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Shin-Young Kim, Kyeong-Dong Park

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# ON THE DEFORMATION RIGIDITY OF SMOOTH PROJECTIVE SYMMETRIC VARIETIES WITH PICARD NUMBER ONE

SHIN-YOUNG KIM AND KYEONG-DONG PARK

**ABSTRACT.** Symmetric varieties are normal equivariant open embeddings of symmetric homogeneous spaces and they are interesting examples of spherical varieties. The principal goal of this article is to study the rigidity under Kähler deformations of smooth projective symmetric varieties with Picard number one.

**Résumé.** Les variétés symétriques sont les plongements ouverts normaux équivariants des espaces homogènes symétriques et ce sont des exemples intéressants de variétés sphériques. L'objectif principal de cet article est d'étudier la rigidité sous les déformations kähleriennes des variétés projectives lisses symétriques de nombre de Picard un.

## 1. INTRODUCTION

For a connected semisimple algebraic group  $G$  over  $\mathbb{C}$  and an involution  $\theta$  of  $G$ , the homogeneous space  $G/H$  is called a *symmetric homogeneous space*, where  $H$  is a closed subgroup of  $G$  such that  $G^\theta \subset H \subset N_G(G^\theta)$  (see Section 2.1 for details). A normal  $G$ -variety  $X$  together with an equivariant open embedding  $G/H \hookrightarrow X$  of a symmetric homogeneous space  $G/H$  is called a *symmetric variety*. Our interest in this paper is the rigidity property under Kähler deformation of smooth projective symmetric varieties with Picard number one.

From the Kodaira-Spencer deformation theory (cf. [15]), the vanishing of the first cohomology group  $H^1(G/P, T_{G/P})$  of a rational homogeneous manifold  $G/P$  for a parabolic subgroup  $P \subset G$  implies the local deformation rigidity of  $G/P$ . The *global deformation rigidity* of a rational homogeneous manifold  $G/P$  with Picard number one was studied by Hwang and Mok in [8], [10], [11], [12], [13]: a rational homogeneous manifold with Picard number one, different from the orthogonal isotropic Grassmannian  $\text{Gr}_q(2, 7)$ , is globally rigid. This result can be generalized to some kinds of quasi-homogeneous varieties, for example, odd Lagrangian Grassmannians [20] and odd symplectic Grassmannians [13] among smooth projective horospherical varieties with Picard number one. It is then natural to ask the same questions about smooth projective symmetric varieties. Recently, the local deformation rigidity has been proven for smooth projective symmetric varieties with Picard number one, whose restricted root system is of type  $A_2$  in [6, Proposition 8.4] or [1, Theorem 1.1]. We obtain the global deformation rigidity of two smooth projective symmetric varieties of type  $A_2$  under the assumption that the central fibers, of their deformation families, are not equivariant compactifications of the vector group  $\mathbb{C}^n$ , where  $n$  is the dimension of the symmetric varieties.

**Theorem 1.1.** *Let  $\pi: \mathcal{X} \rightarrow \Delta$  be a smooth projective morphism from a complex manifold  $\mathcal{X}$  to the unit disc  $\Delta \subset \mathbb{C}$ . Denote by  $S$  the smooth equivariant completion with Picard number one of the symmetric homogeneous space  $\text{SL}(6, \mathbb{C})/\text{Sp}(6, \mathbb{C})$  or  $E_6/F_4$ . Suppose for any  $t \in \Delta \setminus \{0\}$ , the fiber  $\mathcal{X}_t = \pi^{-1}(t)$  is biholomorphic to the smooth projective symmetric variety  $S$ . Then the central fiber  $\mathcal{X}_0$  is biholomorphic to either  $S$  or an equivariant compactification of the vector group  $\mathbb{C}^n$ ,  $n = \dim S$ .*

According to Theorem 2 of [21], when smooth projective symmetric varieties with Picard number one have a restricted root system of type  $G_2$ , they are the smooth equivariant completions of either  $G_2/(\text{SL}(2, \mathbb{C}) \times$

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$\mathrm{SL}(2, \mathbb{C})$  or  $(G_2 \times G_2)/G_2$ . Recently, the smooth equivariant completion of  $G_2/(\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}))$ , which is called the *Cayley Grassmannian*, has been studied by Manivel [18]. Combining geometric descriptions of the Cayley Grassmannian in [18] with the normal exact sequence leads to the local deformation rigidity.

**Theorem 1.2.** *The smooth equivariant completion  $S$  with Picard number one of the symmetric homogeneous space  $G_2/(\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}))$  is locally rigid.*

In Section 2, we will review the classification results of smooth projective symmetric varieties with Picard number one and some general results about the variety of minimal rational tangents (VMRT). Moreover, we will prove the deformation rigidity of VMRT as a projective manifold under the assumption in Theorem 1.1. In Section 3, we relate the automorphism group of a projective variety with the prolongations of the Lie algebra of infinitesimal automorphisms of the cone structure given by its VMRT. By considering the smooth projective symmetric varieties with Picard number one of type  $A_2$  and the affine cone of the VMRT, we can prove Theorem 1.1. In Section 4, we prove Theorem 1.2 using the Koszul complex associated to the Cayley Grassmannian and the Borel-Weil-Bott theorem.

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## 2. SYMMETRIC VARIETIES AND VMRT

**2.1. Smooth projective symmetric varieties with Picard number one.** Let  $G$  be a connected semisimple algebraic group over  $\mathbb{C}$  and  $\theta$  be an *involution* of  $G$ , i.e., a nontrivial automorphism  $\theta: G \rightarrow G$  such that  $\theta^2 = id$ .

**Definition 2.1.** Let  $G^\theta = \{g \in G : \theta(g) = g\}$ .

- (1) When  $H$  is a closed subgroup of  $G$  such that  $G^\theta \subset H \subset N_G(G^\theta)$ , we say that the homogeneous space  $G/H$  is a *symmetric (homogeneous) space*. Here,  $N_G(G^\theta)$  means the normalizer of  $G^\theta$  in  $G$ .
- (2) A normal  $G$ -variety  $X$  together with an equivariant open embedding  $G/H \hookrightarrow X$  of a symmetric space  $G/H$  is called a *symmetric variety*.

**Example.** (1) For  $G = \mathrm{SL}(n, \mathbb{C}) \times \mathrm{SL}(n, \mathbb{C})$  and the involution  $\theta(x, y) = (y, x)$ ,  $G^\theta = \{(x, x) \in \mathrm{SL}(n, \mathbb{C}) \times \mathrm{SL}(n, \mathbb{C})\} \cong \mathrm{SL}(n, \mathbb{C})$ . In particular, if  $n = 2$  and  $H = G^\theta$ , then the symmetric space  $G/H \cong \mathrm{SL}(2, \mathbb{C})$  is a closed subvariety of  $\mathrm{Mat}_{2 \times 2}(\mathbb{C}) \cong \mathbb{C}^4$ . Let's consider an equivariant open embedding of  $G/H$ :

$$\begin{aligned} G/H &\hookrightarrow X := \{[x : t] : \det(x) = t^2\} \subset \mathbb{P}(\mathrm{Mat}_{2 \times 2}(\mathbb{C}) \oplus \mathbb{C}) \\ x &\mapsto [x : 1] \end{aligned}$$

Thus, the symmetric variety  $X$  is the 3-dimensional hyperquadric  $\mathbb{Q}^3 \subset \mathbb{P}^4$ .

(2) For  $G = \mathrm{SL}(3, \mathbb{C})$  and the involution  $\theta(g) = (g^t)^{-1}$ , we get  $G^\theta = \mathrm{SO}(3, \mathbb{C})$ . The irreducible representation  $V_{\mathrm{SL}(3, \mathbb{C})}(2\varpi_1) = \mathrm{Sym}^2 \mathbb{C}^3$  is decomposed into  $\mathrm{Sym}^2 \mathbb{C}^3 \cong V_{\mathrm{SO}(3, \mathbb{C})}(4\varpi_1) \oplus V_{\mathrm{SO}(3, \mathbb{C})}(0) = \mathbb{C}^5 \oplus \mathbb{C}$  as  $\mathrm{SO}(3, \mathbb{C})$ -modules. From this result, we have an equivariant open embedding  $\mathrm{SL}(3, \mathbb{C})/N_G(\mathrm{SO}(3, \mathbb{C})) \hookrightarrow \mathbb{P}(\mathrm{Sym}^2 \mathbb{C}^3) \cong \mathbb{P}^5 = X$ .

Vust [23, Theorem 1 in Section 1.3] proved that a symmetric space  $G/H$  is *spherical*, i.e., it has an open orbit under the action of a Borel subgroup of  $G$ . By using the Luna-Vust theory on spherical varieties, Ruzzi [22] classified the smooth projective symmetric varieties with Picard number one using colored fans.

**Theorem 2.2** (Theorem 1 of [21]). *Let  $X$  be a smooth equivariant completion of a symmetric space  $G/H$  with Picard number one. Then  $X$  is nonhomogeneous if and only if*

- (1) *the restricted root system  $\{\alpha - \theta(\alpha) : \alpha \in R_G\} \setminus \{0\}$  has type either  $A_2$  or  $G_2$ , where  $R_G$  denotes the root system of  $G$ , and*
- (2)  *$H = G^\theta$  (the closed subgroup of invariants of  $\theta$ ).*

Given a symmetric space  $G/H$ , there is at most one embedding of  $G/H$  with these properties. Furthermore, all these varieties are projective and Fano.

Moreover, Ruzzi gave a geometric description of smooth projective symmetric varieties with Picard number one whose restricted root system is of type  $A_2$  (Theorem 3 of [21]): these  $A_2$ -type symmetric varieties are smooth equivariant completions of symmetric homogeneous spaces  $\mathrm{SL}(3, \mathbb{C})/\mathrm{SO}(3, \mathbb{C})$ ,  $(\mathrm{SL}(3, \mathbb{C}) \times \mathrm{SL}(3, \mathbb{C}))/\mathrm{SL}(3, \mathbb{C})$ ,  $\mathrm{SL}(6, \mathbb{C})/\mathrm{Sp}(6, \mathbb{C})$ ,  $E_6/F_4$ , and are isomorphic to a general hyperplane section of rational homogeneous manifolds which are in the third row of the *geometric Freudenthal-Tits magic square* (see [3], [16], and Section 3.5 of [17]), respectively.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$v_4(\mathbb{P}^1)$	$\mathbb{P}(T_{\mathbb{P}^2})$	$\mathrm{Gr}_\omega(2, 6)$	$\mathbb{O}\mathbb{P}_0^2$
$\mathbb{C}$	$v_2(\mathbb{P}^2)$	$\mathbb{P}^2 \times \mathbb{P}^2$	$\mathrm{Gr}(2, 6)$	$\mathbb{O}\mathbb{P}^2$
$\mathbb{H}$	$\mathrm{LGr}(3, 6)$	$\mathrm{Gr}(3, 6)$	$\mathbb{S}_6$	$E_7/P_7$
$\mathbb{O}$	$F_4^{ad}$	$E_6^{ad}$	$E_7^{ad}$	$E_8^{ad}$

The fourth row ( $\mathbb{O}$ -row) of the square consists of the adjoint varieties for the exceptional simple Lie groups except  $G_2$ . Taking the varieties of lines through a point, one obtains the third row which are *Legendre varieties*. The second row is deduced from the third row by the same process, which consists of *Severi varieties*. Then by taking hyperplane sections we get the first row of the square.

**2.2. Variety of minimal rational tangents.** In 1990's Hwang and Mok introduced the notion of the *variety of minimal rational tangents* on uniruled projective manifolds (see [9] and [7]). For the study of Fano manifolds, more generally uniruled manifolds, a basic tool is the deformation of rational curves. The study of the deformation of minimal rational curves leads to their associated variety of minimal rational tangents which is defined as the subvariety of the projectivized tangent bundle  $\mathbb{P}(T_X)$  consisting of tangent directions of minimal rational curves immersed in an uniruled projective manifold  $X$ .

Let  $X$  be a projective manifold of dimension  $n$ . By a *parameterized rational curve* we mean a nonconstant holomorphic map  $f: \mathbb{P}^1 \rightarrow X$  from the projective line  $\mathbb{P}^1$  into  $X$ . We say that a (parameterized) rational curve  $f: \mathbb{P}^1 \rightarrow X$  is *free* if the pullback  $f^*T_X$  of the tangent bundle is nonnegative in the sense that  $f^*T_X$  splits into a direct sum  $\mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$  of line bundles of degree  $a_i \geq 0$  for all  $i = 1, \dots, n$ . For a polarized uniruled projective manifold  $(X, L)$  with an ample line bundle  $L$ , a *minimal rational curve* on  $X$  is a free rational curve of minimal degree among all free rational curves on  $X$ .

Let  $\mathcal{J}$  be a connected component of the space of minimal rational curves and let  $\mathcal{K} := \mathcal{J}/\mathrm{Aut}(\mathbb{P}^1)$  be the quotient space of unparameterized minimal rational curves. We call  $\mathcal{K}$  a *minimal rational component*. For a point  $x \in X$  consider the subvariety  $\mathcal{K}_x$  of  $\mathcal{K}$  consisting of minimal rational curves belonging to  $\mathcal{K}$  marked at  $x$ . Define the (*rational*) *tangent map*  $\tau_x: \mathcal{K}_x \dashrightarrow \mathbb{P}(T_x X)$  by  $\tau_x([f(\mathbb{P}^1)]) = [df(T_o\mathbb{P}^1)]$  sending a member of  $\mathcal{K}_x$  smooth at  $x$  to its tangent direction at  $x$ , where  $f: \mathbb{P}^1 \rightarrow X$  is a minimal rational curve with  $f(o) = x$ . For a general point  $x \in X$ , by Theorem 3.4 of [14], this tangent map induces a morphism  $\tau_x: \mathcal{K}_x \rightarrow \mathbb{P}(T_x X)$  which is finite over its image.

**Definition 2.3.** Let  $X$  be a polarized uniruled projective manifold with a minimal rational component  $\mathcal{K}$ . For a general point  $x \in X$ , the image  $\mathcal{C}_x(X) := \tau_x(\mathcal{K}_x) \subset \mathbb{P}(T_x X)$  is called the *variety of minimal rational tangents* (to be abbreviated as VMRT) of  $X$  at  $x$ . The union of  $\mathcal{C}_x$  over general points  $x \in X$  gives the fibered space  $\mathcal{C} \subset \mathbb{P}(T_X) \rightarrow X$  of varieties of minimal rational tangents associated to  $\mathcal{K}$ .

From now on,  $S$  denotes the smooth equivariant completion with Picard number one of the symmetric homogeneous space  $\mathrm{SL}(6, \mathbb{C})/\mathrm{Sp}(6, \mathbb{C})$  or  $E_6/F_4$ , respectively.

**Proposition 2.4.** *For a general point  $s \in S$ , the VMRT  $\mathcal{C}_s(S)$  of  $S$  is projectively equivalent to  $\mathrm{Gr}_\omega(2, 6) \cong C_3/P_2 \subset \mathbb{P}^{13}$  or  $\mathbb{O}\mathbb{P}_0^2 \cong F_4/P_4 \subset \mathbb{P}^{25}$ , respectively. Here,  $P_k \subset G$  means the  $k$ -th maximal parabolic subgroup of  $G$  following the Bourbaki ordering.*

*Proof.* For a nonsingular projective variety  $X$  covered by lines and a general hyperplane section  $X \cap H$ , if  $\mathcal{C}_x \subset \mathbb{P}(T_x X)$  is the VMRT of  $X$  at a general point  $x \in X \cap H$  and  $\dim \mathcal{C}_x$  is positive, then the VMRT associated to a family of lines covering  $X \cap H$  is  $\mathcal{C}_x \cap \mathbb{P}(T_x H) \subset \mathbb{P}T_x(X \cap H)$  from Lemma 3.3 of [4].

From Theorem 3 of [21], the smooth equivariant completion  $S$  with Picard number one of the symmetric space  $\mathrm{SL}(6, \mathbb{C})/\mathrm{Sp}(6, \mathbb{C})$  is isomorphic to a general hyperplane section of the 15-dimensional spinor variety

$\mathbb{S}_6$ . It is known that the VMRT of a rational homogeneous manifold  $G/P$  associated to a *long* simple root  $\alpha_i$  is isomorphic to the *highest weight variety* defined by the isotropy representation of a Levi factor of  $P$  from Proposition 1 of [10]. Note that, in this case, the VMRT of  $G/P$  at the base point is the homogeneous manifold associated to the marked Dynkin diagram having markings corresponding to the simple roots which are adjacent to  $\alpha_i$  in the Dynkin diagram of the semisimple part of  $P$ . Thus, the VMRT of  $\mathbb{S}_6$  is isomorphic to the Grassmannian  $\text{Gr}(2, 6)$ . Since we have the Plücker embedding of the Grassmannian  $\text{Gr}(2, 6) \hookrightarrow \mathbb{P}(\wedge^2 \mathbb{C}^6) \cong \mathbb{P}^{14}$  and a general hyperplane  $H$  in  $\mathbb{P}(\wedge^2 \mathbb{C}^6)$  is given by the kernel of a nondegenerate skew-symmetric 2-form  $\omega \in (\wedge^2 \mathbb{C}^6)^*$ , the VMRT  $\mathcal{C}_s(S)$  is projectively equivalent to the symplectic isotropic Grassmannian  $\text{Gr}_\omega(2, 6) \subset \mathbb{P}^{13}$ .

Similarly, the smooth equivariant completion with Picard number one of  $E_6/F_4$  is isomorphic to a general hyperplane section of the 27-dimensional Hermitian symmetric space  $E_7/P_7$  of compact type by Theorem 3 of [21]. Because the VMRT of  $E_7/P_7$  is isomorphic to  $E_6/P_6 \cong \mathbb{O}\mathbb{P}^2$ , the result follows from the standard facts on the geometric Freudenthal-Tits square summarized in Section 2.1.  $\square$

**Corollary 2.5.** *Let  $\pi: \mathcal{X} \rightarrow \Delta$  be a smooth projective morphism from a complex manifold  $\mathcal{X}$  to the unit disc  $\Delta \subset \mathbb{C}$ . Suppose for any  $t \in \Delta \setminus \{0\}$ , the fiber  $\mathcal{X}_t = \pi^{-1}(t)$  is biholomorphic to the smooth projective symmetric variety  $S$ . Then the VMRT of the central fiber  $\mathcal{X}_0$  at a general point  $x$  is projectively equivalent to  $\text{Gr}_\omega(2, 6) \subset \mathbb{P}^{13}$  or  $\mathbb{O}\mathbb{P}_0^2 \subset \mathbb{P}^{25}$ , respectively.*

*Proof.* Choose a section  $\sigma: \Delta \rightarrow \mathcal{X}$  of  $\pi$  such that  $\sigma(0) = x$  and  $\sigma(t)$  passes through a general point in  $S$  for  $t \neq 0$ . Let  $\mathcal{K}_{\sigma(t)}$  be the normalized Chow space of minimal rational curves passing through  $\sigma(t)$  in  $\mathcal{X}_t$ . Then the canonical map  $\mu: \mathcal{K}_\sigma \rightarrow \Delta$  given by the family  $\{\mathcal{K}_{\sigma(t)}\}$  is smooth and projective by the same proof as that of Proposition 4 of [8]. The main theorem of [12] says that  $\mathcal{K}_{\sigma(t)}$  is isomorphic to  $\mathcal{K}_s(S)$  for all  $t \in \Delta$  because  $\mathcal{K}_s(S) \cong \mathcal{C}_s(S)$  is a rational homogeneous manifold with Picard number one by Proposition 2.4. Thus it suffices to show that the image of the tangent map for the central fiber is nondegenerate in  $\mathbb{P}(T_x \mathcal{X}_0)$ , that is, the image is not contained in any hyperplane of the projective space  $\mathbb{P}(T_x \mathcal{X}_0)$ .

Since  $\dim \text{Gr}_\omega(2, 6) = 7 > \frac{1}{2} \dim(\text{SL}(6, \mathbb{C})/\text{Sp}(6, \mathbb{C})) - 1 = 6$  and  $\dim \mathbb{O}\mathbb{P}_0^2 = 15 > \frac{1}{2} \dim(E_6/F_4) - 1 = 12$ , the distribution spanned by the VMRTs is integrable by Zak's theorem on tangencies ([25] and Proposition 1.3.2 of [9]). Since the second Betti number  $b_2(\mathcal{X}_0) = 1$ , the VMRT  $\mathcal{C}_x(\mathcal{X}_0)$  at a general point  $x$  is nondegenerate in  $\mathbb{P}(T_x \mathcal{X}_0)$  by Proposition 13 of [8].  $\square$

### 3. PROLONGATIONS OF CONE STRUCTURE DEFINED BY VMRT AND PROOF OF THEOREM 1.1

**3.1. Prolongations of a linear Lie algebra.** Let  $M$  be a differentiable manifold. Fix a vector space  $V$  with  $\dim V = \dim M$ . A *frame* at  $x \in M$  is a linear isomorphism  $\sigma: V \rightarrow T_x M$ . A *frame bundle*  $\mathcal{F}(M)$  on  $M$  is the set of all frames  $\mathcal{F}_x(M) := \text{Isom}(V, T_x M)$  at every point  $x \in M$ . Then  $\mathcal{F}(M)$  is a principal  $\text{GL}(V)$ -bundle on  $M$ . For a closed Lie subgroup  $G \subset \text{GL}(V)$ , a (*geometric*)  $G$ -*structure* on  $M$  is defined as a  $G$ -subbundle  $\mathcal{G} \subset \mathcal{F}(M)$  of the frame bundle. The subbundle  $V \times G$  of the frame bundle  $\mathcal{F}(V) = V \times \text{GL}(V)$  is called the *flat  $G$ -structure* on  $V$  and the  $G$ -structure on  $M$  is *locally flat* if it is locally equivalent to the flat  $G$ -structure on  $V$ . The (*algebraic*) *prolongations*  $\mathfrak{g}^{(k)}$  of a linear Lie algebra  $\mathfrak{g} \subset \mathfrak{gl}(V)$  originate from the higher order derivatives of the infinitesimal automorphisms of the flat  $G$ -structure on  $V$ .

**Definition 3.1.** Let  $V$  be a complex vector space and  $\mathfrak{g} \subset \mathfrak{gl}(V)$  a linear Lie algebra. For an integer  $k \geq 0$ , the space  $\mathfrak{g}^{(k)}$ , called the  *$k$ -th prolongation* of  $\mathfrak{g}$ , is the vector space of symmetric multi-linear homomorphisms  $A: \text{Sym}^{k+1} V \rightarrow V$  such that for any fixed vectors  $v_1, \dots, v_k \in V$ , the endomorphism

$$v \in V \mapsto A(v_1, \dots, v_k, v) \in V$$

belongs to the Lie algebra  $\mathfrak{g}$ . That is,  $\mathfrak{g}^{(k)} = \text{Hom}(\text{Sym}^{k+1} V, V) \cap \text{Hom}(\text{Sym}^k V, \mathfrak{g})$ .

We are interested in the case where a Lie algebra  $\mathfrak{g}$  is relevant to geometric contexts, in particular, the Lie algebra of infinitesimal linear automorphisms of the affine cone of an irreducible projective subvariety.

**Definition 3.2.** Let  $Z \subset \mathbb{P}V$  be an irreducible projective variety. The *projective automorphism group* of  $Z$  is  $\text{Aut}(Z) = \{g \in \text{PGL}(V) : g(Z) = Z\}$  and its Lie algebra is denoted by  $\mathfrak{aut}(Z)$ . Denote by  $\widehat{Z} \subset V$  the affine cone of  $Z$  and by  $T_\alpha \widehat{Z} \subset V$  the affine tangent space at a smooth point  $\alpha \in \widehat{Z}$ . The *Lie algebra of infinitesimal linear automorphisms* of  $\widehat{Z}$  is defined by

$$\mathfrak{aut}(\widehat{Z}) = \{A \in \mathfrak{gl}(V) : \exp(tA)(\widehat{Z}) \subset \widehat{Z}, t \in \mathbb{C}\},$$

where  $\exp(tA)$  denotes the one-parameter group of linear automorphisms of  $V$ . Its  $k$ -th prolongation  $\mathbf{aut}(\widehat{Z})^{(k)}$  will be called the  $k$ -th prolongation of  $Z \subset \mathbb{P}V$ .

In [12], Hwang and Mok studied the prolongations  $\mathbf{aut}(\widehat{Z})^{(k)}$  of a projective variety  $Z \subset \mathbb{P}V$  using the projective geometry of  $Z$  and the deformation theory of rational curves on  $Z$ . In particular, the vanishing of the second prolongation  $\mathbf{aut}(\widehat{Z})^{(2)}$  for an irreducible smooth nondegenerate projective variety  $Z$  embedded in the projective space  $\mathbb{P}V$  was proven.

**Proposition 3.3** (Theorem 1.1.2 of [12]). *Let  $Z \subset \mathbb{P}V$  be an irreducible smooth nondegenerate projective variety. If  $Z \neq \mathbb{P}V$ , then the second prolongation of  $Z$  vanishes, that is,  $\mathbf{aut}(\widehat{Z})^{(2)} = 0$ .*

From the definition of prolongations, it is immediate that  $\mathfrak{g}^{(k)} = 0$  for some  $k \geq 0$  implies  $\mathfrak{g}^{(k+1)} = 0$ . Thus if  $Z \not\subseteq \mathbb{P}V$  is an irreducible smooth nondegenerate projective variety, then  $\mathbf{aut}(\widehat{Z})^{(k)} = 0$  for  $k \geq 2$ .

**3.2. Infinitesimal automorphisms of the cone structures.** A cone structure  $\mathcal{C}$  on a complex manifold  $M$  is a closed analytic subvariety  $\mathcal{C} \subset \mathbb{P}(T_M)$  such that the natural projection  $\pi: \mathcal{C} \rightarrow M$  is proper, flat and surjective with connected fibers. We denote the fiber  $\pi^{-1}(x)$  by  $\mathcal{C}_x$  for a point  $x \in M$ . Let  $\mathcal{C}$  be a cone structure on a complex manifold  $M$ . A germ of holomorphic vector field  $v$  at  $x \in M$  is said to preserve the cone structure if the local one-parameter family of biholomorphisms integrating  $v$  lifts to local biholomorphisms of  $\mathbb{P}(T_M)$  preserving  $\mathcal{C}$ .

**Definition 3.4.** The Lie algebra  $\mathbf{aut}(\mathcal{C}, x)$  of infinitesimal automorphisms of the cone structure  $\mathcal{C}$  at  $x$  is the set of all germs of holomorphic vector fields preserving the cone structure  $\mathcal{C}$  at  $x$ .

The Lie algebra  $\mathbf{aut}(\mathcal{C}, x)$  is naturally filtered by the vanishing order of vector fields at  $x$ . More precisely, for each integer  $k \geq 0$ , let  $\mathbf{aut}(\mathcal{C}, x)_k$  be the subalgebra of  $\mathbf{aut}(\mathcal{C}, x)$  consisting of vector fields that vanish at  $x$  to the order  $\geq k + 1$  for each integer  $k \geq 0$ . The Lie bracket gives the structure of filtration

$$\mathbf{aut}(\mathcal{C}, x) \supset \mathbf{aut}(\mathcal{C}, x)_0 \supset \mathbf{aut}(\mathcal{C}, x)_1 \supset \mathbf{aut}(\mathcal{C}, x)_2 \supset \cdots.$$

Let  $\xi$  be a germ of holomorphic vector field on  $M$  vanishing to order  $\geq k + 1$  at  $x$ . Then its  $(k + 1)$ -jet  $J_x^{k+1}(\xi)$  defines an element of  $\mathrm{Sym}^{k+1}(T_x^*M) \otimes T_xM$ . Because  $J_x^{k+1}(\zeta) = 0$  for a vector field  $\zeta$  vanishing to order  $\geq k + 2$  at  $x$ , this defines the inclusion  $\mathbf{aut}(\mathcal{C}, x)_k / \mathbf{aut}(\mathcal{C}, x)_{k+1} \subset \mathrm{Hom}(\mathrm{Sym}^{k+1}(T_xM), T_xM)$ . The following result follows from Proposition 1.2.1 of [12].

**Proposition 3.5.** *Let  $\mathcal{C} \subset \mathbb{P}(T_M)$  be a cone structure on a complex manifold  $M$  and  $x \in M$  a point. For each  $k \geq 0$ , if the quotient space  $\mathbf{aut}(\mathcal{C}, x)_k / \mathbf{aut}(\mathcal{C}, x)_{k+1}$  is regarded as a subspace of  $\mathrm{Hom}(\mathrm{Sym}^{k+1}(T_xM), T_xM)$ , then we have the inclusion*

$$\mathbf{aut}(\mathcal{C}, x)_k / \mathbf{aut}(\mathcal{C}, x)_{k+1} \subset \mathbf{aut}(\widehat{\mathcal{C}}_x)^{(k)}.$$

From Proposition 3.5, we have the natural inequalities

$$\begin{aligned} \dim \mathbf{aut}(\mathcal{C}, x)_0 &\leq \dim \mathbf{aut}(\widehat{\mathcal{C}}_x) + \dim \mathbf{aut}(\mathcal{C}, x)_1 \\ &\leq \dim \mathbf{aut}(\widehat{\mathcal{C}}_x) + \dim \mathbf{aut}(\widehat{\mathcal{C}}_x)^{(1)} + \dim \mathbf{aut}(\mathcal{C}, x)_2 \leq \cdots \end{aligned}$$

Because the codimension of  $\mathbf{aut}(\mathcal{C}, x)_0$  in  $\mathbf{aut}(\mathcal{C}, x)$  is at most  $\dim M$ , we obtain the following direct consequence (see Proposition 5.10 of [4]).

**Corollary 3.6.** *Let  $\mathcal{C} \subset \mathbb{P}(T_M)$  be a cone structure on a complex manifold  $M$  and  $x \in M$ . If  $\mathbf{aut}(\widehat{\mathcal{C}}_x)^{(k+1)} = 0$  for some  $k \geq 0$ , then*

$$\dim \mathbf{aut}(\mathcal{C}, x) \leq \dim M + \dim \mathbf{aut}(\widehat{\mathcal{C}}_x) + \dim \mathbf{aut}(\widehat{\mathcal{C}}_x)^{(1)} + \cdots + \dim \mathbf{aut}(\widehat{\mathcal{C}}_x)^{(k)}.$$

**3.3. Cone structure defined by VMRT.** Let  $Z \subset \mathbb{P}V$  be a (fixed) projective variety with  $\dim V = \dim M$ . A cone structure  $\mathcal{C} \subset \mathbb{P}(T_M)$  is  $Z$ -isotrivial if for a general point  $x \in M$ , the fiber  $\mathcal{C}_x \subset \mathbb{P}(T_xM)$  is isomorphic to  $Z \subset \mathbb{P}V$  as a projective variety, i.e., there exists a linear isomorphism  $V \rightarrow T_xM$  sending  $Z$  to  $\mathcal{C}_x$ .

For the affine cone  $\widehat{Z} \subset V$  of  $Z$ , let  $G = \mathrm{Aut}(\widehat{Z}) = \{g \in \mathrm{GL}(V) : g(\widehat{Z}) = \widehat{Z}\}$  be the automorphism group of  $\widehat{Z} \subset V$ . A  $Z$ -isotrivial cone structure  $\mathcal{C}$  on  $M$  induces the  $G$ -structure  $\mathcal{G}$  of cone type of which a fiber at general point  $x$  is

$$\mathcal{G}_x = \{\sigma \in \mathrm{Isom}(V, T_xM) : \sigma(\widehat{Z}) = \widehat{\mathcal{C}}_x\}.$$

An isotrivial cone structure  $\mathcal{C}$  on  $M$  is *locally flat* if its associated  $G$ -structure  $\mathcal{G}$  is locally flat. We know that if  $\mathcal{C}$  is a locally flat cone structure on  $M$  with  $\mathbf{aut}(\widehat{\mathcal{C}}_x) = \mathfrak{g}$  and  $k$  is a nonnegative integer such that  $\mathbf{aut}(\widehat{\mathcal{C}}_x)^{(k+1)} = 0$ , then  $\mathbf{aut}(\mathcal{C}, x)$  is isomorphic to the graded Lie algebra  $V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(k)}$ . Conversely, if the equality in Corollary 3.6 holds, then the cone structure  $\mathcal{C}$  is locally flat by Corollary 5.13 of [4].

Now, we are ready to prove Theorem 1.1 by considering the cone structure defined by VMRT which is  $Z$ -isotrivial.

**3.4. Proof of Theorem 1.1.** (i) If  $S$  is the smooth equivariant completion with Picard number one of the symmetric homogeneous space  $\mathrm{SL}(6, \mathbb{C})/\mathrm{Sp}(6, \mathbb{C})$ , then its automorphism group  $\mathrm{Aut}(S)$  is generated by  $\mathrm{PSL}(6, \mathbb{C})$  and the involution  $\theta$  with  $\mathrm{SL}(6, \mathbb{C})^\theta = \mathrm{Sp}(6, \mathbb{C})$  by Proposition 3 of [21].

From Corollary 2.5, we can compute the Lie algebra of infinitesimal automorphisms of the affine cone of VMRT:  $\mathbf{aut}(\widehat{\mathcal{C}}_x(\mathcal{X}_0)) = \mathbf{aut}(\widehat{\mathcal{C}}_s(S)) \cong \mathfrak{sp}(6, \mathbb{C}) \oplus \mathbb{C}$ . Since the variety  $\mathcal{C}_s(S)$  of minimal rational tangents of  $S$  is irreducible smooth nondegenerate and not linear,  $\mathbf{aut}(\widehat{\mathcal{C}}_s)^{(k)} = 0$  for  $k \geq 2$  by Proposition 3.3. Furthermore, the classification of projective varieties with non-zero prolongation in [5] implies that  $\mathbf{aut}(\widehat{\mathcal{C}}_s)^{(k)} = 0$  for all  $k \geq 1$ . Thus, for the cone structure  $\mathcal{C}$  on the fibers given by VMRTs we have the equalities

$$\dim \mathbf{aut}(S) + 1 = \dim S + \dim \mathbf{aut}(\widehat{\mathcal{C}}_s) = \dim \mathcal{X}_0 + \dim \mathbf{aut}(\widehat{\mathcal{C}}_x).$$

Because the Lie algebra  $\mathbf{aut}(\mathcal{X}_0)$  is isomorphic to the space  $H^0(\mathcal{X}_0, T_{\mathcal{X}_0})$  of global sections of the tangent bundle  $T_{\mathcal{X}_0}$ , we know  $h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) = \dim \mathbf{aut}(\mathcal{X}_0)$ . Since the action of  $\mathrm{Aut}(\mathcal{X}_0)$  preserves the VMRT-structure  $\mathcal{C}$  on  $\mathcal{X}_0$ , we have an inclusion  $\mathbf{aut}(\mathcal{X}_0) \subset \mathbf{aut}(\mathcal{C}, x)$ . Hence, from Corollary 3.6 we have the inequality

$$\begin{aligned} h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) = \dim \mathbf{aut}(\mathcal{X}_0) &\leq \dim \mathbf{aut}(\mathcal{C}, x) \\ &\leq \dim \mathcal{X}_0 + \dim \mathbf{aut}(\widehat{\mathcal{C}}_x) \\ &= \dim \mathbf{aut}(S) + 1 = h^0(S, T_S) + 1, \end{aligned}$$

Now, recall the standard fact that the Euler-Poincaré characteristic of the holomorphic tangent bundle  $T_X$  on a Fano manifold  $X$  is given by  $\chi(X, T_X) = h^0(X, T_X) - h^1(X, T_X)$ . In fact, the Serre duality and Kodaira-Nakano vanishing theorem imply that  $H^i(X, T_X) = H^{n-i}(X, T_X^* \otimes K_X)^* = 0$  for  $i \geq 2$ . Since the Euler-Poincaré characteristic is constant in a smooth family and we already know  $h^1(S, T_S) = 0$  by Proposition 8.4 of [6],  $h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = h^0(\mathcal{X}_0, T_{\mathcal{X}_0}) - h^0(S, T_S) \leq 1$ .

Now, it suffices to consider two possible cases. Suppose that the above equality holds. Then we have  $\dim \mathbf{aut}(\mathcal{C}, x) = \dim \mathcal{X}_0 + \dim \mathbf{aut}(\widehat{\mathcal{C}}_x)$ , which implies that the isotrivial cone structure  $\mathcal{C}$  given by VMRT on the central fiber  $\mathcal{X}_0$  should be locally flat by Corollary 5.13 of [4]. Thus  $\mathcal{X}_0$  is an equivariant compactification of the vector group  $\mathbb{C}^{14}$  from Theorem 1.2 of [6]. Next, if  $h^1(\mathcal{X}_0, T_{\mathcal{X}_0}) = 0$ , then the central fiber  $\mathcal{X}_0$  is also biholomorphic to the general fiber  $S$ .

(ii) If  $S$  is the smooth equivariant completion with Picard number one of the symmetric homogeneous space  $E_6/F_4$ , then its automorphism group  $\mathrm{Aut}(S)$  is generated by  $E_6$  and the involution  $\theta$  with  $E_6^\theta = F_4$  by Proposition 3 of [21]. From Corollary 2.5,  $\mathbf{aut}(\widehat{\mathcal{C}}_x(\mathcal{X}_0)) = \mathbf{aut}(\widehat{\mathcal{C}}_s(S)) \cong \mathfrak{f}_4 \oplus \mathbb{C}$ . Because  $\mathbf{aut}(\widehat{\mathcal{C}}_s)^{(k)} = 0$  for all  $k \geq 1$  by [5], we also have the same equality as before:

$$\dim \mathbf{aut}(S) + 1 = \dim S + \dim \mathbf{aut}(\widehat{\mathcal{C}}_s).$$

By Proposition 8.4 of [6], a general hyperplane section of  $E_7/P_7$  is locally rigid, so we see that  $h^1(S, T_S) = 0$ . Therefore, the same argument as (i) works immediately.  $\square$

#### 4. LOCAL RIGIDITY OF SMOOTH PROJECTIVE SYMMETRIC VARIETIES OF TYPE $G_2$

The smooth equivariant completion  $S$  with Picard number one of the symmetric space  $G_2/(\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}))$ , called the *Cayley Grassmannian*, has been studied by Manivel [18]. The Cayley Grassmannian is a smooth projective variety parametrizing four-dimensional subalgebras of the complexified octonions  $\mathbb{O}_{\mathbb{C}}$ . Because all subalgebras contain the unit element, the Cayley Grassmannian is a closed subvariety of the Grassmannian  $\mathrm{Gr}(3, 7)$  by considering only the imaginary parts. It can be also described as a subvariety of the Grassmannian  $\mathrm{Gr}(4, 7)$  by mapping a subalgebra to its orthogonal complement contained in the imaginary part of  $\mathbb{O}_{\mathbb{C}}$ . From now, we will consider the Cayley Grassmannian as a subvariety of  $\mathrm{Gr}(4, 7)$ .

**Proposition 4.1** (Proposition 3.2 of [18]). *The Cayley Grassmannian  $S$  is projectively equivalent to the zero locus of a general global section of the rank four vector bundle  $\wedge^3 \mathcal{U}^*$  on the Grassmannian  $\text{Gr}(4, 7)$ , where  $\mathcal{U}$  denotes the universal subbundle of rank four on  $\text{Gr}(4, 7)$ .*

*Remark 4.2.* The symmetric variety  $S$  is a Fano eightfold of index 4 by the adjunction formula. Indeed,  $K_S = K_{\text{Gr}(4,7)} \otimes \det(\wedge^3 \mathcal{U}^*) = \mathcal{O}(-7) \otimes \mathcal{O}(3) = \mathcal{O}(-4)$ . This implies that the VMRT of  $S$  at a general point is isomorphic to a surface embedded in  $\mathbb{P}^7$ .

From the Kodaira-Spencer deformation theory, it suffices to prove  $H^1(S, T_S) = 0$  for Theorem 1.2. Now, we recall Borel-Weil-Bott theorem to compute the cohomology groups of equivariant vector bundles on a rational homogeneous variety  $G/P$ . Let  $G$  be a simply connected complex semisimple algebraic group and  $P \subset G$  a parabolic subgroup. For an integral dominant weight  $\omega$  with respect to  $P$ , we have an irreducible representation  $V(\omega)$  of  $P$  with the highest weight  $\omega$ , and denote by  $\mathcal{E}_\omega$  the corresponding *irreducible equivariant vector bundle*  $G \times_P V(\omega)$  on  $G/P$ :

$$\mathcal{E}_\omega := G \times_P V(\omega) = (G \times V(\omega))/P,$$

where the equivalence relation is given by  $(g, v) \sim (gp, p^{-1}.v)$  for  $p \in P$ .

**Theorem 4.3** (Borel-Weil-Bott theorem [2]). *Let  $\rho$  denote the sum of fundamental weights of  $G$ .*

- *If a weight  $\omega + \rho$  is singular, that is, it is orthogonal to some (positive) root of  $G$ , then all cohomology groups  $H^i(G/P, \mathcal{E}_\omega)$  vanish for all  $i$ .*
- *Otherwise,  $\omega + \rho$  is regular, that is, it lies in the interior of some Weyl chamber, then  $H^{\ell(w)}(G/P, \mathcal{E}_\omega) = V_G(w(\omega + \rho) - \rho)^*$  and any other cohomology vanishes. Here,  $w \in W$  is a unique element of the Weyl group of  $G$  such that  $w(\omega + \rho)$  is strictly dominant, and  $\ell(w)$  means the length of  $w \in W$ , that is, the minimal integer  $\ell(w)$  such that  $w$  can be expressed as a product of  $\ell(w)$  simple reflections.*

*Proof of Theorem 1.2.* Since  $S$  is the zero locus of a generic global section of  $\wedge^3 \mathcal{U}^*$  on  $\text{Gr}(4, 7)$ , we have the normal exact sequence on  $S$

$$0 \rightarrow T_S \rightarrow T_{\text{Gr}(4,7)}|_S \rightarrow \wedge^3 \mathcal{U}^*|_S \rightarrow 0$$

and the Koszul complex of the structure sheaf  $\mathcal{O}_S$

$$0 \rightarrow \wedge^4(\wedge^3 \mathcal{U}) \rightarrow \wedge^3(\wedge^3 \mathcal{U}) \rightarrow \wedge^2(\wedge^3 \mathcal{U}) \rightarrow \wedge^3 \mathcal{U} \rightarrow \mathcal{O}_{\text{Gr}(4,7)} \rightarrow \mathcal{O}_S \rightarrow 0.$$

Using the isomorphisms  $\wedge^4 \mathcal{U} \cong \mathcal{O}(-1)$  and  $\wedge^3 \mathcal{U} \cong \mathcal{U}^*(-1)$  on  $\text{Gr}(4, 7)$ , we get an exact sequence

$$0 \rightarrow \mathcal{O}_{\text{Gr}(4,7)}(-3) \rightarrow \mathcal{U}(-2) \rightarrow \wedge^2 \mathcal{U}(-1) \rightarrow \wedge^3 \mathcal{U} \rightarrow \mathcal{O}_{\text{Gr}(4,7)} \rightarrow \mathcal{O}_S \rightarrow 0.$$

Indeed,  $\wedge^3(\wedge^3 \mathcal{U}) \cong \wedge^3(\mathcal{U}^*(-1)) = \wedge^3 \mathcal{U}^* \otimes \mathcal{O}(-3) \cong \mathcal{U}(1) \otimes \mathcal{O}(-3) = \mathcal{U}(-2)$ . Taking the tensor product of the Koszul complex with  $\wedge^3 \mathcal{U}^*$ , we have

$$0 \rightarrow \wedge^3 \mathcal{U}^*(-3) \rightarrow \mathcal{U}(-2) \otimes \wedge^3 \mathcal{U}^* \rightarrow \wedge^2 \mathcal{U}(-1) \otimes \wedge^3 \mathcal{U}^* \rightarrow \wedge^3 \mathcal{U} \otimes \wedge^3 \mathcal{U}^* \rightarrow \wedge^3 \mathcal{U}^* \rightarrow \wedge^3 \mathcal{U}^*|_S \rightarrow 0.$$

Let  $\omega_1, \dots, \omega_6$  be the fundamental weights of  $\text{SL}(7, \mathbb{C})$ . Since  $\wedge^3 \mathcal{U}^*(-3)$  is the irreducible equivariant vector bundle associated with the weight  $\omega_3 - 3\omega_4$  and the weight  $\omega_3 - 3\omega_4 + \rho$  is singular, as a straightforward application of the Borel-Weil-Bott theorem, we see that  $H^i(\text{Gr}(4, 7), \wedge^3 \mathcal{U}^*(-3)) = 0$  for all  $i$ . Also, because

$$\mathcal{U}(-2) \otimes \wedge^3 \mathcal{U}^* \cong \mathcal{U}(-2) \otimes \mathcal{U}(1) \cong (\wedge^2 \mathcal{U} \oplus \text{Sym}^2 \mathcal{U}) \otimes \mathcal{O}(-1) = \mathcal{E}_{\omega_2 - 2\omega_4} \oplus \mathcal{E}_{2\omega_3 - 3\omega_4}$$

and both  $\omega_2 - 2\omega_4 + \rho$  and  $2\omega_3 - 3\omega_4 + \rho$  are singular weights, we have that  $H^i(\text{Gr}(4, 7), \mathcal{U}(-2) \otimes \wedge^3 \mathcal{U}^*) = 0$  for all  $i$ . From the Littlewood-Richardson rule (see [24] for details), we can check that  $\wedge^2 \mathcal{U}(-1) \otimes \wedge^3 \mathcal{U}^* \cong \mathcal{E}_{\omega_1 - \omega_4} \oplus \mathcal{E}_{\omega_2 + \omega_3 - 2\omega_4}$  which implies that  $H^i(\text{Gr}(4, 7), \wedge^2 \mathcal{U}(-1) \otimes \wedge^3 \mathcal{U}^*) = 0$  for all  $i$ . Since we know that  $\wedge^3 \mathcal{U} \otimes \wedge^3 \mathcal{U}^* \cong \mathcal{E}_{\omega_1 + \omega_3}(-1) \oplus \mathcal{O}$  by the Littlewood-Richardson rule,  $H^i(\text{Gr}(4, 7), \wedge^3 \mathcal{U} \otimes \wedge^3 \mathcal{U}^*) = 0$  for  $i > 0$  and  $H^0(\text{Gr}(4, 7), \wedge^3 \mathcal{U} \otimes \wedge^3 \mathcal{U}^*) = H^0(\text{Gr}(4, 7), \mathcal{O}) = \mathbb{C}$ . Again, the Borel-Weil theorem says that  $H^0(\text{Gr}(4, 7), \wedge^3 \mathcal{U}^*) = \wedge^3 \mathbb{C}^7$ . Therefore, we conclude  $H^0(S, \wedge^3 \mathcal{U}^*|_S) = \wedge^3 \mathbb{C}^7 / \mathbb{C}$ .

Using the Littlewood-Richardson rule, we get the Koszul complex of the structure sheaf  $\mathcal{O}_S$  tensored with the tangent bundle  $T_{\text{Gr}(4,7)} = \mathcal{U} \otimes \mathcal{Q} = \mathcal{E}_{\omega_1 + \omega_6}$ :

$$\begin{aligned} 0 \rightarrow \mathcal{E}_{\omega_1 - 3\omega_4 + \omega_6} &\rightarrow \mathcal{E}_{\omega_1 + \omega_3 - 3\omega_4 + \omega_6} \oplus \mathcal{E}_{-2\omega_4 + \omega_6} \rightarrow \mathcal{E}_{\omega_1 + \omega_2 - 2\omega_4 + \omega_6} \oplus \mathcal{E}_{\omega_3 - 2\omega_4 + \omega_6} \\ &\rightarrow \mathcal{E}_{2\omega_1 - \omega_4 + \omega_6} \oplus \mathcal{E}_{\omega_2 - \omega_4 + \omega_6} \rightarrow T_{\text{Gr}(4,7)} \rightarrow T_{\text{Gr}(4,7)}|_S \rightarrow 0. \end{aligned}$$



Since all bundles except the last two terms are acyclic, using the Borel-Weil-Bott theorem again, we obtain  $H^0(S, T_{\mathrm{Gr}(4,7)}|_S) = H^0(\mathrm{Gr}(4,7), T_{\mathrm{Gr}(4,7)}) = \mathfrak{sl}_7$  and  $H^1(S, T_{\mathrm{Gr}(4,7)}|_S) = H^1(\mathrm{Gr}(4,7), T_{\mathrm{Gr}(4,7)}) = 0$ .

Then, from the normal exact sequence on  $S$ , we deduce an exact sequence

$$0 \rightarrow H^0(S, T_S) \rightarrow \mathfrak{sl}_7 \rightarrow \wedge^3 \mathbb{C}^7 / \mathbb{C} \rightarrow H^1(S, T_S) \rightarrow 0.$$

Hence,  $H^0(S, T_S) = \mathfrak{aut}(S) = \mathfrak{g}_2$  (Lemma 16 of [21]) implies  $H^1(S, T_S) = 0$  from which the local rigidity of  $S$  follows by the Kodaira-Spencer deformation theory.  $\square$

*Remark 4.4.* By Theorem 2 of [21], the smooth projective symmetric varieties with Picard number one whose restricted root system is of type  $G_2$  are either the Cayley Grassmannian or the smooth equivariant completion of  $(G_2 \times G_2)/G_2$ . Recently, Manivel also studied the latter, called the *double Cayley Grassmannian*, and proved that it is locally rigid in [19]. The double Cayley Grassmannian is projectively equivalent to the zero locus of a general global section of the rank seven vector bundle  $\mathcal{U} \otimes \mathcal{L}$  on the 21-dimensional spinor variety  $\mathbb{S}_{14} = \mathrm{Spin}(14, \mathbb{C})/P_7$ , where  $\mathcal{U}$  denotes the tautological bundle of rank seven on  $\mathbb{S}_{14}$  and  $\mathcal{L}$  is the very ample line bundle defining the minimal embedding  $\mathbb{S}_{14} \hookrightarrow \mathbb{P}^{63}$ . Consequently, we conclude that all smooth projective symmetric varieties with Picard number one are locally rigid. On the other hand, the global deformation rigidity problem on smooth projective symmetric varieties of type  $G_2$  remains open.

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SHIN-YOUNG KIM, INSTITUT FOURIER, GRENOBLE 38058, FRANCE

*Current Address:* CENTER FOR GEOMETRY AND PHYSICS, INSTITUTE FOR BASIC SCIENCE (IBS), POHANG 37673, KOREA

*E-mail address:* [Shinyoung.Kim@univ-grenoble-alpes.fr](mailto:Shinyoung.Kim@univ-grenoble-alpes.fr), [shinyoungkim@ibs.re.kr](mailto:shinyoungkim@ibs.re.kr)

KYEONG-DONG PARK, CENTER FOR GEOMETRY AND PHYSICS, INSTITUTE FOR BASIC SCIENCE (IBS), POHANG 37673, KOREA

*E-mail address:* [kdpark@ibs.re.kr](mailto:kdpark@ibs.re.kr)