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# On some open questions concerning determinantal inequalities

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## Abstract

In 2017, M. Lin formulated two conjectures concerning determinantal inequalities for positive semi-definite matrices  $A$  and  $B$ , and which can be stated as follows

$$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p) \quad \text{for } p \geq 0$$

and

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p) \quad \text{for } 0 \leq p \leq 2.$$

The main goal of this paper is to confirm the first conjecture in a slightly more general setting namely in the case when  $A$  and  $B$  are Hermitian, and also to prove the second conjecture when  $0 \leq p \leq \frac{4}{3}$ . Various related inequalities are then presented and we conclude with an open log-majorization question.

**Keywords:** Determinantal inequalities; Hermitian matrix; Positive semi-definite matrix; Log-majorization; Eigenvalues; Furuta inequality.

**2010 MSC:** 15A45, 15A60, 47A64

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## 1. Introduction

Audenaert [2] proved the following determinantal inequality, for  $n \times n$  positive semi-definite matrices  $A$  and  $B$ ,

$$\det(A^2 + |BA|) \leq \det(A^2 + AB) \quad (1)$$

that answers a question arising in the study of interpolation methods for image processing in diffusion tensor imaging. Recently, Lin [6] generalized Audenaert's result by proving

$$\det(A^2 + |BA|^p) \leq \det(A^2 + A^p B^p), \quad 0 \leq p \leq 2. \quad (2)$$

In the same paper, he complemented (1) by proving that

$$\det(A^2 + |AB|) \geq \det(A^2 + AB). \quad (3)$$

Clearly, Inequalities (1) and (3) imply that

$$\det(A^2 + |AB|) \geq \det(A^2 + |BA|). \quad (4)$$

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8 In the same paper, it was asked whether it is possible to find a generalization of (4) and put forward the  
9 following conjecture.

**Conjecture 1.1.** *Let  $A$  and  $B$  be positive semi-definite matrices. Then, for all  $p \geq 0$ , we have*

$$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p).$$

10 In addition, he introduced the following conjecture which is a generalization of (3) and a complementing  
11 result for (2).

**Conjecture 1.2.** *Let  $A$  and  $B$  be  $n \times n$  positive semi-definite matrices. Then, for  $0 \leq p \leq 2$  it holds that*

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p).$$

12 Lin [6] was able to prove Conjecture 1.1 for  $p = 1$  and for all  $p$  positive even integers, and Conjecture  
13 1.2 for  $p = 0, 1$  and  $2$ . All other cases for both conjectures remain unsolved.

14  
15 The main purpose of this paper is to confirm Conjecture 1.1 in a slightly more general setting; namely in  
16 the case where  $A$  and  $B$  are Hermitian matrices, and also to show that Conjecture 1.2 is valid for  $0 \leq p \leq \frac{4}{3}$ .  
17 In addition, we shall prove that the determinantal inequality of Conjecture 1.2 is also true for all  $2 \leq p \leq 4$ ,  
18 however it remains open for  $\frac{4}{3} < p < 2$ .

19  
To proceed, we first fix some notation. Let  $M_n$  be the space of  $n \times n$  complex matrices where its  
identity matrix is denoted by  $I_n$ . The modulus of a complex matrix  $X$  is defined as  $|X| = (X^* X)^{1/2}$ .  
As usual, we shall write  $X \geq 0$  to indicate that  $X$  is positive semi-definite. Also, for Hermitian matrices  
 $X, Y \in M_n$ , we shall say that  $X \geq Y$  if  $X - Y$  is positive semi-definite matrix. Moreover, if the eigenvalues  
 $\lambda_1(X), \lambda_2(X), \dots, \lambda_n(X)$  of a matrix  $X$  are real, then we shall always assume that they are arranged in  
decreasing order, that is

$$\lambda_1(X) \geq \lambda_2(X) \geq \dots \geq \lambda_n(X).$$

For a Hermitian matrix  $X \in M_n$ , we shall denote

$$\lambda(X) = (\lambda_1(X), \lambda_2(X), \dots, \lambda_n(X))^t$$

20 which is clearly a real vector of order  $n$ .

21  
22 Majorization relations are great tools for deriving determinantal inequalities, see for example [9, Chapter  
23 10] for more details on this subject. If  $\lambda(A), \lambda(B) \in \mathbb{R}_+^n$ , then by  $\lambda(A) \prec_{wlog} \lambda(B)$ , we mean that  $\lambda(A)$  is  
24 weakly log-majorized by  $\lambda(B)$ , that is

$$\prod_{i=1}^k \lambda_i(A) \leq \prod_{i=1}^k \lambda_i(B) \quad \text{for all } k = 1, 2, \dots, n. \quad (5)$$

25 In addition, we shall write  $\lambda(A) \prec_{log} \lambda(B)$  and we will say that  $\lambda(A)$  is log-majorized by  $\lambda(B)$  if (5) is  
26 true and equality holds for  $k = n$ .

27 **2. Comparing  $\det(A^2 + |BA|^p)$  with  $\det(A^2 + A^p B^p)$  when  $A, B \geq 0$ .**

The main purpose here is to study the relation between the two determinantal quantities

$$\det(A^2 + |BA|^p) \quad \text{and} \quad \det(A^2 + A^p B^p) \quad \text{for } p \in [0, +\infty) \text{ and with } A, B \geq 0.$$

28 We shall start with the following lemma which is obtained from a result proved by A. Matsumoto, R.  
29 Nakamoto and M. Fujii [7] by using anti-symmetric tensor product argument.

30 **Lemma 2.1.** *Let  $X$  and  $Y$  be two positive semi-definite matrices. Then,*

- 31 1.  $\lambda(X^{\frac{k+t}{2}} Y^t X^{\frac{k+t}{2}}) \prec_{wlog} \lambda(X^{\frac{k}{2}} (X^{\frac{1}{2}} Y X^{\frac{1}{2}})^t X^{\frac{k}{2}})$  for all  $0 \leq t \leq 1$  and  $k \geq 0$ .
- 32 2.  $\lambda(X^{\frac{k+t}{2}} Y^t X^{\frac{k+t}{2}}) \succ_{wlog} \lambda(X^{\frac{k}{2}} (X^{\frac{1}{2}} Y X^{\frac{1}{2}})^t X^{\frac{k}{2}})$  for all  $t \geq 1$  and  $0 \leq k \leq 1$ .

33 Next, we prove the following elementary lemma which constitutes one of the basis of our main results.  
34 In fact, it is a slight generalization<sup>1</sup> of Lemma A on page 129 of [5].

**Lemma 2.2.** *Let  $X$  and  $Y$  be two invertible matrices. Then, for all  $t \in \mathbb{R}$ ,*

$$(X^* Y^* Y X)^t = X^* Y^* (Y X X^* Y^*)^{t-1} Y X.$$

35 *Proof.* Let  $X^* Y^* = U |X^* Y^*|$  be the polar decomposition of the matrix  $X^* Y^*$ , where  $U$  is unitary. Then,  
36 clearly we obtain  $U = X^* Y^* |X^* Y^*|^{-1}$ ,  $U^* = |X^* Y^*|^{-1} Y X$ , and  $Y X = |X^* Y^*| U^*$ .

$$\begin{aligned} (X^* Y^* Y X)^t &= (U |X^* Y^*| |X^* Y^*| U^*)^t \\ &= (U |X^* Y^*|^{2t} U^*)^t \\ &= U |X^* Y^*|^{2t} U^* \\ &= X^* Y^* |X^* Y^*|^{-1} |X^* Y^*|^{2t} |X^* Y^*|^{-1} Y X \\ &= X^* Y^* (Y X X^* Y^*)^{t-1} Y X. \end{aligned}$$

37 □

38 The next lemma is also needed for our purposes and it shows a close connection between log-majorization  
39 and determinantal inequalities and can be found in [6, (P2)].

**Lemma 2.3.** *Let  $X$  and  $Y$  be two matrices in  $M_n$ . If  $\lambda(X), \lambda(Y)$  are in  $\mathbb{R}_+^n$  such that  $\lambda(X) \prec_{wlog} \lambda(Y)$ , then*

$$\det(I_n + X) \leq \det(I_n + Y).$$

40 As a consequence, we have the following result.

41 **Theorem 2.1.** *Let  $A$  and  $B$  be two positive semi-definite matrices. Then*

- 42 1.  $\det(A^2 + |BA|^p) \geq \det(A^2 + A^p B^p)$  for  $2 \leq p \leq 4$ .
- 43 2. If  $0 \leq p \leq 2$  or  $p \geq 4$ , then  $\det(A^2 + |BA|^p) \leq \det(A^2 + A^p B^p)$ .

---

<sup>1</sup>Thanks to the reviewer for pointing this out.

44 *Proof.* 1. Without loss of generality, we shall assume that  $A$  and  $B$  are positive definite matrices as the  
 45 general case can be then obtained by a continuity argument. For  $2 \leq p \leq 4$ , then we can write

$$\begin{aligned}
 \lambda(A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) &= \lambda\left(A^{-1}(AB(BA^2B)^{\frac{p}{2}-1}BA)A^{-1}\right) && \text{(by Lemma 2.2)} \\
 &= \lambda\left(B(BA^2B)^{\frac{p}{2}-1}B\right) \\
 &= \lambda\left((B^2)^{\frac{1}{2}}\left((B^2)^{\frac{1}{2}}(A^2)(B^2)^{\frac{1}{2}}\right)^{\frac{p}{2}-1}(B^2)^{\frac{1}{2}}\right) \\
 &\succ_{wlog} \lambda\left((B^2)^{1+\frac{p}{2}-1}(A^2)^{\frac{p}{2}-1}\right) && \text{(by Lemma 2.1, Part 1)} \\
 &= \lambda(A^{p-2}B^p).
 \end{aligned}$$

46 So,

$$\lambda(A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \succ_{wlog} \lambda(A^{p-2}B^p). \quad (6)$$

47 Next, applying Lemma 2.3 on (6) gives

$$\det(I_n + A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \geq \det(I_n + A^{p-2}B^p). \quad (7)$$

48 Multiplying both sides of (7) by  $\det(A^2) > 0$  completes the proof of the first part.

49

50 2. The proof for the case where  $0 \leq p \leq 2$ , is due to [6], while the case where  $p \geq 4$  can be done in a  
 51 similar fashion as in the first case by making use this time of Part 2 of Lemma 2.1.  $\square$

### 52 3. Proof of Conjecture 1.1

53 The starting point here is a lemma dealing with a majorization inequality which appears in [8]. For the  
 54 sake of completeness, we reproduce the proof here.

**Lemma 3.1.** *Let  $Y$  be a positive semi-definite matrix and  $X$  be any Hermitian matrix. Then for all  $p, q \in [0, +\infty)$ , it holds that*

$$\lambda(XY^pXY^q) \prec_{wlog} \lambda(X^2Y^{p+q}).$$

*Proof.* By appealing to a standard argument (anti-symmetric product), then it suffices to prove that

$$\lambda_1(XY^pXY^q) \leq \lambda_1(X^2Y^{p+q}).$$

Without loss of generality, we shall assume that  $X$  is invertible as the general case can be done by continuity argument. In addition, we shall assume that  $q \leq p$  and  $\lambda_1(X^2Y^{p+q}) = 1$ . Now, obviously proving our claim is equivalent to showing that

$$\lambda_1(XY^pXY^q) \leq 1.$$

The fact that the largest eigenvalue of the matrix  $X^2Y^{p+q}$  is equal to 1, clearly implies that

$$\lambda_j(X^2Y^{p+q}) \leq 1 \text{ for all } 1 \leq j \leq n.$$

55 But this is equivalent to  $XY^{p+q}X \leq I_n$  which in turn gives

$$Y^{p+q} \leq (X^{-1})^2. \quad (8)$$

56 Next, applying Lowner-Heinz on (8) for a power  $0 \leq \frac{p}{p+q} \leq 1$ , yields

$$Y^p \leq ((X^{-1})^2)^{\frac{p}{p+q}}. \quad (9)$$

Again, taking a power  $0 \leq \frac{q}{p} \leq 1$  in both sides of (9), we get

$$Y^q \leq ((X^{-1})^2)^{\frac{q}{p+q}}.$$

57 Hence,

$$(X^2)^{\frac{q}{p+q}} \leq Y^{-q}. \quad (10)$$

58 Therefore,

$$\begin{aligned} \lambda_1(XY^pXY^q) &= \lambda_1\left(Y^{q/2}XY^pXY^{q/2}\right) \\ &\leq \lambda_1\left(Y^{q/2}X((X^{-1})^2)^{\frac{p}{p+q}}XY^{q/2}\right) \quad (\text{by using (9)}) \\ &= \lambda_1\left((Y^{q/2}(X^2)^{\frac{q}{p+q}}Y^{q/2})\right) \\ &\leq \lambda_1\left(Y^{q/2}Y^{-q}Y^{q/2}\right) \quad (\text{by using (10)}) \\ &= \lambda_1(I_n) \\ &= 1. \end{aligned}$$

59

□

60 As a result, we have the following theorem.

61 **Theorem 3.1.** *Let  $Y$  be a positive definite matrix and  $X$  be a Hermitian matrix. Then for all  $p, q \in [0, \infty)$*

$$\lambda(XY^pXY^{-q}) \succ_{wlog} \lambda(X^2Y^{p-q}). \quad (11)$$

62 *Proof.* The proof will be divided into three cases.

Case 1: Let  $q \geq 2p$ . By Schur's complement we have

$$M = \begin{bmatrix} Y^{-\frac{q}{2}}XY^pXY^{-\frac{q}{2}} & Y^{-\frac{q}{2}}X^2Y^{p-\frac{q}{2}} \\ Y^{p-\frac{q}{2}}X^2Y^{-\frac{q}{2}} & Y^{p-\frac{q}{2}}XY^{-p}XY^{p-\frac{q}{2}} \end{bmatrix} \geq 0.$$

Applying Theorem 10.20 in [9, p. 352] gives

$$\lambda_1\left(Y^{-\frac{q}{2}}XY^pXY^{-\frac{q}{2}}\right) \cdot \lambda_1\left(Y^{p-\frac{q}{2}}XY^{-p}XY^{p-\frac{q}{2}}\right) \geq \lambda_1\left(Y^{-\frac{q}{2}}X^2Y^{p-\frac{q}{2}}\right)^2.$$

63 That is,

$$\lambda_1(XY^pXY^{-q}) \cdot \lambda_1(XY^{2p-q}XY^{-p}) \geq \lambda_1(X^2Y^{p-q})^2. \quad (12)$$

64 In view of Lemma 3.1, it is worthy to observe that

$$\begin{aligned} \lambda(X(Y^{-1})^{q-2p}X(Y^{-1})^p) &\prec_{wlog} \lambda(X^2(Y^{-1})^{q-p}) \\ &= \lambda(X^2Y^{p-q}). \end{aligned}$$

65 Therefore,

$$\begin{aligned}\lambda_1(Y^{2p-q}XY^{-p}X) &= \lambda_1(X(Y^{-1})^{q-2p}X(Y^{-1})^p) \\ &\leq \lambda_1(X^2Y^{p-q}).\end{aligned}$$

Thus, from (12) we obtain

$$\lambda_1(XY^pXY^{-q}) \geq \lambda_1(X^2Y^{p-q}) \quad \text{for } q \geq 2p \geq 0.$$

66 By a standard anti-symmetric tensor product argument, inequality (11) is true for all  $q \geq 2p$ .

67

68 Case 2: Let  $p \leq q \leq 2p$ . The idea of the proof here depends on writing the interval  $[p, 2p] = \bigcup_{k=2}^{\infty} [\frac{k+1}{k}p, 2p]$   
69 and then proving (11) for each subinterval. We start with with case  $k = 2$  i.e. for  $2p \geq q \geq \frac{3p}{2}$ . Then,

$$\begin{aligned}\lambda(Y^pXY^{-q}X) &= \lambda\left(Y^{\frac{3p}{2}}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{p-q}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{\frac{p}{2}}\right) \\ &= \lambda\left(Y^{2p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{-(q-p)}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})\right) \\ &= \lambda\left((Y^{-1})^{-2p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})(Y^{-1})^{q-p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})\right).\end{aligned}$$

70 Now considering this last expression and noticing that  $2p \geq 2(q-p) \geq 0$ , then in view of Case 1;  
71 replacing  $Y$  with  $Y^{-1}$ ,  $X$  with  $Y^{-\frac{p}{2}}XY^{-\frac{p}{2}}$ ,  $2p$  with  $q$  and lastly  $q-p$  with  $p$ , we obtain

$$\begin{aligned}\lambda(Y^pXY^{-q}X) &\succ_{wlog} \lambda\left(Y^{3p-q}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})^2\right) \\ &= \lambda(Y^{2p-q}XY^{-p}X) \\ &\succ_{wlog} \lambda(Y^{p-q}X^2) \quad (\text{again using Case 1 as } p \geq 2(2p-q) \geq 0).\end{aligned}$$

72 Next, using a similar argument, we prove (11) is true for  $k = 3$  i.e. for  $2p \geq q \geq \frac{4p}{3}$ . As earlier, we  
73 can write

$$\begin{aligned}\lambda(Y^pXY^{-q}X) &= \lambda\left(Y^{\frac{3p}{2}}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{p-q}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{\frac{p}{2}}\right) \\ &= \lambda\left(Y^{2p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})Y^{-(q-p)}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})\right) \\ &= \lambda\left((Y^{-1})^{-2p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})(Y^{-1})^{q-p}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})\right) \\ &\succ_{wlog} \lambda\left(Y^{3p-q}(Y^{-\frac{p}{2}}XY^{-\frac{p}{2}})^2\right) \quad (\text{in view of of Case 1 as } 2p \geq 2(q-p) \geq 0) \\ &= \lambda(Y^{2p-q}XY^{-p}X) \\ &\succ_{wlog} \lambda(Y^{p-q}X^2) \quad (\text{similarly in view of Case } k = 2 \text{ as } p \geq \frac{3}{2}(2p-q) \geq 0).\end{aligned}$$

74 Continuing this way, one can easily see (using induction) that for any positive integer  $k$ , inequality  
75 (11) is true for all  $q$  with  $2p \geq q \geq \frac{(k+1)p}{k}$ . Finally, the proof for this case can be achieved by letting  
76  $k$  tends to infinity.  
77

78 **Case 3:** Let  $q \leq p$ . To complete the proof of this case, it suffices to apply the preceding two cases on  $Y^{-1}$ .

79

□

80 As applications, we have the following result.

**Corollary 3.1.** *Let  $A$  and  $B$  be two Hermitian matrices. Then, for all  $p \in [0, \infty)$  we have*

$$\det(A^2 + |BA|^p) \leq \det(A^2 + |AB|^p).$$

81 *Proof.* As usual, we shall assume that  $A$  and  $B$  are invertible, the general case is by continuity argument.

82 Then, for all  $p \in [0, +\infty)$  we have

$$\begin{aligned} \lambda\left(A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}\right) &= \lambda\left(B(BA^2B)^{\frac{p}{2}-1}B\right) && \text{(Using Lemma 2.2)} \\ &= \lambda\left(B^2(BA^2B)^{\frac{p}{2}-1}\right) \\ &\prec_{wlog} \lambda\left(B(BA^2B)^{\frac{p}{2}}B(BA^2B)^{-1}\right) && \text{(Using Theorem 3.1)} \\ &= \lambda\left(A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}\right). \end{aligned}$$

Next, applying Lemma 2.3 gives

$$\det(I_n + A^{-1}(AB^2A)^{\frac{p}{2}}A^{-1}) \leq \det(I_n + A^{-1}(BA^2B)^{\frac{p}{2}}A^{-1}).$$

Finally, multiplying both sides with  $\det(A^2) > 0$  yields

$$\det(A^2 + |BA|^p) \leq \det(A^2 + |AB|^p).$$

83

□

#### 84 **4. Conjecture 1.2**

85 In this section, our purpose is to find for what values of  $k$  and  $t$  the following majorization inequality

$$\lambda(A^{\frac{k}{2}-t}B^t) \prec_{log} \lambda(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}}) \quad (13)$$

86 is valid, where  $A$  and  $B$  are positive definite matrices.

87

88 We shall start here with the following lemma which is needed for our purposes and is well known as  
89 Furuta's inequality [4].

**Lemma 4.1.** *Let  $A, B$  be two positive semi-definite matrices such that  $A \geq B$ . Then, for all  $p \geq 1, r \geq 0$ ,*

$$A^{(p+2r)/p} \geq (A^r B^p A^r)^{1/p}.$$

90 Now we are in a position to prove the next theorem which shows that (13) is valid for all  $0 \leq t \leq 1$  and  
91  $k \geq 4t$ .

**Theorem 4.1.** *Let  $A$  and  $B$  be two positive definite matrices. Then for all  $0 \leq t \leq 1$  and  $k \geq 4t$*

$$\lambda(A^{\frac{k}{2}-t}B^t) \prec_{log} \lambda(A^{\frac{k}{4}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^tA^{\frac{k}{4}}).$$



*Proof.* Let  $0 \leq t \leq 1$  and  $k \geq 4t$ . Using Schur's complement, we know that

$$M = \begin{bmatrix} A^{\frac{k}{4}-t} B^t (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^t B^t A^{\frac{k}{4}-t} & A^{\frac{k}{4}-t} B^t A^{\frac{k}{4}} \\ A^{\frac{k}{4}} B^t A^{\frac{k}{4}-t} & A^{\frac{k}{4}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^t A^{\frac{k}{4}} \end{bmatrix} \geq 0.$$

92 Then

$$\left( \lambda_1(A^{\frac{k}{2}-t} B^t) \right)^2 \leq \lambda_1(A^{\frac{k}{4}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^t A^{\frac{k}{4}}) \cdot \lambda_1(A^{\frac{k}{4}-t} B^t (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^t B^t A^{\frac{k}{4}-t}). \quad (14)$$

93 As mentioned earlier, in order to prove our claim, then it is enough to prove that

$$\lambda_1(A^{\frac{k}{2}-t} B^t) \geq \lambda_1(A^{\frac{k}{4}-t} B^t (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^t B^t A^{\frac{k}{4}-t}), \quad (15)$$

which is in turn equivalent to showing that

$$B^{\frac{t}{2}} A^{\frac{k}{2}-t} B^{\frac{t}{2}} \leq I_n \Rightarrow A^{\frac{k}{4}-t} B^t (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^t B^t A^{\frac{k}{4}-t} \leq I_n.$$

For this, let  $B^{\frac{t}{2}} A^{\frac{k}{2}-t} B^{\frac{t}{2}} \leq I_n$ , then  $A^{\frac{k}{2}-t} \leq B^{-t}$ . First, making use of Löwner-Heinz inequality for  $0 \leq \frac{t}{\frac{k}{2}-t} \leq 1$  gives

$$A^t \leq B^{-\frac{t^2}{\frac{k}{2}-t}}.$$

Next, applying Lemma 4.1 yields

$$\left( B^{-\frac{t^2}{\frac{k}{2}-t}} \right)^{\frac{p+2r}{p}} \geq \left[ \left( B^{-\frac{t^2}{\frac{k}{2}-t}} \right)^r (A^t)^p \left( B^{-\frac{t^2}{\frac{k}{2}-t}} \right)^r \right]^{\frac{1}{p}}.$$

94 Now, replacing  $p$  with  $\frac{1}{t} \geq 1$  and  $r$  with  $\frac{\frac{k}{2}-t}{2t^2} \geq 0$  gives

$$B^{-\frac{\frac{k}{2}-t}{\frac{k}{2}-t}} \geq (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^t.$$

95 Pre-post multiplying both sides by  $B^t > 0$  implies that

$$(B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}} = B^t B^{-\frac{\frac{k}{2}-t}{\frac{k}{2}-t}} B^t \geq B^t (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^t B^t. \quad (16)$$

However,  $A^{\frac{k}{2}-t} \leq B^{-t}$ , so that  $A^{-(\frac{k}{2}-t)} \geq B^t$ . By again appealing to Löwner-Heinz inequality for  $0 \leq \frac{\frac{k}{2}-2t}{\frac{k}{2}-t} \leq 1$  we obtain

$$(A^{-(\frac{k}{2}-t)})^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}} \geq (B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}},$$

96 which gives

$$A^{-(\frac{k}{2}-2t)} \geq (B^t)^{\frac{\frac{k}{2}-2t}{\frac{k}{2}-t}}. \quad (17)$$

97 Now it is worthy to observe that inequalities (16) and (17) yield

$$A^{-(\frac{k}{2}-2t)} \geq B^t (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^t B^t.$$

98 Hence,  $A^{\frac{k}{4}-t} B^t (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^t B^t A^{\frac{k}{4}-t} \leq I_n$ , and therefore, (15) is true for all  $0 \leq t \leq 1$  and  $k \geq 4t$ .  
On the other hand, using (14) and (15) gives

$$\lambda_1(A^{\frac{k}{2}-t} B^t) \leq \lambda_1(A^{\frac{k}{4}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^t A^{\frac{k}{4}}), \quad 0 \leq t \leq 1, \quad k \geq 4t.$$

Thus, by a standard anti-symmetric tensor product argument, we get

$$\lambda(A^{\frac{k}{2}-t} B^t) \prec_{wlog} \lambda(A^{\frac{k}{4}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^t A^{\frac{k}{4}}), \quad 0 \leq t \leq 1, \quad k \geq 4t.$$

Finally, the proof is complete by making use of the fact that

$$\det(A^{\frac{k}{2}-t} B^t) = \det(A^{\frac{k}{4}} (B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}})^t A^{\frac{k}{4}}).$$

99

□

100 Our next goal is to show that (13) is also true for all  $\frac{1}{2} \leq t \leq 1$  and  $k \geq 6t - 2$ . First, we need the  
101 following lemma.

**Lemma 4.2.** *Let  $A$  and  $B$  be two positive definite matrices. Then, for all  $\frac{1}{2} \leq t \leq 1$  and  $k \geq 6t - 2$ , we have*

$$\lambda\left(A^{\frac{k+2t}{4}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{2t} A^{\frac{2t+k}{4}}\right) \succ_{log} \lambda(A^{\frac{k}{2}-t} B^{2t}).$$

*Proof.* As in similar situations, it is enough to prove that for all  $\frac{1}{2} \leq t \leq 1$  and  $k \geq 6t - 2$

$$\lambda_1\left(A^{\frac{k+2t}{4}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{2t} A^{\frac{2t+k}{4}}\right) \geq \lambda_1(A^{\frac{k}{2}-t} B^{2t}).$$

First, in view of Lemma 2.2, we obtain the following equality:

$$A^{\frac{2t+k}{4}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{2t} A^{\frac{2t+k}{4}} = A^{\frac{2t+k-2}{4}} B^{\frac{1}{2}} \left(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}\right)^{2t-1} B^{\frac{1}{2}} A^{\frac{2t+k-2}{4}}.$$

102 For the sake of clarification, we shall use the following notation. For all  $\frac{1}{2} \leq t \leq 1$  and  $k \geq 6t - 2$ , let

$$\begin{aligned} X &= A^{\frac{2t+k-2}{4}} B^{\frac{1}{2}} \left(B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}}\right)^{2t-1} B^{\frac{1}{2}} A^{\frac{2t+k-2}{4}}, \\ Y &= A^{\frac{2t+k-2}{4}} B^{2t} A^{\frac{k-6t+2}{4}}, \quad \text{and} \\ Z &= A^{\frac{k-6t+2}{4}} B^{2t-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1} B^{2t-\frac{1}{2}} A^{\frac{k-6t+2}{4}}. \end{aligned}$$

Now by making use of Schur complement, the following  $2 \times 2$  block matrix

$$M = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix}$$

is positive semi-definite and hence  $\lambda_1(X) \cdot \lambda_1(Z) \geq \lambda_1(Y)^2$ . Our next goal is to show that  $\lambda_1(Z) \leq \lambda_1(Y)$  which in turn gives  $\lambda_1(X) \geq \lambda_1(Y)$ . Noticing that  $\lambda_1(Y) = \lambda_1(B^t A^{\frac{k}{2}-t} B^t)$ , then in order to prove  $\lambda_1(Z) \leq \lambda_1(Y)$ , it suffices to show that

$$B^t A^{\frac{k}{2}-t} B^t \leq I_n \Rightarrow Z \leq I_n.$$

For this purpose, let  $B^t A^{\frac{k}{2}-t} B^t \leq I_n$ , then clearly  $A^{\frac{k}{2}-t} \leq B^{-2t}$ . Using Löwner-Heinz inequality for  $0 \leq \frac{2t-1}{\frac{k}{2}-t} \leq 1$  gives

$$A^{2t-1} \leq B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}}.$$

Now, applying Lemma 4.1 yields

$$\left( B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}} \right)^{\frac{p+2r}{p}} \geq \left[ \left( B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}} \right)^r (A^{2t-1})^p \left( B^{-\frac{2t(2t-1)}{\frac{k}{2}-t}} \right)^r \right]^{\frac{1}{p}}.$$

Next, taking  $p = \frac{1}{2t-1} \geq 1$  and  $r = \frac{\frac{k}{2}-t}{4t(2t-1)} \geq 0$  implies that

$$B^{-\frac{(\frac{k}{2}+t)(2t-1)}{\frac{k}{2}-t}} \geq (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1}.$$

Pre-post multiplying both sides with  $B^{2t-\frac{1}{2}} > 0$  gives

$$B^{2t-\frac{1}{2}} B^{-\frac{(\frac{k}{2}+t)(2t-1)}{\frac{k}{2}-t}} B^{2t-\frac{1}{2}} = (B^{2t})^{\frac{k-6t+2}{\frac{k}{2}-t}} \geq B^{2t-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1} B^{2t-\frac{1}{2}}.$$

Again, noticing that  $A^{\frac{k}{2}-t} \leq B^{-2t}$  implies that  $A^{-(\frac{k}{2}-t)} \geq B^{2t}$ , then by appealing to Löwner-Heinz for  $0 \leq \frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t} \leq 1$  we obtain

$$(A^{-(\frac{k}{2}-t)})^{\frac{\frac{k-6t+2}{2}}{\frac{k}{2}-t}} \geq (B^{2t})^{\frac{k-6t+2}{\frac{k}{2}-t}}$$

which gives

$$A^{-(\frac{k-6t+2}{2})} \geq B^{2t-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1} B^{2t-\frac{1}{2}}.$$

Hence,

$$Z = A^{\frac{k-6t+2}{4}} B^{2t-\frac{1}{2}} (B^{-\frac{1}{2}} A B^{-\frac{1}{2}})^{2t-1} B^{2t-\frac{1}{2}} A^{\frac{k-6t+2}{4}} \leq I_n.$$

Therefore, for all  $\frac{1}{2} \leq t \leq 1$  and  $k \geq 6t - 2$ ,

$$\lambda_1(X) \geq \lambda_1(Y).$$

Thus, by an anti-symmetric tensor product argument and by the fact that  $\det(X) = \det(Y)$ , we get

$$\lambda(X) \succ_{\log} \lambda(Y).$$

103 Finally, noting that  $\lambda(X) = \lambda \left( A^{\frac{k+2t}{4}} \left( A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^{2t} A^{\frac{2t+k}{4}} \right)$  and  $\lambda(Y) = \lambda(A^{\frac{k}{2}-t} B^{2t})$ , the proof is  
 104 complete. □

**Theorem 4.2.** *Let  $A$  and  $B$  be two positive definite matrices. Then, for all  $\frac{1}{2} \leq t \leq 1$  and  $k \geq 6t - 2$*

$$\lambda \left( A^{\frac{k}{4}} \left( B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} \right)^t A^{\frac{k}{4}} \right) \succ_{\log} \lambda(A^{\frac{k}{2}-t} B^t).$$

105 *Proof.* Let  $\frac{1}{2} \leq t \leq 1$  and  $k \geq 6t - 2$ . Then, we can write

$$\begin{aligned}
\lambda \left( A^{\frac{k}{4}} \left( B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} \right)^t A^{\frac{k}{4}} \right) &= \lambda \left( A^{\frac{k}{4}} \left( A^{\frac{1}{2}} (A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} A^{-\frac{1}{2}}) A^{\frac{1}{2}} \right)^t A^{\frac{k}{4}} \right) \\
&\succ_{wlog} \lambda \left( A^{\frac{k}{2}+t} \left( A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} A^{-\frac{1}{2}} \right)^t \right) \quad (\text{by Part 1 of Lemma 2.1}) \\
&= \lambda \left( A^{\frac{k}{2}+t} \left( A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} \right)^{2t} \right) \\
&= \lambda \left( A^{\frac{k+2t}{4}} \left( A^{-\frac{1}{2}} B^{\frac{1}{2}} A^{-\frac{1}{2}} \right)^{2t} A^{\frac{k+2t}{4}} \right) \\
&\succ_{log} \lambda \left( A^{\frac{k}{2}-t} (B^{\frac{1}{2}})^{2t} \right) \quad (\text{by Lemma 4.2}) \\
&= \lambda(A^{\frac{k}{2}-t} B^t).
\end{aligned}$$

106

□

107 As a result of Theorem 4.1 and Theorem 4.2, we have the following corollary.

**Corollary 4.1.** *Let  $A$  and  $B$  be two positive semi-definite matrices. Then, for  $(0 \leq p \leq 2$  and  $k \geq 2p)$  or for  $(1 \leq p \leq 2$  and  $k \geq 3p - 2)$  we have*

$$\det(A^k + |AB|^p) \geq \det(A^k + A^p B^p).$$

108 *Proof.* Without loss of generality we may assume that  $A$  and  $B$  are positive definite matrices, the general  
109 case is by continuity argument. Suppose that  $0 \leq p \leq 2$  and  $k \geq 2p$ . Then, replacing  $A$  with  $A^{-2}$ ,  $B$  with  
110  $B^2$  and  $t$  with  $0 \leq \frac{p}{2} \leq 1$  in Theorem 4.1 gives

$$\lambda(A^{p-k} B^p) \prec_{log} \lambda(A^{-\frac{k}{2}} (BA^2 B)^{\frac{p}{2}} A^{-\frac{k}{2}}). \quad (18)$$

Applying Lemma 2.3 on (18) yields

$$\det(I_n + A^{-\frac{k}{2}} (BA^2 B)^{\frac{p}{2}} A^{-\frac{k}{2}}) \geq \det(I_n + A^{p-k} B^p).$$

Next, multiplying both sides with  $\det(A^k) > 0$ , we obtain

$$\det(A^k + |AB|^p) \geq \det(A^k + A^p B^p), \quad 0 \leq p \leq 2, k \geq 2p.$$

111 For the case when  $1 \leq p \leq 2$  and  $k \geq 3p - 2$ , the proof can be done in a similar fashion by making use  
112 this time of Theorem 4.2. □

113 As an analogue of Theorem 2.1, Corollary 3.1, and Corollary 4.1, we have the following result which  
114 gives a partial answer of Conjecture 1.2.

**Theorem 4.3.** *Let  $A$  and  $B$  be two positive semi-definite matrices. Then, for all  $0 \leq p \leq \frac{4}{3}$  or  $2 \leq p \leq 4$ , the following holds*

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p).$$

*Proof.* Corollary 4.1 is a general case of

$$\det(A^2 + |AB|^p) \geq \det(A^2 + A^p B^p), \quad 0 \leq p \leq \frac{4}{3}.$$

Combining Part 1 of Theorem 2.1 with Corollary 3.1 implies that for all  $2 \leq p \leq 4$

$$\det(A^2 + |AB|^p) \geq \det(A^2 + |BA|^p) \geq \det(A^2 + A^p B^p).$$

115

□

116 We conclude this paper with the following conjecture which is very much related to our work in this  
117 section, and whose validation for  $k = 2$  would imply Conjecture 1.2.

**Conjecture 4.1.** *Let  $A$  and  $B$  be two positive definite matrices. Then for all  $k \geq 0$  and  $0 \leq t \leq 1$ ,*

$$\lambda \left( A^{\frac{k}{4}} \left( B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} \right)^t A^{\frac{k}{4}} \right) \succ_{\log} \lambda(A^{\frac{k}{2}-t} B^t).$$

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## 122 Declaration of Competing Interest

123 The authors declare that they have no competing interests.

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