \det(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}}).$$

□

Our next goal is to show that (13) is also true for all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$. First, we need the following lemma.

Lemma 4.2. *Let A and B be two positive definite matrices. Then, for all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$, we have*

$$\lambda\left(A^{\frac{k+2t}{4}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{2t}A^{\frac{2t+k}{4}}\right) \succ_{log} \lambda(A^{\frac{k}{2}-t}B^{2t}).$$

Proof. As in similar situations, it is enough to prove that for all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$

$$\lambda_1\left(A^{\frac{k+2t}{4}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{2t}A^{\frac{2t+k}{4}}\right) \geq \lambda_1(A^{\frac{k}{2}-t}B^{2t}).$$

First, in view of Lemma 2.2, we obtain the following equality:

$$A^{\frac{2t+k}{4}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{2t}A^{\frac{2t+k}{4}} = A^{\frac{2t+k-2}{4}}B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{2t-1}B^{\frac{1}{2}}A^{\frac{2t+k-2}{4}}.$$

For the sake of clarification, we shall use the following notation. For all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$, let

$$\begin{aligned}
X &= A^{\frac{2t+k-2}{4}}B^{\frac{1}{2}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{2t-1}B^{\frac{1}{2}}A^{\frac{2t+k-2}{4}},\\
Y &= A^{\frac{2t+k-2}{4}}B^{2t}A^{\frac{k-6t+2}{4}}, \;\; \textit{and}\\
Z &= A^{\frac{k-6t+2}{4}}B^{2t-\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{2t-1}B^{2t-\frac{1}{2}}A^{\frac{k-6t+2}{4}}.
\end{aligned}$$

Now by making use of Schur complement, the following 2×2 block matrix

$$M = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$$

is positive semi-definite and hence $\lambda_1(X) \cdot \lambda_1(Z) \geq \lambda_1(Y)^2$. Our next goal is to show that $\lambda_1(Z) \leq \lambda_1(Y)$ which in turn gives $\lambda_1(X) \geq \lambda_1(Y)$. Noticing that $\lambda_1(Y) = \lambda_1(B^tA^{\frac{k}{2}-t}B^t)$, then in order to prove $\lambda_1(Z) \leq \lambda_1(Y)$, it is suffices to show that

$$B^tA^{\frac{k}{2}-t}B^t \leq I_n \Rightarrow Z \leq I_n.$$

For this purpose, let $B^t A^{\frac{k}{2}-t} B^t \leq I_n$, then clearly $A^{\frac{k}{2}-t} \leq B^{-2t}$. Using Löwner-Heinz inequality for $0 \leq \frac{2t-1}{\frac{k}{2}-t} \leq 1$ gives

$$A^{2t-1} \leq B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}.$$

Now, applying Lemma 4.1 yields

$$\left(B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}\right)^{\frac{p+2r}{p}} \geq \left[\left(B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}\right)^r (A^{2t-1})^p \left(B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}\right)^r\right]^{\frac{1}{p}}.$$

Next, taking $p = \frac{1}{2t-1} \geq 1$ and $r = \frac{\frac{k}{2}-t}{4t(2t-1)} \geq 0$ implies that

$$B^{-\frac{(\frac{k}{2}+t)(2t-1)}{\frac{k}{2}-t}} \geq (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1}.$$

Pre-post multiplying both sides with $B^{2t-\frac{1}{2}} > 0$ gives

$$B^{2t-\frac{1}{2}} B^{-\frac{(\frac{k}{2}+t)(2t-1)}{\frac{k}{2}-t}} B^{2t-\frac{1}{2}} = \left(B^{2t}\right)^{\frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t}} \geq B^{2t-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1} B^{2t-\frac{1}{2}}.$$

Again, noticing that $A^{\frac{k}{2}-t} \leq B^{-2t}$ implies that $A^{-(\frac{k}{2}-t)} \geq B^{2t}$, then by appealing to Löwner-Heinz for $0 \leq \frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t} \leq 1$ we obtain

$$\left(A^{-(\frac{k}{2}-t)}\right)^{\frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t}} \geq \left(B^{2t}\right)^{\frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t}}$$

which gives

$$A^{-(\frac{k-6t+2}{2})} \geq B^{2t-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1} B^{2t-\frac{1}{2}}.$$

Hence,

$$Z = A^{\frac{k-6t+2}{4}} B^{2t-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1} B^{2t-\frac{1}{2}} A^{\frac{k-6t+2}{4}} \leq I_n.$$

Therefore, for all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$,

$$\lambda_1(X) \geq \lambda_1(Y).$$

Thus, by an anti-symmetric tensor product argument and by the fact that $\det(X) = \det(Y)$, we get

$$\lambda(X) \succ_{log} \lambda(Y).$$

Finally, noting that $\lambda(X) = \lambda\left(A^{\frac{k+2t}{4}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{2t} A^{\frac{2t+k}{4}}\right)$ and $\lambda(Y) = \lambda(A^{\frac{k}{2}-t} B^{2t})$, the proof is complete. □

Theorem 4.2. *Let A and B be two positive definite matrices. Then, for all $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$*

$$\lambda\left(A^{\frac{k}{4}} \left(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}\right)^t A^{\frac{k}{4}}\right) \succ_{log} \lambda(A^{\frac{k}{2}-t} B^t).$$

Proof. Let $\frac{1}{2} \leq t \leq 1$ and $k \geq 6t-2$. Then, we can write

$$\begin{aligned}
\lambda\left(A^{\frac{k}{4}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^{t}A^{\frac{k}{4}}\right) &= \lambda\left(A^{\frac{k}{4}}\left(A^{\frac{1}{2}}(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{-\frac{1}{2}})A^{\frac{1}{2}}\right)^{t}A^{\frac{k}{4}}\right) \\
&\succ_{wlog} \lambda\left(A^{\frac{k}{2}+t}\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}A^{-\frac{1}{2}}\right)^{t}\right) \quad \text{(by Part 1 of Lemma 2.1)} \\
&= \lambda\left(A^{\frac{k}{2}+t}\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{-\frac{1}{2}}\right)^{2t}\right) \\
&= \lambda\left(A^{\frac{k+2t}{4}}\left(A^{-\frac{1}{2}}B^{\frac{1}{2}}A^{-\frac{1}{2}}\right)^{2t}A^{\frac{k+2t}{4}}\right) \\
&\succ_{log} \lambda\left(A^{\frac{k}{2}-t}(B^{\frac{1}{2}})^{2t}\right) \qquad \text{(by Lemma 4.2)} \\
&= \lambda(A^{\frac{k}{2}-t}B^{t}).
\end{aligned}$$

□

As a result of Theorem 4.1 and Theorem 4.2, we have the following corollary.

Corollary 4.1. *Let A and B be two positive semi-definite matrices. Then, for ($0 \leq p \leq 2$ and $k \geq 2p$) or for ($1 \leq p \leq 2$ and $k \geq 3p-2$) we have*

$$\det(A^k + |AB|^p) \geq \det(A^k + A^pB^p).$$

Proof. Without loss of generality we may assume that A and B are positive definite matrices, the general case is by continuity argument. Suppose that $0 \leq p \leq 2$ and $k \geq 2p$. Then, replacing A with A^{-2}, B with B^2 and t with $0 \leq \frac{p}{2} \leq 1$ in Theorem 4.1 gives

$$\lambda(A^{p-k}B^p) \prec_{log} \lambda(A^{-\frac{k}{2}}(BA^2B)^{\frac{p}{2}}A^{-\frac{k}{2}}). \tag{18}$$

Applying Lemma 2.3 on (18) yields

$$\det(I_n + A^{-\frac{k}{2}}(BA^2B)^{\frac{p}{2}}A^{-\frac{k}{2}}) \geq \det(I_n + A^{p-k}B^p).$$

Next, multiplying both sides with $\det(A^k) > 0$, we obtain

$$\det(A^k + |AB|^p) \geq \det(A^k + A^pB^p), \qquad 0 \leq p \leq 2,\ k \geq 2p.$$

For the case when $1 \leq p \leq 2$ and $k \geq 3p-2$, the proof can be done in a similar fashion by making use this time of Theorem 4.2. □

As an analogue of Theorem 2.1, Corollary 3.1, and Corollary 4.1, we have the following result which gives a partial answer of Conjecture 1.2.

Theorem 4.3. *Let A and B be two positive semi-definite matrices. Then, for all $0 \leq p \leq \frac{4}{3}$ or $2 \leq p \leq 4$, the following holds*

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^pB^p).$$

Proof. Corollary 4.1 is a general case of

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p), \qquad 0 \leq p \leq \frac{4}{3}.$$

Combining Part 1 of Theorem 2.1 with Corollary 3.1 implies that for all $2 \leq p \leq 4$

$$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p) \geq \det(A^2 + A^p B^p).$$

□

We conclude this paper with the following conjecture which is very much related to our work in this section, and whose validation for $k = 2$ would imply Conjecture 1.2.

Conjecture 4.1. *Let A and B be two positive definite matrices. Then for all $k \geq 0$ and $0 \leq t \leq 1$,*

$$\lambda\left(A^{\frac{k}{4}}\left(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}\right)^t A^{\frac{k}{4}}\right) \succ_{log} \lambda(A^{\frac{k}{2}-t}B^t).$$

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Declaration of Competing Interest

The authors declare that they have no competing interests.

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