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Systems of reflected BSDEs with interconnected bilateral obstacles: Existence, Uniqueness and Applications

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Abstract

This paper is related to the study of systems of reflected backward stochastic differential equations with interconnected bilateral obstacles. These systems are connected with zero-sum stochastic switching games. Under appropriate assumptions, we provide either existence or existence and uniqueness of the solution of those systems when the switching costs are Itô processes. The link with systems of PDEs with bilateral interconnected obstacles is also stated via the Feynman-Kac representation when randomness comes from a Markov diffusion process.

Keywords: Systems of reflected BSDEs; Bilateral interconnected obstacles; Zero-sum stochastic switching games; Optimal switching; Systems of PDEs with interconnected obstacles; Feynman-Kac representation; Viscosity solution.

2000 MSC: 93C30, 91A15, 93E20, 49J20

1. Introduction

This paper is related to the study of systems of reflected backward stochastic differential equations (BSDEs in short) with interconnected bilateral obstacles. A solution for such a system is a family of adapted processes $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ such that: For any $(i,j)\in\Gamma$ and $t\leq T$,

$$\begin{cases} Y_t^{ij} = \xi^{ij} + \displaystyle\int_t^T f^{ij}\left(s,\omega,(Y_s^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}, Z_s^{ij}\right) ds - \int_t^T Z_s^{ij} dB_s + \int_t^T (dK_s^{ij,+} - dK_s^{ij,-})\,; \\ L_t^{ij} \leq Y_t^{ij} \leq U_t^{ij}\,; \\ \displaystyle\int_0^T (Y_t^{ij} - L_t^{ij}) dK_t^{ij,+} = 0 \quad \text{and} \quad \int_0^T (U_t^{ij} - Y_t^{ij}) dK_t^{ij,-} = 0, \end{cases} \tag{1.1}$$

where:

a) $\Gamma := \Gamma_1 \times \Gamma_2 = \{1,...,m_1\} \times \{1,...,m_2\}$;

b) $L_t^{ij} := \max\limits_{k\in\Gamma^1-\{i\}} \{Y_t^{kj} - \underline{g}_{ik}(t)\}$ and $U_t^{ij} := \min\limits_{l\in\Gamma^2-\{j\}} \{Y_t^{il} + \overline{g}_{jl}(t)\}$;

c) f^{ij}, ξ^{ij}, $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are given data of the problem which are described precisely later;

d) $K^{ij,\pm}$ are non-decreasing processes such that $K_0^{ij,\pm} = 0$.

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This system introduced first in [16] is related to the zero-sum stochastic switching game, as shown later in some papers including [4, 9]. On the other hand, note that the above BSDEs have two reflecting barriers which depend on the solution $(Y^{ij})_{(i,j)\in\Gamma}$.

A stochastic optimal switching control problem of a system (which can be a portfolio in market, a power plant, etc.) is a discrete stochastic optimal control where a strategy σ is pair of sequences $((\tau_n)_{n\geq 0},(\zeta_n)_{n\geq 0})$ such that for any $n\geq 0$, τ_n is a stopping time such that $\tau_n\leq\tau_{n+1}$ and ζ_n are random variables valued in the set of modes under which the system is run. Roughly speaking at time τ_n the controller decides to switch the system from its current mode to the new one denoted by ζ_n. The switching actions are not free and generate expenditures. When a strategy σ is implemented, it induces a payoff which is equal to $J(\sigma)$ and then the problem is to find a strategy σ^* which realizes $\sup_\sigma J(\sigma)$. This problem is related to systems of reflected backward stochastic differential equations (RBSDEs in short) with interconnected one lower obstacles to which reduces (1.1) in the case when $\overline{g}_{jl}=+\infty$. There are several papers on this topic including [1, 2, 5, 11, 8, 12, 15, 22, 17, 25, 18] (see also the references therein) in connection with energy, finance, etc.

Next, one has a zero-sum switching game if there are two decision makers π_1 and π_2 which intervene on the system by both choosing its joint working mode $(i,j)\in\Gamma$ (π_1 and π_2 choose $i\in\Gamma^1$ and $j\in\Gamma^2$ respectively). The interests of the decision makers are antagonistic, that is to say, when π_1 (resp. π_2) implements the strategy σ_1 (resp. σ_2) there is in-between a payoff $J(\sigma_1,\sigma_2)$ which is a profit (resp. cost) for π_1 (resp. π_2). The zero-sum switching game (especially issues of existence of the value, a saddle point, etc.) is connected with the solutions of systems of reflected BSDEs of types (1.1) (see e.g. [4, 9]). This is the main motivation to study this system (1.1).

There are only very few papers which deal with the problem of existence of a solution for system (1.1). The question of uniqueness is even less studied. According to our best knowledge, system (1.1) is studied in two papers only which are [16] and [4]. In [16], the authors have shown existence of a solution for this system (1.1) when the switching costs $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are constant. The question of uniqueness is not addressed and remained open. On the other hand, in [4], Djehiche et al. have considered system (1.1) in the markovian framework of randomness. By using tools which combine results on partial differential equations (PDEs for short) with results on BSDEs, the authors have shown existence and uniqueness of the solution of system (1.1). The switching costs $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are not constant.

Therefore the main objective of this paper is to complete the existing literature on the problem of existence and uniqueness of a solution for the system of RBSDEs with bilateral interconnected obstacles (1.1) and to provide an application in the field of PDEs. Actually the novelties of this paper are the following:

i) We show that system (1.1) has a solution in the case when the processes $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are of Itô type and under the monotonicity assumption of the functions f^{ij} (see [H5] below) ;

ii) We show that system (1.1) has a unique solution in the case when the processes $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are Itô processes and the functions f^{ij} do no depend on z. We do not require the monotonicity assumption on these latter functions ;

iii) When randomness is Markovian and comes from a diffusion process $X^{t,x}$, we show that the Feynman-Kac representation formula holds for $(Y^{ij})_{(i,j)\in\Gamma}$, the first component of the solution of system (1.1), i.e., there exist deterministic continuous functions $(v^{ij})_{(i,j)\in\Gamma}$ such that for any $(i,j)\in\Gamma$, $s\in[t,T]$, $Y^{ij;t,x}_s=v^{ij}(s,X^{t,x}_s)$. Moreover the functions $(v^{ij})_{(i,j)\in\Gamma}$ are the unique solution of the following system of PDEs with bilateral interconnected obstacles: $\forall(i,j)\in\Gamma$,

$$\left\{\begin{array}{l}\min\{v^{ij}(t,x)-\max\limits_{k\in\Gamma^1-\{i\}}[v^{kj}(t,x)-\underline{g}_{ik}(t,x)];\max[v^{ij}(t,x)-\min\limits_{l\in\Gamma^2-\{j\}}[v^{il}(t,x)+\overline{g}_{jl}(t,x)];\\ \qquad -\partial_t v^{ij}(t,x)-\mathcal{L}^X(v^{ij})(t,x)-f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\big]\}=0;\\ v^{ij}(T,x)=h^{ij}(x).\end{array}\right. \tag{1.2}$$

The monotonicity assumption of the functions $(f^{ij})_{(i,j)\in\Gamma}$ is no longer required as in [3, 4, 13, 24], etc. This result on PDEs improves also substantially the existing literature in this domain (see the previous references). System (1.2) can be seen as the Hamilton-Jacobi-Bellman-Isaacs system associated with the zero-sum switching game when utilities are implicit or depend on the values.

The paper is organized as follows: In Section 2, we give some statements and assumptions. In Section 3 we introduce and analyse, under the monotonicity assumption on the functions $(f^{ij})_{(i,j)\in\Gamma}$, the approximating schemes of (1.1) obtained by penalization. We show that the penalization terms are bounded in appropriate space. We then show that the penalization schemes converge and their limits provide solutions for (1.1). In Section 4, by the zero-sum stochastic representation, we show that, the system (1.1) has a unique solution when $(f^{ij})_{(i,j)\in\Gamma}$ does not depend on z. Finally in Section 5, we deal with application of the result of Section 4 in the field of PDEs. We first show that the processes $(Y^{ij})_{(i,j)\in\Gamma}$ enjoy the Feynman-Kac representation through deterministic continuous with polynomial growth functions $(v^{ij})_{(i,j)\in\Gamma}$. Moreover the functions $(v^{ij})_{(i,j)\in\Gamma}$ are the unique solution in viscosity of system of PDEs with obstacles (1.2) of min-max type. They are also the unique solution of the dual system to (1.2) which is of max-min type.

2. Statements, assumptions and preliminaries

Let $T > 0$ be a fixed real constant. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which carries a d-dimensional Brownian motion $B = (B_t)_{t\in[0,T]}$ whose natural filtration is $\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\}_{0\leq t\leq T}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, then it satisfies the usual conditions, i.e., it is complete and right continuous. On the other hand, we define $\mathcal{P}$ as the σ-algebra on $[0,T]\times\Omega$ of the $\mathbb{F}$-progressively measurable sets. Next, we denote by:

- $\mathcal{S}^2$: the set of $\mathcal{P}$-measurable continuous processes $\phi = (\phi_t)_{t\in[0,T]}$ such that $\mathbb{E}(\sup_{t\in[0,T]} |\phi_t|^2) < \infty$;
- $\mathcal{A}^2$: the subset of $\mathcal{S}^2$ of non-decreasing processes $K = (K_t)_{t\leq T}$ such that $K_0 = 0$;
- $\mathcal{H}^{2,k}(k \geq 1)$: the set of $\mathcal{P}$-measurable, $\mathbb{R}^k$-valued processes $\phi = (\phi_t)_{t\in[0,T]}$ such that $\mathbb{E}(\int_0^T |\phi_t|_k^2 dt) < \infty$.

To proceed, let Γ^1, Γ^2 be the finite sets of the whole switching modes available for the controllers or players. As mentionned previously $\Gamma := \Gamma^1 \times \Gamma^2$ and denote by Λ its cardinal, i.e., $\Lambda := |\Gamma| = |\Gamma^1| \times |\Gamma^2|$. On the other hand for $(i,j) \in \Gamma^1 \times \Gamma^2$, we define $(\Gamma^1)^{-i} := \Gamma^1 - \{i\}$ and $(\Gamma^2)^{-j} := \Gamma^2 - \{j\}$.

Next let us denote by $\vec{y}$ the generic element $(y^{ij})_{(i,j)\in\Gamma}$ of $\mathbb{R}^\Lambda$ and let us introduce the following items: For any $i,k \in \Gamma^1$ and $j,l \in \Gamma^2$,

i) f^{ij}: $(t,\omega,\vec{y},z) \in [0,T]\times\Omega\times\mathbb{R}^\Lambda\times\mathbb{R}^d \mapsto f^{ij}(t,\omega,\vec{y},z) \in \mathbb{R}$;

ii) $\underline{g}_{ik}$: $(t,\omega) \in [0,T]\times\Omega \mapsto \underline{g}_{ik}(t,\omega) \in \mathbb{R}^+$;

iii) $\overline{g}_{jl}$: $(t,\omega) \in [0,T]\times\Omega \mapsto \overline{g}_{jl}(t,\omega) \in \mathbb{R}^+$.

iv) ξ^{ij} is a r.v. valued in $\mathbb{R}$ and $\mathcal{F}_T$-measurable.

Finally let us introduce the following assumptions on $f^{ij}, \overline{g}_{ik}$ and $\underline{g}_{jl}$ for $i,k \in \Gamma^1$ and $j,l \in \Gamma^2$:

[H1] For any $(i,j) \in \Gamma^1 \times \Gamma^2$,

a) There exists a positive constant C and a non negative $\mathcal{P}$-measurable process $(\eta_t)_{t\leq T}$ which satisfies $\mathbb{E}[\sup_{s\leq T}|\eta_s|^2]<\infty$ and such that: $\mathbb{P}$-a.s, $\forall(\vec{y},z)\in\mathbb{R}^{\Lambda+d}$, $t\in[0,T]$,

$$|f^{ij}(t,\vec{y},z)|\leq C(1+\eta_t+|\vec{y}|),$$

where $|\vec{y}|$ refers to the standard Euclidean norm of $\vec{y}$ in $\mathbb{R}^{\Lambda}$ (the same for $|z|$ below). Note that this implies that $\mathbb{E}[\int_0^T|f^{ij}(t,0,0)|^2dt]<\infty$;

b) f^{ij} is Lipschitz continuous with respect to (w.r.t for short) $(\overrightarrow{y},z)$ uniformly in (t,ω), i.e. $\mathbb{P}$-a.s., for any $t\in[0,T]$, $(\overrightarrow{y_1},z_1)$ and $(\overrightarrow{y_2},z_2)$ elements of $\mathbb{R}^{\Lambda+d}$, we have

$$|f^{ij}(t,\overrightarrow{y_1},z_1)-f^{ij}(t,\overrightarrow{y_2},z_2)|\leq C(|\overrightarrow{y_1}-\overrightarrow{y_2}|+|z_1-z_2|)$$

where C is a fixed constant.

[H2] For any $(i,j)\in\Gamma$,

a) $\mathbb{E}(|\xi^{ij}|^2)<\infty$;

b) ξ^{ij}, as the terminal condition at time T of system (1.1), satisfies the following consistency condition: $\mathbb{P}$-a.s.,

$$\max_{k\in(\Gamma^1)^{-i}}\left(\xi^{kj}-\underline{g}_{ik}(T)\right)\leq\xi^{ij}\leq\min_{l\in(\Gamma^2)^{-j}}\left(\xi^{jl}+\overline{g}_{jl}(T)\right).$$

[H3] a) For all $i_1,i_2\in\Gamma^1$ (resp. $j_1,j_2\in\Gamma^2$) and $t\in[0,T]$, the process $\underline{g}_{i_1i_2}$(resp. $\overline{g}_{j_1j_2}$),

(i) is non-negative and continuous;

(ii) For any $k\in\Gamma^1$ (resp. $\ell\in\Gamma^2$) such that $|\{i_1,i_2,k\}|=3$ (resp. $|\{j_1,j_2,\ell\}|=3$) it holds:

$$\mathbb{P}-a.s.,\ \forall t\leq T,\ \underline{g}_{i_1i_2}(t)<\underline{g}_{i_1k}(t)+\underline{g}_{ki_2}(t)\ \left(\text{resp. }\overline{g}_{j_1j_2}(t)<\overline{g}_{j_1\ell}(t)+\overline{g}_{\ell j_2}(t)\right);\tag{2.1}$$

iii) By convention we set $\forall(i,j)\in\Gamma,\underline{g}_{ii}=0$ and $\bar{g}_{jj}=0$. Note that this convention implies the so-called non loop free property (see (4.2) and (4.3)).

[H4] For any $(i,j),(k,\ell)\in\Gamma$, $\underline{g}_{ik}$(resp. $\overline{g}_{j\ell}$) is an Itô process, i.e.,

$$\begin{cases}\underline{g}_{ik}(t)=\underline{g}_{ik}(0)+\displaystyle\int_0^t\underline{b}_{ik}(s)ds+\int_0^t\underline{\sigma}_{ik}(s)dB_s,\ t\leq T,\\ \text{with }\underline{\sigma}_{ik}\in\mathcal{H}^{2,d}\text{ and }\underline{b}_{ik},\ \mathcal{P}\text{-measurable and }\mathbb{E}[\sup_{s\leq T}|\underline{b}_{ik}(s)|^2]<\infty.\end{cases}$$

$$\left(\text{resp.}\begin{cases}\overline{g}_{j\ell}(t)=\overline{g}_{j\ell}(0)+\displaystyle\int_0^t\overline{b}_{j\ell}(s)ds+\int_0^t\overline{\sigma}_{j\ell}(s)dB_s,\ t\leq T,\\ \text{with }\overline{\sigma}_{j\ell}\in\mathcal{H}^{2,d}\text{ and }\overline{b}_{j\ell},\ \mathcal{P}\text{-measurable and }\mathbb{E}[\sup_{s\leq T}|\overline{b}_{j\ell}(s)|^2]<\infty.\end{cases}\right).$$

[H5] Monotonicity:

For any $(i,j)\in\Gamma$ and $(k,l)\in\Gamma^{-ij}:=\Gamma-\{(i,j)\}$, the mapping $y^{kl}\mapsto f^{ij}(t,\overrightarrow{y},z)$ is non-decreasing when the other components $(y^{pq})_{(p,q)\neq(k,l)}$ and z are fixed.

Definition 2.1. *A family* $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ *is said to be a solution of the system of reflected BSDEs with doubly interconnected barriers associated with* $((f^{ij})_{(i,j)\in\Gamma}, (\xi^{ij})_{(i,j)\in\Gamma}, (\underline{g}_{ik})_{i,k\in\Gamma^1}, (\overline{g}^{j,\ell})_{j,\ell\in\Gamma^2})$*, if it satisfies the followings:* $\forall (i,j)\in\Gamma$*,*

$$\begin{cases} Y^{ij}\in\mathcal{S}^2, Z^{ij}\in\mathcal{H}^{2,d}, K^{ij,\pm}\in\mathcal{A}^2\,; \\ Y^{ij}_t=\xi^{ij}+\int_t^T f^{ij}(s,\omega,(Y^{kl}_s)_{(k,l)\in\Gamma^1\times\Gamma^2},Z^{ij}_s)ds- \\ \qquad\qquad \int_t^T Z^{ij}_s dB_s+K^{ij,+}_T-K^{ij,+}_t-(K^{ij,-}_T-K^{ij,-}_t),\ \forall t\le T; \\ L^{ij}_t\le Y^{ij}_t\le U^{ij}_t,\ \forall t\in[0,T]; \\ \int_0^T (Y^{ij}_t-L^{ij}_t)dK^{ij,+}_t=0 \quad \textit{and} \quad \int_0^T (U^{ij}_t-Y^{ij}_t)dK^{ij,-}_t=0, \end{cases} \tag{2.2}$$

where $L^{ij}_t := \max_{k\in(\Gamma^1)^{-i}}\{Y^{kj}_t-\underline{g}_{ik}(t)\}$ *and* $U^{ij}_t := \min_{l\in(\Gamma^2)^{-j}}\{Y^{il}_t+\overline{g}_{jl}(t)\}, \forall t\le T$.

3. Existence under the monotonicity condition [H5]

In this part we prove the existence of a solution for the system of reflected BSDEs (2.2) under Assumptions [H1]-[H5]. For this we first introduce penalization schemes which we analyze and show properties of the penalizing terms. Then by using the monotonicity assumption of the generator $f^{ij}(s,\vec{y},z)$, namely [H5], and comparison of the solutions we prove that the approximating schemes converge and their limits provide solutions of the system of reflected BSDEs with bilateral interconnected obstacles (2.2).

So let us consider the following sequence of BSDEs : $\forall m,n\in\mathbb{N}$, $(i,j)\in\Gamma$,

$$\begin{cases} Y^{ij,m,n}\in\mathcal{S}^2, Z^{ij,m,n}\in\mathcal{H}^{2,d}; \\ Y^{ij,m,n}_t=\xi^{ij}+\displaystyle\int_t^T f^{ij,m,n}\left(s,(Y^{kl,m,n}_s)_{(k,l)\in\Gamma^1\times\Gamma^2},Z^{ij,m,n}_s\right)ds-\int_t^T Z^{ij,m,n}_s dB_s,\ t\le T, \end{cases} \tag{3.1}$$

where

$$f^{ij,m,n}\left(t,(y^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2},z\right)=f^{ij}\left(t,\vec{y},z\right)+n\left\{y^{ij}_t-\max_{k\in(\Gamma^1)^{-i}}\left[y^{kj}_t-\underline{g}_{ik}(t)\right]\right\}^- \\ -m\left\{y^{ij}_t-\min_{l\in(\Gamma^2)^{-j}}\left[y^{il}_t+\overline{g}_{jl}(t)\right]\right\}^+$$

($x^+=x\vee 0$ and $x^-=(-x)\vee 0$, $\forall x\in\mathbb{R}$).

Since (3.1) is a standard BSDE without obstacles, thanks to the results by Pardoux-Peng [19], the solution exists and is unique. Moreover we have the following comparison result based on a paper by Hu-Peng [14] related to comparison of solutions of multi-dimensional BSDEs.

Proposition 3.1 ([3], pp.143)**.** *For any* $(i,j)\in\Gamma$, f^{ij} *satisfies [H1] and [H5],* ξ^{ij} *satisfies [H2] and* $(\underline{g}_{ik})_{i,k\in\Gamma^1}, (\overline{g}_{jl})_{j,l\in\Gamma^2}$ *satisfy [H3]-a), then for* $m,n\ge 0$*, we have*

$$\mathbb{P}-a.s.\quad Y^{ij,m+1,n}\le Y^{ij,m,n}\le Y^{ij,m,n+1}. \tag{3.2}$$

Next we are interested in discussing the limit of $Y^{ij,m,n}$ in $\mathcal{S}^2$ when n goes to $+\infty$ for fixed m. Some similar results are already discussed in [12], [11], [3], [15], etc. Here we apply the same method as in Hamadène et al. [3] to prove the convergence of $Y^{ij,m,n}$ in $\mathcal{S}^2$ as $n\to\infty$ and then we have:

Lemma 3.2. *a) For any* $(i,j)\in\Gamma^1\times\Gamma^2$*, the sequence* $(Y^{ij,m,n},Z^{ij,m,n})_{n\ge 0}$ *converges, as* n *tends to infinity, to* $(\bar{Y}^{ij,m},\bar{Z}^{ij,m})$ *in* $\mathcal{S}^2\times\mathcal{H}^{2,d}$ *;*

b) For any $(i,j) \in \Gamma^1 \times \Gamma^2$ and $m \geq 0$, let $\bar{K}^{ij,m,+}$ be the following limit in $\mathcal{S}^2$ (which exists, one can see [3] for more details):

$$\forall t \leq T,\ \overline{K}_t^{ij,m,+} := \lim_{n\to\infty} \int_0^t n \left\{ Y_s^{ij,m,n} - \max_{k\in(\Gamma^1)^{-i}} \left[Y_s^{kj,m,n} - \underline{g}_{ik}(s) \right] \right\}^- ds$$

Then the triples $(\bar{Y}^{ij,m}, \bar{Z}^{ij,m}, \bar{K}^{ij,m,+})_{(i,j)\in\Gamma}$ is the unique solution of the following system of RBSDEs with lower interconnected obstacles: For any $(i,j) \in \Gamma$ and $t \leq T$,

$$\begin{cases} \overline{Y}^{ij,m} \in \mathcal{S}^2, \overline{Z}^{ij,m} \in \mathcal{H}^2, \overline{K}^{ij,m,+} \in \mathcal{A}^2; \\ \overline{Y}_t^{ij,m} = \xi^{ij} + \int_t^T \overline{f}^{ij,m}\left(s, (\overline{Y}_s^{kl,m})_{(k,l)\in\Gamma}, \overline{Z}_s^{ij,m}\right) ds - \int_t^T \overline{Z}_s^{ij,m} dB_s + \overline{K}_T^{ij,m,+} - \overline{K}_t^{ij,m,+}; \\ \overline{Y}_t^{ij,m} \geq \max_{k\in(\Gamma^1)^{-i}} \left[\overline{Y}_t^{kj,m} - \underline{g}_{ik}(t) \right]; \\ \int_0^T \left\{ \overline{Y}_t^{ij,m} - \max_{k\in(\Gamma^1)^{-i}} [\overline{Y}_t^{kj,m} - \underline{g}_{ik}(t)] \right\} d\overline{K}_t^{ij,m,+} = 0 \end{cases} \tag{3.3}$$

where $\overline{f}^{ij,m}(s, (y^{kl})_{(k,l)\in\Gamma}, z) = f^{ij}\left(s, (y^{kl})_{(k,l)\in\Gamma}, z\right) - m\left(y^{ij} - \min_{l\in(\Gamma^2)^{-j}} \left[y^{il} + \overline{g}_{jl}(s)\right]\right)^+.$

c) For any $m \geq 0$ and $(i,j) \in \Gamma, \overline{Y}^{ij,m} \geq \overline{Y}^{ij,m+1}$.

Let us just point out that the function $(t, \omega, (y^{kl})_{(k,l)\in\Gamma}) \mapsto -m\left\{y^{ij} - \min_{l\in(\Gamma^2)^{-j}}[y^{il} + \overline{g}_{jl}(t)]\right\}^+$ enjoys the same properties as f^{ij} w.r.t $\vec{y}$, hence $\overline{f}^{ij,m}$ keeps the same monotonicity properties as f^{ij} displayed in [H1] and [H5]. Therefore to prove that $(\overline{Y}^{ij,m}, \overline{Z}^{ij,m}, \overline{K}^{ij,m,+})_{(i,j)\in\Gamma^1\times\Gamma^2}$ is the unique solution of the RBSDEs (3.3) can be performed in the same way as in Hamadène and Zhang [12], we then omit the proof. □

Next, we introduce another equivalent approximating scheme defined as follows : for $m \geq 0$, let $(Y^{ij,m}, Z^{ij,m}, K^{ij,m,+})_{(i,j)\in\Gamma}$ be the unique solution of the following system of RBSDEs with lower interconnected obstacle: $\forall (i,j) \in \Gamma$,

$$\begin{cases} Y^{ij,m} \in \mathcal{S}^2, Z^{ij,m} \in \mathcal{H}^2, K^{ij,m,+} \in \mathcal{A}^2; \\ Y_t^{ij,m} = \xi^{ij} + \int_t^T f^{ij,m}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m}) ds - \int_t^T Z_s^{ij,m} dB_s + K_T^{ij,m,+} - K_t^{ij,m,+},\ t \leq T; \\ Y_t^{ij,m} \geq \max_{k\in(\Gamma^1)^{-i}} \left(Y_t^{kj,m} - \underline{g}_{ik}(t)\right),\ t \leq T; \\ \int_0^T \left[Y_t^{ij,m} - \max_{k\in(\Gamma^1)^{-i}} (Y_t^{kj,m} - \underline{g}_{ik}(t)) \right] dK_t^{ij,m,+} = 0 \end{cases} \tag{3.4}$$

where $f^{ij,m}(t, \overrightarrow{y}, z) := f^{ij}(t, \overrightarrow{y}, z) - m \sum_{l\in(\Gamma^2)^{-j}} \left(y^{ij} - y^{il} - \overline{g}_{il}(t)\right)^+$.

To proceed we are going to analyse the properties of this scheme (3.4) and its relationship with system (3.3) as well.

First note that for any $(i,j) \in \Gamma$, the sequence $(f^{ij,m})_{m\geq 0}$ is non decreasing w.r.t. m, since for all $m \geq 0$,

$$f^{ij,m}(t, \vec{y}, z) - f^{ij,m+1}(t, \vec{y}, z) = \sum_{l\in(\Gamma^2)^{-j}} \left(y^{ij} - y^{il} - \overline{g}_{il}(.)\right)^+ \geq 0.$$

Therefore by applying comparison theorem of systems of reflected BSDEs (see [11]) we obtain

$$\forall m \geq 0, (i,j) \in \Gamma^1 \times \Gamma^2, \quad Y^{ij,m} \geq Y^{ij,m+1} \tag{3.5}$$

i.e. $(Y^{ij,m})_{m\geq 0}$ is a non increasing sequence. Besides the following inequalities hold:

$$\overline{f}^{ij,|\Gamma^2|m} = f^{ij}(t,\overrightarrow{y},z) - |\Gamma^2|m\left\{y^{ij} - \min_{l\in(\Gamma^2)^{-j}}[y^{il}+\overline{g}_{jl}(t)]\right\}^+ \leq f^{ij,m} \leq \overline{f}^{ij,m}$$

where $|\Gamma^2|$ is the cardinal of Γ^2. Therefore once more by the comparison result of solutions of systems we have

$$\forall m\geq 0, (i,j)\in\Gamma^1\times\Gamma^2,\ \overline{Y}^{ij,|\Gamma^2|m} \leq Y^{ij,m} \leq \overline{Y}^{ij,m}. \tag{3.6}$$

Consequently, as the sequences $(Y^{ij,m})_{m\geq 0}$ and $(\overline{Y}^{ij,m})_{m\geq 0}$ are decreasing then if one of them converges then is so the other one to the same limit.

Finally we have the following estimate of the penalization term in (3.4). This estimate plays a crucial role in the proof of existence of the solution of (2.2).

Proposition 3.3. *For any $(i,j)\in\Gamma$, $\forall t\leq T$,*

$$m^2\mathbb{E}\left[\sum_{l\in\Gamma^2-\{j\}}\{(Y_t^{ij,m}-Y_t^{il,m}-\overline{g}_{jl}(t))^+\}^2\right] \leq C \tag{3.7}$$

where the constant C is independent of m.

Proof. First let us show that there exists a constant C independent of m such that for any $(i,j)\in\Gamma$,

$$\mathbb{E}\left[\sup_{s\leq T}|Y_s^{ij,m}|^2\right] \leq C. \tag{3.8}$$

Actually taking into account of (3.6), it is enough to show that $\bar{Y}^{ij,m}$ satisfies the same estimate. But from (3.2) we have

$$\mathbb{P}-\text{a.s.}\quad \tilde{Y}^{ij} \leq Y^{ij,m,0} \leq Y^{ij,m,n} \tag{3.9}$$

and the sequences $(Y^{ij,m,0})_{m\geq 0}$, $(i,j)\in\Gamma$, converge in $\mathcal{S}^2$ respectively to $\tilde{Y}^{ij}$ (one can see [3], Prop.3.3, pp.149, for more details) where $(\tilde{Y}^{ij},\tilde{Z}^{ij},\tilde{K}^{ij})_{(i,j)\in\Gamma}$ is the unique solution of the system of reflected BSDEs wih interconnected upper obstacles associated with $\big((f^{ij})_{(i,j)\in\Gamma},(\xi^{ij})_{(i,j)\in\Gamma},(\overline{g}_{jl})_{j,l\in\Gamma^2}\big)$. Now the claim follows since $\bar{Y}^{ij,m}\overset{\mathcal{S}^2}{=}\lim_n Y^{ij,m,n}$ and $Y^{ij,m+1}\leq Y^{ij,m}$.

Next in order to prove the boundedness of the penalized part of (3.4), we rely on the link between solutions of systems of reflected BSDEs with lower interconnected obstacles and optimal stochastic switching, which is well studied in the literature (see e.g. [1, 8, 11, 12, 15] etc). For this purpose, we set $u:=(\sigma_n,\delta_n)_{n\geq 0}$ an admissible strategy of switching, i.e., $(\sigma_n)_{n\geq 0}$ is an increasing sequence of stopping times such that $\mathbb{P}[\sigma_n<T,\forall n\geq 0]=0$, δ_n is Γ^1-valued and $\mathcal{F}_{\sigma_n}-$measurable random variable. Next when u is implemented, we set the cumulative switching cost $A_t^u := \sum_{n\geq 1}\underline{g}_{\delta_{n-1}\delta_n}(\sigma_n)\mathbb{1}_{(\sigma_n\leq t)}$ for $t<T$ and $A_T^u:=\lim_{t\to T}A_t^u$. On the other hand, for $t\leq T$, we set $a_t:=\delta_0\mathbb{1}_{(\sigma_0)}(t)+\sum_{n\geq 1}\delta_{n-1}\mathbb{1}_{(\sigma_{n-1},\sigma_n]}(t)$ which stands for the indicator of the mode in which the system under switching is at time t. Note that a is in bijection with the strategy u. Finally denote by $\mathcal{A}_t^i$ ($t\in[0,T]$ and $i\in\Gamma^1$) the following set:

$$\mathcal{A}_t^i := \{u=(\sigma_n,\delta_n)_{n\geq 0}\text{ admissible strategy such that }\sigma_0=t,\delta_0=i\text{ and }\mathbb{E}\left[(A_T^u)^2\right]<\infty\}.$$

Next for $j \in \Gamma^2$ and $a \in \mathcal{A}_t^i$, let $(U^{aj,m}, V^{aj,m})$ be the unique solution of the following BSDE which is not of standard form since A^a is only rcll: $\forall t \leq T$,

$$\begin{cases} U^{aj,m} \text{ is rcll, } \mathbb{E}\left[\sup_{t\leq T} |U_t^{aj,m}|^2\right] < \infty \text{ and } V^{aj,m} \in \mathcal{H}^{2,d}; \\ U_t^{aj,m} = \xi^{a_T j} + \int_t^T \mathbb{1}_{(s\geq\sigma_0)} \underline{\mathrm{f}}^{aj,m}\left(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, V_s^{aj,m}\right) ds - \int_t^T V_s^{aj,m} dB_s + A_T^a - A_t^a. \end{cases} \tag{3.10}$$

where for any $s \leq T$, $\underline{\mathrm{f}}^{aj,m}$ is defined by:

$$\begin{aligned} \underline{\mathrm{f}}^{aj,m}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, z) = \sum_{n\geq 1} \Bigg(\sum_{q\in\Gamma^1} \{ & f^{qj}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, z) \\ & - m \sum_{l\in(\Gamma^2)^{-j}} (Y_s^{qj,m} - Y_s^{ql,m} - \overline{g}_{jl}(t))^+ \} 1_{\{\delta_{n-1}=q\}} \Bigg) 1_{\{\sigma_{n-1}\leq s<\sigma_n\}} \end{aligned} \tag{3.11}$$

i.e. $\underline{\mathrm{f}}^{aj,m}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, z) = f^{qj,m}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, z)$ if at time s, $a(s) = q$. Let us notice that the arguments of $\underline{\mathrm{f}}^{aj,m}$ are s, ω and z since $(Y_s^{kl,m})_{(k,l)\in\Gamma}$ is already fixed. Then the following representation holds true (see e.g.[11]): $\forall t \in [0, T]$,

$$Y_t^{ij,m} = \operatorname*{ess\,sup}_{a\in\mathcal{A}_t^i} \left(U_t^{aj,m} - A_t^a \right) \tag{3.12}$$

since, mainly, the switching costs verify the non free loop property (4.2).

Indeed let $(\underline{Y}^{ij,m}, \underline{Z}^{ij,m}, \underline{K}^{ij,m})_{(i,j)\in\Gamma}$ be the unique solution of the following system:

$$\begin{cases} \underline{Y}^{ij,m} \in \mathcal{S}^2, \underline{Z}^{ij,m} \in \mathcal{H}^2, \underline{K}^{ij,m,+} \in \mathcal{A}^2; \\ \underline{Y}_t^{ij,m} = \xi^{ij} + \int_t^T \left\{ f^{ij}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, \underline{Z}_s^{ij,m}) - m \sum_{l\in(\Gamma^2)^{-j}} (\underline{Y}_s^{ij,m} - \underline{Y}_s^{il,m} - \overline{g}_{il}(s))^+ \right\} ds \\ \qquad - \int_t^T \underline{Z}_s^{ij,m} dB_s + \underline{K}_T^{ij,m,+} - \underline{K}_t^{ij,m,+}, \ t \leq T; \\ \underline{Y}_t^{ij,m} \geq \max_{k\in(\Gamma^1)^{-i}} \left(\underline{Y}_t^{kj,m} - \underline{g}_{ik}(t) \right), \ t \leq T; \\ \int_0^T \left[\underline{Y}_t^{ij,m} - \max_{k\in(\Gamma^1)^{-i}} \left(\underline{Y}_t^{kj,m} - \underline{g}_{ik}(t) \right) \right] d\underline{K}_t^{ij,m,+} = 0. \end{cases} \tag{3.13}$$

Therefore (see e.g.[11]): $\forall t \in [0, T]$,

$$\underline{Y}_t^{ij,m} = \operatorname*{ess\,sup}_{a\in\mathcal{A}_t^i} (U_t^{aj,m} - A_t^a). \tag{3.14}$$

But $(Y^{ij,m}, Z^{ij,m}, K^{ij,m})_{(i,j)\in\Gamma}$ is also solution of (3.13), then by uniqueness of the solution of system (3.13) we have $Y^{ij,m} = \underline{Y}^{ij,m}$ which combined with (3.14) implies (3.12).

Next as a consequence of (3.12) we have: For any $t \in [0, T]$, $i \in \Gamma^1$ and $j, l \in \Gamma^2$,

$$\left(Y_t^{ij,m} - Y_t^{il,m} - \overline{g}_{jl}(t) \right)^+ \leq \operatorname*{ess\,sup}_{a\in\mathcal{A}_t^i} \left(U_t^{aj,m} - U_t^{al,m} - \overline{g}_{jl}(t) \right)^+. \tag{3.15}$$

Now for $t \leq T$, let us set $W_t^{a,jl,m} := U_t^{aj,m} - U_t^{al,m} - \overline{g}_{jl}(t)$, $W_t^{a,jl,m,+} := (U_t^{aj,m} - U_t^{al,m} - \overline{g}_{jl}(t))^+$ and let θ be a real constant which will be chosen appropriately later. Then applying Itô-Tanaka's

formula with $e^{-\theta t}W_t^{a,jl,m,+}$ yields (note that $W_T^{a,jl,m,+}=0$ by (H2)): $\forall t\le T$,

$$
\begin{aligned}
&e^{-\theta t}W_t^{a,jl,m,+}+\frac{1}{2}\int_t^T e^{-\theta s}dL_s^w\\
&=\theta\int_t^T e^{-\theta s}W_s^{a,jl,m,+}ds\\
&+\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}\left\{\underline{\mathrm{f}}^{aj}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{aj,m})-\underline{\mathrm{f}}^{al}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{al,m})+\overline{b}_{jl}(s)\right\}ds\\
&-\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}\left(V_s^{aj,m}-V_s^{al,m}-\overline{\sigma}_{jl}(s)\right)dB_s\\
&-m\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}\left\{\sum_{k\in(\Gamma^2)^{-j}}W_s^{a,jk,m,+}-\sum_{k\in(\Gamma^2)^{-l}}W_s^{a,lk,m,+}\right\}ds
\end{aligned}
\tag{3.16}
$$

where L^w is the local time of $W^{a,jl,m,+}$ at 0 and $\underline{\mathrm{f}}^{aj}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},z):=\underline{\mathrm{f}}^{aj,0}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},z)$ (see (3.11)). Next let us focus on the last term of the right side of (3.16): $\forall t\le T$

$$
\begin{aligned}
&-m\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}\left\{\sum_{k\in(\Gamma^2)^{-j}}W_s^{a,jk,m,+}-\sum_{k\in\Gamma^2-\{l\}}W_s^{a,lk,m,+}\right\}ds\\
&=m\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}\left\{W_s^{a,lj,m,+}-W_s^{a,jl,m,+}+\sum_{k\in\Gamma^2-\{j,l\}}(W_s^{a,lk,m,+}-W_s^{a,jk,m,+})\right\}ds.
\end{aligned}
\tag{3.17}
$$

Note that $\mathbb{1}_{(W_s^{a,jl,m}>0)}W_s^{a,lj,m,+}=0$ since $\{W_s^{a,jl,m}>0\}\cap\{W_s^{a,lj,m}>0\}=\emptyset$ as $\bar{g}_{jl}\ge 0$. Next by applying the inequality $a^+-b^+\le(a-b)^+$ we have: $\forall s\le T$

$$
\mathbb{1}_{(W_s^{a,jl,m}>0)}\sum_{k\in\Gamma^2-\{j,l\}}\left(W_s^{a,lk,m,+}-W_s^{a,jk,m,+}\right)\le\mathbb{1}_{(W_s^{a,jl,m}>0)}\sum_{k\in\Gamma^2-\{j,l\}}\left(U_s^{al,m}-\overline{g}_{lk}(s)-U_s^{aj,m}+\overline{g}_{jk}(s)\right)^+.
$$

Using the fact that $\overline{g}_{jl}(s)+\overline{g}_{lk}(s)>\overline{g}_{jk}(s)$, by Assumption [H3]-(a),(ii), we deduce that

$$
W_s^{a,jl,m}<U_s^{aj,m}-U_s^{al,m}+\overline{g}_{lk}(s)-\overline{g}_{jk}(s)
$$

and then

$$
\begin{aligned}
0&\le\mathbb{1}_{(W_s^{a,jl,m}>0)}\sum_{k\in\Gamma^2-\{j,l\}}(U_s^{al,m}-\overline{g}_{lk}(s)-U_s^{aj,m}+\overline{g}_{jk}(s))^+\\
&\le\sum_{k\in\Gamma^2-\{j,l\}}\mathbb{1}_{(U_s^{aj,m}-U_s^{al,m}+\overline{g}_{lk}(s)-\overline{g}_{jk}(s)>0)}(U_s^{al,m}-\overline{g}_{lk}(s)-U_s^{aj,m}+\overline{g}_{jk}(s))^+\\
&=0.
\end{aligned}
$$

Now going back to (3.17) we obtain: $\forall t\le T$,

$$
-m\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}\left\{\sum_{k\in(\Gamma^2)^{-j}}W_s^{a,jk,m,+}-\sum_{k\in(\Gamma^2)^{-l}}W_s^{a,lk,m,+}\right\}ds
$$

$$\leq -m\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}W_s^{a,jl,m,+}ds \tag{3.18}$$

and consequently from (3.16) we have: $\forall t \leq T$,

$$\begin{aligned}
&e^{-\theta t}W_t^{a,jl,m,+} + m\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}W_s^{a,jl,m,+}ds + \frac{1}{2}\int_t^T e^{-\theta s}dL_s^w \\
&\quad \leq -\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}(V_s^{aj,m} - V_s^{al,m} - \overline{\sigma}_{jl}(s))dB_s + \theta\int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}W_s^{a,jl,m,+}ds \\
&\qquad + \int_t^T \mathbb{1}_{(W_s^{a,jl,m}>0)}e^{-\theta s}\left\{\underline{\mathrm{f}}^{aj}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{aj,m}) - \underline{\mathrm{f}}^{al}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{al,m}) + \bar{b}_{jl}(s)\right\}ds.
\end{aligned} \tag{3.19}$$

Next by taking $\theta = m$, recall that [H1] implies the boundedness of $(f^{ij}(t,\vec{y},z))_{(i,j)\in\Gamma}$ by $|\vec{y}|$ and [H4] represents $(\bar{g}_{jl})_{jl\in\Gamma^2}$ as Itô process, hence by taking the conditional expectation we deduce: $\forall t \leq T$,

$$\begin{aligned}
W_t^{a,jl,m,+} &\leq \mathbb{E}\left[\int_t^T e^{-m(s-t)}|\underline{\mathrm{f}}^{aj}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{aj,m}) - \underline{\mathrm{f}}^{al}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},V_s^{al,m}) + \bar{b}_{jl}(s)|ds\middle|\mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[C\left\{1 + \sup_{s\leq T}|\eta_s| + \sum_{(k,l)\in\Gamma}\sup_{s\leq T}|Y_s^{kl,m}| + \sup_{s\leq T}|\bar{b}_{jl}(s)|\right\}\int_t^T e^{-m(s-t)}ds\middle|\mathcal{F}_t\right] \\
&= \frac{1}{m}(1-e^{-m(T-t)})\mathbb{E}\left[C\left\{1 + \sup_{s\leq T}|\eta_s| + \sum_{(k,l)\in\Gamma}\sup_{s\leq T}|Y_s^{kl,m}| + \sup_{s\leq T}|\bar{b}_{jl}(s)|\right\}\middle|\mathcal{F}_t\right].
\end{aligned}$$

Now by (3.15), we get

$$\forall t \leq T,\ m(Y_t^{ij,m} - Y_t^{il,m} - \overline{g}_{jl}(t))^+ \leq C\mathbb{E}\left[\left\{1 + \sup_{s\leq T}|\eta_s| + \sum_{(k,l)\in\Gamma}\sup_{s\leq T}|Y_s^{kl,m}| + \sup_{s\leq T}|\bar{b}_{jl}(s)|\right\}\middle|\mathcal{F}_t\right]$$

and then squarring, using conditional Jensen's inequality and finally taking expectation to obtain: $\forall t \leq T$,

$$m^2\mathbb{E}\left[\left\{(Y_t^{ij,m} - Y_t^{il,m} - \overline{g}_{jl}(t))^+\right\}^2\right] \leq C\mathbb{E}\left[1 + \sup_{s\leq T}|\eta_s|^2 + \sum_{(k,l)\in\Gamma}\sup_{s\leq T}|Y_s^{kl,m}|^2 + \sup_{s\leq T}|\bar{b}_{jl}(s)|^2\right]$$

which implies the desired result since the processes η and $\bar{b}_{jl}$ are uniformly square integrable and by estimate (3.8). □

Next we are going to show that $K^{ij,m,+}$ is absolutely continuous w.r.t time and its density $(\frac{dK_s^{ij,m,+}}{ds})_{s\leq T}$ belongs to $\mathcal{H}^{2,1}$ uniformly in m.

Proposition 3.4. *For any $m \geq 0$ and $(i,j) \in \Gamma$, there exists a $\mathcal{P}$-measurable process $(\alpha_t^{ij,m})_{t\leq T}$ such that for any $t \leq T$,*

$$K_t^{ij,m,+} = \textstyle\int_0^t \alpha_s^{ij,m}ds.$$

Moreover there exists a constant C independent of m such that $\mathbb{E}\left[\int_0^T |\alpha_s^{ij,m}|^2ds\right] \leq C$.

Proof. Let us consider the following system of BSDEs: for any $(i,j)\in\Gamma$,

$$\begin{cases} \tilde{Y}^{ij,m,n}\in\mathcal{S}^2, \tilde{Z}^{ij,m,n}\in\mathcal{H}^{2,d}\ ; \\ \tilde{Y}^{ij,m,n}_s=\xi^{ij}+\int_t^T\Big\{f^{ij}(s,(Y^{kl,m}_s)_{(k,l)\in\Gamma},Z^{ij,m}_s)-m\sum_{l\neq j}(Y^{ij,m}_s-Y^{il,m}_s-\overline{g}_{jl}(s))^+ \\ \qquad\qquad +n\sum_{k\in(\Gamma^1)^{-i}}(\tilde{Y}^{ij,m,n}_s-\tilde{Y}^{kj,m,n}_s+\underline{g}_{ik}(s))^-\Big\}ds-\int_t^T\tilde{Z}^{ij,m,n}_s dB_s,\ t\leq T. \end{cases} \tag{3.20}$$

For $(i,j)\in\Gamma$, $m\geq 0$ and $s\leq T$ let us set:

$$\Phi^{ij,m}(s)=f^{ij}(s,(Y^{kl,m}_s)_{(k,l)\in\Gamma},Z^{ij,m}_s)-m\sum_{l\neq j}(Y^{ij,m}_s-Y^{il,m}_s-\overline{g}_{jl}(s))^+.$$

First note that by [H1], (3.7) and (3.8), there exists a constant C independent of m such that

$$\mathbb{E}\left[\int_0^T|\Phi^{ij,m}(s)|^2ds\right]\leq C. \tag{3.21}$$

On the other hand the sequences $\Big(\tilde{Y}^{ij,m,n},\tilde{Z}^{ij,m,n},n\int_0^\cdot\sum_{k\in(\Gamma^1)^{-i}}(\tilde{Y}^{ij,m,n}_s-\tilde{Y}^{kj,m,n}_s+\underline{g}_{ik}(s))^-\}ds\Big)_{n\geq 0}$, $(i,j)\in\Gamma$, converge when n goes to $+\infty$ in $\mathcal{S}^2\times\mathcal{H}^{2,d}\times\mathcal{S}^2$ to $(\tilde{Y}^{ij,m},\tilde{Z}^{ij,m},\tilde{K}^{ij,m,+})$, $(i,j)\in\Gamma$, respectively. Moreover $(\tilde{Y}^{ij,m},\tilde{Z}^{ij,m},\tilde{K}^{ij,m,+})_{(i,j)\in\Gamma}$ (see e.g. [3] for more details) is solution of the following system: $\forall t\leq T$,

$$\begin{cases} \tilde{Y}^{ij,m}_t=\xi^{ij}+\int_t^T f^{ij,m}(s,(Y^{kl,m}_s)_{(k,l)\in\Gamma},Z^{ij,m}_s)ds-\int_t^T\tilde{Z}^{ij,m}_s dB_s+\tilde{K}^{ij,m,+}_T-\tilde{K}^{ij,m,+}_t\ ; \\ \tilde{Y}^{ij,m}_t\geq\max_{k\in(\Gamma^1)^{-i}}\Big(\tilde{Y}^{kj,m}_t-\underline{g}_{ik}(t)\Big)\ ; \\ \int_0^T\Big[\tilde{Y}^{ij,m}_t-\max_{k\in(\Gamma^1)^{-i}}(\tilde{Y}^{kj,m}_t-\underline{g}_{ik}(t))\Big]d\tilde{K}^{ij,m,+}_t=0. \end{cases} \tag{3.22}$$

As the solution of this latter is unique and by (3.4), $(Y^{ij,m},Z^{ij,m},K^{ij,m,+})_{(i,j)\in\Gamma}$ is also a solution then, $\tilde{Y}^{ij,m}=Y^{ij,m}$, $\tilde{Z}^{ij,m}=Z^{ij,m}$ and $\tilde{K}^{ij,m,+}=K^{ij,m,+}$ for any $(i,j)\in\Gamma$.

Next for $s\leq T$, $i,k\in\Gamma^1$ and $j\in\Gamma^2$, let us set

$$\rho^{ikj,m,n}_s:=(\tilde{Y}^{ij,m,n}_s-\tilde{Y}^{kj,m,n}_s+\underline{g}_{ik}(s))^-.$$

Note that by Assumption [H2], $\rho^{ikj,m,n}_T=0$. Now if $(X_s)_{s\leq T}$ is a continuous semimartingale then by the use of Itô-Tanaka formula (see e.g. [20], pp.231) we have that: $\forall t\leq T$,

$$(X^-_t)^2+\int_t^T 1_{\{X_s<0\}}d\langle X\rangle_s=(X^-_T)^2+2\int_t^T X^-_s dX_s.$$

Therefore for any $t\leq T$,

$$\begin{aligned} &(\rho^{ikj,m,n}_t)^2+\int_t^T\mathbb{1}_{\{\tilde{Y}^{ij,m,n}_s-\tilde{Y}^{kj,m,n}_s+\underline{g}_{ik}(s)<0\}}\left(\tilde{Z}^{ij,m,n}_s-\tilde{Z}^{kj,m,n}_s+\underline{\sigma}_{ik}(s)\right)^2ds \\ &=-2\int_t^T\mathbb{1}_{\{\tilde{Y}^{ij,m,n}_s-\tilde{Y}^{kj,m,n}_s+\underline{g}_{ik}(s)<0\}}\rho^{ikj,m,n}_s\left\{\Phi^{ij,m}(s)-\Phi^{kj,m}(s)-\underline{b}_{ik}(s)\right\}ds \\ &+2\int_t^T\mathbb{1}_{\{\tilde{Y}^{ij,m,n}_s-\tilde{Y}^{kj,m,n}_s+\underline{g}_{ik}(s)<0\}}\rho^{ikj,m,n}_s\left(\overline{Z}^{ij,m,n}_s-\overline{Z}^{kj,m,n}_s+\underline{\sigma}_{ik}(s)\right)dB_s \\ &-2n\int_t^T\mathbb{1}_{\{\tilde{Y}^{ij,m,n}_s-\tilde{Y}^{kj,m,n}_s+\underline{g}_{ik}(s)<0\}}\rho^{ikj,m,n}_s\left\{\sum_{l\in(\Gamma^1)^{-i}}\rho^{ilj,m,n}_s-\sum_{l\in(\Gamma^1)^{-k}}\rho^{klj,m,n}_s\right\}ds. \end{aligned} \tag{3.23}$$

We now focus on the last term of (3.23).

$$
\begin{aligned}
&-2n\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{kj,m,n}+\underline{g}_{ik}(s)<0\}}\rho_s^{ikj,m,n}\left\{\sum_{l\in(\Gamma^1)^{-i}}\rho_s^{ilj,m,n}-\sum_{l\in(\Gamma^1)^{-k}}\rho_s^{klj,m,n}\right\}ds\\
&=-2n\int_t^T 1_{\{\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{kj,m,n}+\underline{g}_{ik}(s)<0\}}(\rho_s^{ikj,m,n})^2ds+2n\int_t^T 1_{\{\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{kj,m,n}+\underline{g}_{ik}(s)<0\}}\underbrace{\rho_s^{ikj,m,n}\rho_s^{kij,m,n}}_{=0}ds\\
&\quad+2n\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{kj,m,n}+\underline{g}_{ik}(s)<0\}}\rho_s^{ikj,m,n}\sum_{l\in\Gamma^1-\{i,k\}}\left(-\rho_s^{ilj,m,n}+\rho_s^{klj,m,n}\right)ds
\end{aligned}
\tag{3.24}
$$

since by positiveness of $\underline{g}_{ki}$ and $\underline{g}_{ik}$, $\{\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{kj,m,n}+\underline{g}_{ik}(s)<0\}\cap\{\tilde{Y}_s^{kj,m,n}-\tilde{Y}_s^{ij,m,n}+\underline{g}_{ki}(s)<0\}=\emptyset$. Next by applying the inequality $a^- - b^- \le (a-b)^-$ we have

$$
\begin{aligned}
&\rho_s^{ikj,m,n}\sum_{l\in\Gamma^1-\{i,k\}}\left(\rho_s^{klj,m,n}-\rho_s^{ilj,m,n}\right)\\
&=\rho_s^{ikj,m,n}\sum_{l\in\Gamma^1-\{i,k\}}\left\{\tilde{Y}_s^{kj,m,n}-\tilde{Y}_s^{lj,m,n}+\underline{g}_{kl}(s))^- - (\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{lj,m,n}+\underline{g}_{il}(s))^-\right\}\\
&\le\rho_s^{ikj,m,n}\sum_{l\in\Gamma^1-\{i,k\}}\left(\tilde{Y}_s^{kj,m,n}-\tilde{Y}_s^{ij,m,n}+\underline{g}_{kl}(s)-\underline{g}_{il}(s)\right)^-\\
&=\mathbb{1}_{\{\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{kj,m,n}+\underline{g}_{ik}(s)<0\}}\rho_s^{ikj,m,n}\sum_{l\in\Gamma^1-\{i,k\}}\left(\tilde{Y}_s^{kj,m,n}-\tilde{Y}_s^{ij,m,n}+\underline{g}_{kl}(s)-\underline{g}_{il}(s)\right)^-=0
\end{aligned}
$$

since by Assumption [H3]-(a),(ii), for any $l\in\Gamma^1-\{i,k\}$, $\mathbb{1}_{\{\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{kj,m,n}+\underline{g}_{ik}(s)<0\}}(\tilde{Y}_s^{kj,m,n}-\tilde{Y}_s^{ij,m,n}+\underline{g}_{kl}(s)-\underline{g}_{il}(s))^-=0$. We then deduce from (3.23) that, after taking expectation,

$$
\begin{aligned}
&2n\mathbb{E}\left[\int_t^T \mathbb{1}_{\{\tilde{Y}_s^{ij,m,n}-\tilde{Y}_s^{kj,m,n}+\underline{g}_{ik}(s)<0\}}(\rho_s^{ikj,m,n})^2ds\right]=2n\mathbb{E}\left[\int_t^T(\rho_s^{ikj,m,n})^2ds\right]\\
&\le 2\mathbb{E}\left[\int_t^T\rho_s^{ikj,m,n}|\Phi^{ij,m}(s)-\Phi^{kj,m}(s)-\underline{b}_{ik}(s)|ds\right]\\
&\le n\mathbb{E}\left[\int_t^T(\rho_s^{ikj,m,n})^2ds\right]+\frac{1}{n}\mathbb{E}\left[\int_t^T|\Phi^{ij,m}(s)-\Phi^{kj,m}(s)-\underline{b}_{ik}(s)|^2ds\right]
\end{aligned}
\tag{3.25}
$$

which implies that

$$
n^2\mathbb{E}\left[\int_t^T(\rho_s^{ikj,m,n})^2ds\right]\le C\mathbb{E}\left[\int_t^T\left\{|\Phi^{ij,m}(s)|^2+|\Phi^{kj,m}(s)|^2+|\underline{b}_{ik}(s)|^2\right\}ds\right]. \tag{3.26}
$$

Then by (3.21) and Assumption [H4] on $\underline{b}_{ik}$ we obtain:

$$
n^2\mathbb{E}\left[\int_0^T(\rho_s^{ikj,m,n})^2ds\right]\le\underline{\mathrm{C}}\text{ and }n^2\mathbb{E}\left[\int_0^T(\sum_{k\neq i}\rho_s^{ikj,m,n})^2ds\right]\le\underline{\mathrm{C}}
$$

for some constant $\underline{\mathrm{C}}$ independent of n,m. It implies that for any $(i,j)\in\Gamma$, the sequence $((\alpha_s^{ij,m,n}:=n\sum_{k\in\Gamma^1-\{i\}}\rho_s^{ikj,m,n})_{s\le T})_{n\ge 0}$ is bounded in $\mathcal{H}^{2,1}$. Thus one can substract a subsequence (still

denoted by n) such that for any $(i,j) \in \Gamma$, $((\alpha_s^{ij,m,n})_{s\leq T})_{n\geq 0}$ converges weakly in $\mathcal{H}^{2,1}$ to some $\mathcal{P}$-measurable process $(\alpha_t^{ij,m})_{t\leq T}$ which moreover satisfy: For any $(i,j) \in \Gamma$ and $m \geq 0$,

$$\mathbb{E}\left[\int_0^T (\alpha_s^{ij,m})^2 ds\right] \leq \underline{C}. \tag{3.27}$$

Additionnally for any $(i,j) \in \Gamma$ and any stopping time τ it holds:

$$K_\tau^{ij,m,+} = \int_0^\tau \alpha^{ij,m}(s)ds. \tag{3.28}$$

Actually this is due to the fact that the sequence $(\int_0^\tau \alpha_s^{ij,m,n} ds)_{n\geq 0}$ is also weakly convergent in $L^2_{\mathbb{R}}(\Omega, \mathcal{F}_T, d\mathbb{P})$ and since, as pointed out previously, $K^{ij,m,+} \overset{\mathcal{S}^2}{=} \lim_{n\to\infty} \int_0^\cdot \alpha_s^{ij,m,n} ds$.

Indeed let us show the weak convergence of $(\int_0^\tau \alpha_s^{ij,m,n} ds)_{n\geq 0}$. Let ζ be a random variable of $L^2_{\mathbb{R}}(\Omega, \mathcal{F}_T, d\mathbb{P})$. By the representation property there exists a $\mathcal{P}$-mesurable process $(\bar{\eta}_t)_{t\leq T}$ of $\mathcal{H}^{2,d}$ such that:

$$\forall t \leq T,\ \mathbb{E}[\zeta|\mathcal{F}_t] = \mathbb{E}[\zeta] + \int_0^t \bar{\eta}_s dB_s.$$

Next by Itô's formula we have

$$\mathbb{E}\left[\zeta \int_0^\tau \alpha_s^{ij,m,n} ds\right] = \mathbb{E}\left[\mathbb{E}[\zeta|\mathcal{F}_\tau] \int_0^\tau \alpha_s^{ij,m,n} ds\right] = \mathbb{E}\left[\int_0^\tau \mathbb{E}[\zeta|\mathcal{F}_s]\alpha_s^{ij,m,n} ds\right]$$

since by Burkholder et al.'s inequality ([21], pp.160) $(\int_0^t (\int_0^s \alpha_r^{ij,m,n} dr)\bar{\eta}_s dB_s)_{t\leq T}$ is a martingale due to $\mathbb{E}[\{\int_0^T (\int_0^s \alpha_r^{ij,m,n} dr)^2 |\bar{\eta}_s|^2 ds\}^{\frac{1}{2}}] < \infty$. As the sequence $((\alpha_s^{ij,m,n})_{s\leq T})_{n\geq 0}$ converges weakly in $\mathcal{H}^{2,1}$ to $\alpha^{ij,m}$ then

$$\mathbb{E}\left[\int_0^\tau \mathbb{E}[\zeta|\mathcal{F}_s]\alpha_s^{ij,m,n} ds\right] \longrightarrow_{n\to\infty} \mathbb{E}\left[\int_0^\tau \mathbb{E}[\zeta|\mathcal{F}_s]\alpha_s^{ij,m} ds\right] = \mathbb{E}\left[\zeta \int_0^\tau \alpha_s^{ij,m} ds\right]$$

which is the claim. □

Proposition 3.5. *There exist continuous adapted processes $(Y^{ij})_{(i,j)\in\Gamma}$ and $\mathcal{P}$-measurable processes $(Z^{ij})_{(i,j)\in\Gamma}$, such that for $(i,j) \in \Gamma^1 \times \Gamma^2$:*

i) $(Y^{ij,m})_{m\geq 0}$ uniformly converges to Y^{ij} in $\mathcal{S}^2$.
ii) $(Z^{ij,m})_{m\geq 0}$ converges to Z^{ij} in $\mathcal{H}^{2,d}$.

Proof. First let us recall the process $(Y^{ij,m})_{(i,j)\in\Gamma}$ in (3.4). Next fix $(i,j) \in \Gamma$ and let Y^{ij} be the optional process such that

$$\mathbb{P}\text{-a.s}, \forall t \leq T,\ Y_t^{ij} = \lim_{m\to\infty} Y_t^{ij,m}$$

which exists since the sequence $(Y^{ij,m})_{m\geq 0}$ is decreasing (see (3.5)). On the other hand for any $m \geq 0$ we have: $\forall t \leq T$,

$$Y_t^{ij,m} = \xi^{ij} + \int_t^T f^{ij,m}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m})ds + \int_t^T \alpha^{ij,m}(s)ds - \int_t^T Z_s^{ij,m} dB_s.$$

Then using Itô formula with $(Y^{ij,m})^2$ and taking into account of (3.21)-(3.27), one deduces the existence of a constant C independent of m such that

$$\mathbb{E}\left[\int_0^T \left|Z_s^{ij,m}\right|^2 ds\right] \leq C. \tag{3.29}$$

Next, let $\{m\}$ be a sequence such that:

i) $\left(f^{ij}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},Z_s^{ij,m}))_{s\leq T}\right)_{m\geq 0}$ converges weakly in $\mathcal{H}^{2,1}$ to Φ^{ij} ;

ii) $\left(m\sum_{l\in\Gamma^2-\{j\}}\left(Y_s^{ij,m}-Y_s^{il,m}-\bar{g}_{jl}(s))^+\right)_{s\leq T}\right)_{m\geq 0}$ converges weakly to θ^{ij} is $\mathcal{H}^{2,1}$;

iii) $\left(\alpha^{ij,m}\right)_{m\geq 0}$ converges weakly to α^{ij} is $\mathcal{H}^{2,1}$;

iv) $\left(Z^{ij,m}\right)_{m\geq 0}$ converges weakly to Z^{ij} is $\mathcal{H}^{2,d}$.

This sequence exists thanks to Assumption [H1] on f^{ij} and (3.8), (3.7), (3.27) and finally (3.29). Next let τ be a stopping time. Then as in the proof of Proposition 3.4, the following weak convergences in $L^2(d\mathbb{P})$, as $m\to\infty$, hold true:

$$\begin{aligned}
&\text{a)}\ \int_0^\tau f^{ij}(s,(Y_s^{kl,m})_{(k,l)\in\Gamma},Z_s^{ij,m})ds \rightharpoonup \int_0^\tau \Phi^{ij}(s)ds,\\
&\text{b)}\ \int_0^\tau m\sum_{l\in\Gamma^2-\{j\}}(Y_s^{ij,m}-Y_s^{il,m}-\bar{g}_{jl}(s))^+ds \rightharpoonup \int_0^\tau \theta^{ij}(s)ds,\\
&\text{c)}\ \int_0^\tau \alpha^{ij,m}(s)ds \rightharpoonup \int_0^\tau \alpha^{ij}(s)ds,\\
&\text{d)}\ \int_0^\tau Z_s^{ij,m}dB_s \rightharpoonup \int_0^\tau Z_s^{ij}dB_s.
\end{aligned}$$

Therefore for any stopping time τ, we have:

$$Y_\tau^{ij}=Y_0^{ij}-\int_0^\tau \Phi^{ij}(s)ds+\int_0^\tau \theta^{ij}(s)ds-\int_0^\tau \alpha^{ij}(s)ds-\int_0^\tau Z_s^{ij}dB_s.$$

As Y^{ij} is an optional process and this equality holds for any stopping time then the processes of the left and right-hand side are indistinguishable which means that $\mathbb{P}-a.s.,\forall t\leq T$,

$$Y_t^{ij}=Y_0^{ij}-\int_0^t \Phi^{ij}(s)ds+\int_0^t \theta^{ij}(s)ds-\int_0^t \alpha^{ij}(s)ds-\int_0^t Z_s^{ij}dB_s \tag{3.30}$$

and the process Y^{ij} is continuous. Thus by Dini's Theorem the convergence of the sequence of $(Y^{ij,m})_{m\geq 0}$ to Y^{ij} holds in $\mathcal{S}^2$ i.e. $\lim_{m\to\infty}\mathbb{E}\left[\sup_{t\leq T}|Y_t^{ij,m}-Y_t^{ij}|^2\right]=0$.

Next once more by the use of Itô's formula with $(Y^{ij,m}-Y^{ij,n})^2$ and taking into account of (3.21)-(3.27) one deduces that $(Z^{ij,m})_{m\geq 0}$ is a Cauchy sequence in $\mathcal{H}^{2,d}$ and then $(Z^{ij,m})_{m\geq 0}$ converges strongly to Z^{ij} is $\mathcal{H}^{2,d}$. □

To proceed let us define for any $(i,j)\in\Gamma$, $t\leq T$,

$$K_t^{ij,-}=\int_0^t \theta_s^{ij}ds \text{ and } K_t^{ij,+}=\int_0^t \alpha_s^{ij}ds.$$

We then give the main result of this section:

Theorem 3.6. *The process* $(Y^{ij},Z^{ij},K^{ij,+},K^{ij,-})_{(i,j)\in\Gamma}$ *is a solution of the system of reflected BSDEs* (2.2).

Proof. First note that by (3.30) and since $Y_T^{ij}=\xi^{ij}$ then for any $(i,j)\in\Gamma$,

$$Y_\tau^{ij}=\xi^{ij}+\int_\tau^T \Phi^{ij}(s)ds-\int_\tau^T \theta^{ij}(s)ds+\int_\tau^T \alpha^{ij}(s)ds-\int_\tau^T Z_s^{ij}dB_s$$

Now recall the definition of Φ^{ij} and since the convergences of $(Y^{ij,m})_{m\geq 0}$ and $(Z^{ij,m})_{m\geq 0}$ hold in strong sense then

$$\Phi^{ij}(s) = f^{ij}\left(s, (Y_s^{kl})_{(k,l)\in\Gamma}, Z_s^{ij}\right),\ ds\otimes d\mathbb{P}$$

which implies that for any $(i,j)\in\Gamma$, $\mathbb{P}$-a.s. for any $t\leq T$,

$$Y_t^{ij} = \xi^{ij} + \int_t^T f^{ij}\left(s, (Y_s^{kl})_{(k,l)\in\Gamma}, Z_s^{ij}\right) ds + (K_T^{ij,+} - K_t^{ij,+}) - (K_T^{ij,-} - K_t^{ij,-}) - \int_t^T Z_s^{ij} dB_s.$$

Next from (3.4) we have

$$Y_t^{ij,m} = \xi^{ij} + \int_t^T f^{ij,m}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m})ds - \int_t^T Z_s^{ij,m} dB_s + K_T^{ij,m,+} - K_t^{ij,m,+}$$

which implies in taking expectation

$$\begin{aligned} &m\mathbb{E}\left[\int_0^T \sum_{\ell\in\Gamma-\{i\}} (Y_s^{ij,m} - Y_s^{i\ell,m} - \bar{g}_{j\ell}(s))^+\right] \\ &= \mathbb{E}\left[-Y_0^{ij,m} + \xi^{ij} + \int_0^T f^{ij}(s, (Y_s^{kl,m})_{(k,l)\in\Gamma}, Z_s^{ij,m})ds + K_T^{ij,m,+}\right]. \end{aligned} \tag{3.31}$$

Then by Assumption [H1], (3.8),(3.27) and (3.28), there exists a constant C such that

$$\mathbb{E}\left[\int_0^T \sum_{\ell\in\Gamma^2-\{j\}} (Y_s^{ij,m} - Y_s^{i\ell,m} - \bar{g}_{j\ell}(s))^+\right] \leq Cm^{-1} \tag{3.32}$$

which implies that, in taking the limit as $m\to\infty$, for any $(i,j)\in\Gamma$ and $s\leq T$, $Y_s^{ij}\leq Y_s^{i\ell} + \bar{g}_{j\ell}(s)$ for any $\ell\in\Gamma_2-\{j\}$. Then

$$\mathbb{P}-a.s., \forall s\leq T,\ Y_s^{ij} \leq \min_{\ell\in\Gamma^2-\{j\}} (Y_s^{i\ell} + \bar{g}_{j\ell}(s)).$$

Next

$$\mathbb{E}\left[\int_0^T \left(Y_s^{ij} - \min_{\ell\in\Gamma_2-\{j\}} (Y_s^{i\ell} + \bar{g}_{j\ell}(s))\right) dK_s^{ij,-}\right] = -\mathbb{E}\left[\int_0^T \left(Y_s^{ij} - \min_{\ell\in\Gamma_2-\{j\}} (Y_s^{i\ell} + \bar{g}_{j\ell}(s))\right)^- \alpha_s^{ij} ds\right] \tag{3.33}$$

$$= \lim_{m\to\infty} \mathbb{E}\left[\int_0^T \left(Y_s^{ij,m} - \min_{\ell\in\Gamma_2-\{j\}} (Y_s^{i\ell,m} + \bar{g}_{j\ell}(s))\right)^- \alpha_s^{ij,m} ds\right] = 0$$

since $(\alpha^{ij,m})_m$ is weakly convergent to α^{ij} and $(Y^{ij,m} - \min_{\ell\in\Gamma_2-\{j\}}(Y^{i\ell,m} + \bar{g}_{j\ell}))_m$ converges strongly in $\mathcal{S}^2$ to $Y^{ij} - \min_{\ell\in\Gamma_2-\{j\}}(Y^{i\ell} + \bar{g}_{j\ell}))^-$. As $\int_0^T (Y_s^{ij} - \min_{\ell\in\Gamma_2-\{j\}}(Y_s^{i\ell} + \bar{g}_{j\ell}(s)))dK_s^{ij,-} \leq 0$ then

$$\mathbb{P}-a.s., \int_0^T \left(Y_s^{ij} - \min_{\ell\in\Gamma_2-\{j\}} (Y_s^{i\ell} + \bar{g}_{j\ell}(s))\right) dK_s^{ij,-} = 0.$$

In the same way one can show that

$$\mathbb{P}-a.s., \int_0^T \left(Y_s^{ij} - \max_{k\in\Gamma_1-\{i\}} (Y_s^{kj} - \underline{g}_{kj}(s))\right) dK_s^{ij,+} = 0.$$

Thus the processes $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ is a solution of the system of reflected BSDEs (2.2). □

Remark 3.7.

(i) The constant C such that for any $(i,j) \in \Gamma$,

$$\mathbb{E}\left[\int_0^T (|\alpha_s^{ij}|^2 + |\theta_s^{ij}|^2)ds\right] \leq C$$

depends only on $(f^{ij})_{(i,j)\in\Gamma}$, $(\xi^{ij})_{(i,j)\in\Gamma}$, $(\underline{g}_{ik})_{i,k\in\Gamma^1}$ and $(\bar{g}_{jl})_{j,l\in\Gamma^2}$.

(ii) In our construction of the solution of (2.2) through the penalization scheme (3.4), we have penalized the upper barriers. Had we taken the dual scheme of (3.4) where, instead, the lower barriers are penalized, we would have obtained another solution $(\check{Y}^{ij}, \check{Z}^{ij}, \check{K}^{ij,\pm})_{(i,j)\in\Gamma}$ of system (2.2). Additionally we have $\check{Y}^{ij} \leq Y^{ij}$ for any $(i,j) \in \Gamma$.

(iii) The solutions of systems (2.2) which we have constructed are comparable. Actually let us consider $(\underline{f}^{1,ij})_{(i,j)\in\Gamma}$, $(\underline{\xi}^{1,ij})_{(i,j)\in\Gamma}$, $(\underline{g}^1_{ik})_{i,k\in\Gamma^1}$ and $(\bar{g}^1_{jl})_{j,l\in\Gamma^2}$ items which satisfy the same assumptions (H1)-(H5) repectively as $(f^{ij})_{(i,j)\in\Gamma}$, $(\xi^{ij})_{(i,j)\in\Gamma}$, $(\underline{g}_{ik})_{i,k\in\Gamma^1}$ and $(\bar{g}_{jl})_{j,l\in\Gamma^1}$. Let us denote by $(Y^{1,ij}, Z^{1,ij}, K^{1,ij,+}, K^{1,ij,-})_{(i,j)\in\Gamma}$ the solution of system (2.2) associated with $\{(\underline{f}^{1,ij})_{(i,j)\in\Gamma}, (\underline{\xi}^{1,ij})_{(i,j)\in\Gamma}, (\underline{g}^1_{ik})_{i,k\in\Gamma^1}, (\bar{g}^1_{jl})_{j,l\in\Gamma^2}\}$ (which exists by Theorem 3.6). Assume that for any:

a) $(i,j) \in \Gamma$, $f^{ij} \leq f^{1,ij}$ and $\xi^{ij} \leq \xi^{1,ij}$;
b) $i,k \in \Gamma^1$, $\underline{g}_{ik} \geq \underline{g}^1_{ik}$;
c) $j,l \in \Gamma^2$, $\bar{g}_{ik} \leq \bar{g}^1_{ik}$.

Then we have: For any $(i,j) \in \Gamma$,

$$\mathbb{P} - a.s., Y^{ij} \leq Y^{1,ij}.$$

This is actually a direct consequence of the constructions of Y^{ij} and $Y^{1,ij}$ since for any $(i,j) \in \Gamma$,

$$Y^{ij} = \lim_{m\to\infty} Y^{ij,m} \text{ and } Y^{1,ij} = \lim_{m\to\infty} Y^{1,ij,m}$$

where $(Y^{1,ij,m})_{(i,j)\in\Gamma}$ are defined in the same way as $(Y^{ij,m})_{(i,j)\in\Gamma}$ in (3.3) but with the items $\{(\underline{f}^{1,ij})_{(i,j)\in\Gamma}, (\underline{\xi}^{1,ij})_{(i,j)\in\Gamma}, (\underline{g}^1_{ik})_{i,k\in\Gamma^1}, (\bar{g}^1_{jl})_{j,l\in\Gamma^2}\}$. But by comparison ([11], pp.190 for more details) we have for any $(i,j) \in \Gamma$, $Y^{ij,m} \leq Y^{1,ij,m}$ which implies the result in taking the limit as $m \to \infty$. □

4. Existence and uniqueness without monotonicity

In this section, we focus on the second main result of this paper. Actually we are going to show that system of reflected BSDEs with inter-connected obstacles (2.2) has a unique solution without assuming the monotonicity Assumption [H5] on the functions $(f^{ij})_{(i,j)\in\Gamma}$. Meanwhile in this section we shall need the following assumptions:

[H3] b) The processes $(\underline{g}_{ik})_{i,k\in\Gamma^1}$ and $(\overline{g}_{j,\ell})_{j,\ell\in\Gamma^2}$ verify the non free loop property, that is to say, if $(i_k, j_k)_{k=1,2,\ldots,N}$ is a loop in Γ, i.e., $(i_N, j_N) = (i_1, j_1)$, $\mathbf{card}\{(i_k, j_k)_{k=1,2,\ldots,N}\} = N-1$ and for any $k = 1, 2, ..., N-1$, either $i_{k+1} = i_k$ (resp. $j_{k+1} = j_k$), we have:

$$\mathbb{P} - a.s., \ \forall t \leq T, \quad \sum_{k=1}^{N-1} G_{i_k j_k}(t) \neq 0 \tag{4.1}$$

where $\forall k = 1, ...N-1$, $G_{i_k j_k}(t) = -\underline{g}_{i_k i_{k+1}}(t)\mathbb{1}_{i_k \neq i_{k+1}} + \overline{g}_{j_k j_k}(t)\mathbb{1}_{j_k \neq j_{k+1}}$. This assumption makes sure that any instantaneous loop in the switching mode set $\Gamma^1 \times \Gamma^2$, of the players (or decision makers), is not free i.e. one of the controllers needs to pay something when the system is switched and comes back instantaneously to the initial mode. Note that (4.1) also implies: For any $(i_1, ..., i_N) \in (\Gamma^1)^N$ such that $i_N = i_1$ and $\mathbf{card}\{i_1, i_2, ..., i_N\} = N-1$,

$$\mathbb{P}\left[\sum_{k=1}^{N-1} \underline{g}_{i_k i_{k+1}}(t) = 0\right] = 0, \ \forall t \leq T, \tag{4.2}$$

and for any $(j_1, ..., j_N) \in (\Gamma^2)^N$ such that $j_N = j_1$ and $\mathbf{card}\{j_1, j_2, ..., j_N\} = N-1$,

$$\mathbb{P}\left[\sum_{k=1}^{N-1} \overline{g}_{j_k j_{k+1}}(t) = 0\right] = 0, \ \forall t \leq T. \tag{4.3}$$

[H6] For any $(i,j) \in \Gamma$, the function f^{ij} does not depend on z.

We highlight that in this section, the generator $(f^{ij})_{(i,j)\in\Gamma}$ is not monotonic any more, i.e. it does not verify [H5].

First let us temporirally assume that for any $(i,j) \in \Gamma$, the function f^{ij} does not depend on $(\vec{y}, z)$. Therefore by Theorem 3.6, there is a solution $(\underline{Y}^{ij}, \underline{Z}^{ij}, \underline{K}^{ij,\pm})_{(i,j)\in\Gamma}$ of the following system: $\forall (i,j) \in \Gamma$,

$$\begin{cases} \underline{Y}^{ij} \in \mathcal{S}^2, \underline{Z}^{ij} \in \mathcal{H}^{2,d}, \underline{K}^{ij,+} \in \mathcal{A}^2, \underline{K}^{ij,-} \in \mathcal{A}^2; \\ \underline{Y}^{ij}_t = \xi^{ij} + \int_t^T f^{ij}(s)ds - \int_t^T \underline{Z}^{ij}_s dB_s + \underline{K}^{ij,+}_T - \underline{K}^{ij,+}_t - (\underline{K}^{ij,-}_T - \underline{K}^{ij,-}_t), \ \forall t \leq T; \\ \underline{L}^{ij}_t \leq \underline{Y}^{ij}_t \leq \underline{U}^{ij}_t, \ \forall t \in [0,T]; \\ \int_0^T \left(\underline{Y}^{ij}_t - \underline{L}^{ij}_t\right) d\underline{K}^{ij,+}_t = 0 \text{ and } \int_0^T \left(\underline{U}^{ij}_t - \underline{Y}^{ij}_t\right) d\underline{K}^{ij,-}_t = 0. \end{cases} \tag{4.4}$$

where $\underline{L}^{ij}_t := \max_{k\in(\Gamma^1)^{-i}} \left\{\underline{Y}^{kj}_t - \underline{g}_{ik}(t)\right\}$ *and* $\underline{U}^{ij}_t := \min_{l\in(\Gamma^2)^{-j}} \left\{\underline{Y}^{il}_t + \overline{g}_{jl}(t)\right\}$, $t \leq T$.

As pointed out previously we are going to represent the process $\underline{Y}^{ij}$ as the value function of a zero-sum switching game which we describe briefly now.

Let us consider a system which has $\Lambda = |\Gamma^1 \times \Gamma^2|$ working modes indexed by $\Gamma^1 \times \Gamma^2$. It means that a working mode is a pair (i,j) such that $i \in \Gamma^1$ and $j \in \Gamma^2$. This system is controlled by two agents or players P1 and P2 by choosing their own appropriate working mode of the system and switch to another one when they make the decision to do so (e.g. according to profitability, etc.). The player P1 (resp. P2) chooses her modes in Γ^1 (resp. Γ^2). The features of the system is that when it works in mode (i,j) from time t to $t+dt$, it comes with a payoff which amounts to $f^{ij}(t)dt$ and which is a profit (resp. cost) for P1 (resp. P2). On the other hand when the player P1 (resp. P2) makes the decision at time t to switch from mode i (resp. j) to $k \in \Gamma^1 - \{i\}$ (resp. $l \in \Gamma^2 - \{j\}$), she pays an amount which equals to $\underline{g}_{ik}(t)$ (resp. $\overline{g}_{jl}(t)$). Therefore a switching control for P1 (resp. P2), denoted by u (resp. v) is a sequence of pairs $u := (\sigma_n, \delta_n)_{n\geq 0}$ (resp. $v := (\tau_n, \zeta_n)_{n\geq 0}$) such that: $\forall n \geq 0$,

i) σ_n is an $\mathbb{F}$-stopping time such that $\sigma_n \leq \sigma_{n+1}$ and δ_n is a r.v. with values in Γ^1 and $\mathcal{F}_{\sigma_n}$-measurable (resp. τ_n is an $\mathbb{F}$-stopping time such that $\tau_n \leq \tau_{n+1}$ and ζ_n is a r.v. with values in Γ^2 and $\mathcal{F}_{\tau_n}$-measurable) ;

ii) $\mathbb{P}[\sigma_n < T, \forall n \geq 0] = 0$ (resp. $\mathbb{P}[\tau_n < T, \forall n \geq 0] = 0$) ;

iii) Let us define the process A^u (resp. B^v) by

$$A_t^u := \sum_{n\geq 1} \underline{g}_{\delta_{n-1}\delta_n}(\sigma_n)\mathbb{1}_{(\sigma_n\leq t)} \text{ for } t<T \text{ and } A_T^u := \lim_{t\to T} A_t^u$$

(resp. $B_t^v := \sum_{n\geq 1} \overline{g}_{\zeta_{n-1}\zeta_n}(\tau_n)\mathbb{1}_{(\zeta_n\leq t)}$ for $t<T$ and $B_T^v := \lim_{t\to T} B_t^v$)
then $\mathbb{E}[|A_T^u|^2]<\infty$ (resp. $\mathbb{E}[|B_T^v|^2]<\infty$).

A control which satisfies the properties i)-iii) is called admissible.

Next let $\mathcal{A}_t^i$ (resp. $\mathcal{B}_t^j$) be the set of admissible controls $u := (\sigma_n,\delta_n)_{n\geq 0}$ (resp. $v := (\tau_n,\zeta_n)_{n\geq 0}$) for P1 (resp. P2) satisfying $\sigma_0 = t, \delta_0 = i$ (resp. $\tau_0 = t, \xi_0 = j$).

To proceed let $(u,v)\in \mathcal{A}_t^i\times\mathcal{B}_t^j$ be a pair of switching controls of the players. We define the coupling of (u,v) by $\gamma(u,v) = (\rho_n,\gamma_n)_{n\geq 0}$ as the modes under which the system is run along with time after t when P1 (resp. P2) implements u (resp. v). In our definition we give the priority of switching to player P1 in the case when both players make the decision to switch at the same time.

Precisely let:

i) $r_0 = s_0 = 1, r_1 = s_1 = 1$ and for $n\geq 2$,

$$r_n = r_{n-1} + \mathbb{1}_{(\sigma_{r_{n-1}}\leq\tau_{s_{n-1}})}, \quad s_n = s_{n-1} + \mathbb{1}_{(\tau_{s_{n-1}}<\sigma_{r_{n-1}})};$$

ii)

$$\forall n\geq 0, \rho_n = \sigma_{r_n}\wedge\tau_{s_n};$$

iii) $(\gamma_n := (\gamma_n^{(1)},\gamma_n^{(2)}))_{n\geq 0}$ is a sequence of $\Gamma-$valued random variables defined as follows: $\gamma_0 = (\delta_0,\zeta_0)$ and for all $n\geq 1$,

$$\gamma_n = \begin{cases} (\delta_{r_n},\gamma_{n-1}^{(2)}) & \text{if } \sigma_{r_n}\leq\tau_{s_n} \text{and } \sigma_{r_n}<T; \\ (\gamma_{n-1}^{(1)},\zeta_{s_n}) & \text{if } \tau_{s_n}<\sigma_{r_n}; \\ \gamma_{n-1} & \text{if } \tau_{s_n}=\sigma_{r_n}=T. \end{cases}$$

We associate with $\gamma(u,v)_t$ the following process $(\pi_s)_{s\in[t,T]}$ which indicates in which pair of modes the system is along with time: $\forall s\in[t,T]$,

$$\pi_s = \gamma_0\mathbb{1}_{[\rho_0,\rho_1]}(s) + \sum_{n\geq 1}\gamma_n\mathbb{1}_{(\rho_n,\rho_{n+1}]}(s)$$

where $(\rho_n,\rho_{n+1}] = \emptyset$ on $\{\rho_n = \rho_{n+1}\}$.

Finally when the player P1 (resp. P2) implements the control $u\in\mathcal{A}_t^i$ (resp. $v\in\mathcal{B}_t^j$), the payoff in-between, which is a reward for P1 and a cost for P2, is given by:

$$J_t^{ij}(\gamma(u,v)) = \mathbb{E}\left[\xi^{\pi_T} + \int_t^T f^\pi(s)ds - \sum_{n\geq 1}\left(\underline{g}_{\gamma_{n-1}^{(1)}\gamma_n^{(1)}}(\rho_n) - \overline{g}_{\gamma_{n-1}^{(2)}\gamma_n^{(2)}}(\rho_n)\right)\middle|\mathcal{F}_t\right] \tag{4.5}$$

where $\xi^{\pi_T} = \xi^{ij}$ if at time T, $\pi_T = (i,j)$ and $f^\pi(s) = f^{ij}$ if at time s, $\pi(s) = (i,j)$, for any $s\leq T$.

The following result is stated in [9]:

Theorem 4.1. *([9], Theorem 3.1) For any $t\in[0,T]$ and $(i,j)\in\Gamma$,*

$$\underline{Y}_t^{ij} = \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i}\operatorname*{ess\,inf}_{v\in\mathcal{B}_t^j} J_t^{ij}(\gamma(u,v)) = \operatorname*{ess\,inf}_{v\in\mathcal{B}_t^j}\operatorname*{ess\,sup}_{u\in\mathcal{A}_t^i} J_t^{ij}(\gamma(u,v)).$$

As a by-product of this result we have the following one related to uniqueness of the solution of system (4.4) which stems from the above characterization of the component Y^{ij} as the value function of the zero-sum switching game.

Corollary 4.2. *Let* $(\underline{Y}_1^{ij}, \underline{Z}_1^{ij}, \underline{K}_1^{ij,\pm})_{(i,j)\in\Gamma}$ *be another solution of sytem* (4.4)*, then for any* $(i,j) \in \Gamma$*,*

$$\underline{Y}^{ij} = \underline{Y}_1^{ij},\ \underline{Z}^{ij} = \underline{Z}_1^{ij} \text{ and } \underline{K}_1^{ij,+} - \underline{K}_1^{ij,-} = \underline{K}^{ij,+} - \underline{K}^{ij,-}.$$

Finally thanks to Theorems 3.6 and 4.1, we will prove the existence and uniqueness of the solution for the system of reflected BSDEs with bilateral interconnected obstacles (2.2) without assuming Assumption (H5) on monotonicity and we instead assume [H6].

Theorem 4.3. *Assume that [H1]-[H4] and [H6] are fulfilled. Then system of reflected BSDEs* (2.2) *has a solution* $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$*, i.e., for any* $(i,j) \in \Gamma$ *and* $t \leq T$*,*

$$\begin{cases} Y^{ij} \in \mathcal{S}^2, Z^{ij} \in \mathcal{H}^{2,d}, K^{ij,\pm} \in \mathcal{A}^2\ ; \\ Y_t^{ij} = \xi^{ij} + \int_t^T f^{ij}\left(s,\omega,(Y_s^{kl})_{(k,l)\in\Gamma^1\times\Gamma^2}\right)ds - \int_t^T Z_s^{ij}dB_s + K_T^{ij,+} - K_t^{ij,+} - (K_T^{ij,-} - K_t^{ij,-})\,; \\ L_t^{ij} \leq Y_t^{ij} \leq U_t^{ij}\,; \\ \int_0^T \left(Y_t^{ij} - L_t^{ij}\right)dK_t^{ij,+} = 0 \text{ and } \int_0^T \left(U_t^{ij} - Y_t^{ij}\right)dK_t^{ij,-} = 0, \end{cases} \tag{4.6}$$

where $L_t^{ij} := \max_{k\in(\Gamma^1)^{-i}} \left\{Y_t^{kj} - \underline{g}_{ik}(t)\right\}$ *and* $U_t^{ij} := \min_{l\in(\Gamma^2)^{-j}} \left\{Y_t^{il} + \overline{g}_{jl}(t)\right\}$*. Moreover it is unique in the following sense: If* $(\overline{Y}^{ij}, \overline{Z}^{ij}, \overline{K}^{ij,+}, \overline{K}^{ij,-})_{(i,j)\in\Gamma^1\times\Gamma^2}$ *is another solution of* (4.6)*, then for any* $(i,j) \in \Gamma$*,*

$$\overline{Y}^{ij} = Y^{ij},\ \overline{Z}^{ij} = Z^{ij},\ \overline{K}^{ij,+} - \overline{K}^{ij,-} = K^{ij,+} - K^{ij,-}.$$

Proof. First let us define the following operator:

$$\begin{aligned} \Phi :\quad & \mathcal{H}^{2,\Lambda} \to \mathcal{H}^{2,\Lambda} \\ \vec{\phi} := (\phi^{ij})_{(i,j)\in\Gamma} \mapsto & \Phi(\vec{\phi}) := \left(Y^{\phi,ij}\right)_{(i,j)\in\Gamma} \end{aligned} \tag{4.7}$$

where $(Y^{\phi,ij}, Z^{\phi,ij}, K^{\phi,ij,\pm})_{(i,j)\in\Gamma}$ is the solution of the following system (this solution exists and is unique by Theorem 3.6 and Corollary 4.2): $\forall (i,j) \in \Gamma$,

$$\begin{cases} Y^{\phi,ij} \in \mathcal{S}^2, Z^{\phi,ij} \in \mathcal{H}^{2,d}, K^{\phi,ij,\pm} \in \mathcal{A}^2; \\ Y_t^{\phi,ij} = \xi^{ij} + \int_t^T f^{ij}(s,\vec{\phi}(s))ds - \int_t^T Z_s^{\phi,ij}dB_s + K_T^{\phi,ij,+} - K_t^{\phi,ij,+} - (K_T^{\phi,ij,-} - K_t^{\phi,ij,-}),\ \forall t \leq T; \\ L_t^{\phi,ij} \leq Y_t^{\phi,ij} \leq U_t^{\phi,ij},\ \forall t \in [0,T]; \\ \int_0^T \left(Y_t^{\phi,ij} - L_t^{\phi,ij}\right)dK_t^{\phi,ij,+} = 0 \quad \text{and} \quad \int_0^T \left(U_t^{\phi,ij} - Y_t^{\phi,ij}\right)dK_t^{\phi,ij,-} = 0 \end{cases} \tag{4.8}$$

where $L^{\phi,ij}$ and $U^{\phi,ij}$ are defined as previously but with the processes $(Y^{\phi,ij})_{(i,j)\in\Gamma}$. Let $\vec{\psi} := (\psi^{ij})_{(i,j)\in\Gamma}$ be another element of $\mathcal{H}^{2,\Lambda}$ and let $(Y^{\psi,ij}, Z^{\psi,ij}, K^{\psi,ij,\pm})_{(i,j)\in\Gamma}$ be defined as in (4.8) but where $\vec{\phi}$ is replaced with $\vec{\psi}$.

Next let us introduce the following norm on $\mathcal{H}^{2,\Lambda}$, denoted by $\| \,.\, \|_{2,\alpha}$, and defined by

$$\| y \|_{2,\alpha} := \left[\mathbb{E}\left(\int_0^T e^{\alpha t}|y_t|^2 dt\right)\right]^{\frac{1}{2}}.$$

The space $(\mathcal{H}^{2,\Lambda}, \| \,.\, \|_{2,\alpha})$ is of Banach type. If the map Φ is a contraction on $(\mathcal{H}^{2,\Lambda}, \| \,.\, \|_{2,\alpha})$, then it has a fixed point which is the unique solution of (4.6). So let us show that Φ is a contraction. By Theorem 4.1, the following representation holds true: $\forall (i,j) \in \Gamma$ and $t \leq T$,

$$Y_t^{\phi,ij} = \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^i} \operatorname*{ess\,inf}_{v \in \mathcal{B}_t^j} J_t^{\phi,ij}(\gamma(u,v)) = \operatorname*{ess\,inf}_{v \in \mathcal{B}_t^j} \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^i} J_t^{\phi,ij}(\gamma(u,v))$$

where

$$J_t^{\phi,ij}(\gamma(u,v)) = \mathbb{E}\left[\xi^{\pi_T} + \int_t^T f^\pi(s,\vec{\phi}(s))ds - \sum_{n\geq 1} (\underline{g}_{\gamma_{n-1}^{(1)}\gamma_n^{(1)}}(\rho_n) - \overline{g}_{\gamma_{n-1}^{(2)}\gamma_n^{(2)}}(\rho_n)) \middle| \mathcal{F}_t \right]. \tag{4.9}$$

Next let $\vec{\psi} := (\psi^{ij})_{(i,j)\in\Gamma}$ be another element of $\mathcal{H}^{2,\Lambda}$. Once again, for any $(i,j) \in \Gamma$ and $t \leq T$,

$$Y_t^{\psi,ij} = \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^i} \operatorname*{ess\,inf}_{v \in \mathcal{B}_t^j} J_t^{\psi,ij}(\gamma(u,v)) = \operatorname*{ess\,inf}_{v \in \mathcal{B}_t^j} \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^i} J_t^{\psi,ij}(\gamma(u,v))$$

where $J_t^{\psi,ij}$ is defined similarly as $J_t^{\phi,ij}$ but with $\vec{\psi}$ instead of $\vec{\phi}$. Therefore

$$\forall t \leq T,\ |Y_t^{\psi,ij} - Y_t^{\phi,ij}| \leq \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^i} \operatorname*{ess\,sup}_{v \in \mathcal{B}_t^j} \left| J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v)) \right|. \tag{4.10}$$

First, by the martingale representation theorem, there exists a predictable process $\Delta Z^{\psi,\phi,\pi} \in \mathcal{H}^{2,d}$ (π depends on (i,j)) which is adapted with respect to $(\mathcal{F}_t)_{t\leq T}$ such that: $\forall t \leq T$,

$$\begin{aligned} J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v)) &= \mathbb{E}\left[\int_t^T (f^\pi(s,\vec{\psi}(s)) - f^\pi(s,\vec{\phi}(s)))ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}\left[\int_0^T (f^\pi(s,\vec{\psi}(s)) - f^\pi(s,\vec{\phi}(s)))ds \middle| \mathcal{F}_t \right] - \int_0^t (f^\pi(s,\vec{\psi}(s)) - f^\pi(s,\vec{\phi}(s)))ds \\ &= C^{\psi,\phi,\pi} + \int_0^t \Delta Z_s^{\psi,\phi,\pi} dB_s - \int_0^t (f^\pi(s,\vec{\psi}(s)) - f^\pi(s,\vec{\phi}(s)))ds. \end{aligned} \tag{4.11}$$

where $C^{\psi,\phi,\pi} := \mathbb{E}\left[\int_0^T (f^\pi(s,\vec{\psi}(s)) - f^\pi(s,\vec{\phi}(s)))ds\right]$. Thus $\forall t \leq T$,

$$d(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v))) = -(f^\pi(t,\vec{\psi}(t)) - f^\pi(t,\vec{\phi}(t)))dt + \Delta Z_t^{\psi,\phi,\pi} dB_t. \tag{4.12}$$

Next by applying Itô's formula, one has: $\forall t \leq T$,

$$\begin{aligned} d\left[e^{\alpha t} \left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v)) \right)^2 \right] &= \alpha e^{\alpha t} \left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v)) \right)^2 dt \\ &+ 2\left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v)) \right) \left\{ -\left(f^\pi(t,\vec{\psi}(t)) - f^\pi(t,\vec{\phi}(t)) \right) dt + \Delta Z_t^{\psi,\phi,\pi} dB_t \right\}. \end{aligned} \tag{4.13}$$

Now let $t \in [0,T]$ fixed. By integrating in (4.13) from t to T we obtain:

$$\begin{aligned} & e^{\alpha t} \left(J_t^{\psi,ij}(\gamma(u,v)) - J_t^{\phi,ij}(\gamma(u,v)) \right)^2 + \int_t^T e^{\alpha s} |\Delta Z_s^{\psi,\phi,\pi}|^2 ds \\ &= -\alpha \int_t^T e^{\alpha s} \left(J_s^{\psi,ij}(\gamma(u,v)) - J_s^{\phi,ij}(\gamma(u,v)) \right)^2 ds \end{aligned} \tag{4.14}$$

$$+2\int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v))-J_s^{\phi,ij}(\gamma(u,v))\right)\left(f^\pi(s,\vec{\psi}(s))-f^\pi(s,\vec{\phi}(s))\right)ds$$
$$-2\int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v))-J_s^{\phi,ij}(\gamma(u,v))\right)\Delta Z_s^{\psi,\phi,\pi}dB_s. \tag{4.15}$$

Now let us apply the inequality $2ab\le \alpha a^2+\alpha^{-1}b^2, \forall\alpha>0, a,b\in\mathbb{R}$, then (4.14) yields

$$e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v))-J_t^{\phi,ij}(\gamma(u,v))\right)^2+\int_t^T e^{\alpha s}|\Delta Z_s^{\psi,\phi,\pi}|^2ds$$
$$\le\frac{1}{\alpha}\int_t^T e^{\alpha s}\left(f^\pi(s,\vec{\psi}(s))-f^\pi(s,\vec{\phi}(s))\right)^2ds-2\int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v))-J_s^{\phi,ij}(\gamma(u,v))\right)\Delta Z_s^{\psi,\phi,\pi}dB_s.$$

Then by Lipschitz condition of f we have

$$e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v))-J_t^{\phi,ij}(\gamma(u,v))\right)^2\le\frac{C^2(f)}{\alpha}\int_t^T e^{\alpha s}|\vec{\psi}(s)-\vec{\phi}(s)|^2ds$$
$$-2\int_t^T e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v))-J_s^{\phi,ij}(\gamma(u,v))\right)\Delta Z_s^{\psi,\phi,\pi}dB_s \tag{4.16}$$

where $C(f)=\sum_{(i,j)\in\Gamma}C_{ij}$ with C_{ij} is the Lipschitz constant w.r.t. f^{ij}.
Next $(\int_t^s e^{\alpha r}(J_r^{\psi,ij}(\gamma(u,v))-J_r^{\phi,ij}(\gamma(u,v)))\Delta Z_r^{\psi,\phi,\pi}dB_r)_{s\in[t,T]}$ is a martingale. Then by taking the conditional expectation on both sides of (4.16) we obtain

$$\mathbb{E}\left[e^{\alpha s}\left(J_s^{\psi,ij}(\gamma(u,v))-J_s^{\phi,ij}(\gamma(u,v))\right)^2\Big|\mathcal{F}_t\right]\le\frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_s^T e^{\alpha r}|\vec{\psi}(r)-\vec{\phi}(r)|^2dr\Big|\mathcal{F}_t\right]. \tag{4.17}$$

Take now the limit as $s\to t$ in (4.17) yields

$$e^{\alpha t}\left(J_t^{\psi,ij}(\gamma(u,v))-J_t^{\phi,ij}(\gamma(u,v))\right)^2\le\frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha r}|\vec{\psi}(r)-\vec{\phi}(r)|^2dr\Big|\mathcal{F}_t\right],\ \forall t\le T. \tag{4.18}$$

Let us recall now (4.10), then (4.18) implies that: $\forall t\le T$,

$$e^{\alpha t}\left(Y_t^{\psi,ij}-Y_t^{\phi,ij}\right)^2\le\frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}|\vec{\psi}(s)-\vec{\phi}(s)|^2ds\Big|\mathcal{F}_t\right] \tag{4.19}$$

Next take the expectation in both hand-sides of (4.20) (and replace t with 0 in the right one) to obtain:

$$\mathbb{E}\left[e^{\alpha t}\left(Y_t^{\psi,ij}-Y_t^{\phi,ij}\right)^2\right]\le\frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_0^T e^{\alpha s}|\vec{\psi}(s)-\vec{\phi}(s)|^2ds\right]. \tag{4.20}$$

Finally by integrating (4.19) from 0 to T and summing over $(i,j)\in\Gamma$ we get

$$\int_0^T\sum_{(i,j)\in\Gamma}e^{\alpha t}\left(Y_t^{\psi,ij}-Y_t^{\phi,ij}\right)^2dt\le\frac{C^2(f)T\Lambda}{\alpha}\mathbb{E}\left[\int_0^T e^{\alpha s}|\vec{\psi}(s)-\vec{\phi}(s)|^2ds\right] \tag{4.21}$$

Now if we take $\alpha>C^2(f)T\Lambda$ then $\dfrac{C^2(f)T\Lambda}{\alpha}<1$. This implies that Φ is a contraction from $\mathcal{H}^{2,\Lambda}$ into itself, and then it has a fixed point which is the unique solution of (4.6). The proof is complete. □

As a by-product of the above result we also have:

Corollary 4.4. *The Λ-tuple of processes $(Y^{ij})_{(i,j)\in\Gamma}$ is the unique fixed point of the mapping Φ on $\mathcal{H}^{2,\Lambda}$.*

Remark 4.5. *Assume that for any $(i,j)\in\Gamma$, the function f^{ij} does not depend on z and verify the monotonicity Assumption [H5], then the solution constructed in Section 3, Theorem 3.6, is unique.*

5. Connection with systems of PDEs with bilateral interconnected obstacles

It is well-known that BSDEs, through the Feynman-Kac representation of solutions in the Markovian framework of randomness, provide solutions for partial differential equations. Similarly, in this section we are going to show that, in this very Markovian framework, the component $(Y^{ij})_{ij\in\Gamma}$ of the solution of system (4.6), has a Feynman-Kac representation which, besides, provides a unique solution in viscosity sense of the following system of PDEs with bilateral interconnected obstacles: For any $(i,j)\in\Gamma$,

$$\begin{cases} \min\Big\{v^{ij}(t,x)-\max_{k\in(\Gamma^1)^{-i}}[v^{kj}(t,x)-\underline{g}_{ik}(t,x)];\max\big[v^{ij}(t,x)-\min_{l\in(\Gamma^2)^{-j}}[v^{il}(t,x)+\overline{g}_{jl}(t,x)]; \\ -\partial_t v^{ij}(t,x)-\mathcal{L}^X(v^{ij})(t,x)-f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\big]\Big\}=0; \\ \\ v^{ij}(T,x)=h^{ij}(x). \end{cases} \tag{5.1}$$

So first let us fix the framework:

i) A function $\varrho:(t,x)\in[0,T]\mapsto\varrho(t,x)\in\mathbb{R}^m$ $(m\geq 1)$ has of polynomial growth if there exist two non-negative real constants C and γ such that $\forall(t,x)\in[0,T]\times\mathbb{R}^k$,

$$|\varrho(t,x)|\leq C(1+|x|^\gamma).$$

Hereafter this class of functions is denoted by Π_g.

ii) Let $\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k)$(or $\mathcal{C}^{1,2}$ for short) denote the set of real-valued functions defined on $[0,T]\times\mathbb{R}^k$ which are respectively once and twice differentiable w.r.t. t and x, with continuous derivatives.

iii) Let $b(t,x)$ and $\sigma(t,x)$ be two functions from $[0,T]\times\mathbb{R}^k$ into $\mathbb{R}^k$ jointly continuous and Lipschitz w.r.t x, i.e., for any $(t,x,x')\in[0,T]\times\mathbb{R}^{k+k}$, there exists a non-negative constant C such that

$$|\sigma(t,x)-\sigma(t,x')|+|b(t,x)-b(t,x')|\leq C|x-x'|. \tag{5.2}$$

Therefore b and σ are of linear growth w.r.t x, i.e.,

$$|b(t,x)|+|\sigma(t,x)|\leq C(1+|x|). \tag{5.3}$$

Under (5.2)-(5.3), for any $(t,x)\in[0,T]\times\mathbb{R}^k$, there exists a unique process $X^{t,x}$ solution of the following standard SDE:

$$\begin{array}{l} dX_s^{t,x}=b(s,X_s^{t,x})ds+\sigma(s,X_s^{t,x})dB_s,\ s\in[t,T]; \\ X_s^{t,x}=x,\ \forall s\leq t. \end{array} \tag{5.4}$$

Besides, $X^{t,x}$ satisfies the following estimates: $\forall\gamma\geq 1$,

$$\mathbb{E}\left[\sup_{s\leq T}|X_s^{t,x}|^\gamma\right]\leq C(1+|x|^\gamma) \tag{5.5}$$

and its infinitesimal generator $\mathcal{L}^X$ is given by: for any $(t,x) \in [0,T] \times \mathbb{R}^k, \phi \in \mathcal{C}^{1,2}$ ($(.)^\top$ is the transpose),

$$\mathcal{L}^X \phi(t,x) := \frac{1}{2} \sum_{i,j=1}^{k} (\sigma\sigma^\top(t,x))_{i,j} \partial^2_{x_i x_j} \phi(t,x) + \sum_{i=1}^{k} b_i(t,x) \partial_{x_i} \phi(t,x). \tag{5.6}$$

We are now going to decline the assumptions [H1]-[H4] of Section 2 in this markovian framework of randomness. So let us introduce deterministic functions $f^{ij}(t,x,\vec{y})$, $h^{ij}(x)$, $\underline{g}_{ik}(t,x)$ and $\bar{g}_{jl}(t,x)$, $i,k \in \Gamma^1$, $j,l \in \Gamma^2$ and $t,x,\vec{y}$ in $[0,T]$, $\mathbb{R}^k$ and $\mathbb{R}^\Lambda$ respectively.

[H1b]: For any $(i,j) \in \Gamma$,

i) There exist non negative constants C and γ such that

$$|f^{ij}(t,x,\vec{y})| \leq C(1 + |x|^\gamma + |\vec{y}\,|).$$

ii) f^{ij} is Lipschitz continuous w.r.t. $\vec{y}$ uniformly in (t,x), i.e. there exists a constant C such that for any $\vec{y}_1, \vec{y}_2 \in \mathbb{R}^\Lambda$,

$$|f^{ij}(t,x,\vec{y}_1) - f^{ij}(t,x,\vec{y}_2)| \leq C|\vec{y}_1 - \vec{y}_2|.$$

[H2b]: For any $(i,j) \in \Gamma$, the function h^{ij}, which stands for the terminal condition, is continuous w.r.t. x, belongs to class Π_g and satisfies the following consistency condition: $\forall (i,j) \in \Gamma$ and $x \in \mathbb{R}^k$,

$$\max_{k \in (\Gamma^1)^{-i}} (h^{kj}(x) - \underline{g}_{ik}(T,x)) \leq h^{ij}(x) \leq \min_{l \in (\Gamma^2)^{-j}} (h^{il}(x) + \overline{g}_{jl}(T,x)). \tag{5.7}$$

[H3b]: For all $i_1, i_2 \in \Gamma^1$ (resp. $j_1, j_2 \in \Gamma^2$), the function $\underline{g}_{i_1 i_2}$(resp. $\overline{g}_{j_1 j_2}$)

iii) is non-negative, continuous and belong to Π_g ;

iv) For any $k \in \Gamma^1$ (resp. $\ell \in \Gamma^2$) such that $|\{i_1, i_2, k\}| = 3$ (resp. $|\{j_1, j_2, \ell\}| = 3$) it holds: $\forall (t,x) \in [0,T] \times \mathbb{R}^k$,

$$\underline{g}_{i_1 i_2}(t,x) < \underline{g}_{i_1 k}(t,x) + \underline{g}_{k i_2}(t,x) \ \big(\text{resp. } \overline{g}_{j_1 j_2}(t,x) < \overline{g}_{j_1 \ell}(t,x) + \overline{g}_{\ell j_2}(t,x)\big)\,; \tag{5.8}$$

v) The functions $(\underline{g}_{ik})_{i,k \in \Gamma^1}$ and $(\overline{g}_{jl})_{j,l \in \Gamma^2}$ verify the non free loop property, that is to say, if $(i_k, j_k)_{k=1,2,...,N}$ is a loop in Γ, i.e., $(i_N, j_N) = (i_1, j_1)$, $\mathbf{card}\,\{(i_k, j_k)_{k=1,2,...,N}\} = N-1$ and for any $k = 1, 2, ..., N-1$, either $i_{k+1} = i_k$ or $j_{k+1} = j_k$, we have:

$$\forall t \leq T, \quad \sum_{k=1}^{N-1} G_{i_k j_k}(t,x) \neq 0 \tag{5.9}$$

where $\forall k = 1, ...N-1, \quad G_{i_k j_k}(t,x) = -\underline{g}_{i_k i_{k+1}}(t,x) \mathbb{1}_{(i_k \neq i_{k+1})} + \overline{g}_{j_k j_k}(t,x) \mathbb{1}_{(j_k \neq j_{k+1})}$. This assumption makes sure that any instantaneous loop in the switching mode set $\Gamma^1 \times \Gamma^2$ is not free, i.e. one of the controllers needs to pay something when the system is switched and comes back instantaneously to the initial mode.

Note that (5.9) also implies: For any $(i_1, ..., i_N) \in (\Gamma^1)^N$ such that $i_N = i_1$ and $\mathbf{card}\{i_1, i_2, ..., i_N\} = N-1$,

$$\sum_{k=1}^{N-1} \underline{g}_{i_k i_{k+1}}(t,x) > 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^k$$

and for any $(j_1, ..., j_N) \in (\Gamma^2)^N$ such that $j_N = j_1$ and $\mathbf{card}\{j_1, j_2, ..., j_N\} = N-1$,

$$\sum_{k=1}^{N-1} \overline{g}_{j_k j_{k+1}}(t,x) > 0, \ \forall (t,x) \in [0,T] \times \mathbb{R}^k.$$

[H4b] For any $i, k \in \Gamma^1$ (resp. $j, l \in \Gamma^2$), $\underline{g}_{ik}$(resp. $\overline{g}_{jl}$) is $\mathcal{C}^{1,2}$ and $D_x \underline{g}_{ik}, D^2_{xx} \underline{g}_{ik}$ (resp. $D_x \overline{g}_{jl}, D^2_{xx} \overline{g}_{jl}$) belong to Π_g. Thus by Itô's formula we have:

$$\begin{cases} \underline{g}_{ik}(s, X_s^{t,x}) = \underline{g}_{ik}(t,x) + \int_t^s \mathcal{L}^X(\underline{g}_{ik})(r, X_r^{t,x}) dr + \int_t^s D_x \underline{g}_{ik}(r, X_r^{t,x}) \sigma(r, X_r^{t,x}) dB_r, \ s \in [t,T]; \\ \underline{g}_{ik}(s, X_s^{t,x}) = \underline{g}_{ik}(s,x), \ s \leq t. \end{cases}$$

$$\left(\text{resp.} \begin{cases} \overline{g}_{jl}(s, X_s^{t,x}) = \overline{g}_{jl}(t,x) + \int_t^s \mathcal{L}^X(\overline{g}_{jl})(r, X_r^{t,x}) dr \int_t^s D_x \overline{g}_{jl}(r, X_r^{t,x}) \sigma(r, X_r^{t,x}) dB_r, \ s \in [t,T]; \\ \overline{g}_{jl}(s, X_s^{t,x}) = \overline{g}_{jl}(s,x), \ s \leq t. \end{cases} \right).$$

Remark 5.1. *Since $D_x \underline{g}_{ik}, D^2_{xx} \underline{g}_{ik}$ (resp. $D_x \overline{g}_{jl}, D^2_{xx} \overline{g}_{jl}$) belong to Π_g, taking into account of assumptions* (5.3) *on linear growth of b and σ and finally estimate* (5.5)*, one gets that* $\sup_{s \leq T} |D_x \underline{g}_{ik}(s, X_s^{t,x})|$ *(resp.* $\sup_{s \leq T} |D_x \overline{g}_{jl}(s, X_s^{t,x})|$*) belongs to $L^2(d\mathbb{P})$.* □

To begin with we first give the following result which stems from Theorem 4.3 under assumptions [H1b]-[H4b].

Proposition 5.2. *Assume that Assumptions [H1b]-[H4b] are fulfilled. Then for any $(t,x) \in [0,T] \times \mathbb{R}^k$, there exist processes $(Y^{ij;t,x}, Z^{ij;t,x}, K^{ij,+;t,x}, K^{ij,-;t,x})_{(i,j)\in\Gamma}$ unique solution of system of reflected BSDEs with bilateral interconnected obstacles associated with $(f^{ij}, h^{ij}, \underline{g}_{ik}, \overline{g}_{jl})$, i.e., for any $(i,j) \in \Gamma$ and $s \in [0,T]$,*

$$\begin{cases} Y^{ij;t,x} \in \mathcal{S}^2, Z^{ij;t,x} \in \mathcal{H}^{2,1}, K^{ij,\pm;t,x} \in \mathcal{A}^2 \ ; \\ Y_s^{ij;t,x} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij}\left(r, X_r^{t,x}, (Y_r^{kl;t,x})_{(k,l)\in\Gamma}\right) dr - \int_s^T Z_r^{ij;t,x} dB_r + K_T^{ij,+;t,x} - K_s^{ij,+;t,x} \\ \qquad -(K_T^{ij,-;t,x} - K_s^{ij,-;t,x}); \\ L_s^{ij;t,x} \leq Y_s^{ij;t,x} \leq U_s^{ij;t,x}; \\ \int_0^T (Y_s^{ij;t,x} - L_s^{ij;t,x}) dK_s^{ij,+;t,x} = 0 \ \text{and} \ \int_0^T (Y_s^{ij;t,x} - U_s^{ij;t,x}) dK_s^{ij,-;t,x} = 0 \end{cases} \tag{5.10}$$

where $L_s^{ij;t,x} := \max_{k \in (\Gamma^1)^{-i}} \left[Y_s^{kj;t,x} - \underline{g}_{ik}(s, X_s^{t,x}) \right]$ *and* $U_s^{ij;t,x} := \min_{l \in (\Gamma^2)^{-j}} \left[Y_s^{il;t,x} + \overline{g}_{jl}(s, X_s^{t,x}) \right], s \in [0,T]$.

We are now going to focus on the properties of $(Y^{ij;t,x})_{(i,j)\in\Gamma}$. For simplicity reasons the quadruple of processes $(Y^{ij;t,x}, Z^{ij;t,x}, K^{ij,+;t,x}, K^{ij,-;t,x})$ will be sometimes simply denoted by $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})$.

Theorem 5.3. *Assume that Assumptions [H1b]-[H4b] are fulfilled. Then there exist deterministic continuous functions $(v^{ij})_{(i,j)\in\Gamma}$ of polynomial growth, defined on $[0,T] \times \mathbb{R}^k$ such that for any $(i,j) \in \Gamma$, $(t,x) \in [0,T] \times \mathbb{R}^k$,*

$$\mathbb{P} - a.s., \forall s \in [t,T], \quad Y_s^{ij;t,x} = v^{ij}(s, X_s^{t,x}). \tag{5.11}$$

Proof. The proof is given in several steps.

A) We first assume that $\forall (i,j)\in\Gamma, (t,x)\in[0,T]\times\mathbb{R}^k, f^{ij}(t,x,0,0)$ and $h^{ij}(x)$ are bounded.

We will prove that for any $(i,j)\in\Gamma$, for a fixed δ_1 there exists a bounded continuous deterministic function v^{ij} defined on $[T-\delta_1,T]\times\mathbb{R}^k$ such that for any $(t,x)\in[T-\delta_1,T]\times\mathbb{R}^k$ we have:

$$\mathbb{P}-\text{ a.s. for any } s\in[t,T], Y_s^{ij}=v^{ij}(s,X_s^{t,x}).$$

Let us recall the system (5.10) and let $(\bar{Y},\bar{Z})$ be the unique solution in $\mathcal{S}^2\times\mathcal{H}^{2,d}$ of the following BSDE (it depends on t,x which we omit as there is no confusion):

$$\bar{Y}_s=\bar{h}(X_T^{t,x})+\int_s^T\Psi(\bar{Y}_r)dr-\int_s^T\bar{Z}_rdB_r,\ s\le T,$$

where $\bar{h}(x)=\sum_{(i,j)\in\Gamma}|h^{ij}(x)|$ and $\Psi(y):=\Lambda^2C^\sharp(1+|y|)$ where $C^\sharp=\max\{C(f),\underline{C}\}$ with $\underline{C}$ is a uniform constant of boundedness of $|f^{ij}(t,x,0)|$. It is well-known that there exists a bounded deterministic continuous function $\bar{v}$ such that $\mathbb{P}$-a.s., $\forall s\in[t,T]$, $\bar{Y}_s=\bar{v}(s,X_s^{t,x})$ (see e.g. [6]). Finally note that $\bar{Y}\ge 0$ and then $\bar{v}\ge 0$.

Now for any $(i,j)\in\Gamma$, we set $(\hat{Y}^{ij},\hat{Z}^{ij},\hat{K}^{ij,+},\hat{K}^{ij,-}):=(\bar{Y},\bar{Z},0,0)$. Therefore $(\hat{Y}^{ij},\hat{Z}^{ij},\hat{K}^{ij,+},\hat{K}^{ij,-})$ is the unique solution of doubly reflected BSDEs associated with $(\bar{h}^{ij},\hat{\Psi},(\underline{g}_{ik})_{k\in(\Gamma^1)^{-i}},(\bar{g}_{jl})_{l\in(\Gamma^2)^{-j}})$ where $\hat{\Psi}(y):=\Lambda^2C^\sharp(1+(y)^+)$. This actually holds in taking into account of: i) the backward equation satisfied by $(\bar{Y},\bar{Z})$; ii) the fact that $\underline{g}_{ik}$ and $\overline{g}_{jl}$ are non-negative ; iii) the fact that $\bar{Y}\ge 0$ and then $|\bar{Y}|=\bar{Y}^+$. Lastly let us notice that by Theorem 4.3, the solution of this system exists and is unique and then it is equal to $(\bar{Y},\bar{Z},0,0)_{(i,j)\in\Gamma}$. Hence we also have $\mathbb{P}$-a.s., for any $s\in[t,T]$, $\hat{Y}_s^{ij}=\bar{v}(s,X_s^{t,x})$.

In the same way, setting $(\tilde{Y}^{ij},\tilde{Z}^{ij},\tilde{K}^{ij,+},\tilde{K}^{ij,-})=(-\bar{Y},-\bar{Z},0,0)$ for any $(i,j)\in\Gamma$, we obtain that the family $(\tilde{Y}^{ij},\tilde{Z}^{ij},\tilde{K}^{ij,+},\tilde{K}^{ij,-})_{(i,j)\in\Gamma}$ is the unique solution of reflected BSDEs associated with $(-\bar{h}^{ij},\hat{\Psi}_2,(\underline{g}_{ik})_{k\in(\Gamma^1)^{-i}},(\bar{g}_{jl})_{l\in(\Gamma^2)^{-j}})$ where $\hat{\Psi}_2(y)=-C^\sharp\Lambda^2(1+(y)^-)$. Next let us consider the following Picard iterations: for any $(i,j)\in\Gamma, Y^{ij,0;t,x}=0$ and for all $n\ge 1$, $(Y^{ij,n;t,x})_{(i,j)\in\Gamma}=\Phi((Y^{ij,n-1;t,x})_{(i,j)\in\Gamma})$, where Φ is defined in (4.7). In other words the family $(Y^{ij,n;t,x},Z^{ij,n;t,x},K^{ij,n,+;t,x},K^{ij,n,-;t,x})_{(i,j)\in\Gamma}$ (which sometimes is simply denoted by $(Y^{ij,n},Z^{ij,n},K^{ij,n,+},K^{ij,n,-})_{(i,j)\in\Gamma}$ as no confusion is possible) is the unique solution of the following system of BSDEs: $\forall(i,j)\in\Gamma$ and $s\in[0,T]$,

$$\begin{cases} Y_s^{ij,n;t,x}=h^{ij}(X_T^{t,x})+\displaystyle\int_s^T f^{ij}(r,X_r^{t,x},(Y_r^{kl,n-1;t,x})_{(k,l)\in\Gamma})dr-\int_s^T Z_r^{ij,n;t,x}dB_r \\ \qquad\qquad +K_T^{ij,n,+;t,x}-K_s^{ij,n,+;t,x}-(K_T^{ij,n,-;t,x}-K_s^{ij,n,-;t,x}); \\ \max\limits_{k\in(\Gamma^1)^{-i}}\left[Y_s^{kj,n;t,x}-\underline{g}_{ik}(s,X_s^{t,x})\right]\le Y_s^{ij,n;t,x}\le\min\limits_{l\in(\Gamma^2)^{-j}}\left[Y_s^{il,n;t,x}+\overline{g}_{jl}(s,X_s^{t,x})\right]; \\ \displaystyle\int_0^T\left\{Y_s^{ij,n;t,x}-\max_{k\in(\Gamma^1)^{-i}}\left[Y_s^{kj,n;t,x}-\underline{g}_{ik}(s,X_s^{t,x})\right]\right\}dK_s^{ij,n,+;t,x}=0; \\ \displaystyle\int_0^T\left\{Y_s^{ij,n;t,x}-\min_{l\in(\Gamma^2)^{-j}}\left[Y_s^{il,n;t,x}+\overline{g}_{jl}(s,X_s^{t,x})\right]\right\}dK_s^{ij,n,-;t,x}=0. \end{cases} \tag{5.12}$$

Then we have the following inequalities: for any $n\ge 0,(i,j)\in\Gamma$,

$$-\bar{Y}\le Y^{ij,n}\le\bar{Y} \tag{5.13}$$

Indeed when $n=0$, (5.13) holds true since for any $(i,j)\in\Gamma,(t,x)\in[0,T]\times\mathbb{R}^k,-\bar{Y}\le 0\le\bar{Y}$. Next we assume that (5.13) holds for some $n-1$, i.e. for any $(i,j)\in\Gamma$, $\tilde{Y}^{ij}=-\bar{Y}\le Y^{ij,n-1}\le\hat{Y}^{ij}=\bar{Y}$.

Then by [H1b]-ii), the boundedness of $f^{ij}(t,x,\vec{0})$ and the induction hypothesis we have:

$$f^{ij}(s,X_s^{t,x},(Y_s^{kl,n-1})_{(k,l)\in\Gamma})) \leq C^\sharp(1+\sum_{(k,l)\in\Gamma}|Y_s^{kl,n-1}|) \leq \Psi(\hat{Y}_s^{ij}).$$

As $h^{ij}(x) \leq \bar{h}(x)$, then by the comparison result (Remark 3.7, iii)) between the solutions of equations $(\hat{Y}^{ij})_{ij}$ and (5.12), one deduces that for any $(i,j)\in\Gamma$, $Y^{ij,n} \leq \hat{Y}^{ij}$. Similarly by the induction steps, one deduces that for any $(i,j)\in\Gamma$, $Y^{ij,n} \geq \tilde{Y}^{ij} = -\bar{Y}$. The proof of the claim (5.13) is complete.

Next once more by induction, using the result by Djehiche et al. [4] there exist deterministic continuous functions $(v^{ij,n})_{(i,j)\in\Gamma}$, $n\geq 0$, such that $\forall (i,j)\in\Gamma, (t,x)\in[0,T]\times\mathbb{R}^k$ we have

$$\mathbb{P}-a.s.,\ \forall s\in[t,T], Y_s^{ij,n} = v^{ij,n}(s,X_s^{t,x}). \tag{5.14}$$

Therefore from (5.13), we deduce that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $-\bar{v}(t,x)\leq v^{ij,n}(t,x)\leq\bar{v}(t,x)$, for any $(i,j)\in\Gamma$. As a by-product the sequence $(v^{ij,n}(t,x))_{n\geq 0}$ is uniformly bounded since $\bar{v}$ is so. Afterwards we just need to prove that $((v^{ij,n})_{(i,j)\in\Gamma})_{n\geq 0}$ is a Cauchy sequence for the uniform convrgence norm.

Actually as shown in the proof of Theorem 4.3, the sequence $((Y^{ij,n})_{(i,j)\in\Gamma})_{n\geq 0}$ converges in $\mathcal{H}^{2,d}_{[0,T]}$ to $(Y^{ij})_{(i,j)\in\Gamma}$ since $(Y^{ij})_{(i,j)\in\Gamma}$ is the fixed point in $\mathcal{H}^{2,\Lambda}$. On the other hand, for any $t\in[0,T]$ and $x\in\mathbb{R}^k$, by (4.19) we have:

$$e^{\alpha t}|v^{ij,n}(t,x)-v^{ij,q}(t,x)|^2 \leq \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T\sum_{(i,j)\in\Gamma}|Y_s^{ij,n-1;t,x}-Y_s^{ij,q-1;t,x}|^2ds\right]. \tag{5.15}$$

But, as mentioned previously, the last term converges to 0 as n,q go to infinite. It follows that for any $(i,j)\in\Gamma$, the sequence $(v^{ij,n})_{n\geq 0}$ is of Cauchy type pointwise on $[0,T]\times\mathbb{R}^k$. Therefore there exists a function v^{ij} defined on $[0,T]\times\mathbb{R}^k$ such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $v^{ij}(t,x) = \lim_{n\to\infty}v^{ij,n}(t,x)$. Moreover, $-\bar{v}(t,x)\leq v^{ij}(t,x)\leq\bar{v}(t,x)$ which implies that the function v^{ij} is bounded. Finally we have

$$\forall\ (i,j)\in\Gamma,\ Y_s^{ij;t,x} = v^{ij}(s,X_s^{t,x}),\ ds\otimes d\mathbb{P} \text{ on } [t,T]\times\mathbb{R}^k.$$

Next by the inequality (4.19) and taking expectation to obtain: For any $(t,x)\in[0,T]\times\mathbb{R}^k$, $(i,j)\in\Gamma$ and $n,q\geq 1$,

$$|v^{ij,n}(t,x)-v^{ij,q}(t,x)|^2 \leq \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha(s-t)}\sum_{(i,j)\in\Gamma}|v^{ij,n-1}(s,X_s^{t,x})-v^{ij,q-1}(s,X_s^{t,x})|^2ds\right]. \tag{5.16}$$

Recall (5.16), for any $(i,j)\in\Gamma$ and $t\in[T-\delta_1,T]$ we have

$$\begin{aligned}\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|^2_{\infty,\delta_1} &\leq \frac{C^2(f)\Lambda}{\alpha}e^{\alpha(T-\delta_1)}\int_{T-\delta_1}^T e^{\alpha s}ds\sum_{(i,j)\in\Gamma}\|v^{ij,n-1}(t,x)-v^{ij,q-1}(t,x)\|^2_{\infty,\delta_1}\\ &= \frac{C^2(f)\Lambda(e^{\alpha\delta_1}-1)}{\alpha^2}\sum_{(i,j)\in\Gamma}\|v^{ij,n-1}-v^{ij,q-1}\|^2_{\infty,\delta_1}\end{aligned} \tag{5.17}$$

Choose now δ_1 such that $\dfrac{C^2(f)\Lambda(e^{\alpha\delta_1}-1)}{\alpha^2} = \dfrac{3}{4}$, then, as a result, the sequence of continuous functions $(v^{ij,n})_{(i,j)\in\Gamma}$ is uniformly convergent on $[T-\delta_1,T]\times\mathbb{R}^k$ which implies that $(v^{ij})_{(i,j)\in\Gamma}$ is continuous on $[T-\delta_1,T]\times\mathbb{R}^k$.

Next by (4.19) and since $Y_t^{ij,n}$ is deterministic then for any $t\in[0,T-\delta_1]$, $x\in\mathbb{R}^k$, we have:

$$|v^{ij,n}(t,x)-v^{ij,q}(t,x)|^2 = \mathbb{E}[|Y_t^{ij,n}-Y_t^{ij,q}|^2] \tag{5.18}$$

$$\leq \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha(s-t)}\sum_{(i,j)\in\Gamma}|v^{ij,n-1}(s,X_s^{t,x})-v^{ij,q-1}(s,X_s^{t,x})|^2ds\right]$$

$$\leq \frac{C^2(f)}{\alpha}\mathbb{E}\left[\int_t^{T-\delta_1} e^{\alpha(s-t)}\sum_{(i,j)\in\Gamma}|v^{ij,n-1}(s,X_s^{t,x})-v^{ij,q-1}(s,X_s^{t,x})|^2ds\right]$$

$$+\frac{3}{4}\sum_{(i,j)\in\Gamma}\|v^{ij,n-1}-v^{ij,q-1}\|^2_{\infty,\delta_1} \tag{5.19}$$

The last inequality is valid thanks to (5.17). Now let $(t,x)\in[T-2\delta_1,T-\delta_1]\times\mathbb{R}^k$. Taking the supremum on (t,x) in (5.16) and summing over $(i,j)\in\Gamma$, yields:

$$\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|^2_{\infty,2\delta_1} \leq \frac{3}{4}\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|^2_{\infty,\delta_1}+\frac{3}{4}\sum_{(i,j)\in\Gamma}\|v^{ij,n-1}-v^{ij,q-1}\|^2_{\infty,2\delta_1}.$$

But we know that $\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|_{\infty,\delta_1}\to 0$ as $n,q\to\infty$, therefore we have also:

$$\sum_{(i,j)\in\Gamma}\|v^{ij,n}-v^{ij,q}\|_{\infty,2\delta_1}\to 0, \text{ as } n,q\to\infty.$$

It follows that for any $(i,j)\in\Gamma$, the sequence $(v^{ij,n})_n$ converges uniformly to v^{ij} in $[T-2\delta_1,T-\delta_1]\times\mathbb{R}^k$. Consequently v^{ij} is continuous in $[T-2\delta_1,T-\delta_1]\times\mathbb{R}^k$ and then also on $[T-2\delta_1,T]\times\mathbb{R}^k$ since we have already shown that it continuous on $[T-\delta_1,T]\times\mathbb{R}^k$. Repeating now this procedure as many times as necessary on $[T-3\delta_1,T-2\delta_1]\times\mathbb{R}^k,[T-4\delta_1,T-3\delta_1]\times\mathbb{R}^k$ and so on, we obtain that for any $(i,j)\in\Gamma$, v^{ij} is continuous on $[0,T]\times\mathbb{R}^k$ and then the processes $(Y_s^{ij;t,x})_{s\in[0,T]}$ and $(v^{ij}(s,X_s^{t,x}))_{s\in[0,T]}$ are indistinguishable, i.e.,

$$\forall\ (i,j)\in\Gamma,\ \mathbb{P}-a.s., \forall s\in[0,T], Y_s^{ij;t,x}=v^{ij}(s,X_s^{t,x}).$$

<u>B) The general case</u>: The functions $f^{ij}(t,x,0)$ and $h^{ij}(x)$, $(i,j)\in\Gamma$, are of polynomial growth.

Let γ be a positive constant such that for any $(i,j)\in\Gamma$,

$$|f^{ij}(t,x,0)|+|h^{ij}(x)|+|\underline{g}_{ij}(t,x)|+|\overline{g}_{ij}(t,x)|\leq C(1+|x|^\gamma).$$

Let $\rho(x):=(1+|x|^2)^{-\gamma}$, $x\in\mathbb{R}^k$, and for any $(i,j)\in\Gamma, s\in[t,T]$, set

$$\tilde{Y}_s^{ij}:=Y_s^{ij}\rho(X_s^{t,x}). \tag{5.20}$$

Then by Itô's formula we have: $\forall s\in[t,T]$,

$$\begin{aligned} d\tilde{Y}_s^{ij} &= Y_s^{ij}d\rho(X_s^{t,x})+\rho(X_s^{t,x})dY_s^{ij}+d\langle Y^{ij},\rho(X^{t,x})\rangle_s \\ &= \left[Y_s^{ij}\mathcal{L}^X\rho(X_s^{t,x})-\rho(X_s^{t,x})f^{ij}(s,X_s^{t,x},(Y_s^{kl})_{(k,l)\in\Gamma})+D_x\rho(X_s^{t,x})\sigma(s,X_s^{t,x})Z_s^{ij}\right]ds \\ &\quad+\left[Y_s^{ij}D_x\rho(X_s^{t,x})\sigma(s,X_s^{t,x})+\rho(X_s^{t,x})Z_s^{ij}\right]dB_s-\rho(X_s^{t,x})dK_s^{ij,+}+\rho(X_s^{t,x})dK_s^{ij,-}. \end{aligned} \tag{5.21}$$

Next for $(i,j) \in \Gamma$ and $s \in [t,T]$, let us set:

a) $\tilde{Z}_s^{ij} := Y_s^{ij} D_x\rho(X_s^{t,x})\sigma(s, X_s^{t,x}) + \rho(X_s^{t,x}) Z_s^{ij}$;

b) $d\tilde{K}_s^{ij,+} := \rho(X_s^{t,x}) dK_s^{ij,+}$ and $d\tilde{K}_s^{ij,-} := \rho(X_s^{t,x}) dK_s^{ij,-}$;

c) $\tilde{f}^{ij}(s, X_s^{t,x}, \overrightarrow{y}) := \rho(X_s^{t,x}) f^{ij}(s, X_s^{t,x}, (\rho^{-1}(X_s^{t,x}) y^{kl})_{(k,l)\in\Gamma}) - \rho^{-1}(X_s^{t,x}) y^{ij} \mathcal{L}^X \rho(X_s^{t,x})$
$\quad - D_x\rho(X_s^{t,x})\sigma(s, X_s^{t,x})\rho^{-1}(X_s^{t,x})[\tilde{Z}_s^{ij} - y^{ij}\rho^{-1}(X_s^{t,x}) D_x\rho(X_s^{t,x})\sigma(s, X_s^{t,x})]$;

d) $\tilde{\underline{g}}_{ij}(s, X_s^{t,x}) := \rho(X_s^{t,x}) \underline{g}_{ij}(s, X_s^{t,x})$ and $\tilde{g}_{ij}(s, X_s^{t,x}) := \rho(X_s^{t,x}) \overline{g}_{ij}(s, X_s^{t,x})$;

e) $\tilde{h}^{ij}(X_T^{t,x}) := \rho(X_T^{t,x}) h^{ij}(X_T^{t,x})$.

Then the family $(\tilde{Y}^{ij}, \tilde{Z}^{ij}, \tilde{K}^{ij,+}, \tilde{K}^{ij,-})_{(i,j)\in\Gamma}$ is the unique solution of the system of reflected BSDEs associated with $((\tilde{f}^{ij})_{ij}, (\tilde{h}^{ij})_{ij}, (\tilde{\underline{g}}_{ik})_{i,k\in\Gamma^1}, (\tilde{g}_{jl})_{j,l\in\Gamma^2})$.

But for any $(i,j) \in \Gamma$, $\tilde{h}^{ij}, \tilde{f}^{ij}(t,x,0), \tilde{\underline{g}}_{ik}, \tilde{g}_{jl}$ are bounded. Then thanks to the previous step, for any $(i,j) \in \Gamma$, one can find continuous bounded functions $(\tilde{v}^{ij})_{(i,j)\in\Gamma}$ defined on $[0,T] \times \mathbb{R}^k$ such that $\tilde{Y}_s^{ij,t,x} = \tilde{v}^{ij}(s, X_s^{t,x})$, $\forall s \in [t,T]$. Therefore in setting, for $(i,j) \in \Gamma$ and $(t,x) \in [0,T] \times \mathbb{R}^k$, $v^{ij}(t,x) = \rho^{-1}(x)\tilde{v}^{ij}(t,x)$ makes that $(v^{ij}(t,x))_{(i,j)\in\Gamma}$ is continuous on $[0,T] \times \mathbb{R}^k$, is of polynomial growth and verifies for any $(i,j) \in \Gamma$, $Y_s^{ij;t,x} = v^{ij}(s, X_s^{t,x})$, $\forall s \in [t,T]$. The proof is now complete. □

We are now ready to give the main result of this section.

Theorem 5.4. *Assume that Assumptions [H1b]-[H4b] and [H6] are fulfilled. Then the Λ-tuple of continuous functions $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution (see Appendix for the definition) of the following system of variational inequalities with bilateral interconnected obstacles: For any $(i,j) \in \Gamma$,*

$$\begin{cases} \min\{v^{ij}(t,x) - \max_{k\in(\Gamma^1)^{-i}}[v^{kj}(t,x) - \underline{g}_{ik}(t,x)]; \max\{v^{ij}(t,x) - \min_{l\in(\Gamma^2)^{-j}}[v^{il}(t,x) + \overline{g}_{jl}(t,x)]; \\ \qquad -\partial_t v^{ij}(t,x) - \mathcal{L}^X(v^{ij})(t,x) - f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\}\} = 0; \\ \\ v^{ij}(T,x) = h^{ij}(x). \end{cases} \tag{5.22}$$

Moreover it is unique in the class of continuous functions which belong to Π_g.

Proof. We first prove that $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution, then we prove the uniqueness.

<u>*Step 1*</u>: $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of (5.22).

For convenience we recall the unique solution $(Y^{ij}, Z^{ij}, K^{ij,+}, K^{ij,-})_{(i,j)\in\Gamma}$ of (5.10): For any $(i,j) \in \Gamma$ and $s \le T$,

$$\begin{cases} Y_s^{ij} = h^{ij}(X_T^{t,x}) + \int_s^T f^{ij}\left(r, X_r^{t,x}, (Y_r^{kl})_{(k,l)\in\Gamma}\right) dr - \int_s^T Z_r^{ij} dB_r + \int_s^T d(K_s^{ij,+} - dK_s^{ij,-}); \\ \\ L_s^{ij} \le Y_s^{ij} \le U_s^{ij}; \\ \\ \int_0^T \left(Y_s^{ij} - L_s^{ij}\right) dK_s^{ij,+} = 0 \text{ and } \int_0^T \left(Y_s^{ij} - U_s^{ij}\right) dK_s^{ij,-} = 0. \end{cases} \tag{5.23}$$

By (5.11), the system (5.23) can be decoupled as follows: for any $(i,j)\in\Gamma$ and $s\in[t,T]$,

$$\begin{cases} Y_s^{ij}=h^{ij}(X_T^{t,x})+\int_s^T f^{ij}(r,X_r^{t,x},(v^{kl}(r,X_r^{t,x}))_{(k,l)\in\Gamma})dr-\int_s^T Z_r^{ij}dB_r+\int_s^T d(K_s^{ij,+}-dK_s^{ij,-});\\ \max_{k\in(\Gamma^1)^{-i}}\left[v^{kj}(s,X_s^{t,x})-\underline{g}_{ik}(s,X_s^{t,x})\right]\le Y_s^{ij}\le\min_{l\in(\Gamma^2)^{-j}}\left[v^{il}(s,X_s^{t,x})+\overline{g}_{jl}(s,X_s^{t,x})\right];\\ \int_t^T\left\{Y_s^{ij}-\max_{k\in(\Gamma^1)^{-i}}[v^{kj}(s,X_s^{t,x})-\underline{g}_{ik}(s,X_s^{t,x})]\right\}dK_s^{ij,+}=0\\ \int_t^T\left\{Y_s^{ij}-\min_{l\in(\Gamma^2)^{-j}}[v^{il}(s,X_s^{t,x})+\overline{g}_{jl}(s,X_s^{t,x})]\right\}dK_s^{ij,-}=0. \end{cases} \tag{5.24}$$

Applying Theorem 6.2 in [7] (see also Theorem A.3 in [4]), for any arbitrary (i,j) in Γ, v^{ij} is a viscosity solution of

$$\begin{cases} \min\{v^{ij}(t,x)-\max_{k\in(\Gamma^1)^{-i}}[v^{kj}(t,x)-\underline{g}_{ik}(t,x)];\max\left\{v^{ij}(t,x)-\min_{l\in(\Gamma^2)^{-j}}[v^{il}(t,x)+\overline{g}_{jl}(t,x)];\right.\\ \qquad\left.-\partial_t v^{ij}(t,x)-\mathcal{L}^X(v^{ij})(t,x)-f^{ij}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\right\}\}=0;\\ v^{ij}(T,x)=h^{ij}(x). \end{cases}$$

As (i,j) is arbitrary then $(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of (5.22).

Step 2: Uniqueness

Firstly let us suppose the existence of another solution $(\tilde{v}^{ij})_{(i,j)\in\Gamma}$ of system (5.22) which is continuous and of polynomial growth. Next let $(\tilde{y}^{ij})_{(i,j)\in\Gamma}$ be the process of $\mathcal{H}^{2,\Lambda}$ such that for any $(i,j)\in\Gamma$ and $s\le T$,

$$\tilde{y}_s^{ij}=\tilde{v}^{ij}(s,X_s^{t,x}) \tag{5.25}$$

We can now define another process $(\tilde{Y}^{ij})_{(i,j)\in\Gamma}$ via the mapping Φ of (4.7) as follows:

$$(\tilde{Y}^{ij})_{(i,j)\in\Gamma}:=\Phi\left((\tilde{y}^{ij})_{(i,j)\in\Gamma}\right) \tag{5.26}$$

By the definition of Φ, $(\tilde{Y}^{ij})_{(i,j)\in\Gamma}$ is the first component of the unique solution of following doubly RBSDEs: For any $(i,j)\in\Gamma$ and $s\le T$,

$$\begin{cases} \tilde{Y}_s^{ij}=h^{ij}(X_T^{t,x})+\int_s^T f^{ij}(r,X_r^{t,x},(\tilde{v}_r^{kl}(r,X_r^{t,x}))_{(k,l)\in\Gamma})dr-\int_s^T \tilde{Z}_r^{ij}dB_r+\int_s^T d(\tilde{K}_s^{ij,+}-\tilde{K}_s^{ij,-});\\ \max_{k\in(\Gamma^1)^{-i}}\left[\tilde{Y}_s^{kj}-\underline{g}_{ik}(s,X_s^{t,x})\right]\le\tilde{Y}_s^{ij}\le\min_{l\in(\Gamma^2)^{-j}}\left[\tilde{Y}_s^{il}+\overline{g}_{jl}(s,X_s^{t,x})\right];\\ \int_0^T\left\{\tilde{Y}_s^{ij}-\max_{k\in(\Gamma^1)^{-i}}\left[\tilde{Y}_s^{kj}-\underline{g}_{ik}(s,X_s^{t,x})\right]\right\}d\tilde{K}_s^{ij,+}=0;\\ \int_0^T\left\{\tilde{Y}_s^{ij}-\min_{l\in(\Gamma^2)^{-j}}\left[\tilde{Y}_s^{il}+\overline{g}_{jl}(s,X_s^{t,x})\right]\right\}d\tilde{K}_s^{ij,-}=0. \end{cases}$$

As a result, by Theorem 5.3, there exist deterministic functions of polynomial growth, denoted $(u^{ij})_{(i,j)\in\Gamma}$, such that for any $(t,x)\in[0,T]\times\mathbb{R}^k$, $(i,j)\in\Gamma$ and $s\in[t,T]$,

$$\tilde{Y}_s^{ij}=u^{ij}(s,X_s^{t,x}).$$

Moreover by the result of Step 1, $(u^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of the following system of variational inequalities with bilateral interconnected obstacles: $\forall(i,j)\in\Gamma$,

$$\begin{cases} \min\{u^{ij}(t,x)-\max_{k\in(\Gamma^1)^{-i}}[u_t^{kj}-\underline{g}_{ik}(t,x);\max[u^i(t,x)-\min_{l\in(\Gamma^2)^{-j}}[u_t^{il}+\overline{g}_{jl}(t,x);\\ \qquad -\partial_t u^{ij}(t,x)-\mathcal{L}^X u^{ij}(t,x)-f^{ij}(t,x,(\tilde{v}^{kl}(t,x))_{(k,l)\in\Gamma})]\}=0;\\ u^{ij}(T,x)=h^{ij}(x) \end{cases} \tag{5.27}$$

since the generators $f^{ij}(t,x,(\tilde{v}^{kl}(t,x))_{(k,l)\in\Gamma})$, $(i,j)\in\Gamma$, do not depend on the solution $(u^{ij})_{(i,j)\in\Gamma}$. But the solution of system (5.27) is unique in the class of continuous functions of Π_g (see Theorem 3.2 in [3] for more details) and $(\tilde{v}^{ij})_{(i,j)\in\Gamma}$ is a solution in this class. Therefore, for any $(i,j)\in\Gamma$, $u^{ij}=\tilde{v}^{ij}$ and then

$$\mathbb{P}-a.s., \forall s\in[t,T], \tilde{y}^{ij}_s=\tilde{Y}^{ij}_s, \forall (i,j)\in\Gamma. \tag{5.28}$$

Next by (5.26) we obtain on $[t,T]$,

$$(\tilde{y}^{ij})_{(i,j)\in\Gamma} := \Phi\left((\tilde{y}^{ij})_{(i,j)\in\Gamma}\right)$$

However, by Corollary 4.4, $(Y^{ij})_{(i,j)\in\Gamma}$ is the only fixed point of Φ in $(\mathcal{H}^{2,\Lambda}_{[t,T]}, \|.\|_2)$. Therefore for any $(i,j)\in\Gamma, (t,x)\in[0,T]\times\mathbb{R}^k$, $\mathbb{P}-a.s.$, $\forall\ s\in[t,T]$,

$$\tilde{v}^{ij}(s,X^{t,x}_s)=\tilde{y}^{ij}_s=Y^{ij}_s=v^{ij}(s,X^{t,x}_s). \tag{5.29}$$

Take now $s=t$, leads to $\tilde{v}^{ij}(t,x)=v^{ij}(t,x)$ for any $(i,j)\in\Gamma$ which means that the solution is unique. □

Remark 5.5. *The functions $(v^{ij})_{(i,j)\in\Gamma}$ are also the unique solution in the class of continuous functions which belong to Π_g, of the following system which is of max-min type and dual to* (5.22)*:* $\forall (i,j)\in\Gamma$,

$$\begin{cases} \max\{v^{ij}(t,x)-\max\limits_{k\in(\Gamma^1)^{-i}}[v^{kj}(t,x)-\underline{g}_{ik}(t,x)]; \min[v^{ij}(t,x)-\min\limits_{l\in(\Gamma^2)^{-j}}[v^{il}(t,x)+\overline{g}_{jl}(t,x)]; \\ \qquad -\partial_t v^{ij}(t,x)-\mathcal{L}^X(v^{ij})(t,x)-f^{ij}(t,x,(v^{kl}(t,x))_{kl\in\Gamma})]\}=0; \\ v^{ij}(T,x)=h^{ij}(x). \end{cases} \tag{5.30}$$

This can be shown in considering $(-Y^{ij},-Z^{ij},K^{ij,\pm})_{(i,j)\in\Gamma}$ which is the solution of the system of reflected BSDEs with inter-connected bilateral obstacles associated with $((-f^{ij}(t,x,-\vec{y}))_{(i,j)\in\Gamma}$, $(-h^{ij}(x))_{(i,j)\in\Gamma}, (\overline{g}_{jl}(t,x))_{j,l\in\Gamma^2}, (\underline{g}_{ik}(t,x))_{i,k\in\Gamma^1})$ and then use the result of the previous Theorem 5.4 with $(-v^{ij})_{(i,j)\in\Gamma}$ which implies that $(v^{ij})_{(i,j)\in\Gamma}$ is also he unique solution of (5.30)*.*

6. Appendix

The definiton of the viscosity solution of system (5.22) is the following:

Definition 6.1. *Let $\vec{v}:=(v^{ij})_{(i,j)\in\Gamma}$ be a Λ-tuple of continuous functions on $[0,T]\times\mathbb{R}^k$.*

A) We say that $\vec{v}$ is a viscosity supersolution (resp. subsolution) of (5.22) *if for any fixed (i_0,j_0) in Γ, $v^{i_0j_0}$ is a viscosity supersolution (resp. subsolution) of the following PDE with bilateral obstacles:*

$$\begin{cases} \min\{v^{i_0j_0}(t,x)-\max_{k\in(\Gamma^1)^{-i_0}}[v^{kj_0}(t,x)-\underline{g}_{i_0k}(t,x)]; \max\left\{v^{i_0j_0}(t,x)-\min_{l\in(\Gamma^2)^{-j_0}}[v^{i_0l}(t,x)+\overline{g}_{j_0l}(t,x)]; \right. \\ \qquad \left. -\partial_t v^{i_0j_0}(t,x)-\mathcal{L}^X(v^{i_0j_0})(t,x)-f^{i_0j_0}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\right\}\}=0; \\ \\ v^{i_0j_0}(T,x)=h^{i_0j_0}(x), \end{cases} \tag{6.1}$$

that is to say:

i) $v^{i_0j_0}(T,x)\geq h^{i_0j_0}(x)$ (resp. $v^{i_0j_0}(T,x)\leq h^{i_0j_0}(x)$) ;
ii) if $(t,x)\in[0,T)\times\mathbb{R}^k$ and $\phi\in\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k)$ such that (t,x) is a local minimum (resp. maximum) point of $v^{i_0j_0}-\phi$ then

$$\begin{cases} \min\{v^{i_0j_0}(t,x)-\max_{k\in(\Gamma^1)^{-i_0}}[v^{kj_0}(t,x)-\underline{g}_{i_0k}(t,x)]; \max\left\{v^{i_0j_0}(t,x)-\min_{l\in(\Gamma^2)^{-j_0}}[v^{i_0l}(t,x)+\overline{g}_{j_0l}(t,x)]; \right. \\ \qquad \left. -\partial_t\phi(t,x)-\mathcal{L}^X(\phi)(t,x)-f^{i_0j_0}(t,x,(v^{kl}(t,x))_{(k,l)\in\Gamma})\right\}\}\geq 0 \ (resp.\ \leq 0). \end{cases} \tag{6.2}$$

B) We say that $\vec{v}:=(v^{ij})_{(i,j)\in\Gamma}$ is a viscosity solution of (5.22) *if it is both a supersolution and subsolution of* (5.22)*.*

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References

[1] Jean François Chassagneux, Romuald Elie, Idris Kharroubi: A note on existence and uniqueness for solutions of multidimensional reflected BSDEs. *Electronic Communications in Probability*, 16:120–128, 2011.

[2] René Carmona, and Michael Ludkovski: Valuation of energy storage: An optimal switching approach. *Quantitative finance 10.4 (2010): 359-374.*

[3] Boualem Djehiche, Said Hamadène, and Marie-Amélie Morlais: Viscosity solutions of systems of variational inequalities with interconnected bilateral obstacles. *Funkcialaj Ekvacioj*, 58(1):135–175, 2015.

[4] Boualem Djehiche, Said Hamadène, Marie-Amélie Morlais, and Xuzhe Zhao: On the equality of solutions of max–min and min–max systems of variational inequalities with interconnected bilateral obstacles. *Journal of Mathematical Analysis and Applications*, 452(1):148–175, 2017.

[5] Boualem Djehiche, Said Hamadène, and Alexandre Popier: A finite horizon optimal multiple switching problem. *SIAM Journal on Control and Optimization*, 48(4):2751–2770, 2009.

[6] Nicole El Karoui, Shige Peng, and Marie Claire Quenez: Backward stochastic differential equations in finance. *Mathematical finance*, 7(1):1–71, 1997.

[7] Said Hamadène and Mohammed Hassani: BSDEs with two reflecting barriers: the general result. *Probability Theory and Related Fields*, 132(2):237–264, 2005.

[8] Said Hamadène and Monique Jeanblanc: On the starting and stopping problem: application in reversible investments. *Mathematics of Operations Research*, 32(1):182–192, 2007.

[9] Said Hamadène, Randall Martyr, and John Moriarty: A probabilistic verification theorem for the finite horizon two-player zero-sum optimal switching game in continuous time. *arXiv preprint arXiv:1806.00345*, 2018.

[10] Said Hamadène, Mohamed Mnif, and Sarah Neffati: Viscosity solutions of systems of PDEs with interconnected obstacles and switching problem without monotonicity condition. *arXiv preprint arXiv:1802.04747*, 2018.

[11] Said Hamadène and Marie-Amélie Morlais: Viscosity solutions of systems of PDEs with interconnected obstacles and switching problem. *Applied Mathematics & Optimization*, 67(2):163–196, 2013.

[12] Said Hamadène and Jianfeng Zhang: Switching problem and related system of reflected backward SDEs. *Stochastic Processes and their applications*, 120(4):403–426, 2010.

[13] Hitoshi Ishii and Shigeaki Koike: Viscosity solutions of a system of nonlinear second-order elliptic PDEs arising in switching games. *Funkcial. Ekvac 34.1 (1991): 143-155.*

[14] Ying Hu and Shige Peng: On the comparison theorem for multidimensional BSDEs. *C. R. Acad. Sci. Paris, Ser.*, 343(I):135–140, 2006.

[15] Ying Hu and Shanjian Tang: Multi-dimensional BSDE with oblique reflection and optimal switching. *Probability Theory and Related Fields*, 147(1):89–121, 2010.

[16] Ying Hu and Shanjian Tang: Switching game of backward stochastic differential equations and associated system of obliquely reflected backward stochastic differential equations. *Discrete and Continuous Dynamical Systems-Series A*, 35(11):5447–5465, 2015.

[17] Timothy C. Johnson and Mihail Zervos: The explicit solution to a sequential switching problem with non-smooth data.*Stochastics An International Journal of Probability and Stochastics Processes 82.1 (2010): 69-109.*

[18] Vathana Ly Vath, Huyen Pham, and Stephane Villeneuve: A mixed singular/switching control problem for a dividend policy with reversible technology investment. *The Annals of Applied Probability 18.3 (2008) : 1164-1200.*

[19] Étienne Pardoux and Shige Peng: Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.

[20] Philip Protter: Stochastic integration and differential equations. *Math. Appl., Second edition, Springer-Verlag*, 2004.

[21] Daniel Revuz and Marc Yor: Continuous martingales and Brownian motion. *Vol. 293. Springer Science & Business Media, 2013.*

[22] Shanjian Tang, Wei Zhong, and Hyeng Keun Koo: Optimal switching of one-dimensional reflected bsdes and associated multidimensional BSDEs with oblique reflection. *SIAM Journal on Control and Optimization*, 49(6):2279–2317, 2011.

[23] Naoki Yamada: Viscosity solutions for a system of elliptic inequalities with bilateral obstacles. *Funkcial. Ekvac*, 30(2-3):417–425, 1987.

[24] Naoki Yamada: A system of elliptic variational inequalities associated with a stochastic switching game. *Hiroshima Mathematical Journal*, 13(1):109–132, 1983.

[25] Mihail Zervos: A problem of sequential entry and exit decisions combined with discretionary stopping. *SIAM Journal on Control and Optimization 42.2 (2003) : 397-421.*