

# Convergence of a family of perturbed conservation laws with diffusion and non-positive dispersion

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**Abstract:** We consider a family of conservation laws with convex flux perturbed by vanishing diffusion and non-positive dispersion of the form  $u_t + f(u)_x = \varepsilon u_{xx} - \delta(|u_{xx}|^n)_x$ . Convergence of the solutions  $\{u^{\varepsilon, \delta}\}$  to the entropy weak solution of the hyperbolic limit equation  $u_t + f(u)_x = 0$ , for all real numbers  $1 \leq n \leq 2$  is proved if  $\delta = o(\varepsilon^{\frac{3n-1}{2}}; \varepsilon^{\frac{5n-1}{2(2n-1)}})$ .

**Keywords:** diffusion, nonlinear dispersion, KdV-Burgers equation, hyperbolic conservation laws, entropy measure-valued solutions.

## 1 Introduction

We consider the initial value problem

$$u_t + f(u)_x = \varepsilon u_{xx} + \delta g(u_{xx})_x \quad (1.1)$$

$$u(x, 0) = u_0(x) \quad (1.2)$$

where  $f$  is a convex function, (i.e.  $f'' \geq 0$ ),  $\varepsilon$  and  $\delta$  are small positive parameters,  $g$  is a non-positive function of the form, with  $1 \leq n \leq 2$ ,

$$g(v) = -|v|^n.$$

When  $\delta = 0$ , equation (1.1) reduces to the generalized viscous Burgers equation

$$u_t + f(u)_x = \varepsilon u_{xx}$$

and we know, as  $\varepsilon \searrow 0$ , the corresponding solution  $u^{\varepsilon, 0}$  converges to the entropy weak solution of the hyperbolic problem (the *vanishing viscosity method*, see, e.g., Whitham [13] or Kružkov [6])

$$u_t + f(u)_x = 0 \quad (1.3)$$

$$u(x, 0) = u_0(x). \quad (1.4)$$

On the other hand, when  $\varepsilon = 0$ , if we consider the flux function  $f(u) = u^2$  and the linear dispersion, i.e., if we consider the Korteweg-de Vries equation

$$u_t + (u^2/2)_x = \delta u_{xxx},$$

the solution  $u^{0,\delta}$  does not converge in a strong topology (see Lax-Levermore [7]). We are concerned with singular limits.

In the general case, as parameters  $\varepsilon > 0$  and  $\delta \neq 0$  vanish, to ensure convergence we wait that a dominant dissipation regime is necessary. The pioneer study of these singular limits was given in 1982 by Schonbek [10] for the generalized Korteweg-de Vries-Burgers equation

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta u_{xxx}.$$

For a convex flux function  $f(u)$ , she proved the convergence of the solutions  $u^{\varepsilon,\delta}$  to the entropy weak solution of (1.3)-(1.4) under the condition  $\delta = \mathcal{O}(\varepsilon^3)$  and, for the particular flux function  $f(u) = u^2$ , under the condition  $\delta = \mathcal{O}(\varepsilon^2)$ . While for the last case, according to Perthame-Ryzhik [9], the sharp condition could be  $\delta = o(\varepsilon)$ .

In “From Boltzmann to Euler: Hilbert’s 6th problem revisited” (Slemrod [11]) one may see the analogy between the singular limit for the Korteweg-de Vries-Burgers equation and the hydrodynamic limit of the kinetic Boltzmann equation for a rarefied gas to the continuum Euler equations of compressible gas dynamics as the Knudsen number approaches zero.

In 1998, LeFloch-Natalini [8] proved the convergence in the case of a nonlinear viscosity function  $\beta$  and linear capillarity

$$u_t + f(u)_x = \varepsilon \beta(u_x)_x - \delta u_{xxx},$$

and Correia-LeFloch [3] dealt with the multidimensional equation. An up-to-date setting is given in Correia [4]. Whereas in those cases the dominant dissipation regime is ensured by the nonlinear viscosity. In this work we consider the reverse situation, where the nonlinearity concerns the dispersive term.

In general when  $\varepsilon = 0$ , the divergent behaviour is expected as we are considering “pure-dispersive equations” similar to the Korteweg-de Vries equation. Nonetheless, Brenier-Levy [2] studied the nonlinear equation

$$u_t + f(u)_x = \delta g(u_{xx})_x,$$

where  $g(v) = -|v|^n$  (for  $n = 1, 2$ ), as a nonlinear generalization of the Korteweg-de Vries equation. Such nonlinear dispersion significantly affects the dispersive behaviour of the solutions. In particular they conjectured in [2] that, with strictly convex flux functions  $f$ , the solution of the initial value problem for the equation

$$u_t + f(u)_x = \delta g(u_{xx})_x - \varepsilon u_{xxxx}$$

converges under the condition  $\varepsilon = o(\delta)$ .

Recently in [1], we have studied the problem (1.1)-(1.2) for the particular values  $n = 1$  and  $n = 2$ . In the case of  $n = 1$ , *i.e.*

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta |u_{xx}|_x,$$

the convergence to the entropy weak solution of (1.3)-(1.4) is obtained if  $\delta = o(\varepsilon^2)$ . And in the case of  $n = 2$ , *i.e.*

$$u_t + f(u)_x = \varepsilon u_{xx} - \delta (u_{xx}^2)_x,$$

under the convergence rate  $\delta = o(\varepsilon^{3/2})$ .

In this paper the intermediate power  $1 < n < 2$  is tackled. The convergence is obtained by combining the estimates obtained in [1], for  $n = 1$  and  $n = 2$ . The paper is organized as follows. In Section 2, the main convergence result is stated. In Section 3, we provide uniform estimates for the solutions  $u^{\varepsilon,\delta}$  of the perturbed problem (1.1)-(1.2). Finally, Section 4 is devoted to the convergence proof of those  $u^{\varepsilon,\delta}$  to the entropy weak solution of problem (1.3)-(1.4) when both  $\varepsilon$  and  $\delta$  go down to zero, under suitable conditions.

## 2 Main Result

The main convergence result is stated. When  $g(v) = -|v|^n$  with  $1 \leq n \leq 2$ , we prove the following theorem.

**Theorem 2.1** *Let  $\varepsilon > 0$ , and  $\delta = o(\varepsilon^{\frac{3n-1}{2}}; \varepsilon^{\frac{5n-1}{2(2n-1)}})$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a flux function satisfying  $0 \leq f''(u) \leq C(1 + |u|^\beta)$ , where  $0 \leq \beta < \frac{4-n}{n^2}$ . Then, the family solutions  $\{u^{\varepsilon, \delta}\}$  of (1.1) – (1.2) converges to the entropy solution of (1.3) – (1.4).*

In the particular cases  $n = 1$  and  $n = 2$ , the convergence is found for  $0 \leq \beta < 3, \delta = o(\varepsilon^2)$  and  $0 \leq \beta < \frac{1}{2}, \delta = o(\varepsilon^{5/2})$  respectively. It has been proved in a previous paper [1].

Here, a better rate, equal to  $\delta = o(\varepsilon^k)$  with  $k < 2$ , is obtained when  $1 < n < 5/3$ . As shown in Figure 1, the optimal rate is achieved for  $n = \frac{5+\sqrt{13}}{6}$  and  $\delta = o(\varepsilon^{\frac{3+\sqrt{13}}{6}})$ .

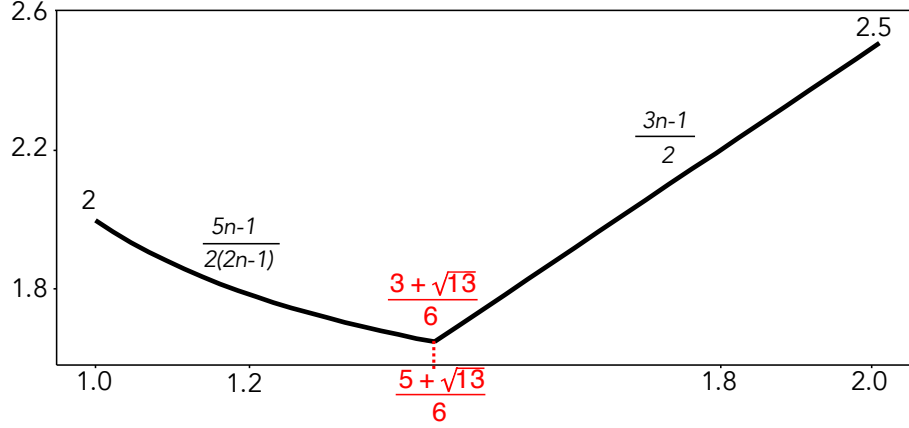


Figure 1: Rate of convergence  $\max(\frac{3n-1}{2}; \frac{5n-1}{2(2n-1)})$  for  $1 \leq n \leq 2$ .

## 3 A priori Estimates

Assume that  $\eta$  is a regular function and  $q$  a function defined by  $q' = \eta' f'$ , and let us multiply (1.1) by  $\eta'(u)$ . We obtain

$$\begin{aligned} \eta(u)_t + q(u)_x &= \varepsilon (\eta'(u) u_x)_x - \varepsilon \eta''(u) u_x^2 \\ &\quad + \delta (\eta'(u) g(u_{xx}))_x - \delta \eta''(u) u_x g(u_{xx}). \end{aligned} \quad (3.1)$$

Integrate over  $\mathbb{R} \times [0, t]$  with  $\eta(u) = |u|^{\alpha+1}$ . The conservative terms vanish and we obtain the following lemma.

**Lemma 3.1** *Let  $\alpha \geq 1$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be any dispersion function. Each solution of (1.1) satisfies for  $t \in [0, T]$*

$$\begin{aligned} \int_{\mathbb{R}} |u(t)|^{\alpha+1} dx &+ (\alpha+1) \alpha \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds \\ &+ (\alpha+1) \alpha \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x g(u_{xx}) dx ds \\ &= \int_{\mathbb{R}} |u_0|^{\alpha+1} dx. \end{aligned} \quad (3.2)$$

Taking  $\alpha = 1$  in (3.2), provides *a priori*  $L^2$  first energy estimates. Nevertheless, this estimate is useless, the sign  $\delta u_x$  can not be controlled.

Let us introduce the functions  $\mathcal{G}$ , and  $G$  defined by  $\mathcal{G}'' = G' = g$ , i.e,

$$G(u) = -\frac{1}{n+1}|u|^n u, \quad \text{and} \quad \mathcal{G}(u) = -\frac{1}{(n+1)(n+2)}|u|^{n+2}.$$

Also, we define for  $t \geq 0$  the sets

$$\mathcal{U}_t^+ = \{x \in \mathbb{R}, u_x(x, t) > 0\},$$

and

$$\mathcal{U}_t^- = \{x \in \mathbb{R}, u_x(x, t) < 0\}.$$

We obtain the following estimate.

**Lemma 3.2** *Let  $q$  be a odd number. Then, each solution of (1.1) satisfies for  $t \in [0, T]$*

$$\begin{aligned} & \int_{\mathcal{U}_t^+} |u_x(t)|^{q+2} dx + \varepsilon (q+2)(q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q u_{xx}^2 dx ds \\ & + \delta (q+2)(q+1) q (n+2) n \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |\mathcal{G}(u_{xx})| dx ds \\ & + (q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u) dx ds \\ & = \int_{\mathcal{U}_0^+} |u'_0|^{q+2} dx, \end{aligned} \tag{3.3}$$

where the last left hand-side term can be replaced by

$$- (q+2)(q+1) \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+1} f'(u) u_{xx} dx ds. \tag{3.4}$$

**Proof.** Multiplying (1.1) by  $(q+2)(|u_x|^q u_x)_x$  we have

$$\begin{aligned} ((q+2) u_t |u_x|^q u_x)_x &= (|u_x|^{q+2})_t \\ &= -(q+2)(q+1) |u_x|^q u_{xx} f'(u) u_x \\ &\quad + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ &\quad + \delta (q+2)(q+1) |u_x|^q u_{xx} g'(u_{xx}) u_{xxx} \\ &= -(q+1) (|u_x|^{q+2})_x f'(u) \\ &\quad + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ &\quad + \delta (q+2)(q+1) n |u_x|^q G(u_{xx})_x, \end{aligned}$$

and we get the estimate

$$\begin{aligned} ((q+2) u_t |u_x|^q u_x)_x &= (|u_x|^{q+2})_t \\ &= -((q+1) |u_x|^{q+2} f'(u))_x \\ &\quad + (q+1) |u_x|^{q+2} u_x f''(u) \\ &\quad + \varepsilon (q+2)(q+1) |u_x|^q u_{xx}^2 \\ &\quad + (\delta (q+2)(q+1) n |u_x|^q G(u_{xx}))_x \\ &\quad - \delta (q+2)(q+1) q (n+2) n |u_x|^{q-2} u_x \mathcal{G}(u_{xx}). \end{aligned} \tag{3.5}$$

Similarly, using the multiplier  $(q+2)(u_x^{q+1})_x$  in (1.1), it comes

$$\begin{aligned}
((q+2) u_t u_x^{q+1})_x &= (u_x^{q+2})_t \\
&= -(q+2)(q+1) u_x^{q+1} f'(u) u_{xx} \\
&\quad + \varepsilon (q+2)(q+1) u_x^q u_{xx}^2 \\
&\quad + (\delta (q+2)(q+1) n u_x^q G(u_{xx}))_x \\
&\quad - \delta (q+2)(q+1) q (n+2) n u_x^{q-1} \mathcal{G}(u_{xx}),
\end{aligned}$$

and one gets

$$\begin{aligned}
((q+2) u_t u_x^{q+1})_x &= (u_x^{q+2})_t \\
&= -((q+1) u_x^{q+2} f'(u))_x \\
&\quad + (q+1) u_x^{q+3} f''(u) \\
&\quad + \varepsilon (q+2)(q+1) u_x^q u_{xx}^2 \\
&\quad + (\delta (q+2)(q+1) n u_x^q G(u_{xx}))_x \\
&\quad - \delta (q+2)(q+1) q (n+2) n u_x^{q-1} \mathcal{G}(u_{xx}).
\end{aligned} \tag{3.6}$$

Integrating (3.5) and (3.6) over  $\mathbb{R} \times [0, t]$ , we obtain respectively

$$\begin{aligned}
&\int_{\mathbb{R}} |u_x(t)|^{q+2} dx + \varepsilon (q+2)(q+1) \int_0^t \int_{\mathbb{R}} |u_x|^q u_{xx}^2 dx ds \\
&= \int_{\mathbb{R}} |u'_0|^{q+2} dx - (q+1) \int_0^t \int_{\mathbb{R}} u_x |u_x|^{q+2} f''(u) dx ds \\
&\quad + \delta (q+2)(q+1) q (n+2) n \int_0^t \int_{\mathbb{R}} u_x |u_x|^{q-2} \mathcal{G}(u_{xx}) dx ds,
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} u_x(t)^{q+2} dx + \varepsilon (q+2)(q+1) \int_0^t \int_{\mathbb{R}} u_x^q u_{xx}^2 dx ds \\
&= \int_{\mathbb{R}} (u'_0)^{q+2} dx - (q+1) \int_0^t \int_{\mathbb{R}} u_x^{q+3} f''(u) dx ds \\
&\quad + \delta (q+2)(q+1) q (n+2) n \int_0^t \int_{\mathbb{R}} u_x^{q-1} \mathcal{G}(u_{xx}) dx ds.
\end{aligned} \tag{3.8}$$

Adding (3.8) and (3.7) for  $q$  odd, we obtain (3.3).  $\square$

Combining Lemmas 3.1 and 3.2 we obtain the following estimate.

**Proposition 3.1** *Let  $\varepsilon, \delta > 0$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex flux function, i.e.  $f'' \geq 0$ . Then, the family solutions  $\{u_{\varepsilon, \delta}\}$  of (1.1) – (1.2) satisfies the uniform estimate for all  $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$*

$$\int_{\mathbb{R}} |u(t)|^{\alpha+1} dx + \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} |u_x| |u_{xx}|^n dx ds \leq C_0. \tag{3.9}$$

**Proof.** When  $g(v) = -|v|^n$ , equation (3.2) writes

$$\begin{aligned}
&\int_{\mathbb{R}} |u(t)|^{\alpha+1} dx + \alpha(\alpha+1) \varepsilon \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x^2 dx ds = \\
&\quad \|u_0\|_{\alpha+1}^{\alpha+1} + \alpha(\alpha+1) \delta \int_0^t \int_{\mathbb{R}} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds.
\end{aligned} \tag{3.10}$$

Since  $f$  is convex, inequality (3.3) is rewritten for  $q \geq 1$  odd as

$$\begin{aligned} \int_{\mathcal{U}_t^+} |u_x(t)|^{q+2} dx + \varepsilon \int_0^t \int_{\mathcal{U}_s^+} |u_x|^q u_{xx}^2 dx ds \\ + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q-1} |u_{xx}|^{n+2} dx ds + \int_0^t \int_{\mathcal{U}_s^+} |u_x|^{q+3} f''(u) dx ds \leq C_0. \end{aligned} \quad (3.11)$$

Using Young's inequality, we get

$$\begin{aligned} \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \\ = \int_0^t \int_{\mathcal{U}_s^+} \left( \frac{1}{c t^{\frac{\alpha-1}{\alpha+1}}} |u|^{\alpha-1} \right) \left( c t^{\frac{\alpha-1}{\alpha+1}} |u_x| \right) (\delta |u_{xx}|^n) dx ds \\ \leq \frac{1}{t c^{\frac{\alpha+1}{\alpha-1}}} \frac{\alpha-1}{\alpha+1} \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} dx ds + \frac{c^k t^k \frac{\alpha-1}{\alpha+1}}{k} \int_0^t \int_{\mathcal{U}_s^+} |u_x|^k dx ds \\ + \frac{n}{n+2} \delta^{1+\frac{2}{n}} \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} dx ds, \end{aligned} \quad (3.12)$$

where  $c$  and  $k$  are two positive constants such that

$$c^{\frac{\alpha+1}{\alpha-1}} = 4\alpha(\alpha-1), \text{ and } \frac{1}{k} + \frac{n}{n+2} + \frac{\alpha-1}{\alpha+1} = 1.$$

Thus,

$$k = \frac{(n+2)(\alpha+1)}{(4+n) - n\alpha},$$

and  $k \geq 3$  if  $\frac{5+n}{2n+1} \leq \alpha < \frac{4+n}{n}$ .

Now, we choose  $q$  odd such that  $2+q \geq k$  and

$$|u_x|^k \leq |u_x|^3 + |u_x|^{q+2}.$$

From (3.11) with  $q = 1$  and  $q \geq k-2$  odd, we deduce

$$\int_0^t \int_{\mathcal{U}_s^+} |u_x|^k dx ds + \delta \int_0^t \int_{\mathcal{U}_s^+} |u_{xx}|^{n+2} dx ds \leq C_0. \quad (3.13)$$

Integrating (3.10) over  $[0, t]$  gives

$$\begin{aligned} \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha+1} dx ds &\leq \int_0^t \int_{\mathbb{R}} |u|^{\alpha+1} dx ds \\ &\leq t C_0 + \alpha(\alpha+1) t \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds. \end{aligned} \quad (3.14)$$

Now, injecting (3.13) and (3.14) into (3.12), it comes

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \leq C_0 + \frac{1}{4} \delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \quad (3.15)$$

that yields

$$\delta \int_0^t \int_{\mathcal{U}_s^+} |u|^{\alpha-1} u_x |u_{xx}|^n dx ds \leq C_0. \quad (3.16)$$

Finally, it is enough to merge (3.16) in (3.10) to obtain (3.9).  $\square$

Now, let us prove the following lemma.

**Lemma 3.3** *Assume that  $f$  satisfies  $0 \leq f''(u) \leq C(1 + |u|^\beta)$ , with  $0 \leq \beta < \frac{4-n}{n^2}$  and  $1 \leq n \leq 2$ . Then, we have the estimate*

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C_0 \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \\ &+ C_0(n-1) \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \left( \int_0^t \int_{\mathbb{R}} u^2 dx ds + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \right). \end{aligned} \quad (3.17)$$

In particular, when  $n = 1$ , we have

$$\left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| \leq \frac{C_0}{\delta}.$$

**Proof.** Note that, since  $f''(u) \leq C(1 + |u|^\beta)$ , then

$$|f'(u) - f'(0)| \leq C(|u| + |u|^{\beta+1}).$$

Thus, we derive

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq C \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}| dx ds \\ &+ C \int_0^t \int_{\mathbb{R}} |u|^{\beta+1} |u_x| |u_{xx}| dx ds \\ &\leq \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \int_0^t \int_{\mathbb{R}} \left( \delta^{\frac{1}{n}} |u|^{\frac{1}{n}} |u_x|^{\frac{1}{n}} |u_{xx}| \right) \left( |u|^{\frac{n-1}{n}} \right) \left( \epsilon^{\frac{n-1}{2n}} |u_x|^{\frac{n-1}{n}} \right) dx ds \\ &+ \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \int_0^t \int_{\mathbb{R}} \left( \delta^{\frac{1}{n}} |u|^{\beta+\frac{1}{n}} |u_x|^{\frac{1}{n}} |u_{xx}| \right) \left( |u|^{\frac{n-1}{n}} \right) \left( \epsilon^{\frac{n-1}{2n}} |u_x|^{\frac{n-1}{n}} \right) dx ds. \end{aligned}$$

Thanks to the Young inequality and the relation

$$\frac{1}{n} + \frac{n-1}{2n} + \frac{n-1}{2n} = 1,$$

one gets

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds \right| &\leq \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \left( C_0 \delta \int_0^t \int_{\mathbb{R}} |u|^{n\beta+1} |u_x| |u_{xx}|^n dx ds \right. \\ &+ C_0 \delta \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}|^n dx ds \\ &\left. + C_0(n-1) \int_0^t \int_{\mathbb{R}} u^2 dx ds + C_0(n-1) \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \right). \end{aligned} \quad (3.18)$$

It comes from (3.9) with  $1 \leq n \leq 2$ ,  $\alpha = 2$  and  $\alpha = n\beta + 2 < 1 + \frac{4}{n}$

$$\delta \int_0^t \int_{\mathbb{R}} |u| |u_x| |u_{xx}|^n dx ds + \delta \int_0^t \int_{\mathbb{R}} |u|^{n\beta+1} |u_x| |u_{xx}|^n dx ds \leq C_0. \quad (3.19)$$

Finally, inequalities (3.19) and (3.18) allow us to conclude the proof of Lemma 3.3  $\square$

We will use this lemma to prove the following proposition.

**Proposition 3.2** *Let  $\varepsilon > 0$ ,  $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$  and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a flux function satisfying  $0 \leq f''(u) \leq C(1+|u|^\beta)$ , where  $0 \leq \beta < \frac{4-n}{n^2}$ . Then the family solutions  $\{u_{\varepsilon,\delta}\}$  of (1.1) – (1.2) satisfies the estimates*

$$\int_{\mathbb{R}} u_x(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq C_0 \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}}, \quad (3.20)$$

$$\delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds \leq C_0 \left( \frac{\delta}{\varepsilon^{\frac{5n-1}{2(2n-1)}}} \right)^{\frac{2n-1}{2n}}, \quad (3.21)$$

$$\int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds + \delta \int_0^t \int_{\mathbb{R}} |u_x| |u_{xx}|^n dx ds \leq \|u_0\|_2^2 + C_0 \left( \frac{\delta}{\varepsilon^{\frac{5n-1}{2(2n-1)}}} \right)^{\frac{2n-1}{2n}}. \quad (3.22)$$

**Proof.** First, equality (3.2) for  $1 \leq n \leq 2$  and  $\alpha = 1$  reads as

$$\int_{\mathbb{R}} u(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds = \|u_0\|_2^2 + 2\delta \int_0^t \int_{\mathbb{R}} u_x |u_{xx}|^n dx ds. \quad (3.23)$$

Since  $1 \leq n \leq 2$ , we have

$$\delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds \leq \delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}| dx ds + \delta \int_0^t \int_{\mathcal{U}_s^+} u_x u_{xx}^2 dx ds. \quad (3.24)$$

From (3.3) with  $q = 1$ , we get

$$\varepsilon \int_0^t \int_{\mathcal{U}_s^+} u_x u_{xx}^2 dx ds \leq C_0, \quad (3.25)$$

which is equivalent to

$$\delta \int_0^t \int_{\mathcal{U}_s^+} u_x u_{xx}^2 dx ds \leq C_0 \frac{\delta}{\varepsilon}. \quad (3.26)$$

We can also write

$$\begin{aligned} \delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}| dx ds &= \delta^{\frac{2n-1}{2n}} \varepsilon^{-\frac{n-1}{4n}} \int_0^t \int_{\mathcal{U}_s^+} u_x \left( \delta^{\frac{1}{2n}} \varepsilon^{\frac{n-1}{4n}} |u_{xx}| \right) dx ds \\ &\leq \frac{\delta^{\frac{2n-1}{2n}} \varepsilon^{-\frac{n-1}{4n}}}{2\varepsilon} \left( \varepsilon \int_0^t \int_{\mathcal{U}_s^+} u_x^2 dx ds + \varepsilon \delta^{\frac{1}{n}} \varepsilon^{\frac{n-1}{2n}} \int_0^t \int_{\mathcal{U}_s^+} u_{xx}^2 dx ds \right) \\ &\leq \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \int_0^t \int_{\mathcal{U}_s^+} \varepsilon u_x^2 dx ds + \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathcal{U}_s^+} u_{xx}^2 dx ds. \end{aligned} \quad (3.27)$$

Putting (3.27) and (3.26) in (3.24) produces

$$\begin{aligned} \delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds &\leq C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \\ &\quad + C_0 \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \int_0^t \int_{\mathbb{R}} \varepsilon u_x^2 dx ds. \end{aligned} \quad (3.28)$$

Since  $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$ , by injecting (3.28) in (3.23), we obtain

$$\int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \leq \|u_0\|_2^2 + C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds. \quad (3.29)$$



Inequality (3.7) is rewritten with  $q = 0$  as

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds & \\ = \int_{\mathbb{R}} (u'_0)^2 dx + 2 \int_0^t \int_{\mathbb{R}} f'(u) u_x u_{xx} dx ds. \end{aligned} \quad (3.30)$$

Lemma 3.3 points to

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds &\leq C_0 + C_0 \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \\ &+ C_0(n-1) \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \left( \int_0^t \int_{\mathbb{R}} u^2 dx ds + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \right), \end{aligned} \quad (3.31)$$

and inequality (3.29) implies

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds &\leq C_0 + C_0 \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \\ &+ C_0(n-1) \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \left( \|u_0\|_2^2 + C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \varepsilon^{\frac{n-1}{4n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right). \end{aligned} \quad (3.32)$$

Note that if  $\delta = o(\varepsilon)$ , this inequality reduces to

$$\begin{aligned} \int_{\mathbb{R}} u_x(t)^2 dx + 2\varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds &\leq C_0 \delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}} \\ &+ C_0 \delta^{\frac{2n-1}{2n}} \epsilon^{-\frac{5n-1}{4n}} \left( \varepsilon \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \right). \end{aligned} \quad (3.33)$$

Here,  $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$  which is equivalent to  $\delta^{\frac{2n-1}{2n}} = o(\varepsilon^{\frac{5n-1}{4n}})$  offers (3.20).

Now, injecting (3.20) in (3.29) gives

$$\begin{aligned} \int_{\mathbb{R}} u(t)^2 dx + \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds &\leq \|u_0\|_2^2 + C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \epsilon^{\frac{n-1}{4n}} \left( \frac{\delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}}}{\varepsilon} \right) \\ &\leq \|u_0\|_2^2 + \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\leq \|u_0\|_2^2 + C_0 \left( \delta / \varepsilon^{\frac{5n-1}{2(2n-1)}} \right)^{\frac{2n-1}{2n}} \\ &\leq C_0. \end{aligned} \quad (3.34)$$

Finally, combining (3.20), (3.34) and (3.28), we have

$$\begin{aligned} \delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds &\leq C_0 \frac{\delta}{\varepsilon} + C_0 \delta^{\frac{2n+1}{2n}} \epsilon^{\frac{n-1}{4n}} \left( \frac{\delta^{-\frac{1}{n}} \epsilon^{-\frac{n-1}{2n}}}{\varepsilon} \right) + C_0 \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\leq C_0 \frac{\delta}{\varepsilon} + 2 C_0 \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \\ &\leq C_0 \frac{\delta^{\frac{2n-1}{2n}}}{\varepsilon^{\frac{5n-1}{4n}}} \left( 2 + \varepsilon^{\frac{n-1}{4n}} \delta^{\frac{1}{2n}} \right), \end{aligned} \quad (3.35)$$

and  $\delta$  and  $\varepsilon$  being bounded, inequality (3.21) is found.

Inequality (3.22) directly follows from (3.21) and (3.23), and this achieves the proof of Proposition 3.2.  $\square$

**Proposition 3.3** *Let  $\varepsilon > 0$ , and  $\delta = o(\varepsilon^{\frac{5n-1}{2(2n-1)}})$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a flux function satisfying  $0 \leq f''(u) \leq C(1 + |u|^\beta)$ , where  $0 \leq \beta < \frac{4-n}{n^2}$ . Then, the family solutions  $\{u^{\varepsilon, \delta}\}$  of (1.1) – (1.2) satisfies*

$$\begin{aligned} (a) \quad & \varepsilon \int_0^t \int_{\mathbb{R}} u_x^2 dx ds \leq C_0, \\ (b) \quad & \delta \int_0^t \int_{\mathbb{R}} u_x^- |u_{xx}|^n dx ds \leq C_0, \\ (c) \quad & \delta \int_0^t \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds \leq C_0 \left( \delta / \varepsilon^{\frac{5n-1}{2(2n-1)}} \right)^{\frac{2n-1}{2n}}, \\ (d) \quad & \delta^{\frac{2}{n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds \leq C_0 \left( \delta / \varepsilon^{\frac{3n-1}{2}} \right)^{\frac{1}{n}}, \end{aligned}$$

where  $u_x^- = \max(0, -u_x)$ .

**Proof.** Assertions (a) and (b) come from (3.22), while (c) corresponds to (3.21). Thanks to (3.20), we have

$$\begin{aligned} \delta^{\frac{2}{n}} \int_0^t \int_{\mathbb{R}} u_{xx}^2 dx ds & \leq C_0 \delta^{\frac{2}{n}} \varepsilon^{-1} \delta^{-\frac{1}{n}} \varepsilon^{-\frac{n-1}{2n}} \\ & \leq C_0 \delta^{\frac{1}{n}} \varepsilon^{-\frac{3n-1}{2n}}, \end{aligned}$$

that is (d).  $\square$

## 4 Proof of convergence

We have now all the tools to prove the main result given in Theorem 2.1. We first remind the notion of entropy measure-valued solution [5].

**Definition 4.1** *Assume that  $u_0 \in L^1(\mathbb{R}) \cap L^q(\mathbb{R})$  and  $f \in \mathcal{C}(\mathbb{R})$  satisfies the growth condition*

$$|f(u)| \leq \mathcal{O}(|u|^m) \text{ as } |u| \rightarrow \infty, \quad \text{for some } m \in [0, q). \quad (4.1)$$

*A Young measure  $\nu$  is called an entropy measure-valued (e.m.-v.) solution of (1.3)-(1.4) if*

$$\langle \nu, |u - k| \rangle_t + \langle \nu, \operatorname{sgn}(u - k)(f(u) - f(k)) \rangle_x \leq 0, \quad \text{for all } k \in \mathbb{R}, \quad (4.2)$$

*in the sense of distributions on  $\mathbb{R} \times (0, T)$ , and*

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle dx ds = 0, \quad \text{for all compact set } K \subseteq \mathbb{R}. \quad (4.3)$$

We will apply the following Young measure representation theorem [10] in a suitable  $L^q$  space to show that  $\nu$  satisfies (4.2).

**Lemma 4.1** *Let  $\{u_n\}_{n \in \mathbb{N}}$  be a bounded sequence in  $L^\infty((0, T); L^q(\mathbb{R}))$ . Then there exists a subsequence denoted by  $\{\tilde{u}_n\}_{n \in \mathbb{N}}$  and a weakly- $\star$  measurable mapping  $\nu : \mathbb{R} \times (0, T) \rightarrow \text{Prob}(\mathbb{R})$  such that, for all functions  $h \in \mathcal{C}(\mathbb{R})$  satisfying (4.1),  $\langle \nu_{(x,t)}, h \rangle$  belongs to  $L^\infty((0, T); L_{loc}^{q/m}(\mathbb{R}))$  and the following limit representation holds:*

$$\iint_{\mathbb{R} \times (0, T)} \langle \nu_{(x,t)}, h \rangle \phi(x, t) \, dx dt = \lim_{n \rightarrow \infty} \iint_{\mathbb{R} \times (0, T)} h(\tilde{u}_n(x, t)) \phi(x, t) \, dx dt, \quad (4.4)$$

for all  $\phi \in L^1(\mathbb{R} \times (0, T)) \cap L^\infty(\mathbb{R} \times (0, T))$ .

Conversely, given  $\nu$ , there exists a sequence  $\{u_n\}$  satisfying the same conditions as above and such that (4.4) holds for any  $h$  satisfying (4.1).

Let us start proving (4.2) thanks to Proposition 3.3 and assuming that, in the regime  $\delta = o(\varepsilon^{\frac{3n-1}{2}}; \varepsilon^{\frac{5n-1}{2(2n-1)}})$ , initial data  $u_0^{\varepsilon, \delta}$  converge to  $u_0$  as follows

$$\begin{cases} \lim_{\varepsilon \rightarrow 0} u_0^{\varepsilon, \delta} = u_0 & \text{in } L^1(\mathbb{R}) \cap L^q(\mathbb{R}) \\ \|u_0^{\varepsilon, \delta}\|_{L^2} \leq \|u_0\|_{L^2}. \end{cases} \quad (4.5)$$

Standard regularizations of  $\text{sgn}(u - k)(f(u) - f(k))$  and  $|u - k|$  ( $k \in \mathbb{R}$ ) are used, to show that it is enough to prove the existence of a bounded measure  $\mu \leq 0$  such that

$$\eta(u)_t + q(u)_x \longrightarrow \mu \quad \text{in } \mathcal{D}'(\mathbb{R} \times (0, T)) \quad (4.6)$$

for an arbitrary convex function  $\eta$  (we assume that  $\eta'$  and  $\eta''$  are bounded on  $\mathbb{R}$ ).

To prove (4.6), we rewrite the formula (3.1) as

$$\eta(u)_t + q(u)_x = \mu_1 + \mu_2 + \mu_3 + \mu_4, \quad (4.7)$$

with

$$\begin{aligned} \mu_1 : &= \varepsilon (\eta'(u) u_x)_x, \\ \mu_2 : &= -\varepsilon \eta''(u) u_x^2, \\ \mu_3 : &= \delta (\eta'(u) g(u_{xx}))_x, \\ \mu_4 : &= -\delta \eta''(u) u_x g(u_{xx}). \end{aligned}$$

Let's  $\theta \in C_0^\infty(\mathbb{R} \times (0, T))$  a test function. We have

$$\begin{aligned} | \langle \mu_1, \theta \rangle | &\leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) u_x| \, dx ds \\ &\leq C \varepsilon \int_0^T \int_{\mathbb{R}} |\theta_x u_x| \, dx ds \\ &\leq C \|\theta_x\|_{L^2} \varepsilon^{1/2} \left( \varepsilon \int_0^T \int_{\mathbb{R}} u_x^2 \, dx ds \right)^{1/2} \\ &\leq C \|\theta_x\|_{L^2} \varepsilon^{1/2} \left( \varepsilon \int_0^T \int_{\mathbb{R}} u_x^2 \, dx ds \right)^{1/2}. \end{aligned} \quad (4.8)$$

According to Proposition 3.3-(a),

$$| \langle \mu_1, \theta \rangle | \leq C \|\theta_x\|_{L^2} \varepsilon^{1/2} \quad (4.9)$$

and  $\mu_1 \rightarrow 0$  when  $\varepsilon \rightarrow 0$ .  
Thereafter,

$$\begin{aligned} | \langle \mu_2, \theta \rangle | &\leq \varepsilon \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^2| dx ds \\ &\leq \varepsilon C \|\theta\|_{L^\infty} \int_0^T \int_{\mathbb{R}} u_x^2 dx ds. \end{aligned} \quad (4.10)$$

and  $\mu_2$  is bounded thanks to Proposition 3.3–(a). Moreover, since  $\eta$  is convex, we have for a non negative function  $\theta$

$$\langle \mu_2, \theta \rangle = -\varepsilon \int_0^T \int_{\mathbb{R}} \theta \eta''(u) u_x^2 dx ds \leq 0. \quad (4.11)$$

Concerning  $\mu_3$ , we have

$$\begin{aligned} | \langle \mu_3, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta_x \eta'(u) |u_{xx}|^n| dx ds \\ &\leq C \delta \int_0^T \int_{\mathbb{R}} |\theta_x| |u_{xx}|^n dx ds \\ &\leq C \|\theta_x\|_{L^{\frac{2}{2-n}}} \left( \delta^{\frac{2}{n}} \int_0^T \int_{\mathbb{R}} u_{xx}^2 dx ds \right)^{\frac{n}{2}}. \end{aligned} \quad (4.12)$$

Remark that when  $n = 2$ , the space  $L^{\frac{2}{2-n}}$  is replaced by  $L^\infty$ . From Proposition 3.3–(d), one gets

$$| \langle \mu_3, \theta \rangle | \leq C_0 \|\theta_x\|_{L^{\frac{2}{2-n}}} \left( \delta / \varepsilon^{\frac{3n-1}{2}} \right)^{\frac{1}{2}},$$

and  $\mu_3 \rightarrow 0$  because  $\delta = o(\varepsilon^{\frac{3n-1}{2}})$ .

Finally,  $\mu_4$  is split as

$$\begin{aligned} \mu_4 &= \delta \eta''(u) u_x^+ |u_{xx}|^n - \delta \eta''(u) u_x^- |u_{xx}|^n \\ &=: \mu_4^+ + \mu_4^- \end{aligned}$$

where  $u_x^+ = \max(0, u_x)$  and  $u_x^- = \max(0, -u_x)$ , and we have  $u_x = u_x^+ - u_x^-$ . On the one hand,

$$\begin{aligned} | \langle \mu_4^-, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta \eta''(u) u_x^- |u_{xx}|^n| dx ds \\ &\leq C \delta \int_0^T \int_{\mathbb{R}} |\theta u_x^-| |u_{xx}|^n dx ds \\ &\leq C \|\theta\|_{L^\infty} \delta \int_0^T \int_{\mathbb{R}} u_x^- |u_{xx}|^n dx ds, \end{aligned} \quad (4.13)$$

and according to Proposition 3.3–(b),

$$| \langle \mu_4^-, \theta \rangle | \leq C_0 \|\theta\|_{L^\infty}.$$

Again,  $\eta'' \geq 0$ , we have

$$\langle \mu_4^-, \theta \rangle = -\delta \int_0^T \int_{\mathbb{R}} \theta \eta''(u) u_x^- |u_{xx}|^n dx ds \leq 0. \quad (4.14)$$

Similarly,

$$\begin{aligned}
| \langle \mu_4^+, \theta \rangle | &\leq \delta \int_0^T \int_{\mathbb{R}} |\theta| \eta''(u) u_x^+ |u_{xx}|^n dx ds \\
&\leq C \|\theta\|_{L^\infty} \delta \int_0^T \int_{\mathbb{R}} u_x^+ |u_{xx}|^n dx ds \\
&\leq C \|\theta\|_{L^\infty} \delta \int_0^T \int_{\mathcal{U}_s^+} u_x |u_{xx}|^n dx ds.
\end{aligned} \tag{4.15}$$

Proposition 3.3-(c) implies

$$| \langle \mu_4^+, \theta \rangle | \leq C \|\theta\|_{L^\infty} (\delta / \varepsilon^{\frac{5n-1}{2(2n-1)}})^{\frac{2n-1}{2n}}, \tag{4.16}$$

and  $\mu_4^+ \rightarrow 0$  because  $\delta = o(\varepsilon^{\frac{n(n+1)}{2}})$ .

Condition (4.6) is proved and  $\mu$  is non positive bounded measure.

To show (4.3), we follow DiPerna [5] and Szepessy [12] arguments. We check that, for each compact set  $K$  of  $\mathbb{R}$ ,

$$\begin{aligned}
\lim_{t \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K \langle \nu_{(x,s)}, |u - u_0(x)| \rangle dx ds &= \\
\lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K |u^{\varepsilon, \delta}(x, s) - u_0(x)| dx ds &= 0.
\end{aligned}$$

Thanks to Jensen's inequality, for  $m(K)$  the Lebesgue measure of  $K$ ,

$$\frac{1}{t} \int_0^t \int_K |u^{\varepsilon, \delta}(x, s) - u_0(x)| dx ds \leq m(K)^{1/2} \left( \frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(x, s) - u_0(x))^2 dx ds \right)^{1/2}$$

and we claim that

$$\lim_{t \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(x, s) - u_0(x))^2 dx ds = 0.$$

Indeed, let  $K_i \subset K_{i+1}$  ( $i = 0, 1, \dots$ ) be an increasing sequence of compact sets such that  $K_0 = K$  and  $\cup_{i \geq 0} K_i = \mathbb{R}$ . Using the identity  $u^2 - u_0^2 - 2u_0(u - u_0) = (u - u_0)^2$ , we get for all  $i = 0, 1, \dots$

$$\begin{aligned}
&\frac{1}{t} \int_0^t \int_K (u^{\varepsilon, \delta}(\cdot, s) - u_0)^2 dx ds \\
&\leq \frac{1}{t} \int_0^t \left( \int_{K_i} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{K_i} u_0^2 dx - 2 \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right) ds \\
&\leq \int_{\mathbb{R} \setminus K_i} u_0^2 dx + \frac{1}{t} \int_0^t \left( \int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx \right) \\
&\quad + \frac{2}{t} \int_0^t \left| \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right| ds.
\end{aligned} \tag{4.17}$$

Clearly, the first term of the right hand side verifies

$$\lim_{i \rightarrow \infty} \int_{\mathbb{R} \setminus K_i} u_0^2 dx = 0.$$

Using (3.22) and (4.5), we obtain

$$\begin{aligned}
\int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} u_0^2 dx &= \int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} (u_0^{\delta, \varepsilon})^2 dx + \int_{\mathbb{R}} (u_0^{\delta, \varepsilon})^2 dx - \int_{\mathbb{R}} u_0^2 dx \\
&\leq \int_{\mathbb{R}} |u^{\varepsilon, \delta}(\cdot, s)|^2 dx - \int_{\mathbb{R}} (u_0^{\delta, \varepsilon})^2 dx \\
&\leq C_0 (\delta / \varepsilon^{\frac{5n-1}{2(2n-1)}})^{\frac{2n-1}{2n}},
\end{aligned}$$

and the second term of right hand side goes to zero when  $\varepsilon \rightarrow 0$ .

To estimate the last term of (4.17), we consider  $\{\theta_k\}_{k \in \mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R})$  being such that

$$\lim_{k \rightarrow \infty} \theta_k = u_0 \quad \text{in } L^2(\mathbb{R}).$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned} \left| \int_{K_i} u_0 (u^{\varepsilon, \delta}(\cdot, s) - u_0) dx \right| &\leq \int_{K_i} |u_0 - \theta_k| |u^{\varepsilon, \delta}(\cdot, s) - u_0| dx \\ &\quad + \left| \int_{K_i} \theta_k (u_0^{\varepsilon, \delta} - u_0) + \int_{K_i} \theta_k (u^{\varepsilon, \delta}(\cdot, s) - u_0^{\varepsilon, \delta}) dx \right| \\ &\leq \|u_0 - \theta_k\|_{L^2(\mathbb{R})} (\|u^{\varepsilon, \delta}(\cdot, s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})}) \\ &\quad + \|\theta_k\|_{L^2(\mathbb{R})} \|u_0^{\varepsilon, \delta} - u_0\|_{L^2(\mathbb{R})} + \left| \int_0^s \int_{K_i} \theta_k \partial_s u^{\varepsilon, \delta} dx d\tau \right|. \end{aligned}$$

We deduce from (3.22)

$$\|u_0 - \theta_k\|_{L^2(\mathbb{R})} (\|u^{\varepsilon, \delta}(\cdot, s)\|_{L^2(\mathbb{R})} + \|u_0\|_{L^2(\mathbb{R})}) \leq (\|u_0\|_{L^2(\mathbb{R})} + C) \|u_0 - \theta_k\|_{L^2(\mathbb{R})},$$

which goes to zero when  $k \rightarrow \infty$  because  $\lim_{\varepsilon \rightarrow 0+} \|u_0^{\varepsilon, \delta} - u_0\|_{L^2(\mathbb{R})} = 0$  from (4.5). It remains to prove that

$$\lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_k \partial_s u^{\varepsilon, \delta} dx d\tau \right| ds = 0.$$

The equation (1.1) provides

$$\begin{aligned} \left| \int_0^s \int_{K_i} \theta_k \partial_s u^{\varepsilon, \delta} dx d\tau \right| &= \left| \int_0^s \int_{K_i} \theta_k (-f(u^{\varepsilon, \delta})_x + \varepsilon u_{xx}^{\varepsilon, \delta} + \delta g(u_{xx}^{\varepsilon, \delta})_x) dx d\tau \right| \\ &\leq \int_0^s \int_{K_i} |(\theta_k)_x f(u^{\varepsilon, \delta})| dx d\tau \\ &\quad + \varepsilon \int_0^s \int_{K_i} |(\theta_k)_x u_x^{\varepsilon, \delta}| dx d\tau \\ &\quad + \delta \int_0^s \int_{K_i} |(\theta_k)_x g(u_{xx}^{\varepsilon, \delta})| dx d\tau \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

To deal with  $I_1$ , the function  $f$  being such that

$$0 \leq f''(u) \leq C(1 + |u|^\beta),$$

with  $\beta < \frac{4-n}{n^2}$ , then for  $m = \beta + 2$

$$|f(u)| \leq C(1 + |u|^m).$$

Thus, Proposition 3.1 with  $\alpha = m - 1 = \beta + 1 < \frac{4-n}{n^2} + 1 < \frac{4}{n^2} - \frac{1}{n} + 1 < \frac{4}{n} + 1$  is applied

$$\int_0^s \int_{K_i} |u^{\varepsilon, \delta}|^m dx d\tau \leq \int_0^s \int_{\mathbb{R}} |u^{\varepsilon, \delta}|^m dx d\tau \leq C s. \quad (4.18)$$

Then we can write

$$\begin{aligned}
I_1 &= \int_0^s \int_{K_i} |(\theta_k)_x| |f(u^{\varepsilon, \delta})| dx d\tau \\
&\leq C \int_0^s \int_{K_i} |(\theta_k)_x| dx d\tau + C \int_0^s \int_{K_i} |(\theta_k)_x| |u^{\varepsilon, \delta}|^m dx d\tau \\
&\leq C \int_0^s \int_{K_i} |(\theta_k)_x| dx d\tau + C \|(\theta_k)_x\|_{L^\infty(\mathbb{R})} \int_0^s \int_{K_i} |u^{\varepsilon, \delta}|^m dx d\tau \\
&\leq C s \|(\theta_k)_x\|_{L^1(\mathbb{R})} + C s \|(\theta_k)_x\|_{L^\infty(\mathbb{R})}.
\end{aligned} \tag{4.19}$$

Again, thanks to Proposition 3.3–(a) we have

$$\begin{aligned}
I_2 &= \varepsilon \int_0^s \int_{K_i} |(\theta_k)_x| |u_x^{\varepsilon, \delta}| dx d\tau \\
&\leq \left[ \varepsilon \int_0^s \int_{K_i} |(\theta_k)_x|^2 dx d\tau \right]^{\frac{1}{2}} \left[ \varepsilon \int_0^s \int_{K_i} |u_x^{\varepsilon, \delta}|^2 dx d\tau \right]^{\frac{1}{2}} \\
&\leq C \varepsilon^{\frac{1}{2}} s^{\frac{1}{2}} \|(\theta_k)_x\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{4.20}$$

Finally, we have

$$\begin{aligned}
I_3 &= \delta \int_0^s \int_{K_i} |(\theta_k)_x g(u_{xx}^{\varepsilon, \delta})| dx d\tau \\
&= \int_0^s \int_{K_i} \delta |(\theta_k)_x| |u_{xx}^{\varepsilon, \delta}|^n dx d\tau \\
&\leq \left[ \int_0^s \int_{K_i} |(\theta_k)_x|^{\frac{2}{2-n}} dx d\tau \right]^{\frac{2-n}{2}} \left[ \delta^{\frac{2}{n}} \int_0^s \int_{\mathcal{K}_{t, \varepsilon, \delta-}} |u_{xx}^{\varepsilon, \delta}|^2 dx d\tau \right]^{\frac{n}{2}} \\
&\leq s^{\frac{2-n}{2}} \|(\theta_k)_x\|_{L^{\frac{2}{2-n}}(\mathbb{R})} \left[ \delta^{\frac{2}{n}} \int_0^s \int_{\mathcal{K}_{t, \varepsilon, \delta-}} |u_{xx}^{\varepsilon, \delta}|^2 dx d\tau \right]^{\frac{n}{2}}
\end{aligned}$$

and using Proposition 3.3–(d), one gets

$$I_3 \leq C s^{\frac{2-n}{2}} \left( \frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \|(\theta_k)_x\|_{L^{\frac{2}{2-n}}(\mathbb{R})}. \tag{4.21}$$

Combining (4.19), (4.20) and (4.21) produces

$$\begin{aligned}
&\lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{1}{t} \int_0^t \left| \int_0^s \int_{K_i} \theta_k u_s^{\varepsilon, \delta} dx d\tau \right| ds \\
&\leq \lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} \frac{C}{t} \left( t^2 (\|(\theta_k)_x\|_{L^1(\mathbb{R})} + \|(\theta_k)_x\|_{L^\infty(\mathbb{R})}) + \varepsilon^{\frac{1}{2}} t^{\frac{3}{2}} \|(\theta_k)_x\|_{L^2(\mathbb{R})} \right. \\
&\quad \left. + t^{\frac{4-n}{2}} \left( \frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \|(\theta_k)_x\|_{L^{\frac{2}{2-n}}(\mathbb{R})} \right) \\
&\leq \lim_{t \rightarrow 0+} \lim_{\varepsilon \rightarrow 0+} C \left( t + \varepsilon^{\frac{1}{2}} t^{\frac{1}{2}} + t^{\frac{2-n}{2}} \left( \frac{\delta}{\varepsilon^{\frac{3n-1}{2}}} \right)^{\frac{1}{2}} \right),
\end{aligned}$$

and since  $\delta = o(\varepsilon^{\frac{3n-1}{2}})$ , we obtain the desired conclusion, and Theorem 2.1 is proved.

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