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Convergence of a family of perturbed conservation laws with diffusion and non-positive dispersion

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Abstract: We consider a family of conservation laws with convex flux perturbed by vanishing diffusion and non-positive dispersion of the form $u_t + f(u)_x = \varepsilon u_{xx} - \delta(|u_{xx}|^n)_x$. Convergence of the solutions $\{u^{\varepsilon, \delta}\}$ to the entropy weak solution of the hyperbolic limit equation $u_t + f(u)_x = 0$, for all real numbers $1 \leq n \leq 2$ is proved if $\delta = o(\varepsilon^{\frac{3n-1}{2}}; \varepsilon^{\frac{5n-1}{2(2n-1)}})$.

Keywords: diffusion, nonlinear dispersion, KdV-Burgers equation, hyperbolic conservation laws, entropy measure-valued solutions.

1 Introduction

We consider the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} + \delta g(u_{xx})_x \quad (1.1)$$

$$u(x, 0) = u_0(x) \quad (1.2)$$

where f is a convex function, (i.e. $f'' \geq 0$), ε and δ are small positive parameters, g is a non-positive function of the form, with $1 \leq n \leq 2$,

$$g(v) = -|v|^n.$$

When $\delta = 0$, equation (1.1) reduces to the generalized viscous Burgers equation

$$u_t + f(u)_x = \varepsilon u_{xx}$$

and we know, as $\varepsilon \searrow 0$, the corresponding solution $u^{\varepsilon, 0}$ converges to the entropy weak solution of the hyperbolic problem (the *vanishing viscosity method*, see, e.g., Whitham [13] or Kruzkov [6])

$$u_t + f(u)_x = 0 \quad (1.3)$$

$$u(x, 0) = u_0(x). \quad (1.4)$$

On the other hand, when $\varepsilon = 0$, if we consider the flux function $f(u) = u^2$ and the linear dispersion, i.e., if we consider the Korteweg-de Vries equation

$$u_t + (u^2/2)_x = \delta u_{xxx},$$

the solution $u^{0,\delta}$ does not converge in a strong topology (see Lax-Levermore [7]). We are concerned with singular limits.

In the general case, as parameters $\varepsilon > 0$ and $\delta \neq 0$ vanish, to ensure convergence we wait that a dominant dissipation regime is necessary. The pioneer study of these singular limits was given in 1982 by Schonbek [10] for the generalized Körteweg-de Vries-Burgers equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta u_{xxx}.$$

For a convex flux function $f(u)$, she proved the convergence of the solutions $u^{\varepsilon,\delta}$ to the entropy weak solution of (1.3)-(1.4) under the condition $\delta = \mathcal{O}(\varepsilon^3)$ and, for the particular flux function $f(u) = u^2$, under the condition $\delta = \mathcal{O}(\varepsilon^2)$. While for the last case, according to Perthame-Ryzhik [9], the sharp condition could be $\delta = o(\varepsilon)$.

In “From Boltzmann to Euler: Hilbert’s 6th problem revisited” (Slemrod [11]) one may see the analogy between the singular limit for the Körteweg-de Vries-Burgers equation and the hydrodynamic limit of the kinetic Boltzmann equation for a rarefied gas to the continuum Euler equations of compressible gas dynamics as the Knudsen number approaches zero.

In 1998, LeFloch-Natalini [8] proved the convergence in the case of a nonlinear viscosity function β and linear capillarity

$$u_t + f(u)_x = \varepsilon \beta(u_x)_x - \delta u_{xxx},$$

and Correia-LeFloch [3] dealt with the multidimensional equation. An up-to-date setting is given in Correia [4]. Whereas in those cases the dominant dissipation regime is ensured by the nonlinear viscosity. In this work we consider the reverse situation, where the nonlinearity concerns the dispersive term .

In general when $\varepsilon = 0$, the divergent behaviour is expected as we are considering “pure-dispersive equations” similar to the Körteweg-de Vries equation. Nonetheless, Brenier-Levy [2] studied the nonlinear equation

$$u_t + f(u)_x = \delta g(u_{xx})_x,$$

where $g(v) = -|v|^n$ (for $n = 1, 2$), as a nonlinear generalization of the Körteweg-de Vries equation. Such nonlinear dispersion significantly affects the dispersive behaviour of the solutions. In particular they conjectured in [2] that, with strictly convex flux functions f , the solution of the initial value problem for the equation

$$u_t + f(u)_x = \delta g(u_{xx})_x - \varepsilon u_{xxxx}$$

converges under the condition $\varepsilon = o(\delta)$.

Recently in [1], we have studied the problem (1.1)-(1.2) for the particular values $n = 1$ and $n = 2$. In the case of $n = 1$, *i.e.*

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta |u_{xx}|_x,$$

the convergence to the entropy weak solution of (1.3)-(1.4) is obtained if $\delta = o(\varepsilon^2)$. And in the case of $n = 2$, *i.e.*

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta (u_{xx}^2)_x,$$

under the convergence rate $\delta = o(\varepsilon^{3/2})$.

In this paper the intermediate power $1 < n < 2$ is tackled. The convergence is obtained by combining the estimates obtained in [1], for $n = 1$ and $n = 2$. The paper is organized as follows. In Section 2, the main convergence result is stated. In Section 3, we provide uniform estimates for the solutions $u^{\varepsilon,\delta}$ of the perturbed problem (1.1)-(1.2). Finally, Section 4 is devoted to the convergence proof of those $u^{\varepsilon,\delta}$ to the entropy weak solution of problem (1.3)-(1.4) when both ε and δ go down to zero, under suitable conditions.

2 Main Result

The main convergence result is stated. When $g(v) = -|v|^n$ with $1 \leq n \leq 2$, we prove the following theorem.

Theorem 2.1 *Let $\varepsilon > 0$, and $\delta = o(\varepsilon^{\frac{3n-1}{2}}; \varepsilon^{\frac{5n-1}{2(2n-1)}})$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a flux function satisfying $0 \leq f''(u) \leq C(1 + |u|^\beta)$, where $0 \leq \beta < \frac{4-n}{n^2}$. Then, the family solutions $\{u^{\varepsilon, \delta}\}$ of (1.1) – (1.2) converges to the entropy solution of (1.3) – (1.4).*

In the particular cases $n = 1$ and $n = 2$, the convergence is found for $0 \leq \beta < 3, \delta = o(\varepsilon^2)$ and $0 \leq \beta < \frac{1}{2}, \delta = o(\varepsilon^{5/2})$ respectively. It has been proved in a previous paper [1].

Here, a better rate, equal to $\delta = o(\varepsilon^k)$ with $k < 2$, is obtained when $1 < n < 5/3$. As shown in Figure 1, the optimal rate is achieved for $n = \frac{5+\sqrt{13}}{6}$ and $\delta = o(\varepsilon^{\frac{3+\sqrt{13}}{6}})$.

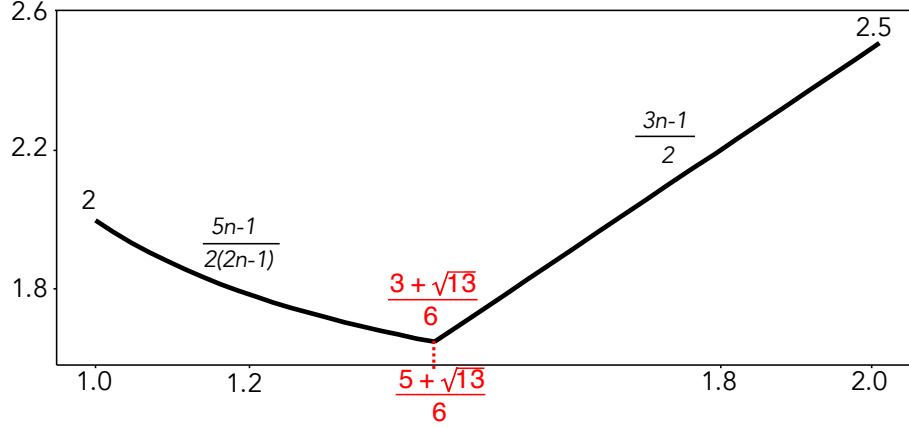


Figure 1: Rate of convergence $\max(\frac{3n-1}{2}; \frac{5n-1}{2(2n-1)})$ for $1 \leq n \leq 2$.

3 A priori Estimates

Assume that η is a regular function and q a function defined by $q' = \eta' f'$, and let us multiply (1.1) by $\eta'(u)$. We obtain

$$\begin{aligned} \eta(u)_t + q(u)_x &= \varepsilon (\eta'(u) u_x)_x - \varepsilon \eta''(u) u_x^2 \\ &\quad + \delta (\eta'(u) g(u_{xx}))_x - \delta \eta''(u) u_x g(u_{xx}). \end{aligned} \quad (3.1)$$

Integrate over $\mathbb{R} \times [0, t]$ with $\eta(u) = |u|^{\alpha+1}$. The conservative terms vanish and we obtain the following lemma.

Lemma 3.1 *Let $\alpha \geq 1$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be any dispersion function. Each solution of (1.1) satisfies for $t \in [0, T]$*

$$\begin{aligned} \int_{\mathbb{R}} |u(t)|^{\alpha+1} dx &+ (\alpha + 1) \alpha \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds \\ &+ (\alpha + 1) \alpha \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x g(u_{xx}) dx ds \\ &= \int_{\mathbb{R}} |u_0|^{\alpha+1} dx. \end{aligned} \quad (3.2)$$

Taking $\alpha = 1$ in (3.2), provides *a priori* L^2 first energy estimates. Nevertheless, this estimate is useless, the sign δu_x can not be controlled.

Let us introduce the functions \mathcal{G} , and G defined by $\mathcal{G}'' = G' = g$, i.e,

$$G(u) = -\frac{1}{n+1}|u|^n u, \quad \text{and} \quad \mathcal{G}(u) = -\frac{1}{(n+1)(n+2)}|u|^{n+2}.$$

Also, we define for $t \geq 0$ the sets

$$\mathcal{U}_t^+ = \{x \in \mathbb{R}, u_x(x, t) > 0\},$$

and

$$\mathcal{U}_t^- = \{x \in \mathbb{R}, u_x(x, t) < 0\}.$$

We obtain the following estimate.

Lemma 3.2 *Let q be a odd number. Then, each solution of (1.1) satisfies for $t \in [0, T]$*

$$\begin{aligned} & \int_{\mathcal{U}_t^+} |u_x(t)|^{q+2} dx + \varepsilon (q+2)(q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q u_{xx}^2 dx ds \\ & + \delta (q+2)(q+1) q (n+2) n \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |\mathcal{G}(u_{xx})| dx ds \\ & + (q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u) dx ds \\ & = \int_{\mathcal{U}_0^+} |u_0'|^{q+2} dx, \end{aligned} \tag{3.3}$$

where the last left hand-side term can be replaced by

$$-(q+2)(q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+1} f'(u) u_{xx} dx ds. \tag{3.4}$$

Proof. Multiplying (1.1) by $(q+2)(|u_x|^q u_x)_x$ we have

$$\begin{aligned} ((q+2) u_t |u_x|^q u_x)_x & - (|u_x|^{q+2})_t \\ & = -(q+2)(q+1) |u_x|^q u_{xx} f'(u) u_x \\ & + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ & + \delta (q+2)(q+1) |u_x|^q u_{xx} g'(u_{xx}) u_{xxx} \\ & = -(q+1) (|u_x|^{q+2})_x f'(u) \\ & + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ & + \delta (q+2)(q+1) n |u_x|^q G(u_{xx})_x, \end{aligned}$$

and we get the estimate

$$\begin{aligned} ((q+2) u_t |u_x|^q u_x)_x & - (|u_x|^{q+2})_t \\ & = -((q+1) |u_x|^{q+2} f'(u))_x \\ & + (q+1) |u_x|^{q+2} u_x f''(u) \\ & + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ & + (\delta (q+2)(q+1) n |u_x|^q G(u_{xx}))_x \\ & - \delta (q+2)(q+1) q (n+2) n |u_x|^{q-2} u_x \mathcal{G}(u_{xx}). \end{aligned} \tag{3.5}$$

Similarly, using the multiplier $(q+2)(u_x^{q+1})_x$ in (1.1), it comes

$$\begin{aligned}
((q+2) u_t u_x^{q+1})_x &= (u_x^{q+2})_t \\
&= -(q+2)(q+1) u_x^{q+1} f'(u) u_{xx} \\
&\quad + \varepsilon (q+2)(q+1) u_x^q u_{xx}^2 \\
&\quad + (\delta (q+2)(q+1) n u_x^q G(u_{xx}))_x \\
&\quad - \delta (q+2)(q+1) q (n+2) n u_x^{q-1} \mathcal{G}(u_{xx}),
\end{aligned}$$

and one gets

$$\begin{aligned}
((q+2) u_t u_x^{q+1})_x &= (u_x^{q+2})_t \\
&= -((q+1) u_x^{q+2} f'(u))_x \\
&\quad + (q+1) u_x^{q+3} f''(u) \\
&\quad + \varepsilon (q+2)(q+1) u_x^q u_{xx}^2 \\
&\quad + (\delta (q+2)(q+1) n u_x^q G(u_{xx}))_x \\
&\quad - \delta (q+2)(q+1) q (n+2) n u_x^{q-1} \mathcal{G}(u_{xx}).
\end{aligned} \tag{3.6}$$

Integrating (3.5) and (3.6) over $\mathbb{R} \times [0, t]$, we obtain respectively

$$\begin{aligned}
&\int_{\mathbb{R}} |u_x(t)|^{q+2} dx + \varepsilon (q+2)(q+1) \int_0^t \int_{\mathbb{R}} |u_x|^q u_{xx}^2 dx ds \\
&= \int_{\mathbb{R}} |u'_0|^{q+2} dx - (q+1) \int_0^t \int_{\mathbb{R}} u_x |u_x|^{q+2} f''(u) dx ds \\
&\quad + \delta (q+2)(q+1) q (n+2) n \int_0^t \int_{\mathbb{R}} u_x |u_x|^{q-2} \mathcal{G}(u_{xx}) dx ds,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} u_x(t)^{q+2} dx + \varepsilon (q+2)(q+1) \int_0^t \int_{\mathbb{R}} u_x^q u_{xx}^2 dx ds \\
&= \int_{\mathbb{R}} (u'_0)^{q+2} dx - (q+1) \int_0^t \int_{\mathbb{R}} u_x^{q+3} f''(u) dx ds \\
&\quad + \delta (q+2)(q+1) q (n+2) n \int_0^t \int_{\mathbb{R}} u_x^{q-1} \mathcal{G}(u_{xx}) dx ds.
\end{aligned} \tag{3.8}$$

Adding (3.8) and (3.7) for q odd, we obtain (3.3). \square

Combining Lemmas 3.1 and 3.2 we obtain the following estimate.

Proposition 3.1 *Let $\varepsilon, \delta > 0$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a convex flux function, i.e. $f'' \geq 0$. Then, the family solutions $\{u_{\varepsilon, \delta}\}$ of (1.1) – (1.2) satisfies the uniform estimate for all $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$*

$$\int_{\mathbb{R}} |u(t)|^{\alpha+1} dx + \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} |u_x| |u_{xx}|^n dx ds \leq C_0. \tag{3.9}$$

Proof. When $g(v) = -|v|^n$, equation (3.2) writes

$$\begin{aligned}
&\int_{\mathbb{R}} |u(t)|^{\alpha+1} dx + \alpha(\alpha+1) \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds = \\
&\quad \|u_0\|_{\alpha+1}^{\alpha+1} + \alpha(\alpha+1) \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds.
\end{aligned} \tag{3.10}$$

Since f is convex, inequality (3.3) is rewritten for $q \geq 1$ odd as

$$\begin{aligned} & \int_{\mathcal{U}_t^+} |u_x(t)|^{q+2} dx + \varepsilon \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q u_{xx}^2 dx ds \\ & + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |u_{xx}|^{n+2} dx ds + \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u) dx ds \leq C_0. \end{aligned} \quad (3.11)$$

Using Young's inequality, we get

$$\begin{aligned} & \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \\ & = \int_0^t \int_{\mathcal{U}_s^+} \left(\frac{1}{c t^{\frac{\alpha-1}{\alpha+1}}} |u|^{\alpha-1} \right) \left(c t^{\frac{\alpha-1}{\alpha+1}} |u_x| \right) (\delta |u_{xx}|^n) dx ds \\ & \leq \frac{1}{t c^{\frac{\alpha+1}{\alpha-1}}} \frac{\alpha-1}{\alpha+1} \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} dx ds + \frac{c^k t^k \frac{\alpha-1}{\alpha+1}}{k} \int_0^t \int_{\mathcal{U}_s^+} |u_x|^k dx ds \\ & \quad + \frac{n}{n+2} \delta^{1+\frac{2}{n}} \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} dx ds, \end{aligned} \quad (3.12)$$

where c and k are two positive constants such that

$$c^{\frac{\alpha+1}{\alpha-1}} = 4\alpha(\alpha-1), \text{ and } \frac{1}{k} + \frac{n}{n+2} + \frac{\alpha-1}{\alpha+1} = 1.$$

Thus,

$$k = \frac{(n+2)(\alpha+1)}{(4+n) - n\alpha},$$

and $k \geq 3$ if $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$.

Now, we choose q odd such that $2+q \geq k$ and

$$|u_x|^k \leq |u_x|^3 + |u_x|^{q+2}.$$

From (3.11) with $q = 1$ and $q \geq k - 2$ odd, we deduce

$$\int_0^t \int_{\mathcal{U}_s^+} |u_x|^k dx ds + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} dx ds \leq C_0. \quad (3.13)$$

Integrating (3.10) over $[0, t]$ gives

$$\begin{aligned} & \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} dx ds \leq \int_0^t \int_{\mathbb{R}} |u|^{\alpha+1} dx ds \\ & \leq t C_0 + \alpha(\alpha+1) t \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds. \end{aligned} \quad (3.14)$$

Now, injecting (3.13) and (3.14) into (3.12), it comes

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \leq C_0 + \frac{1}{4} \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \quad (3.15)$$

that yields

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \leq C_0. \quad (3.16)$$

Finally, it is enough to merge (3.16) in (3.10) to obtain (3.9). □

Now, let us prove the following lemma.

Lemma 3.3 *Assume that f satisfies $0 \leq f''(u) \leq C(1 + |u|^\beta)$, with $0 \leq \beta < \frac{4-n}{n^2}$ and $1 \leq n \leq 2$. Then, we have the estimate*

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C_0 \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \\ &+ C_0(n-1) \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \left(\int_0^t \int_{\mathbb{R}} u^2 dx ds + \epsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \right). \end{aligned} \quad (3.17)$$

In particular, when $n = 1$, we have

$$\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| \leq \frac{C_0}{\delta}.$$

Proof. Note that, since $f''(u) \leq C(1 + |u|^\beta)$, then

$$|f'(u) - f'(0)| \leq C(|u| + |u|^{\beta+1}).$$

Thus, we derive

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}| dx ds \\ &+ C \int_0^t \int_{\mathbb{R}} |u|^{\beta+1} |u_x| |u_{xx}| dx ds \\ &\leq \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \int_0^t \int_{\mathbb{R}} \left(\delta^{\frac{1}{n}} |u|^{\frac{1}{n}} |u_x|^{\frac{1}{n}} |u_{xx}| \right) \left(|u|^{\frac{n-1}{n}} \right) \left(\epsilon^{\frac{n-1}{2n}} |u_x|^{\frac{n-1}{n}} \right) dx ds \\ &+ \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \int_0^t \int_{\mathbb{R}} \left(\delta^{\frac{1}{n}} |u|^{\beta+\frac{1}{n}} |u_x|^{\frac{1}{n}} |u_{xx}| \right) \left(|u|^{\frac{n-1}{n}} \right) \left(\epsilon^{\frac{n-1}{2n}} |u_x|^{\frac{n-1}{n}} \right) dx ds. \end{aligned}$$

Thanks to the Young inequality and the relation

$$\frac{1}{n} + \frac{n-1}{2n} + \frac{n-1}{2n} = 1,$$

one gets

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \left(C_0 \delta \int_0^t \int_{\mathbb{R}} |u|^{n\beta+1} |u_x| |u_{xx}|^n dx ds \right. \\ &+ C_0 \delta \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}|^n dx ds \\ &\left. + C_0(n-1) \int_0^t \int_{\mathbb{R}} u^2 dx ds + C_0(n-1) \epsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \right). \end{aligned} \quad (3.18)$$

It comes from (3.9) with $1 \leq n \leq 2$, $\alpha = 2$ and $\alpha = n\beta + 2 < 1 + \frac{4}{n}$

$$\delta \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}|^n dx ds + \delta \int_0^t \int_{\mathbb{R}} |u|^{n\beta+1} |u_x| |u_{xx}|^n dx ds \leq C_0. \quad (3.19)$$

Finally, inequalities (3.19) and (3.18) allow us to conclude the proof of Lemma 3.3 □

We will use this lemma to prove the following proposition.

Proposition 3.2 *Let $\varepsilon > 0$, $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a flux function satisfying $0 \leq f''(u) \leq C(1+|u|^\beta)$, where $0 \leq \beta < \frac{4-n}{n^2}$. Then the family solutions $\{u_{\varepsilon,\delta}\}$ of (1.1) – (1.2) satisfies the estimates*

$$\int_{\mathbb{R}} u_x(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq C_0 \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}}, \quad (3.20)$$

$$\delta \int_0^t \int_{U_s^+} u_x |u_{xx}|^n dx ds \leq C_0 \left(\frac{\delta}{\varepsilon^{\frac{5n-1}{2(2n-1)}}} \right)^{\frac{2n-1}{2n}}, \quad (3.21)$$

$$\int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}|^n dx ds \leq \|u_0\|_2^2 + C_0 \left(\frac{\delta}{\varepsilon^{\frac{5n-1}{2(2n-1)}}} \right)^{\frac{2n-1}{2n}}. \quad (3.22)$$

Proof. First, equality (3.2) for $1 \leq n \leq 2$ and $\alpha = 1$ reads as

$$\int_{\mathbb{R}} u(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds = \|u_0\|_2^2 + 2\delta \int_0^t \int_{\mathbb{R}} u_x |u_{xx}|^n dx ds. \quad (3.23)$$

Since $1 \leq n \leq 2$, we have

$$\delta \int_0^t \int_{U_s^+} u_x |u_{xx}|^n dx ds \leq \delta \int_0^t \int_{U_s^+} u_x |u_{xx}| dx ds + \delta \int_0^t \int_{U_s^+} u_x u_{xx}^2 dx ds. \quad (3.24)$$

From (3.3) with $q = 1$, we get

$$\varepsilon \int_0^t \int_{U_s^+} u_x u_{xx}^2 dx ds \leq C_0, \quad (3.25)$$

which is equivalent to

$$\delta \int_0^t \int_{U_s^+} u_x u_{xx}^2 dx ds \leq C_0 \frac{\delta}{\varepsilon}. \quad (3.26)$$

We can also write

$$\begin{aligned} \delta \int_0^t \int_{U_s^+} u_x |u_{xx}| dx ds &= \delta^{\frac{2n-1}{2n}} \varepsilon^{-\frac{n-1}{4n}} \int_0^t \int_{U_s^+} u_x \left(\delta^{\frac{1}{2n}} \varepsilon^{\frac{n-1}{4n}} |u_{xx}| \right) dx ds \\ &\leq \frac{\delta^{\frac{2n-1}{2n}} \varepsilon^{-\frac{n-1}{4n}}}{2\varepsilon} \left(\varepsilon \int_0^t \int_{U_s^+} u_x^2 dx ds + \varepsilon \delta^{\frac{1}{n}} \varepsilon^{\frac{n-1}{2n}} \int_0^t \int_{U_s^+} u_{xx}^2 dx ds \right) \\ &\leq \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \int_0^t \int_{U_s^+} \varepsilon u_x^2 dx ds + \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{U_s^+} u_{xx}^2 dx ds. \end{aligned} \quad (3.27)$$

Putting (3.27) and (3.26) in (3.24) produces

$$\begin{aligned} \delta \int_0^t \int_{U_s^+} u_x |u_{xx}|^n dx ds &\leq C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \\ &\quad + C_0 \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \int_0^t \int_{\mathbb{R}} \varepsilon u_x^2 dx ds. \end{aligned} \quad (3.28)$$

Since $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$, by injecting (3.28) in (3.23), we obtain

$$\int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \leq \|u_0\|_2^2 + C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds. \quad (3.29)$$

Inequality (3.7) is rewritten with $q = 0$ as

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds & \\ &= \int_{\mathbb{R}} (u'_0)^2 dx + 2 \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds. \end{aligned} \quad (3.30)$$

Lemma 3.3 points to

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds &\leq C_0 + C_0 \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}} \\ &+ C_0(n-1) \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}} \left(\int_0^t \int_{\mathbb{R}} u^2 dx ds + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \right), \end{aligned} \quad (3.31)$$

and inequality (3.29) implies

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds &\leq C_0 + C_0 \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}} \\ &+ C_0(n-1) \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}} \left(\|u_0\|_2^2 + C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right). \end{aligned} \quad (3.32)$$

Note that if $\delta = o(\varepsilon)$, this inequality reduces to

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds &\leq C_0 \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}} \\ &+ C_0 \delta^{\frac{2n-1}{2n}} \varepsilon^{-\frac{5n-1}{4n}} \left(\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right). \end{aligned} \quad (3.33)$$

Here, $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$ which is equivalent to $\delta^{\frac{2n-1}{2n}} = o(\varepsilon^{\frac{5n-1}{4n}})$ offers (3.20).

Now, injecting (3.20) in (3.29) gives

$$\begin{aligned} \int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds &\leq \|u_0\|_2^2 + C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \left(\frac{\delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}}}{\varepsilon} \right) \\ &\leq \|u_0\|_2^2 + \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\leq \|u_0\|_2^2 + C_0 \left(\delta / \varepsilon^{\frac{5n-1}{2(2n-1)}} \right)^{\frac{2n-1}{2n}} \\ &\leq C_0. \end{aligned} \quad (3.34)$$

Finally, combining (3.20), (3.34) and (3.28), we have

$$\begin{aligned} \delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds &\leq C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \left(\frac{\delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}}}{\varepsilon} \right) + C_0 \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\leq C_0 \frac{\delta}{\varepsilon} + 2C_0 \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\leq C_0 \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \left(2 + \varepsilon^{\frac{n-1}{4n}} \delta^{\frac{1}{2n}} \right), \end{aligned} \quad (3.35)$$

and δ and ε being bounded, inequality (3.21) is found.

Inequality (3.22) directly follows from (3.21) and (3.23), and this achieves the proof of Proposition 3.2. □

Proposition 3.3 *Let $\varepsilon > 0$, and $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$, and $f : \mathbb{R} \rightarrow \mathbb{R}$ be a flux function satisfying $0 \leq f''(u) \leq C(1 + |u|^\beta)$, where $0 \leq \beta < \frac{4-n}{n^2}$. Then, the family solutions $\{u^{\varepsilon, \delta}\}$ of (1.1) – (1.2) satisfies*

$$\begin{aligned}
(a) \quad & \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \leq C_0, \\
(b) \quad & \delta \int_0^t \int_{\mathbb{R}} u_x^- |u_{xx}|^n dx ds \leq C_0, \\
(c) \quad & \delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds \leq C_0 \left(\delta / \varepsilon^{\frac{5n-1}{2(2n-1)}} \right)^{\frac{2n-1}{2n}}, \\
(d) \quad & \delta^{\frac{2}{n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq C_0 \left(\delta / \varepsilon^{\frac{3n-1}{2}} \right)^{\frac{1}{n}},
\end{aligned}$$

where $u_x^- = \max(0, -u_x)$.

Proof. Assertions (a) and (b) come from (3.22), while (c) corresponds to (3.21). Thanks to (3.20), we have

$$\begin{aligned}
\delta^{\frac{2}{n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds & \leq C_0 \delta^{\frac{2}{n}} \varepsilon^{-1} \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}} \\
& \leq C_0 \delta^{\frac{1}{n}} \varepsilon^{-\frac{3n-1}{2n}},
\end{aligned}$$

that is (d). □

4 Proof of convergence

We have now all the tools to prove the main result given in Theorem 2.1. We first remind the notion of entropy measure-valued solution [5].

Definition 4.1 *Assume that $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R})$ satisfies the growth condition*

$$|f(u)| \leq \mathcal{O}(|u|^m) \text{ as } |u| \rightarrow \infty, \quad \text{for some } m \in [0, q). \quad (4.1)$$

A Young measure ν is called an entropy measure-valued (e.m.-v.) solution of (1.3)-(1.4) if

$$\langle \nu, |u - k| \rangle_t + \langle \nu, \text{sgn}(u - k)(f(u) - f(k)) \rangle_x \leq 0, \quad \text{for all } k \in \mathbb{R}, \quad (4.2)$$

in the sense of distributions on $\mathbb{R} \times (0, T)$, and

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle dx ds = 0, \quad \text{for all compact set } K \subseteq \mathbb{R}. \quad (4.3)$$

We will apply the following Young measure representation theorem [10] in a suitable L^q space to show that ν satisfies (4.2).

Lemma 4.1 *Let $\{u_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $L^\infty((0, T); L^q(\mathbb{R}))$. Then there exists a subsequence denoted by $\{\tilde{u}_n\}_{n \in \mathbb{N}}$ and a weakly- \star measurable mapping $\nu : \mathbb{R} \times (0, T) \rightarrow \text{Prob}(\mathbb{R})$ such that, for all functions $h \in \mathcal{C}(\mathbb{R})$ satisfying (4.1), $\langle \nu_{(x,t)}, h \rangle$ belongs to $L^\infty((0, T); L^q_{loc}(\mathbb{R}))$ and the following limit representation holds:*

$$\iint_{\mathbb{R} \times (0, T)} \langle \nu_{(x,t)}, h \rangle \phi(x, t) \, dx dt = \lim_{n \rightarrow \infty} \iint_{\mathbb{R} \times (0, T)} h(\tilde{u}_n(x, t)) \phi(x, t) \, dx dt, \quad (4.4)$$

for all $\phi \in L^1(\mathbb{R} \times (0, T)) \cap L^\infty(\mathbb{R} \times (0, T))$.

Conversely, given ν , there exists a sequence $\{u_n\}$ satisfying the same conditions as above and such that (4.4) holds for any h satisfying (4.1).

Let us start proving (4.2) thanks to Proposition 3.3 and assuming that, in the regime $\delta = o(\varepsilon^{\frac{3n-1}{2}}; \varepsilon^{\frac{5n-1}{2(2n-1)}})$, initial data $u_0^{\varepsilon, \delta}$ converge to u_0 as follows

$$\begin{cases} \lim_{\varepsilon \rightarrow 0} u_0^{\varepsilon, \delta} = u_0 \text{ in } L^1(\mathbb{R}) \cap L^q(\mathbb{R}) \\ \|u_0^{\varepsilon, \delta}\|_{L^2} \leq \|u_0\|_{L^2}. \end{cases} \quad (4.5)$$

Standard regularizations of $\text{sgn}(u - k)(f(u) - f(k))$ and $|u - k|$ ($k \in \mathbb{R}$) are used, to show that it is enough to prove the existence of a bounded measure $\mu \leq 0$ such that

$$\eta(u)_t + q(u)_x \longrightarrow \mu \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, T)) \quad (4.6)$$

for an arbitrary convex function η (we assume that η' and η'' are bounded on \mathbb{R}). To prove (4.6), we rewrite the formula (3.1) as

$$\eta(u)_t + q(u)_x = \mu_1 + \mu_2 + \mu_3 + \mu_4, \quad (4.7)$$

with

$$\begin{aligned} \mu_1 &: = \varepsilon (\eta'(u) u_x)_x, \\ \mu_2 &: = -\varepsilon \eta''(u) u_x^2, \\ \mu_3 &: = \delta (\eta'(u) g(u_{xx}))_x, \\ \mu_4 &: = -\delta \eta''(u) u_x g(u_{xx}). \end{aligned}$$

Let's $\theta \in C_0^\infty(\mathbb{R} \times (0, T))$ a test function. We have

$$\begin{aligned} |\langle \mu_1, \theta \rangle| &\leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) u_x| \, dx ds \\ &\leq C \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x u_x| \, dx ds \\ &\leq C \|\theta_x\|_{L^2} \varepsilon^{1/2} \left(\varepsilon \int_0^T \int_{\mathbb{R}} u_x^2 \, dx ds \right)^{1/2} \\ &\leq C \|\theta_x\|_{L^2} \varepsilon^{1/2} \left(\varepsilon \int_0^T \int_{\mathbb{R}} u_x^2 \, dx ds \right)^{1/2}. \end{aligned} \quad (4.8)$$

According to Proposition 3.3-(a),

$$|\langle \mu_1, \theta \rangle| \leq C \|\theta_x\|_{L^2} \varepsilon^{1/2} \quad (4.9)$$

and $\mu_1 \rightarrow 0$ when $\varepsilon \rightarrow 0$.
Thereafter,

$$\begin{aligned} |\langle \mu_2, \theta \rangle| &\leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^2| dx ds \\ &\leq \varepsilon C \|\theta\|_{L^\infty} \int_0^T \int_{\mathbb{R}} u_x^2 dx ds. \end{aligned} \quad (4.10)$$

and μ_2 is bounded thanks to Proposition 3.3–(a). Moreover, since η is convex, we have for a non negative function θ

$$\langle \mu_2, \theta \rangle = -\varepsilon \int_0^T \int_{\mathbb{R}} \theta \eta''(u) u_x^2 dx ds \leq 0. \quad (4.11)$$

Concerning μ_3 , we have

$$\begin{aligned} |\langle \mu_3, \theta \rangle| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) |u_{xx}|^n| dx ds \\ &\leq C \delta \int_0^T \int_{\mathbb{R}} |\theta_x| |u_{xx}|^n dx ds \\ &\leq C \|\theta_x\|_{L^{\frac{2}{2-n}}} \left(\delta^{\frac{2}{n}} \int_0^T \int_{\mathbb{R}} u_{xx}^2 dx ds \right)^{\frac{n}{2}}. \end{aligned} \quad (4.12)$$

Remark that when $n = 2$, the space $L^{\frac{2}{2-n}}$ is replaced by L^∞ . From Proposition 3.3–(d), one gets

$$|\langle \mu_3, \theta \rangle| \leq C_0 \|\theta_x\|_{L^{\frac{2}{2-n}}} \left(\delta / \varepsilon^{\frac{3n-1}{2}} \right)^{\frac{1}{2}},$$

and $\mu_3 \rightarrow 0$ because $\delta = o(\varepsilon^{\frac{3n-1}{2}})$.

Finally, μ_4 is split as

$$\begin{aligned} \mu_4 &= \delta \eta''(u) u_x^+ |u_{xx}|^n - \delta \eta''(u) u_x^- |u_{xx}|^n \\ &=: \mu_4^+ + \mu_4^- \end{aligned}$$

where $u_x^+ = \max(0, u_x)$ and $u_x^- = \max(0, -u_x)$, and we have $u_x = u_x^+ - u_x^-$. On the one hand,

$$\begin{aligned} |\langle \mu_4^-, \theta \rangle| &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^- |u_{xx}|^n| dx ds \\ &\leq C \delta \int_0^T \int_{\mathbb{R}} |\theta u_x^- |u_{xx}|^n| dx ds \\ &\leq C \|\theta\|_{L^\infty} \delta \int_0^T \int_{\mathbb{R}} u_x^- |u_{xx}|^n dx ds, \end{aligned} \quad (4.13)$$

and according to Proposition 3.3–(b),

$$|\langle \mu_4^-, \theta \rangle| \leq C_0 \|\theta\|_{L^\infty}.$$

Again, $\eta'' \geq 0$, we have

$$\langle \mu_4^-, \theta \rangle = -\delta \int_0^T \int_{\mathbb{R}} \theta \eta''(u) u_x^- |u_{xx}|^n dx ds \leq 0. \quad (4.14)$$

Similarly,

$$\begin{aligned}
| \langle \mu_4^+, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta| \eta''(u) u_x^+ |u_{xx}|^n dx ds \\
&\leq C \|\theta\|_{L^\infty} \delta \int_0^T \int_{\mathbb{R}} u_x^+ |u_{xx}|^n dx ds \\
&\leq C \|\theta\|_{L^\infty} \delta \int_0^T \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds.
\end{aligned} \tag{4.15}$$

Proposition 3.3–(c) implies

$$| \langle \mu_4^+, \theta \rangle | \leq C \|\theta\|_{L^\infty} (\delta / \varepsilon^{\frac{5n-1}{2(2n-1)}})^{\frac{2n-1}{2n}}, \tag{4.16}$$

and $\mu_4^+ \rightarrow 0$ because $\delta = o(\varepsilon^{\frac{n(n+1)}{2}})$.

Condition (4.6) is proved and μ is non positive bounded measure.

To show (4.3), we follow DiPerna [5] and Szepessy [12] arguments. We check that, for each compact set K of \mathbb{R} ,

$$\begin{aligned}
\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle dx ds &= \\
\lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K |u^{\varepsilon, \delta}(x, s) - u_0(x)| dx ds &= 0.
\end{aligned}$$

Thanks to Jensen's inequality, for $m(K)$ the Lebesgue measure of K ,

$$\frac{1}{t} \int_0^t \int_K |u^{\varepsilon, \delta}(x, s) - u_0(x)| dx ds \leq m(K)^{1/2} \left(\frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(x, s) - u_0(x))^2 dx ds \right)^{1/2}$$

and we claim that

$$\lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(x, s) - u_0(x))^2 dx ds = 0.$$

Indeed, let $K_i \subset K_{i+1}$ ($i = 0, 1, \dots$) be an increasing sequence of compact sets such that $K_0 = K$ and $\cup_{i \geq 0} K_i = \mathbb{R}$. Using the identity $u^2 - u_0^2 - 2u_0(u - u_0) = (u - u_0)^2$, we get for all $i = 0, 1, \dots$

$$\begin{aligned}
&\frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(\cdot, s) - u_0)^2 dx ds \\
&\leq \frac{1}{t} \int_0^t \left(\int_{K_i} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{K_i} u_0^2 dx - 2 \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right) ds \\
&\leq \int_{\mathbb{R} \setminus K_i} u_0^2 dx + \frac{1}{t} \int_0^t \left(\int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx \right) \\
&\quad + \frac{2}{t} \int_0^t \left| \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right| ds.
\end{aligned} \tag{4.17}$$

Clearly, the first term of the right hand side verifies

$$\lim_{i \rightarrow \infty} \int_{\mathbb{R} \setminus K_i} u_0^2 dx = 0.$$

Using (3.22) and (4.5), we obtain

$$\begin{aligned}
\int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx &= \int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} (u_0^{\delta, \varepsilon})^2 dx + \int_{\mathbb{R}} (u_0^{\delta, \varepsilon})^2 dx - \int_{\mathbb{R}} u_0^2 dx \\
&\leq \int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} (u_0^{\delta, \varepsilon})^2 dx \\
&\leq C_0 (\delta / \varepsilon^{\frac{5n-1}{2(2n-1)}})^{\frac{2n-1}{2n}},
\end{aligned}$$

and the second term of right hand side goes to zero when $\varepsilon \rightarrow 0$.

To estimate the last term of (4.17), we consider $\{\theta_k\}_{k \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R})$ being such that

$$\lim_{k \rightarrow \infty} \theta_k = u_0 \quad \text{in } L^2(\mathbb{R}).$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned} \left| \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right| &\leq \int_{K_i} |u_0 - \theta_k| |u^{\varepsilon, \delta}(\cdot, s) - u_0| dx \\ &\quad + \left| \int_{K_i} \theta_k (u_0^{\varepsilon, \delta} - u_0) + \int_{K_i} \theta_k (u^{\varepsilon, \delta}(\cdot, s) - u_0^{\varepsilon, \delta}) dx \right| \\ &\leq \|u_0 - \theta_k\|_{L^2(\mathbb{R})} (\|u^{\varepsilon, \delta}(\cdot, s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})}) \\ &\quad + \|\theta_k\|_{L^2(\mathbb{R})} \|u_0^{\varepsilon, \delta} - u_0\|_{L^2(\mathbb{R})} + \left| \int_0^s \int_{K_i} \theta_k \partial_s u^{\varepsilon, \delta} dx d\tau \right|. \end{aligned}$$

We deduce from (3.22)

$$\|u_0 - \theta_k\|_{L^2(\mathbb{R})} (\|u^{\varepsilon, \delta}(\cdot, s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})}) \leq (\|u_0\|_{L^2(\mathbb{R})} + C) \|u_0 - \theta_k\|_{L^2(\mathbb{R})},$$

which goes to zero when $k \rightarrow \infty$ because $\lim_{\varepsilon \rightarrow 0+} \|u_0^{\varepsilon, \delta} - u_0\|_{L^2(\mathbb{R})} = 0$ from (4.5). It remains to prove that

$$\lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_k \partial_s u^{\varepsilon, \delta} dx d\tau \right| ds = 0.$$

The equation (1.1) provides

$$\begin{aligned} \left| \int_0^s \int_{K_i} \theta_k \partial_s u^{\varepsilon, \delta} dx d\tau \right| &= \left| \int_0^s \int_{K_i} \theta_k (-f(u^{\varepsilon, \delta})_x + \varepsilon u_{xx}^{\varepsilon, \delta} + \delta g(u_{xx}^{\varepsilon, \delta})_x) dx d\tau \right| \\ &\leq \int_0^s \int_{K_i} |(\theta_k)_x f(u^{\varepsilon, \delta})| dx d\tau \\ &\quad + \varepsilon \int_0^s \int_{K_i} |(\theta_k)_x u_x^{\varepsilon, \delta}| dx d\tau \\ &\quad + \delta \int_0^s \int_{K_i} |(\theta_k)_x g(u_{xx}^{\varepsilon, \delta})| dx d\tau \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

To deal with I_1 , the function f being such that

$$0 \leq f''(u) \leq C(1 + |u|^\beta),$$

with $\beta < \frac{4-n}{n^2}$, then for $m = \beta + 2$

$$|f(u)| \leq C(1 + |u|^m).$$

Thus, Proposition 3.1 with $\alpha = m - 1 = \beta + 1 < \frac{4-n}{n^2} + 1 < \frac{4}{n^2} - \frac{1}{n} + 1 < \frac{4}{n} + 1$ is applied

$$\int_0^s \int_{K_i} |u^{\varepsilon, \delta}|^m dx d\tau \leq \int_0^s \int_{\mathbb{R}} |u^{\varepsilon, \delta}|^m dx d\tau \leq C s. \quad (4.18)$$

Then we can write

$$\begin{aligned}
I_1 &= \int_0^s \int_{K_i} |(\theta_k)_x| |f(u^{\varepsilon, \delta})| dx d\tau \\
&\leq C \int_0^s \int_{K_i} |(\theta_k)_x| dx d\tau + C \int_0^s \int_{K_i} |(\theta_k)_x| |u^{\varepsilon, \delta}|^m dx d\tau \\
&\leq C \int_0^s \int_{K_i} |(\theta_k)_x| dx d\tau + C \|(\theta_k)_x\|_{L^\infty(\mathbb{R})} \int_0^s \int_{K_i} |u^{\varepsilon, \delta}|^m dx d\tau \\
&\leq C s \|(\theta_k)_x\|_{L^1(\mathbb{R})} + C s \|(\theta_k)_x\|_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{4.19}$$

Again, thanks to Proposition 3.3–(a) we have

$$\begin{aligned}
I_2 &= \varepsilon \int_0^s \int_{K_i} |(\theta_k)_x| |u_x^{\varepsilon, \delta}| dx d\tau \\
&\leq \left[\varepsilon \int_0^s \int_{K_i} |(\theta_k)_x|^2 dx d\tau \right]^{\frac{1}{2}} \left[\varepsilon \int_0^s \int_{K_i} |u_x^{\varepsilon, \delta}|^2 dx d\tau \right]^{\frac{1}{2}} \\
&\leq C \varepsilon^{\frac{1}{2}} s^{\frac{1}{2}} \|(\theta_k)_x\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{4.20}$$

Finally, we have

$$\begin{aligned}
I_3 &= \delta \int_0^s \int_{K_i} |(\theta_k)_x g(u_{xx}^{\varepsilon, \delta})| dx d\tau \\
&= \int_0^s \int_{K_i} \delta |(\theta_k)_x| |u_{xx}^{\varepsilon, \delta}|^n dx d\tau \\
&\leq \left[\int_0^s \int_{K_i} |(\theta_k)_x|^{\frac{2}{2-n}} dx d\tau \right]^{\frac{2-n}{2}} \left[\delta^{\frac{2}{n}} \int_0^s \int_{\mathcal{K}_{t, \varepsilon, \delta^-}} |u_{xx}^{\varepsilon, \delta}|^2 dx d\tau \right]^{\frac{n}{2}} \\
&\leq s^{\frac{2-n}{2}} \|(\theta_k)_x\|_{L^{\frac{2}{2-n}}(\mathbb{R})} \left[\delta^{\frac{2}{n}} \int_0^s \int_{\mathcal{K}_{t, \varepsilon, \delta^-}} |u_{xx}^{\varepsilon, \delta}|^2 dx d\tau \right]^{\frac{n}{2}}
\end{aligned}$$

and using Proposition 3.3–(d), one gets

$$I_3 \leq C s^{\frac{2-n}{2}} \left(\frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \|(\theta_k)_x\|_{L^{\frac{2}{2-n}}(\mathbb{R})}. \tag{4.21}$$

Combining (4.19), (4.20) and (4.21) produces

$$\begin{aligned}
&\lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_k u_s^{\varepsilon, \delta} dx d\tau \right| ds \\
&\leq \lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{C}{t} \left(t^2 (\|(\theta_k)_x\|_{L^1(\mathbb{R})} + \|(\theta_k)_x\|_{L^\infty(\mathbb{R})}) + \varepsilon^{\frac{1}{2}} t^{\frac{3}{2}} \|(\theta_k)_x\|_{L^2(\mathbb{R})} \right. \\
&\quad \left. + t^{\frac{4-n}{2}} \left(\frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \|(\theta_k)_x\|_{L^{\frac{2}{2-n}}(\mathbb{R})} \right) \\
&\leq \lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} C \left(t + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + t^{\frac{2-n}{2}} \left(\frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \right),
\end{aligned}$$

and since $\delta = o(\varepsilon^{\frac{3n-1}{2}})$, we obtain the desired conclusion, and Theorem 2.1 is proved.

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