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Convergence of a family of perturbed conservation laws with diffusion and non-positive dispersion

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Abstract: We consider a family of conservation laws with convex flux perturbed by vanishing diffusion and non-positive dispersion of the form $u_t + f(u)_x = \varepsilon\, u_{xx} - \delta(|u_{xx}|^n)_x$. Convergence of the solutions $\{u^{\varepsilon,\delta}\}$ to the entropy weak solution of the hyperbolic limit equation $u_t + f(u)_x = 0$, for all real numbers $1 \leq n \leq 2$ is proved if $\delta = o(\varepsilon^{\frac{3n-1}{2}}; \varepsilon^{\frac{5n-1}{2(2n-1)}})$.

Keywords: diffusion, nonlinear dispersion, KdV-Burgers equation, hyperbolic conservation laws, entropy measure-valued solutions.

1 Introduction

We consider the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} + \delta g(u_{xx})_x \tag{1.1}$$
$$u(x,0) = u_0(x) \tag{1.2}$$

where f is a convex function, (i.e. $f'' \geq 0$), ϵ and δ are small positive parameters, g is a non-positive function of the form, with $1 \leq n \leq 2$,

$$g(v) = -|v|^n.$$

When $\delta = 0$, equation (1.1) reduces to the generalized viscous Burgers equation

$$u_t + f(u)_x = \varepsilon u_{xx}$$

and we know, as $\varepsilon \searrow 0$, the corresponding solution $u^{\varepsilon,0}$ converges to the entropy weak solution of the hyperbolic problem (the *vanishing viscosity method*, see, e.g., Whitham [13] or Kružkov [6])

$$u_t + f(u)_x = 0 \tag{1.3}$$
$$u(x,0) = u_0(x). \tag{1.4}$$

On the other hand, when $\varepsilon = 0$, if we consider the flux function $f(u) = u^2$ and the linear dispersion, i.e., if we consider the Körteweg-de Vries equation

$$u_t + \left(u^2/2\right)_x = \delta\, u_{xxx},$$

the solution $u^{0,\delta}$ does not converge in a strong topology (see Lax-Levermore [7]). We are concerned with singular limits.

In the general case, as parameters $\varepsilon > 0$ and $\delta \neq 0$ vanish, to ensure convergence we wait that a dominant dissipation regime is necessary. The pioneer study of these singular limits was given in 1982 by Schonbek [10] for the generalized Körteweg-de Vries-Burgers equation

$$u_t + f(u)_x = \varepsilon\, u_{xx} - \delta\, u_{xxx}.$$

For a convex flux function $f(u)$, she proved the convergence of the solutions $u^{\varepsilon,\delta}$ to the entropy weak solution of (1.3)-(1.4) under the condition $\delta = \mathcal{O}(\varepsilon^3)$ and, for the particular flux function $f(u) = u^2$, under the condition $\delta = \mathcal{O}(\varepsilon^2)$. While for the last case, according to Perthame-Ryzhik [9], the sharp condition could be $\delta = o(\varepsilon)$.

In "From Boltzmann to Euler: Hilbert's 6th problem revisited"(Slemrod [11]) one may see the analogy between the singular limit for the Körteweg-de Vries-Burgers equation and the hydrodynamic limit of the kinetic Boltzmann equation for a rarefied gas to the continuum Euler equations of compressible gas dynamics as the Knudsen number approaches zero.

In 1998, LeFloch-Natalini [8] proved the convergence in the case of a nonlinear viscosity function β and linear capillarity

$$u_t + f(u)_x = \varepsilon\, \beta(u_x)_x - \delta\, u_{xxx},$$

and Correia-LeFloch [3] dealt with the multidimensional equation. An up-to-date setting is given in Correia [4]. Whereas in those cases the dominant dissipation regime is ensured by the nonlinear viscosity. In this work we consider the reverse situation, where the nonlinearity concerns the dispersive term .

In general when $\varepsilon = 0$, the divergent behaviour is expected as we are considering "pure-dispersive equations" similar to the Körteweg-de Vries equation. Nonetheless, Brenier-Levy [2] studied the nonlinear equation

$$u_t + f(u)_x = \delta g(u_{xx})_x,$$

where $g(v) = -|v|^n$ (for $n = 1, 2$), as a nonlinear generalization of the Körteweg-de Vries equation. Such nonlinear dispersion significantly affects the dispersive behaviour of the solutions. In particular they conjectured in [2] that, with strictly convex flux functions f, the solution of the initial value problem for the equation

$$u_t + f(u)_x = \delta g(u_{xx})_x - \varepsilon\, u_{xxxx}$$

converges under the condition $\varepsilon = o(\delta)$.

Recently in [1], we have studied the problem (1.1)-(1.2) for the particular values $n = 1$ and $n = 2$. In the case of $n = 1$, *i.e.*

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta |u_{xx}|_x,$$

the convergence to the entropy weak solution of (1.3)-(1.4) is obtained if $\delta = o(\varepsilon^2)$. And in the case of $n = 2$, *i.e.*

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta (u_{xx}^2)_x,$$

under the convergence rate $\delta = o(\varepsilon^{3/2})$.

In this paper the intermediate power $1 < n < 2$ is tackled. The convergence is obtained by combining the estimates obtained in [1], for $n = 1$ and $n = 2$. The paper is organized as follows. In Section 2, the main convergence result is stated. In Section 3, we provide uniform estimates for the solutions $u^{\varepsilon,\delta}$ of the perturbed problem (1.1)-(1.2). Finally, Section 4 is devoted to the convergence proof of those $u^{\varepsilon,\delta}$ to the entropy weak solution of problem (1.3)-(1.4) when both ε and δ go down to zero, under suitable conditions.

2 Main Result

The main convergence result is stated. When $g(v) = -|v|^n$ with $1 \leq n \leq 2$, we prove the following theorem.

Theorem 2.1 *Let $\varepsilon > 0$, and $\delta = o(\varepsilon^{\frac{3n-1}{2}}; \varepsilon^{\frac{5n-1}{2(2n-1)}})$, and $f : \mathbb{R} \to \mathbb{R}$ be a flux function satisfying $0 \leq f''(u) \leq C(1 + |u|^\beta)$, where $0 \leq \beta < \frac{4-n}{n^2}$. Then, the family solutions $\{u^{\varepsilon,\delta}\}$ of (1.1) − (1.2) converges to the entropy solution of (1.3) − (1.4).*

In the particular cases $n = 1$ and $n = 2$, the convergence is found for $0 \leq \beta < 3, \delta = o(\varepsilon^2)$ and $0 \leq \beta < \frac{1}{2}, \delta = o(\varepsilon^{5/2})$ respectively. It has been proved in a previous paper [1].

Here, a better rate, equal to $\delta = o(\varepsilon^k)$ with $k < 2$, is obtained when $1 < n < 5/3$. As shown in Figure 1, the optimal rate is achieved for $n = \frac{5+\sqrt{13}}{6}$ and $\delta = o(\varepsilon^{\frac{3+\sqrt{13}}{6}})$.

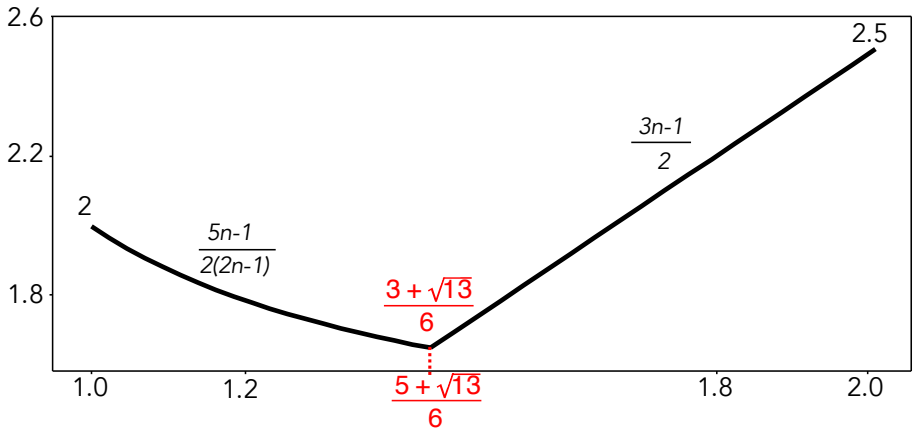

Figure 1: Rate of convergence $\max(\frac{3n-1}{2}; \frac{5n-1}{2(2n-1)})$ for $1 \leq n \leq 2$.

3 A priori Estimates

Assume that η is a regular function and q a function defined by $q' = \eta' f'$, and let us multiply (1.1) by $\eta'(u)$. We obtain

$$\begin{aligned} \eta(u)_t + q(u)_x &= \varepsilon\,(\eta'(u)\,u_x)_x - \varepsilon\,\eta''(u)\,u_x^2 \\ &\quad + \delta\,\big(\eta'(u)\,g(u_{xx})\big)_x - \delta\,\eta''(u)\,u_x\,g(u_{xx})\,. \end{aligned} \tag{3.1}$$

Integrate over $\mathbb{R} \times [0,t]$ with $\eta(u) = |u|^{\alpha+1}$. The conservative terms vanish and we obtain the following lemma.

Lemma 3.1 *Let $\alpha \geq 1$ and $g : \mathbb{R} \to \mathbb{R}$ be any dispersion function. Each solution of (1.1) satisfies for $t \in [0,T]$*

$$\begin{aligned} \int_{\mathbb{R}} |u(t)|^{\alpha+1}\,dx &+ (\alpha+1)\,\alpha\,\varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1}\,u_x^2\,dxds \\ &+ (\alpha+1)\,\alpha\,\delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1}\,u_x\,g(u_{xx})\,dxds \\ &= \int_{\mathbb{R}} |u_0|^{\alpha+1}\,dx\,. \end{aligned} \tag{3.2}$$

Taking $\alpha = 1$ in (3.2), provides *a priori* L^2 first energy estimates. Nevertheless, this estimate is useless, the sign δu_x can not be controlled.
Let us introduce the functions $\mathcal{G}$, and G defined by $\mathcal{G}'' = G' = g$, i.e,

$$G(u) = -\frac{1}{n+1}|u|^n u, \quad \text{and} \quad \mathcal{G}(u) = -\frac{1}{(n+1)(n+2)}|u|^{n+2}.$$

Also, we define for $t \geq 0$ the sets

$$\mathcal{U}_t^+ = \{x \in \mathbb{R},\, u_x(x,t) > 0\},$$

and

$$\mathcal{U}_t^- = \{x \in \mathbb{R},\, u_x(x,t) < 0\}.$$

We obtain the following estimate.

Lemma 3.2 *Let q be a odd number. Then, each solution of* (1.1) *satisfies for $t \in [0,T]$*

$$\begin{aligned}
&\int_{\mathcal{U}_t^+} |u_x(t)|^{q+2}\,dx + \varepsilon\,(q+2)(q+1)\int_0^t\int_{\mathcal{U}_s^+} |u_x|^q\,u_{xx}^2\,dxds \\
&\qquad +\delta\,(q+2)(q+1)\,q\,(n+2)\,n\int_0^t\int_{\mathcal{U}_s^+} |u_x|^{q-1}\,|\mathcal{G}(u_{xx})|\,dxds \\
&\qquad +(q+1)\int_0^t\int_{\mathcal{U}_s^+} |u_x|^{q+3}\,f''(u)\,dxds \\
&\qquad =\int_{\mathcal{U}_0^+} |u_0'|^{q+2}\,dx\,,
\end{aligned} \tag{3.3}$$

where the last left hand-side term can be replaced by

$$-(q+2)(q+1)\int_0^t\int_{\mathcal{U}_s^+} |u_x|^{q+1}\,f'(u)\,u_{xx}\,dxds\,. \tag{3.4}$$

Proof. Multiplying (1.1) by $(q+2)(|u_x|^q u_x)_x$ we have

$$\begin{aligned}
\big((q+2)\,u_t\,|u_x|^q\,u_x\big)_x &- (|u_x|^{q+2})_t \\
&= -(q+2)(q+1)|u_x|^q\,u_{xx}\,f'(u)\,u_x \\
&\quad +\varepsilon\,(q+2)(q+1)\,|u_x|^q\,u_{xx}^2 \\
&\quad +\delta\,(q+2)(q+1)\,|u_x|^q\,u_{xx}\,g'(u_{xx})\,u_{xxx} \\
&= -(q+1)(|u_x|^{q+2})_x\,f'(u) \\
&\quad +\varepsilon\,(q+2)(q+1)\,|u_x|^q\,u_{xx}^2 \\
&\quad +\delta\,(q+2)(q+1)\,n\,|u_x|^q\,G(u_{xx})_x,
\end{aligned}$$

and we get the estimate

$$\begin{aligned}
\big((q+2)\,u_t\,|u_x|^q\,u_x\big)_x &- (|u_x|^{q+2})_t \\
&= -\big((q+1)|u_x|^{q+2}f'(u)\big)_x \\
&\quad +(q+1)|u_x|^{q+2}\,u_x\,f''(u) \\
&\quad +\varepsilon\,(q+2)(q+1)\,|u_x|^q\,u_{xx}^2 \\
&\quad +\big(\delta\,(q+2)(q+1)\,n\,|u_x|^q\,G(u_{xx})\big)_x \\
&\quad -\delta\,(q+2)(q+1)\,q\,(n+2)\,n\,|u_x|^{q-2}\,u_x\,\mathcal{G}(u_{xx}).
\end{aligned} \tag{3.5}$$

Similarly, using the multiplier $(q+2)(u_x^{q+1})_x$ in (1.1), it comes

$$\begin{aligned}\big((q+2)\,u_t\,u_x^{q+1}\big)_x &- (u_x^{q+2})_t \\ &= -(q+2)(q+1)\,u_x^{q+1}\,f'(u)\,u_{xx} \\ &\quad +\varepsilon\,(q+2)(q+1)\,u_x^q\,u_{xx}^2 \\ &\quad +\big(\delta\,(q+2)(q+1)\,n\,u_x^q\,G(u_{xx})\big)_x \\ &\quad -\delta\,(q+2)(q+1)\,q\,(n+2)\,n\,u_x^{q-1}\,\mathcal{G}(u_{xx}),\end{aligned}$$

and one gets

$$\begin{aligned}\big((q+2)\,u_t\,u_x^{q+1}\big)_x &- (u_x^{q+2})_t \\ &= -\big((q+1)\,u_x^{q+2}f'(u)\big)_x \\ &\quad +(q+1)\,u_x^{q+3}\,f''(u) \\ &\quad +\varepsilon\,(q+2)(q+1)\,u_x^q\,u_{xx}^2 \\ &\quad +\big(\delta\,(q+2)(q+1)\,n\,u_x^q\,G(u_{xx})\big)_x \\ &\quad -\delta\,(q+2)(q+1)\,q\,(n+2)\,n\,u_x^{q-1}\,\mathcal{G}(u_{xx}).\end{aligned} \tag{3.6}$$

Integrating (3.5) and (3.6) over $\mathbb{R}\times[0,t]$, we obtain respectively

$$\begin{aligned}&\int_{\mathbb{R}}|u_x(t)|^{q+2}\,dx+\varepsilon\,(q+2)(q+1)\int_0^t\!\!\int_{\mathbb{R}}|u_x|^q\,u_{xx}^2\,dxds \\ &\qquad=\int_{\mathbb{R}}|u_0'|^{q+2}\,dx-(q+1)\int_0^t\!\!\int_{\mathbb{R}}u_x\,|u_x|^{q+2}\,f''(u)\,dxds \\ &\qquad\quad+\delta\,(q+2)(q+1)\,q\,(n+2)\,n\int_0^t\!\!\int_{\mathbb{R}}u_x\,|u_x|^{q-2}\,\mathcal{G}(u_{xx})\,dxds\,,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&\int_{\mathbb{R}}u_x(t)^{q+2}\,dx+\varepsilon\,(q+2)(q+1)\int_0^t\!\!\int_{\mathbb{R}}u_x^q\,u_{xx}^2\,dxds \\ &\qquad=\int_{\mathbb{R}}(u_0')^{q+2}\,dx-(q+1)\int_0^t\!\!\int_{\mathbb{R}}u_x^{q+3}\,f''(u)\,dxds \\ &\qquad\quad+\delta\,(q+2)(q+1)\,q\,(n+2)\,n\int_0^t\!\!\int_{\mathbb{R}}u_x^{q-1}\,\mathcal{G}(u_{xx})\,dxds\,.\end{aligned} \tag{3.8}$$

Adding (3.8) and (3.7) for q odd, we obtain (3.3). □

Combining Lemmas 3.1 and 3.2 we obtain the following estimate.

Proposition 3.1 *Let $\varepsilon,\delta>0$, and $f:\mathbb{R}\to\mathbb{R}$ be a convex flux function, i.e. $f''\geq 0$. Then, the family solutions $\{u_{\varepsilon,\delta}\}$ of (1.1) − (1.2) satisfies the uniform estimate for all $\frac{5+n}{2n+1}\leq\alpha<\frac{4+n}{n}$*

$$\int_{\mathbb{R}}|u(t)|^{\alpha+1}\,dx+\varepsilon\int_0^t\!\!\int_{\mathbb{R}}|u|^{\alpha-1}\,u_x^2\,dxds+\delta\int_0^t\!\!\int_{\mathbb{R}}|u|^{\alpha-1}\,|u_x|\,|u_{xx}|^n\,dxds\leq C_0. \tag{3.9}$$

Proof. When $g(v)=-|v|^n$, equation (3.2) writes

$$\begin{aligned}&\int_{\mathbb{R}}|u(t)|^{\alpha+1}\,dx+\alpha(\alpha+1)\,\varepsilon\int_0^t\!\!\int_{\mathbb{R}}|u|^{\alpha-1}\,u_x^2\,dxds= \\ &\qquad\|u_0\|_{\alpha+1}^{\alpha+1}+\alpha(\alpha+1)\,\delta\int_0^t\!\!\int_{\mathbb{R}}|u|^{\alpha-1}\,u_x\,|u_{xx}|^n\,dxds.\end{aligned} \tag{3.10}$$

Since f is convex, inequality (3.3) is rewritten for $q \geq 1$ odd as

$$\int_{\mathcal{U}_t^+} |u_x(t)|^{q+2}\, dx + \varepsilon \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q\, u_{xx}^2\, dxds \tag{3.11}$$
$$+ \delta \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1}\, |u_{xx}|^{n+2}\, dxds + \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3}\, f''(u)\, dxds \leq C_0.$$

Using Young's inequality, we get

$$\begin{aligned}
&\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n\, dxds \\
&\quad = \int_0^t \int_{\mathcal{U}_s^+} \left(\frac{1}{c\, t^{\frac{\alpha-1}{\alpha+1}}} |u|^{\alpha-1} \right) \left(c\, t^{\frac{\alpha-1}{\alpha+1}}\, |u_x| \right) (\delta |u_{xx}|^n)\, dxds \\
&\quad \leq \frac{1}{t\, c^{\frac{\alpha+1}{\alpha-1}}} \frac{\alpha-1}{\alpha+1} \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1}\, dxds + \frac{c^k\, t^{k\frac{\alpha-1}{\alpha+1}}}{k} \int_0^t \int_{\mathcal{U}_s^+} |u_x|^k\, dxds \\
&\qquad + \frac{n}{n+2} \delta^{1+\frac{2}{n}} \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2}\, dxds,
\end{aligned} \tag{3.12}$$

where c and k are two positive constants such that

$$c^{\frac{\alpha+1}{\alpha-1}} = 4\alpha(\alpha-1), \text{ and } \frac{1}{k} + \frac{n}{n+2} + \frac{\alpha-1}{\alpha+1} = 1.$$

Thus,

$$k = \frac{(n+2)(\alpha+1)}{(4+n) - n\,\alpha},$$

and $k \geq 3$ if $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$.
Now, we choose q odd such that $2 + q \geq k$ and

$$|u_x|^k \leq |u_x|^3 + |u_x|^{q+2}.$$

From (3.11) with $q = 1$ and $q \geq k-2$ odd, we deduce

$$\int_0^t \int_{\mathcal{U}_s{}^+} |u_x|^k\, dxds + \delta \int_0^t \int_{\mathcal{U}_s{}^+} |u_{xx}|^{n+2}\, dxds \leq C_0. \tag{3.13}$$

Integrating (3.10) over $[0, t]$ gives

$$\int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1}\, dxds \leq \int_0^t \int_{\mathbb{R}} |u|^{\alpha+1}\, dxds$$

$$\leq t\, C_0 + \alpha(\alpha+1)\, t\, \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n\, dxds. \tag{3.14}$$

Now, injecting (3.13) and (3.14) into (3.12), it comes

$$\delta \int_0^t \int_{\mathcal{U}_s{}^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n\, dxds \leq C_0 + \frac{1}{4}\delta \int_0^t \int_{\mathcal{U}_s{}^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n\, dxds \tag{3.15}$$

that yields

$$\delta \int_0^t \int_{\mathcal{U}_s{}^+} |u|^{\alpha-1}\, u_x\, |u_{xx}|^n\, dxds \leq C_0. \tag{3.16}$$

Finally, it is enough to merge (3.16) in (3.10) to obtain (3.9).

□

Now, let us prove the following lemma.

Lemma 3.3 *Assume that f satisfies $0 \le f''(u) \le C(1+|u|^\beta)$, with $0 \le \beta < \frac{4-n}{n^2}$ and $1 \le n \le 2$. Then, we have the estimate*

$$\left|\int_0^t\int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds\right| \le C_0\,\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}} + C_0(n-1)\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}\left(\int_0^t\int_{\mathbb{R}} u^2\,dxds + \varepsilon\int_0^t\int_{\mathbb{R}} u_x^2\,dxds\right). \tag{3.17}$$

In particular, when $n=1$, we have

$$\left|\int_0^t\int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds\right| \le \frac{C_0}{\delta}.$$

Proof. Note that, since $f''(u) \le C(1+|u|^\beta)$, then

$$|f'(u)-f'(0)| \le C(|u|+|u|^{\beta+1}).$$

Thus, we derive

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds\right| &\le C\int_0^t\int_{\mathbb{R}} |u||u_x||u_{xx}|\,dxds \\ &\quad + C\int_0^t\int_{\mathbb{R}} |u|^{\beta+1}|u_x||u_{xx}|\,dxds \\ &\le \delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}\int_0^t\int_{\mathbb{R}}\left(\delta^{\frac{1}{n}}|u|^{\frac{1}{n}}|u_x|^{\frac{1}{n}}|u_{xx}|\right)\left(|u|^{\frac{n-1}{n}}\right)\left(\epsilon^{\frac{n-1}{2n}}|u_x|^{\frac{n-1}{n}}\right)\,dxds \\ &\quad + \delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}\int_0^t\int_{\mathbb{R}}\left(\delta^{\frac{1}{n}}|u|^{\beta+\frac{1}{n}}|u_x|^{\frac{1}{n}}|u_{xx}|\right)\left(|u|^{\frac{n-1}{n}}\right)\left(\epsilon^{\frac{n-1}{2n}}|u_x|^{\frac{n-1}{n}}\right)\,dxds.\end{aligned}$$

Thanks to the Young inequality and the relation

$$\frac{1}{n}+\frac{n-1}{2n}+\frac{n-1}{2n}=1,$$

one gets

$$\begin{aligned}\left|\int_0^t\int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds\right| \;\le\; \delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}\Big(&C_0\,\delta\int_0^t\int_{\mathbb{R}}|u|^{n\,\beta+1}|u_x||u_{xx}|^n\,dxds \\ &+\; C_0\,\delta\int_0^t\int_{\mathbb{R}}|u||u_x||u_{xx}|^n\,dxds \\ &+\; C_0(n-1)\int_0^t\int_{\mathbb{R}}u^2\,dxds + C_0(n-1)\,\varepsilon\int_0^t\int_{\mathbb{R}}u_x^2\,dxds\Big).\end{aligned} \tag{3.18}$$

It comes from (3.9) with $1 \le n \le 2, \alpha = 2$ and $\alpha = n\,\beta + 2 < 1 + \frac{4}{n}$

$$\delta\int_0^t\int_{\mathbb{R}}|u||u_x||u_{xx}|^n\,dxds + \delta\int_0^t\int_{\mathbb{R}}|u|^{n\,\beta+1}|u_x||u_{xx}|^n\,dxds \le C_0. \tag{3.19}$$

Finally, inequalities (3.19) and (3.18) allow us to conclude the proof of Lemma 3.3 □

We will use this lemma to prove the following proposition.

Proposition 3.2 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$ and $f : \mathbb{R} \to \mathbb{R}$ be a flux function satisfying $0 \leq f''(u) \leq C(1+|u|^\beta)$, where $0 \leq \beta < \frac{4-n}{n^2}$. Then the family solutions $\{u_{\varepsilon,\delta}\}$ of $(1.1)-(1.2)$ satisfies the estimates*

$$\int_{\mathbb{R}} u_x(t)^2\,dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \leq C_0\,\delta^{-\frac{1}{n}}\varepsilon^{-\frac{n-1}{2n}}, \tag{3.20}$$

$$\delta \int_0^t \int_{\mathcal{U}_s^+} u_x\,|u_{xx}|^n\,dxds \leq C_0\,\left(\frac{\delta}{\varepsilon^{\frac{5n-1}{2(2n-1)}}}\right)^{\frac{2n-1}{2n}}, \tag{3.21}$$

$$\int_{\mathbb{R}} u(t)^2\,dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds + \delta \int_0^t \int_{\mathbb{R}} |u_x|\,|u_{xx}|^n\,dxds \leq \|u_0\|_2^2 + C_0\,\left(\frac{\delta}{\varepsilon^{\frac{5n-1}{2(2n-1)}}}\right)^{\frac{2n-1}{2n}}. \tag{3.22}$$

Proof. First, equality (3.2) for $1 \leq n \leq 2$ and $\alpha = 1$ reads as

$$\int_{\mathbb{R}} u(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds = \|u_0\|_2^2 + 2\,\delta \int_0^t \int_{\mathbb{R}} u_x\,|u_{xx}|^n\,dxds. \tag{3.23}$$

Since $1 \leq n \leq 2$, we have

$$\delta \int_0^t \int_{\mathcal{U}_s^+} u_x\,|u_{xx}|^n\,dxds \leq\ \delta \int_0^t \int_{\mathcal{U}_s^+} u_x\,|u_{xx}|\,dxds + \delta \int_0^t \int_{\mathcal{U}_s^+} u_x\,u_{xx}^2\,dxds. \tag{3.24}$$

From (3.3) with $q = 1$, we get

$$\varepsilon \int_0^t \int_{\mathcal{U}_s^+} u_x u_{xx}^2\,dxds \leq C_0, \tag{3.25}$$

which is equivalent to

$$\delta \int_0^t \int_{\mathcal{U}_s{}^+} u_x u_{xx}^2\,dxds \leq C_0\,\frac{\delta}{\varepsilon}. \tag{3.26}$$

We can also write

$$\begin{aligned}
\delta \int_0^t \int_{\mathcal{U}_s^+} u_x\,|u_{xx}|\,dxds &= \delta^{\frac{2n-1}{2n}}\varepsilon^{-\frac{n-1}{4n}} \int_0^t \int_{\mathcal{U}_s^+} u_x\,\left(\delta^{\frac{1}{2n}}\varepsilon^{\frac{n-1}{4n}}|u_{xx}|\right)\,dxds. \\
&\leq \frac{\delta^{\frac{2n-1}{2n}}\varepsilon^{-\frac{n-1}{4n}}}{2\varepsilon}\left(\varepsilon \int_0^t \int_{\mathcal{U}_s{}^+} u_x^2\,dxds + \varepsilon\delta^{\frac{1}{n}}\varepsilon^{\frac{n-1}{2n}} \int_0^t \int_{\mathcal{U}_s^+} u_{xx}^2\,dxds\right) \\
&\leq \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \int_0^t \int_{\mathcal{U}_s^+} \varepsilon\,u_x^2\,dxds + \delta^{\frac{2n+1}{2n}}\varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathcal{U}_s^+} u_{xx}^2\,dxds.
\end{aligned} \tag{3.27}$$

Putting (3.27) and (3.26) in (3.24) produces

$$\begin{aligned}
\delta \int_0^t \int_{\mathcal{U}_s^+} u_x\,|u_{xx}|^n\,dxds &\leq C_0\,\frac{\delta}{\varepsilon} + C_0\,\delta^{\frac{2n+1}{2n}}\varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \\
&\quad + C_0\,\frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \int_0^t \int_{\mathbb{R}} \varepsilon\,u_x^2\,dxds.
\end{aligned} \tag{3.28}$$

Since $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$, by injecting (3.28) in (3.23), we obtain

$$\int_{\mathbb{R}} u(t)^2\,dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2\,dxds \leq \|u_0\|_2^2 + C_0\frac{\delta}{\varepsilon} + C_0\,\delta^{\frac{2n+1}{2n}}\varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds. \tag{3.29}$$

Inequality (3.7) is rewritten with $q = 0$ as

$$\int_{\mathbb{R}} u_x(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \tag{3.30}$$
$$= \int_{\mathbb{R}} (u_0')^2\,dx + 2\int_0^t \int_{\mathbb{R}} f'(u)\,u_x\,u_{xx}\,dxds.$$

Lemma 3.3 points to

$$\int_{\mathbb{R}} u_x(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \le C_0 + C_0\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}$$
$$+ C_0(n-1)\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}\left(\int_0^t \int_{\mathbb{R}} u^2\,dxds + \varepsilon\int_0^t \int_{\mathbb{R}} u_x^2\,dxds\right), \tag{3.31}$$

and inequality (3.29) implies

$$\int_{\mathbb{R}} u_x(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \le C_0 + C_0\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}$$
$$+ C_0(n-1)\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}\left(\|u_0\|_2^2 + C_0\frac{\delta}{\varepsilon} + C_0\,\delta^{\frac{2n+1}{2n}}\varepsilon^{\frac{n-1}{4n}}\int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds\right). \tag{3.32}$$

Note that if $\delta = o(\varepsilon)$, this inequality reduces to

$$\begin{aligned}\int_{\mathbb{R}} u_x(t)^2\,dx + 2\,\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds \quad &\le \quad C_0\,\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}} \\ &+ C_0\,\delta^{\frac{2n-1}{2n}}\epsilon^{-\frac{5n-1}{4n}}\left(\varepsilon\int_0^t \int_{\mathbb{R}} u_{xx}^2\,dxds\right).\end{aligned} \tag{3.33}$$

Here, $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$ which is equivalent to $\delta^{\frac{2n-1}{2n}} = o(\varepsilon^{\frac{5n-1}{4n}})$ offers (3.20).

Now, injecting (3.20) in (3.29) gives

$$\begin{aligned}\int_{\mathbb{R}} u(t)^2\,dx + \varepsilon\int_0^t \int_{\mathbb{R}} u_x^2\,dxds \quad &\le \quad \|u_0\|_2^2 + C_0\,\frac{\delta}{\varepsilon} + C_0\,\delta^{\frac{2n+1}{2n}}\epsilon^{\frac{n-1}{4n}}\left(\frac{\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}}{\varepsilon}\right) \\ &\le \quad \|u_0\|_2^2 + \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\le \quad \|u_0\|_2^2 + C_0\,\left(\delta/\varepsilon^{\frac{5n-1}{2(2n-1)}}\right)^{\frac{2n-1}{2n}} \\ &\le \quad C_0.\end{aligned} \tag{3.34}$$

Finally, combining (3.20), (3.34) and (3.28), we have

$$\begin{aligned}\delta\int_0^t \int_{\mathcal{U}_s^+} u_x\,|u_{xx}|^n\,dxds \quad &\le \quad C_0\,\frac{\delta}{\varepsilon} + C_0\,\delta^{\frac{2n+1}{2n}}\epsilon^{\frac{n-1}{4n}}\left(\frac{\delta^{-\frac{1}{n}}\epsilon^{-\frac{n-1}{2n}}}{\varepsilon}\right) + C_0\,\frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\le \quad C_0\,\frac{\delta}{\varepsilon} + 2\,C_0\,\frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\le \quad C_0\,\frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}}\left(2 + \varepsilon^{\frac{n-1}{4n}}\delta^{\frac{1}{2n}}\right),\end{aligned} \tag{3.35}$$

and δ and ε being bounded, inequality (3.21) is found.

Inequality (3.22) directly follows from (3.21) and (3.23), and this achieves the proof of Proposition 3.2.

□

Proposition 3.3 *Let $\varepsilon > 0$, and $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$, and $f : \mathbb{R} \to \mathbb{R}$ be a flux function satisfying $0 \le f''(u) \le C(1 + |u|^\beta)$, where $0 \le \beta < \frac{4-n}{n^2}$. Then, the family solutions $\{u^{\varepsilon,\delta}\}$ of (1.1) − (1.2) satisfies*

$$
\begin{aligned}
&(a) \qquad \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 \, dxds \le C_0, \\
&(b) \qquad \delta \int_0^t \int_{\mathbb{R}} u_x^- \, |u_{xx}|^n \, dxds \le C_0, \\
&(c) \qquad \delta \int_0^t \int_{\mathcal{U}_s^+} u_x \, |u_{xx}|^n \, dxds \le C_0 \left(\delta / \varepsilon^{\frac{5n-1}{2(2n-1)}} \right)^{\frac{2n-1}{2n}}, \\
&(d) \qquad \delta^{\frac{2}{n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dxds \le C_0 \left(\delta / \varepsilon^{\frac{3n-1}{2}} \right)^{\frac{1}{n}},
\end{aligned}
$$

where $u_x^- = \max(0, -u_x)$.

Proof. Assertions (a) and (b) come from (3.22), while (c) corresponds to (3.21).
Thanks to (3.20), we have

$$
\begin{aligned}
\delta^{\frac{2}{n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 \, dxds &\le C_0 \, \delta^{\frac{2}{n}} \varepsilon^{-1} \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}} \\
&\le C_0 \, \delta^{\frac{1}{n}} \varepsilon^{-\frac{3n-1}{2n}},
\end{aligned}
$$

that is (d).

□

4 Proof of convergence

We have now all the tools to prove the main result given in Theorem 2.1. We first remind the notion of entropy measure-valued solution [5].

Definition 4.1 *Assume that $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R})$ satisfies the growth condition*

$$
|f(u)| \le \mathcal{O}(|u|^m) \text{ as } |u| \to \infty, \qquad \text{for some } m \in [0, q). \tag{4.1}
$$

A Young measure ν is called an entropy measure-valued (e.m.-v.) solution of (1.3)-(1.4) *if*

$$
\langle \nu, |u - k| \rangle_t + \langle \nu, \operatorname{sgn}(u - k)(f(u) - f(k)) \rangle_x \le 0, \qquad \text{for all } k \in \mathbb{R}, \tag{4.2}
$$

in the sense of distributions on $\mathbb{R} \times (0, T)$, and

$$
\lim_{t \to 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle \, dxds = 0, \qquad \text{for all compact set } K \subseteq \mathbb{R}. \tag{4.3}
$$

We will apply the following Young measure representation theorem [10] in a suitable L^q space to show that ν satisfies (4.2).

Lemma 4.1 *Let $\{u_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty((0,T);L^q(\mathbb{R}))$. Then there exists a subsequence denoted by $\{\tilde{u}_n\}_{n\in\mathbb{N}}$ and a weakly-$\star$ measurable mapping $\nu:\mathbb{R}\times(0,T)\to Prob(\mathbb{R})$ such that, for all functions $h\in\mathcal{C}(\mathbb{R})$ satisfying (4.1), $\langle\nu_{(x,t)},h\rangle$ belongs to $L^\infty((0,T);L^{q/m}_{loc}(\mathbb{R}))$ and the following limit representation holds:*

$$\iint_{\mathbb{R}\times(0,T)}\langle\nu_{(x,t)},h\rangle\ \phi(x,t)\ dxdt=\lim_{n\to\infty}\iint_{\mathbb{R}\times(0,T)}h(\tilde{u}_n(x,t))\,\phi(x,t)\ dxdt, \tag{4.4}$$

for all $\phi\in L^1(\mathbb{R}\times(0,T))\cap L^\infty(\mathbb{R}\times(0,T))$.
Conversely, given ν, there exists a sequence $\{u_n\}$ satisfying the same conditions as above and such that (4.4) holds for any h satisfying (4.1).

Let us start proving (4.2) thanks to Proposition 3.3 and assuming that, in the regime $\delta=o(\varepsilon^{\frac{3n-1}{2}};\varepsilon^{\frac{5n-1}{2(2n-1)}})$, initial data $u_0^{\varepsilon,\delta}$ converge to u_0 as follows

$$\begin{cases}\lim_{\varepsilon\to0}u_0^{\varepsilon,\delta}=u_0\ \text{ in } L^1(\mathbb{R})\cap L^q(\mathbb{R})\\ ||u_0^{\varepsilon,\delta}||_{L^2}\le||u_0||_{L^2}.\end{cases} \tag{4.5}$$

Standard regularizations of $\mathrm{sgn}(u-k)(f(u)-f(k))$ and $|u-k|$ $(k\in\mathbb{R})$ are used, to show that it is enough to prove the existence of a bounded measure $\mu\le0$ such that

$$\eta(u)_t+q(u)_x\longrightarrow\mu\quad\text{ in }\ \mathcal{D}'(\mathbb{R}\times(0,T)) \tag{4.6}$$

for an arbitrary convex function η (we assume that η' and η'' are bounded on $\mathbb{R}$).
To prove (4.6), we rewrite the formula (3.1) as

$$\eta(u)_t+q(u)_x=\mu_1+\mu_2+\mu_3+\mu_4, \tag{4.7}$$

with

$$\begin{aligned}\mu_1:&=\varepsilon\left(\eta'(u)\,u_x\right)_x,\\ \mu_2:&=-\varepsilon\,\eta''(u)\,u_x^2,\\ \mu_3:&=\delta\left(\eta'(u)\,g(u_{xx})\right)_x,\\ \mu_4:&=-\delta\ \eta''(u)\,u_x\,g(u_{xx}).\end{aligned}$$

Let's $\theta\in C_0^\infty(\mathbb{R}\times(0,T))$ a test function. We have

$$\begin{aligned}|<\mu_1,\theta>|&\le\varepsilon\int_0^T\int_{\mathbb{R}}|\theta_x\,\eta'(u)\,u_x|\,dxds\\ &\le C\,\varepsilon\int_0^T\int_{\mathbb{R}}|\theta_x\,u_x|\,dxds\\ &\le C||\theta_x||_{L^2}\varepsilon^{1/2}\left(\varepsilon\int_0^T\int_{\mathbb{R}}u_x^2\,dxds\right)^{1/2}\\ &\le C||\theta_x||_{L^2}\varepsilon^{1/2}\left(\varepsilon\int_0^T\int_{\mathbb{R}}u_x^2\,dxds\right)^{1/2}.\end{aligned} \tag{4.8}$$

According to Proposition 3.3$-(a)$,

$$|<\mu_1,\theta>|\ \le\ C||\theta_x||_{L^2}\varepsilon^{1/2} \tag{4.9}$$

and $\mu_1 \to 0$ when $\varepsilon \to 0$.
Thereafter,

$$\begin{aligned} |<\mu_2, \theta>| &\leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta\, \eta''(u)\, u_x^2|\, dxds \\ &\leq \varepsilon\, C||\theta||_{L^\infty} \int_0^T \int_{\mathbb{R}} u_x^2\, dxds. \end{aligned} \tag{4.10}$$

and μ_2 is bounded thanks to Proposition 3.3$-(a)$. Moreover, since η is convex, we have for a non negative function θ

$$<\mu_2, \theta> \;=\; -\varepsilon \int_0^T \int_{\mathbb{R}} \theta\, \eta''(u)\, u_x^2\, dxds \leq 0. \tag{4.11}$$

Concerning μ_3, we have

$$\begin{aligned} |<\mu_3, \theta>| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta_x\, \eta'(u)\, |u_{xx}|^n|\, dxds \\ &\leq C\, \delta \int_0^T \int_{\mathbb{R}} |\theta_x|\, |u_{xx}|^n\, dxds \\ &\leq C||\theta_x||_{L^{\frac{2}{2-n}}} \left(\delta^{\frac{2}{n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2\, dxds \right)^{\frac{n}{2}}. \end{aligned} \tag{4.12}$$

Remark that when $n = 2$, the space $L^{\frac{2}{2-n}}$ is replaced by L^∞. From Proposition 3.3$-(d)$, one gets

$$|<\mu_3, \theta>| \leq C_0||\theta_x||_{L^{\frac{2}{2-n}}} \left(\delta/\varepsilon^{\frac{3n-1}{2}} \right)^{\frac{1}{2}},$$

and $\mu_3 \to 0$ because $\delta = o(\varepsilon^{\frac{3n-1}{2}})$.

Finally, μ_4 is split as

$$\begin{aligned} \mu_4 &= \delta\ \eta''(u)\, u_x^+\, |u_{xx}|^n - \delta\ \eta''(u)\, u_x^-\, |u_{xx}|^n. \\ &=: \mu_4^+ + \mu_4^- \end{aligned}$$

where $u_x^+ = \max(0, u_x)$ and $u_x^- = \max(0, -u_x)$, and we have $u_x = u_x^+ - u_x^-$. On the one hand,

$$\begin{aligned} |<\mu_4^-, \theta>| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta\, \eta''(u)\, u_x^- |u_{xx}|^n|\, dxds \\ &\leq C\, \delta \int_0^T \int_{\mathbb{R}} |\theta\, u_x^-\, |u_{xx}|^n|\, dxds \\ &\leq C||\theta||_{L^\infty} \delta \int_0^T \int_{\mathbb{R}} u_x^-\, |u_{xx}|^n\, dxds, \end{aligned} \tag{4.13}$$

and according to Proposition 3.3$-(b)$,

$$|<\mu_4^-, \theta>| \leq C_0\, ||\theta||_{L^\infty}.$$

Again, $\eta'' \geq 0$, we have

$$<\mu_4^-, \theta> \;=\; -\delta \int_0^T \int_{\mathbb{R}} \theta\, \eta''(u)\, u_x^- |u_{xx}|^n\, dxds \leq 0. \tag{4.14}$$

Similarly,

$$\begin{aligned} |<\mu_4^+,\theta>| &\leq \delta\int_0^T\int_{\mathbb{R}} |\theta|\,\eta''(u)\,u_x^+|u_{xx}|^n\,dxds \\ &\leq C\,||\theta||_{L^\infty}\delta\int_0^T\int_{\mathbb{R}} u_x^+|u_{xx}|^n\,dxds \\ &\leq C\,||\theta||_{L^\infty}\delta\int_0^T\int_{\mathcal{U}_s^+} u_x|u_{xx}|^n\,dxds. \end{aligned} \tag{4.15}$$

Proposition 3.3−(c) implies

$$|<\mu_4^+,\theta>| \leq C\,||\theta||_{L^\infty}\,(\delta/\varepsilon^{\frac{5n-1}{2(2n-1)}})^{\frac{2n-1}{2n}}, \tag{4.16}$$

and $\mu_4^+\to 0$ because $\delta=o(\varepsilon^{\frac{n(n+1)}{2}})$.
Condition (4.6) is proved and μ is non positive bounded measure.

To show (4.3), we follow DiPerna [5] and Szepessy [12] arguments. We check that, for each compact set K of $\mathbb{R}$,

$$\lim_{t\to 0+}\frac{1}{t}\int_0^t\int_K \langle\nu_{(x,s)},|u-u_0(x)|\,\rangle\,dxds \quad = \quad \lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\int_K \left|u^{\varepsilon,\delta}(x,s)-u_0(x)\right|\,dxds=0.$$

Thanks to Jensen's inequality, for $m(K)$ the Lebesgue measure of K,

$$\frac{1}{t}\int_0^t\int_K \left|u^{\varepsilon,\delta}(x,s)-u_0(x)\right|\,dxds \leq m(K)^{1/2}\left(\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(x,s)-u_0(x)\right)^2\,dxds\right)^{1/2}$$

and we claim that

$$\lim_{t\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(x,s)-u_0(x)\right)^2\,dxds=0.$$

Indeed, let $K_i\subset K_{i+1}$ $(\,i=0,1,...\,)$ be an increasing sequence of compact sets such that $K_0=K$ and $\cup_{i\geq 0}K_i=\mathbb{R}$. Using the identity $u^2-u_0^2-2u_0(u-u_0)=(u-u_0)^2$, we get for all $i=0,1,\ldots$

$$\begin{aligned} &\frac{1}{t}\int_0^t\int_K \left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)^2\,dxds \\ &\leq \frac{1}{t}\int_0^t\left(\int_{K_i}|u^{\varepsilon,\delta}(\cdot,s)|^2\,dx-\int_{K_i}u_0^2\,dx-2\int_{K_i}u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\,dx\right)ds \\ &\leq \int_{\mathbb{R}\setminus K_i}u_0^2\,dx+\frac{1}{t}\int_0^t\left(\int_{\mathbb{R}}|u^{\varepsilon,\delta}(\cdot,s)|^2\,dx-\int_{\mathbb{R}}u_0^2\,dx\right) \\ &\qquad +\frac{2}{t}\int_0^t\left|\int_{K_i}u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\,dx\right|ds. \end{aligned} \tag{4.17}$$

Clearly, the first term of the right hand side verifies

$$\lim_{i\to\infty}\int_{\mathbb{R}\setminus K_i}u_0^2\,dx=0.$$

Using (3.22) and (4.5), we obtain

$$\begin{aligned} \int_{\mathbb{R}}|u^{\varepsilon,\delta}(\cdot,s)|^2\,dx-\int_{\mathbb{R}}u_0^2\,dx &= \int_{\mathbb{R}}|u^{\varepsilon,\delta}(\cdot,s)|^2\,dx-\int_{\mathbb{R}}(u_0^{\delta,\varepsilon})^2\,dx+\int_{\mathbb{R}}(u_0^{\delta,\varepsilon})^2\,dx-\int_{\mathbb{R}}u_0^2\,dx \\ &\leq \int_{\mathbb{R}}|u^{\varepsilon,\delta}(\cdot,s)|^2\,dx-\int_{\mathbb{R}}(u_0^{\delta,\varepsilon})^2\,dx \\ &\leq C_0\,(\delta/\varepsilon^{\frac{5n-1}{2(2n-1)}})^{\frac{2n-1}{2n}}, \end{aligned}$$

and the second term of right hand side goes to zero when $\varepsilon \to 0$.
To estimate the last term of (4.17), we consider $\{\theta_k\}_{k\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R})$ being such that

$$\lim_{k\to\infty} \theta_k = u_0 \quad \text{in} \quad L^2(\mathbb{R}).$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned}\left|\int_{K_i} u_0\left(u^{\varepsilon,\delta}(\cdot,s)-u_0\right)\,dx\right| &\le \int_{K_i}|u_0-\theta_k|\left|u^{\varepsilon,\delta}(\cdot,s)-u_0\right|\,dx\\ &\quad+\left|\int_{K_i}\theta_k\left(u_0^{\varepsilon,\delta}-u_0\right)+\int_{K_i}\theta_k\left(u^{\varepsilon,\delta}(\cdot,s)-u_0^{\varepsilon,\delta}\right)\,dx\right|\\ &\le \|u_0-\theta_k\|_{L^2(\mathbb{R})}\left(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(\mathbb{R})}+\|u_0\|_{L^2(\mathbb{R})}\right)\\ &\quad+\|\theta_k\|_{L^2(\mathbb{R})}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(\mathbb{R})}+\left|\int_0^s\int_{K_i}\theta_k\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|.\end{aligned}$$

We deduce from (3.22)

$$\|u_0-\theta_k\|_{L^2(\mathbb{R})}\left(\|u^{\varepsilon,\delta}(\cdot,s)\|_{L^2(\mathbb{R})}+\|u_0\|_{L^2(\mathbb{R})}\right)\le\left(\|u_0\|_{L^2(\mathbb{R})}+C\right)\|u_0-\theta_k\|_{L^2(\mathbb{R})},$$

which goes to zero when $k\to\infty$ because $\lim_{\varepsilon\to0+}\|u_0^{\varepsilon,\delta}-u_0\|_{L^2(\mathbb{R})}=0$ from (4.5). It remains to prove that

$$\lim_{t\to0+}\lim_{\varepsilon\to0+}\frac{1}{t}\int_0^t\left|\int_0^s\int_{K_i}\theta_k\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right|ds=0.$$

The equation (1.1) provides

$$\begin{aligned}\left|\int_0^s\int_{K_i}\theta_k\,\partial_s u^{\varepsilon,\delta}\,dxd\tau\right| &= \left|\int_0^s\int_{K_i}\theta_k\big(-f(u^{\varepsilon,\delta})_x+\varepsilon\,u_{xx}^{\varepsilon,\delta}+\delta\,g\big(u_{xx}^{\varepsilon,\delta}\big)_x\big)\,dxd\tau\right|\\ &\le \int_0^s\int_{K_i}\left|(\theta_k)_x\,f\big(u^{\varepsilon,\delta}\big)\right|\,dxd\tau\\ &\quad+\varepsilon\int_0^s\int_{K_i}\left|(\theta_k)_x\,u_x^{\varepsilon,\delta}\right|dxd\tau\\ &\quad+\delta\int_0^s\int_{K_i}\left|(\theta_k)_x\,g\big(u_{xx}^{\varepsilon,\delta}\big)\right|\,dxd\tau\\ &=: I_1+I_2+I_3.\end{aligned}$$

To deal with I_1, the function f being such that

$$0\le f''(u)\le C(1+|u|^\beta),$$

with $\beta<\frac{4-n}{n^2}$, then for $m=\beta+2$

$$|f(u)|\le C(1+|u|^m).$$

Thus, Proposition 3.1 with $\alpha=m-1=\beta+1<\frac{4-n}{n^2}+1<\frac{4}{n^2}-\frac{1}{n}+1<\frac{4}{n}+1$ is applied

$$\int_0^s\int_{K_i}|u^{\varepsilon,\delta}|^m\,dx\,d\tau\le\int_0^s\int_{\mathbb{R}}|u^{\varepsilon,\delta}|^m\,dx\,d\tau\le C\,s. \tag{4.18}$$

Then we can write

$$\begin{aligned} I_1 &= \int_0^s \int_{K_i} |(\theta_k)_x| \left| f\left(u^{\varepsilon,\delta}\right)\right| \, dx \, d\tau \\ &\leq C \int_0^s \int_{K_i} |(\theta_k)_x| \, dx \, d\tau + C \int_0^s \int_{K_i} |(\theta_k)_x| \, |u^{\varepsilon,\delta}|^m \, dx \, d\tau \\ &\leq C \int_0^s \int_{K_i} |(\theta_k)_x| \, dx \, d\tau + C \| (\theta_k)_x \|_{L^\infty(\mathbb{R})} \int_0^s \int_{K_i} |u^{\varepsilon,\delta}|^m \, dx \, d\tau \\ &\leq C \, s \, \| (\theta_k)_x \|_{L^1(\mathbb{R})} + C \, s \, \| (\theta_k)_x \|_{L^\infty(\mathbb{R})}. \end{aligned} \tag{4.19}$$

Again, thanks to Proposition 3.3$-(a)$ we have

$$\begin{aligned} I_2 &= \varepsilon \int_0^s \int_{K_i} |(\theta_k)_x| \left| u_x^{\varepsilon,\delta}\right| dx \, d\tau \\ &\leq \left[\varepsilon \int_0^s \int_{K_i} |(\theta_k)_x|^2 \, dx d\tau \right]^{\frac{1}{2}} \left[\varepsilon \int_0^s \int_{K_i} \left| u_x^{\varepsilon,\delta}\right|^2 dx d\tau \right]^{\frac{1}{2}} \\ &\leq C \, \varepsilon^{\frac{1}{2}} \, s^{\frac{1}{2}} \, \| (\theta_k)_x \|_{L^2(\mathbb{R})}. \end{aligned} \tag{4.20}$$

Finally, we have

$$\begin{aligned} I_3 &= \delta \int_0^s \int_{K_i} \left| (\theta_k)_x \, g\left(u_{xx}^{\varepsilon,\delta}\right)\right| \, dx d\tau \\ &= \int_0^s \int_{K_i} \delta \, |(\theta_k)_x| \left| u_{xx}^{\varepsilon,\delta}\right|^n \, dx d\tau \\ &\leq \left[\int_0^s \int_{K_i} |(\theta_k)_x|^{\frac{2}{2-n}} \, dx d\tau \right]^{\frac{2-n}{2}} \left[\delta^{\frac{2}{n}} \int_0^s \int_{\mathcal{K}_{t,\varepsilon,\delta^-}} \left| u_{xx}^{\varepsilon,\delta}\right|^2 \, dx d\tau \right]^{\frac{n}{2}} \\ &\leq \, s^{\frac{2-n}{2}} \, \| (\theta_k)_x \|_{L^{\frac{2}{2-n}}(\mathbb{R})} \left[\delta^{\frac{2}{n}} \int_0^s \int_{\mathcal{K}_{t,\varepsilon,\delta^-}} \left| u_{xx}^{\varepsilon,\delta}\right|^2 \, dx d\tau \right]^{\frac{n}{2}} \end{aligned}$$

and using Proposition 3.3$-(d)$, one gets

$$I_3 \leq C \, s^{\frac{2-n}{2}} \left(\frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \| (\theta_k)_x \|_{L^{\frac{2}{2-n}}(\mathbb{R})}. \tag{4.21}$$

Combining (4.19), (4.20) and (4.21) produces

$$\begin{aligned} &\lim_{t\to 0+} \lim_{\varepsilon \to 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_k \, u_s^{\varepsilon,\delta} \, dx d\tau \right| ds \\ &\quad \leq \lim_{t\to 0+} \lim_{\varepsilon \to 0+} \frac{C}{t} \left(t^2 \left(\| (\theta_k)_x \|_{L^1(\mathbb{R})} + \| (\theta_k)_x \|_{L^\infty(\mathbb{R})} \right) + \varepsilon^{\frac{1}{2}} t^{\frac{3}{2}} \| (\theta_k)_x \|_{L^2(\mathbb{R})} \right. \\ &\qquad\qquad \left. + \, t^{\frac{4-n}{2}} \left(\frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \| (\theta_k)_x \|_{L^{\frac{2}{2-n}}(\mathbb{R})} \right) \\ &\quad \leq \lim_{t\to 0+} \lim_{\varepsilon \to 0+} C \left(t + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + t^{\frac{2-n}{2}} \left(\frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \right), \end{aligned}$$

and since $\delta = o(\varepsilon^{\frac{3n-1}{2}})$, we obtain the desired conclusion, and Theorem 2.1 is proved.

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