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Abstract

This paper provides a general and abstract approach to compute invariant distributions for Feller processes. More precisely, we show that the recursive algorithm presented in [10] and based on simulation algorithms of stochastic schemes with decreasing steps can be used to build invariant measures for general Feller processes. We also propose various applications: Approximation of Markov Brownian diffusion stationary regimes with a Milstein or an Euler scheme and approximation of a Markov switching Brownian diffusion stationary regimes using an Euler scheme.
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1 Introduction

In this paper, we propose a generic method to compute (recursively) the invariant distribution (denoted ν) of an ergodic homogeneous Feller processes $X = (X_t)_{t\geqslant 0}$ with semi-group of transitions $(P_t)_{t\geq 0}$. Invariant distributions are crucial objects in the study of the long term behavior of continuous dynamical systems like, among others, (mean-reverting) SDE with or without jumps, regimes switching SDEs, PDMP, etc, which all share a homogeneous Markov property. We invite the reader to refer to [9] and [5] for an overview of the subject. Usually, no closed forms for invariant distributions are available with the noticeable exceptions of one dimensional Brownian diffusions (see [6]), dissipative gradient diffusions with a constant diffusion coefficient. Other examples, typically borrowed from Hamiltonian Mechanics, see [23], lead to explicit exact expressions of the invariant density distribution for some solutions of Stochastic Differential Equations are given. However, in many cases there is no explicit formula for ν and even there some, computing it by remains a serious issue especially in higher dimension like for dissipative gradients SDEs.

For a continuous time ergodic Feller process $(X_t)_{t\geq 0}$ with (unique) invariant distribution ν, pointwise Birkhoff's ergodic theorem implies that

$$\nu(dx)\text{-}a.s. \quad \mathbb{P}(d\omega)\text{-}a.s. \quad \nu_t(dx,\omega) := \frac{1}{t}\int_0^t \delta_{X_s^x(\omega)}(dx)ds \xrightarrow{\mathrm{L}^1(\nu)} \nu(dx)$$

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where X_t^x denotes the process starting from x at time 0. Under more stringent assumptions of mean-reverting nature, one shows that this convergence holds *for every* $x \in \mathbb{R}^d$ and not only $\nu(dx)$-$a.s$ and, usually, one even obtains that, $\mathbb{P}(d\omega)$-$a.s.$, $\mathcal{W}_p(\nu_t(\omega, dx), \nu) \to 0$ as $t \to +\infty$ for some $p \geq 1$. Such a property is often called *stability* of the Markov process. If the semi-group P_t is strongly Feller, i.e. $P_t f$ is continuous for every bounded Borel function $f : \mathbb{R}^d \to \mathbb{R}$, then

$$\mathbb{P}(d\omega)\text{-}a.s. \quad \nu_t(\omega, f) \to \nu(f) \quad \text{as} \quad t \to +\infty.$$

For more details we refer to [15] and the references therein. As for the (weak) rate of this convergence, let us me-cite in the framework of Brownian diffusions Bhattacharia's Central Limit Theorem (CLT, see [2]) which shows that for functions $f$which are smooth enough ν-coboundaries of the infinitesimal generator A of the diffusion – i.e. such that the Poisson equation $Ag = f - \nu(f)$ has a solution – , the following CLT holds

$$\sqrt{t}\,\big(\nu_t(f) - \nu(f)\big) = \sqrt{t}\,\nu_t(Ag) \stackrel{w}{\longrightarrow} \mathcal{N}\Big(0, \int_{\mathbb{R}^d} |\sigma^* g|^2\Big)$$

where σ denotes the diffusion coefficient of the diffusion.

A first natural approach to approximate ν, in a weak sense, that is through the integrals $\nu(f)$ for a wide class of functions f, relies on the following two facts: $X_t^x \stackrel{w}{\to} \nu$ as $t \to +\infty$ or, equivalently, $P_t(x, dy) \stackrel{w}{\to} \nu$. For weak convergence results of X_t toward ν (and rates) we refer e.g. to [7]. Then, for a large enough fixed T, on may assume that $X_T^x \stackrel{d}{\simeq} \nu$ and perform a regular Monte Carlo simulation to compute $\mathbb{E}[f(X_T^x)]$. This approach introduces two errors: an approximation one and a statistical one, not to mention the fact that exact simulation of X_t^x is non standard situation (think about diffusions). If X_T^x is not simulable, it can be replaced by its Euler (or any other) scheme, inducing a third source of error.

In [24], is introduced, still in a Brownian diffusion framework, a potentially more generic method still based on a standard time discretization scheme (Euler or higher order schemes) $(\bar{X}_n^{\gamma_0})_{n\geq 0}$ with step γ_0 (so that at iteration n, time is $\Gamma_n = n\gamma_0$) but which takes advantage of the pathwise properties of stability. If γ_0 is small enough, it is shown that the scheme shares stability properties properties with the diffusion $(X_t^x)_{t\geq 0}$ so that

$$\bar{\nu}_n(dx, \omega) = \frac{1}{n}\sum_{k=1}^n \delta_{\bar{X}_k^{\gamma_0}(\omega)}(dx) = \frac{1}{\Gamma_n}\sum_{k=1}^n \gamma_0 \delta_{\bar{X}_k^{\gamma_0}(\omega)}(dx) \stackrel{w}{\longrightarrow} \bar{\nu}^{\gamma_0}(dx) \quad \mathbb{P}(d\omega)\text{-}a.s.$$

This means that, taking advantage of the ergodic properties of the dynamics (and of its time discretization), it is possible to approximate the invariant distribution ν using a long enough simulation of a single path of the scheme and, with the possibility to do it recursively. Moreover, it is shown, still in [24], that $\bar{\nu}^{\gamma_0}(f) \to \nu(f)$ at a $O(\gamma_0)$-rate (for the Euler scheme) for a wide class of smooth test functions f. Then

$$\bar{\nu}_n(f) - \nu(f) = \big(\bar{\nu}_n(f) - \bar{\nu}^{\gamma_0}(f)\big) + \big(\underbrace{\bar{\nu}^{\gamma_0}(f) - \nu(f)}_{O(\gamma_0)}\big).$$

If a CLT (and/or an L^2-convergence) rules the convergence of the first bracket on the right hand side at rate $\sqrt{\Gamma_n} = \sqrt{\gamma_0 n}$, then one has to make the balance between these two errors i.e. minimize (in γ_0) a term of the form

$$\frac{C_1}{\sqrt{n\gamma_0}} + C_2\gamma_0, \qquad \text{with } C_1,\, C_2 > 0 \text{ real constants.}$$

This leads to set, n being fixed, $\gamma_0 \asymp n^{-1/3}$ which of course annihilates the recursive feature of the method.

Another way to exploit the mean-reverting properties of a diffusion was proposed in [1], still for Brownian diffusions, and can be used as a numerical method. The authors directly prove that the Euler scheme with decreasing step $(\bar{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$, with step sequence $\gamma = (\gamma_n)_{n\in\mathbb{N}}$ going to 0 and $\Gamma_n = \gamma_1 + \ldots + \gamma_n$, weakly converges toward the invariant distribution ν, supposed to be unique, of X. It avoids the asymptotic analysis between ν^{γ_1} and ν but the the resulting error is still made of two terms. The first one is due to the rate of this weak convergence and the second one to the Monte Carlo error involved in the computation of the law of $\bar{X}^{\gamma}_{\Gamma_n}$, for a fixed large enough n and the recursiveness is lost. For recent result of convergence in total variation of $\bar{X}^{\gamma}_{\Gamma_n}$ toward ν in the case of the Langevin over damped diffusion, we refer to e.g. [4].

In the above two approaches the results are established under a strong mean-reversion assumption of the form $AV \leqslant \beta - \alpha V$ $(\alpha > 0)$ where A denotes the infinitesimal generator of the diffusion and $V : \mathbb{R}^d \to R_+$ is an essentially sub-quadratic $\mathcal{C}^2$ function with a bounded Hessian going to infinity at infinity often called *Lyapunov function* of the dynamics (here a diffusion).

To take full advantage of both the ergodicity/stability properties and preserve the recursiveness of the algorithm, it was necessary to have ν – in practice $\nu(f)$ – as a direct target of the procedure and consider a single path. In the early 2000's, emerged the idea to directly mimic the Ergodic theorem by considering the empirical/occupation measure of this Euler scheme with decreasing step. Doing so, one preserves this recursiveness like in [24] and has now ν itself as a target instead of ν^{γ_0}. This lead to investigate the *a.s.* convergence properties of the weighted empirical measure of the Euler scheme with decreasing step $(\bar{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ at discretization times, namely

$$\forall\, \omega \in \Omega,\ \forall\, n \geq 1, \quad \nu^{\gamma}_n(\omega, dx) = \frac{1}{\Gamma_n}\sum_{k=1}^{n} \gamma_k \delta_{\bar{X}^{\gamma}_{\Gamma_{k-1}}(\omega)}(dx) = \frac{1}{\Gamma_n}\int_0^{\Gamma_n} \delta_{\bar{X}^{\gamma}_{\underline{s}}(\omega)}(dx)ds,$$

where $\Gamma_n = \sum_{k=1}^{n} \gamma_k$ and $\underline{s} = \Gamma_{k-1}$ on $[\Gamma_{k-1}, \Gamma_k)$. Then, the stability of this *non-homogeneous Markov chain* is proved under various mean-reverting assumptions and frameworks. This approach was first introduced for strongly mean-reverting diffusions in [10] and then developed in a series of papers [11] (weakly mean-reverting setting of the form $AV \leqslant \beta - \alpha V^a$, $a \in (0,1)$), [13] (exponential convergence), [19] (jump diffusions), [16] (functional versions), [17] (functional convergence rate) or [14] (regime switching diffusions), see also [15] for a review... One important asset of this approach is that it does not require uniqueness of the invariant measure since it is shown that, under appropriate mean-reverting assumptions, $\nu_n(dx, \omega)$ always *a.s.* weakly converges to the non empty set weakly compact $\mathcal{V}$ of invariant distributions of the underlying SDE, proc-bing on its way the existence of at least an invariant distributions, all sharing some moment finiteness of polynomial of even exponential nature (see [13]). In case of uniqueness of ν, making these power moments converge implies the above (*a.s.*) weak convergence holds for the L^p-Wasserstein distance for some $p \geq 1$.

The aim of this paper is to extend the above purely ergodic approach based on a *Langevin Monte Carlo* estimator to a wider class of Feller processes. To this end, we will establish a somewhat abstract version inspired by the the above convergence result of the empirical measure of an Euler scheme with decreasing step.

The starting idea is to consider a non-homogeneous discrete time Markov process which can be simulated using a family of *simulable* transitions kernels $(Q_\gamma)_{\gamma>0}$ approximating the transitions P_γ of the Feller process X as $\gamma \to 0$ in a sense involving its infinitesimal generator A to be specified later on. Usually – think about the Euler transitions with step $\gamma > 0$ – the price to be paid for the simulability of $Q_\gamma(x, dy)$ is that the family $(Q_\gamma)_{\gamma>0}$ no longer makes up a semi-group.

Then, we introduce the weighted empirical measure

$$\nu^{\gamma}_n(dx) = \frac{1}{\Gamma_n}\sum_{k=1}^{n} \gamma_k \delta_{\bar{X}^{\gamma}_{\Gamma_{k-1}}}(dx), \qquad \Gamma_n = \sum_{k=1}^{n} \gamma_k, \tag{1}$$

and aim at proving its *a.s.* weak/Wasserstein convergence toward the set $\mathcal{V}$ of invariant distributions of X under some mean-reverting assumptions. For convenience (see further on), this condition will be specified on the pseudo-infinitesimal generators $A_\gamma = \frac{Q_\gamma - Id}{\gamma}$ associated to the family $(Q_\gamma)_{\gamma>0}$ rather than on the generator A of X itself. In fact we will introduce more general weights η_n than γ_n and deal with the random measures ν^η in prevision on future works on the (weak) convergence rate which require the convergence of such random measures for weights of the form $\eta_n = \gamma_n^r$ for some $r \geq 0$.

The paper is organized as follows. In a first step, we present an abstract framework adapted to the computation of invariant distributions for Feller processes under general mean-reverting assumptions (including weakly mean-reverting assumptions). Then, we establish *a.s* weak convergence of $(\nu_n^\gamma)_{n\in\mathbb{N}^*}$. Moreover, when the invariant distribution ν is unique we obtain $\lim_n \nu_n^\gamma f = \nu f$ *a.s.* for a generic class of continuous test functions f (adapted among other to polynomial and exponential test functions f).

Then, in a second step, we apply this abstract results to concrete cases and obtain original results. Notice that the existing results mentioned above can be recovered from our abstract framework. We begin by providing Wasserstein convergence results concerning Euler and Milstein schemes of Brownian diffusion processes in a weakly mean-reverting setting. Then, we propose a detailed application concerning the Euler scheme of a Markov Switching diffusion for test functions f with polynomial growth (Wasserstein convergence) or exponential growth. Here, we extend the convergence results from [14] where the authors adapted the algorithm from [10] under strong ergodicity assumptions for the Wasserstein convergence. Notice that this generic aproach is used in the companion paper [18] where we study the Milstein scheme for Brownian diffusions and the Euler scheme for Jump diffusion processes with censored jumps (that extend Levy jump processes). The Milstein study is the first one out of the scope of the Euler scheme while the censored jump study extends results from [20] concerning the Levy processes. A main interest of censored jump diffusions is that, controversly to Levy processes, the intensity of jumps may depend on the spatial position of the underlying process.

NOTATIONS.
• Let $(E, |.|)$ be a locally compact separable metric space. $\mathcal{C}(E)$ will denote the set of continuous functions on E and $\mathcal{C}_0(E)$ the set of continuous functions that vanish at infinity. We equip this space with the sup norm $\|f\|_\infty = \sup_{x\in E} |f(x)|$ so that $(\mathcal{C}_0(E), \|\cdot\|_\infty)$ is a Banach space. $\mathcal{B}(E)$ will denote the σ-algebra of Borel subsets of E and $\mathcal{P}(E)$ the family of Borel probability measures on E. We will denote by $\mathcal{K}_E$ the set of compact subsets of E.
• Convergence of a Borel function $f : E \to \mathbb{R}$ at infinity should be understood with respect to the convergence filter base made up with complements of compact sets.

2 Convergence to invariant distributions: a general approach

In this section, we present the framework devised to show that the empirical measures formally defined by (1) and built from a numerical scheme with decreasing step $(\bar{X}_{\Gamma_n}^\gamma)_{n\in\mathbb{N}}$ of a Feller process $(X_t)_{t\geqslant 0}$ (which are not specified explicitly), where the step sequence $\gamma = (\gamma_n)_{n\in\mathbb{N}^*}$ converges to 0, *a.s.* weakly converges to the set $\mathcal{V}$ of the invariant distributions of $(X_t)_{t\geqslant 0}$. To this end, we will provide as weak as possible mean-reverting assumptions on the pseudo-generator of $(\bar{X}_{\Gamma_n}^\gamma)_{n\in\mathbb{N}}$ on the one hand and appropriate rate conditions on the step sequence $(\gamma_n)_{n\in\mathbb{N}^*}$ on the other hand.

We first develop an abstract framework. Then, we establish the abstract convergence results of the empirical measures. Finally, we provide convergence results for the specific but nevertheless classic case of the Euler scheme of a SDE.

2.1 Presentation of the abstract framework

2.1.1 Construction of the random measures

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. We consider a Feller process $(X_t)_{t\geqslant 0}$ (see [6] for details) on $(\Omega, \mathcal{G}, \mathbb{P})$ taking values in a locally compact and separable metric space E. We denote by $(P_t)_{t\geqslant 0}$ the Feller semi-group (see [21]) of this process. We recall that $(P_t)_{t\geqslant 0}$ is a family of linear operators from $\mathcal{C}_0(E)$ to itself such that $P_0 f = f$, $P_{t+s} f = P_t P_s f$, $t, s \geqslant 0$ (semi-group property) and $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$ (Feller property). Using this semi-group, we can introduce the infinitesimal generator of $(X_t)_{t\geqslant 0}$ as a linear operator A defined on a subspace $\mathcal{D}(A)$ of $\mathcal{C}_0(E)$, satisfying: For every $f \in \mathcal{D}(A)$,

$$Af = \lim_{t\to 0} \frac{P_t f - f}{t}$$

exists for the $\|.\|_\infty$-norm. The operator $A : \mathcal{D}(A) \to \mathcal{C}_0(E)$ is thus well defined and $\mathcal{D}(A)$ is called the domain of A. From the Echeverria-Weiss theorem (see Theorem 2.1), the set of invariant distributions for $(X_t)_{t\geqslant 0}$ can be characterized in the following way:

$$\mathcal{V} = \{\nu \in \mathcal{P}(E), \forall t \geqslant 0, P_t \nu = \nu\} = \{\nu \in \mathcal{P}(E), \forall f \in \mathcal{D}(A), \nu(Af) = 0\}.$$

The starting point of our reasoning is thus to consider an approximation of A. First, we introduce the family of transition kernels $(\mathcal{Q}_\gamma)_{\gamma>0}$ from $\mathcal{C}_0(E)$ to itself. Now, let us define the family of linear operators $\widetilde{A} := (\widetilde{A}_\gamma)_{\gamma>0}$ from $\mathcal{C}_0(E)$ into itself, as follows

$$\forall f \in \mathcal{C}_0(E), \quad \gamma > 0, \qquad \widetilde{A}_\gamma f = \frac{\mathcal{Q}_\gamma f - f}{\gamma}.$$

The family $\widetilde{A}$ is usually called the pseudo-generator of the transition kernels $(\mathcal{Q}_\gamma)_{\gamma>0}$ and is an approximation of A as γ tends to zero. From a practical viewpoint, the main interest of our approach is that we can consider that there exists $\bar{\gamma} > 0$ such that for every $x \in E$ and every $\gamma \in [0, \bar{\gamma}]$, $\mathcal{Q}_\gamma(x, dy)$ is simulable at a reasonable computational cost. We use the family $(\mathcal{Q}_\gamma)_{\gamma>0}$, to build $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ (this notation replaces $(\bar{X}^\gamma_{\Gamma_n})_{n\in\mathbb{N}}$ from now on for clarity in the writing) as the non-homogeneous Markov approximation of the Feller process $(X_t)_{t\geqslant 0}$. It is defined at times grid $\Gamma_n = \sum_{k=1}^{n} \gamma_k$, $n \in \mathbb{N}$ with the sequence $\gamma := (\gamma_n)_{n\in\mathbb{N}^*}$ of time steps satisfying

$$\forall n \in \mathbb{N}^*, \quad 0 < \gamma_n \leqslant \bar{\gamma} := \sup_{n\in\mathbb{N}^*} \gamma_n < +\infty, \quad \lim_n \gamma_n = 0 \quad \text{and} \quad \lim_n \Gamma_n = +\infty.$$

Its transition probability distributions are given by $\mathcal{Q}_{\gamma_n}(x, dy), n \in \mathbb{N}^*, x \in E$, *i.e.* :

$$\mathbb{P}(\bar{X}_{\Gamma_{n+1}} \in dy \,|\, \bar{X}_{\Gamma_n}) = \mathcal{Q}_{\gamma_{n+1}}(\bar{X}_{\Gamma_n}, dy), \quad n \in \mathbb{N}.$$

We can canonically extend $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ into a *càdlàg* process by setting $\bar{X}(t, \omega) = \bar{X}_{\Gamma_{n(t)}}(\omega)$ with $n(t) = \inf\{n \in \mathbb{N}, \Gamma_{n+1} > t\}$. Then $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ is a simulable (as soon as $\bar{X}_0$ is) non-homogeneous Markov chain with transitions

$$\forall m \leqslant n, \qquad \bar{P}_{\Gamma_m, \Gamma_n}(x, dy) = \mathcal{Q}_{\gamma_{m+1}} \circ \cdots \circ \mathcal{Q}_{\gamma_n}(x, dy),$$

and law

$$\mathcal{L}(\bar{X}_{\Gamma_n} \,|\, \bar{X}_0 = x) = \bar{P}_{\Gamma_n}(x, dy) = \mathcal{Q}_{\gamma_1} \circ \cdots \circ \mathcal{Q}_{\gamma_n}(x, dy).$$

We use $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ to design a Langevin Monte Carlo algorithm. Notice that this approach is generic since the approximation transition kernels $(\mathcal{Q}_\gamma)_{\gamma>0}$ are not explicitly specified and then, it can be used in many different configurations including among others, weak numerical schemes or exact

simulation *i.e.* $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}} = (X_{\Gamma_n})_{n\in\mathbb{N}}$. In particular, using high weak order schemes for $(X_t)_{t\geqslant 0}$ may lead to higher rates of convergence for the empirical measures. The approach we use to build the empirical measures is quite more general than in (1) as we consider some general weights which are not necessarily equal to the time steps. We define this weight sequence. Let $\eta := (\eta_n)_{n\in\mathbb{N}^*}$ be such that

$$\forall n\in\mathbb{N}^*,\quad \eta_n \geqslant 0,\quad \lim_n H_n = +\infty,\qquad \text{with}\qquad H_n = \sum_{k=1}^{n}\eta_k.$$

Now we present our algorithm adapted from the one introduced in [10] designed with a Euler scheme with decreasing steps $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ of a Brownian diffusion process $(X_t)_{t\geqslant 0}$. For $x\in E$, let δ_x denote the Dirac mass at point x. For every $n\in\mathbb{N}^*$, we define the random weighted empirical random measures as follows

$$\nu_n^\eta(dx) = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\delta_{\bar{X}_{\Gamma_{k-1}}}(dx). \tag{2}$$

This paper is dedicated to show that *a.s.* every weak limiting distribution of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ belongs to $\mathcal{V}$. In particular when the invariant measure of $(X_t)_{t\geqslant 0}$ is unique, *i.e.* $\mathcal{V}=\{\nu\}$ so that $\mathbb{P}$-*a.s.* $\lim_n \nu_n^\eta f = \nu f$, for a generic class of continuous test functions f.

2.1.2 Assumptions on the random measures

In this part, we present the necessary assumptions on the pseudo-generator $\widetilde{A} = (\widetilde{A}_\gamma)_{\gamma>0}$ in order to prove the convergence of the empirical measures $(\nu_n^\eta)_{n\in\mathbb{N}^*}$.
Our approach consists in two parts. First, we consider a mean-reverting control assumption satisfied by the pseudo-generators $\widetilde{A}_\gamma$ and a *Lyapunov function* to show the *a.s.* tightness of the empirical measures. Secondly, we consider an infinitesimal generator approximation assumption used to characterize these weak limiting distributions as invariant distributions of the Markov process of interest X. To be more precise, we will show that any such limiting distribution, say ν, satisfies $\int Ag d\nu = 0$ for test functions $g\in\mathcal{D}(A)$ which is a characterization of invariant distributions following Echeveria-Weiss Theorem (see [5], Theorem 2.1). In both steps, it is necessary to prove convergence of martingale which will be obtain using growth control and step-weight assumptions presented at the end of this section.

Mean-reverting recursive control The mean-reverting assumption reads as follows: There exists a Borel function V such that

$$\mathcal{L}_V \quad\equiv\quad V : E\to[v_*,+\infty), v_*>0 \quad\text{ and }\quad \lim_{x\to\infty} V(x) = +\infty. \tag{3}$$

Let $\bar{\gamma}>0$, $s\geqslant 1$ and let $\psi,\phi:[v_*,\infty)\to(0,+\infty)$ Borel functions, let $\alpha>0$ and $\beta\in\mathbb{R}$. We assume

$$\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s) \quad\equiv \begin{cases} (i)\ \widetilde{A}_\gamma\psi\circ V \text{ exists for every } \gamma\in(0,\bar{\gamma}], \\ (ii)\ \forall\, x\in E,\quad \sup_{\gamma\in(0,\bar{\gamma}]}\widetilde{A}_\gamma\psi\circ V(x) \leqslant \frac{\psi\circ V(x)}{V(x)}(\beta-\alpha\phi\circ V(x)), \\ (iii)\quad \lim_{v\to+\infty}\frac{\phi(v)\psi(v)^{1/s}}{v} = +\infty \text{ and } \liminf_{v\to+\infty}\phi(v) > \beta/\alpha. \end{cases} \tag{4}$$

A function V is a Lyapunov function for transitions $(Q_\gamma)_{\gamma>0}$ if it satisfies Assumption $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta)$, which in turn is called the (weakly) mean-reverting recursive control assumption of the pseudo-generators (by V).

Comments.
• The assumption $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)$ is devised to be directly checked on the transitions of the time discretization schemes with decreasing steps γ_n of the Feller process X. This can be seen as a

variant of an *a.s.* tightness criterion for the occupation measure $\frac{1}{t}\int_0^t \delta_{X_s^x} ds$ in which (ii) would be replaced by

$$\forall\, x \in E, \quad \sup_{\gamma\in(0,\bar\gamma]} A\,\psi\circ V(x) \leqslant \frac{\psi\circ V(x)}{V(x)}\big(\beta - \alpha\phi\circ V(x)\big). \tag{5}$$

which appears as a very general tightness criterion for the above continuous time empirical measure (see [15] for a review). Thus if we set $\psi = Id_{[v_*,+\infty)}$ and $\phi(v) = v^a$, for some $a\in(0,1]$, then the mean-reverting condition (5) reduces to

$$A\,V \leqslant \beta - \alpha V^a,$$

which is a classical mean-reverting condition (see [9], [5] or [15], for a review) to establish the existence of an invariant distribution for a Feller process. It is also extensively used to control the Euler scheme with decreasing steps of Brownian diffusions with infinitesimal generator A and its weighted empirical measures : see, when $a = 1$, the seminal paper [10] when $a = 1$ (see also [1]) and [11] when $a\in(0,1)$). When $a = 1$ this condition is known as a *strong mean-reverting* assumption, whereas it is called *weak* when $a\in(0,1)$ or, more generally when $\lim_{v\to+\infty}\phi(v)/v = 0$.

In fact, in these papers the measures ν_n are investigated under (5) rather than (4). It turns out that $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)$ is more convenient for an abstract result and more flexible to cope with a wide range of examples. However, under the generator approximation condition (7) given later, one easily checks that (4) follows from (5).

• The condition $\mathcal{RC}_{Q,V}(v^p, I_d,\alpha,\beta,s)(ii)$, $p,s\geqslant 1$, is considered – for A – in the seminal paper [10] (and then in [11] with $\phi(v) = v^a$, $a\in(0,1]$, $v\in[v_*,\infty)$) concerning the Wasserstein convergence of the weighted empirical measures of the Euler scheme with decreasing steps of a Brownian diffusions. When $\phi = I_d$, the Euler scheme is also studied for Markov switching Brownian diffusions in [14]. Notice also that $\mathcal{RC}_{Q,V}(I_d,\phi,\alpha,\beta,s)(ii)$ with ϕ concave appears in [3] to prove sub-geometrical ergodicity of Markov chains. In [12], a similar hypothesis to $\mathcal{RC}_{Q,V}(I_d,\phi,\alpha,\beta,s)(ii)$ (with ϕ not necessarily concave and $\widetilde{A}_\gamma$ replaced by A), is also used to study the Wasserstein but also exponential convergence of the weighted empirical measures (2) for the Euler scheme of a Brownian diffusions. Finally, in [20] similar properties as $\mathcal{RC}_{Q,V}(v^p,v^a,\alpha,\beta,s)(ii)$, $a\in(0,1]$, $p>0$, are developed in the study of the Euler scheme for Lévy processes.

• The function ψ is related to the set of possibly unbounded (ν-*a.s.*) continuous functions f for which the convergence $\nu_n^\eta(f)\to\nu(f)$ *a.s.* holds, when ν is the unique invariant distribution of the underlying Feller process. Let us be more specific on that point: for an $s\geqslant 1$ – which is related to step-weight assumption on the sequences γ and η – we will prove in Theorem 2.4 point B. that the sets of functions for which this above *a.s.* convergence holds contains:

$$\mathcal{C}_{\widetilde{V}_{\psi,\phi,s}}(E) = \Big\{f\in\mathcal{C}(E),\ |f(x)| = \underset{x\to\infty}{o}(\widetilde{V}_{\psi,\phi,s}(x))\Big\}, \tag{6}$$

$$\text{where}\quad \widetilde{V}_{\psi,\phi,s} : E\to\mathbb{R}_+ \text{ is defined by } \widetilde{V}_{\psi,\phi,s}(x) := \frac{\phi\circ V(x)\psi\circ V(x)^{1/s}}{V(x)}.$$

Infinitesimal generator approximation We present now the main assumption that enables to characterize the limiting distributions of the *a.s.* tight sequence $(\nu_n^\eta(dx,\omega))_{n\in\mathbb{N}^*}$. The strategy is to show that, *a.s.*, any limiting distribution $\nu_\infty(\omega,dx)$ of the sequence of random measures $(\nu_n(\omega,dx))_{n\geq1}$ satisfies $\int\widetilde{A}_{\gamma_n}f(x)\nu_n(\omega,dx)\to 0$, then derive that $\int\widetilde{A}_{\gamma_n}f(x)\nu_\infty(\omega,dx) = 0$ for every f in a dense subspace of the domain of A and conclude by the Echeveria-Weiss Theorem that $\nu_\infty(\omega,dx)$ is invariant for the semi-group $(P_t)_{t\geq0}$.

We thus introduce a hypothesis concerning the distance between $(\widetilde{A}_\gamma)_{\gamma>0}$, the pseudo-generator of $(\mathcal{Q}_\gamma)_{\gamma>0}$, and A, the infinitesimal generator of $(P_t)_{t\geqslant0}$. We assume that there exists $\mathcal{D}(A)_0\subset\mathcal{D}(A)$ with $\mathcal{D}(A)_0$ dense in $\mathcal{C}_0(E)$ such that:

$$\mathcal{E}(\widetilde{A},A,\mathcal{D}(A)_0) \quad\equiv\quad \forall\gamma\in(0,\bar\gamma],\ \forall f\in\mathcal{D}(A)_0,\forall\, x\in E,$$
$$|\widetilde{A}_\gamma f(x) - Af(x)| \leqslant \Lambda_f(x,\gamma), \tag{7}$$

where $\Lambda_f : E \times \mathbb{R}_+ \to \mathbb{R}_+$ can be represented in the following way: Let $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{P}})$ be a probability space. Let $g : E \to \mathbb{R}_+^q$, $q \in \mathbb{N}$, be a locally bounded Borel measurable function and let $\widetilde{\Lambda}_f : (E \times \mathbb{R}_+ \times \widetilde{\Omega}, \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \widetilde{\mathcal{G}}) \to \mathbb{R}_+^q$ be a measurable function such that

$$\sup_{i\in\{1,\dots,q\}} \widetilde{\mathbb{E}}\big[\sup_{x\in E}\sup_{\gamma\in(0,\bar{\gamma}]} \widetilde{\Lambda}_{f,i}(x,\gamma,\widetilde{\omega})\big] < +\infty,$$

and

$$\forall\, x\in E, \forall \gamma \in (0,\bar{\gamma}], \qquad \Lambda_f(x,\gamma) = \big\langle g(x), \widetilde{\mathbb{E}}[\widetilde{\Lambda}_f(x,\gamma,\widetilde{\omega})]\big\rangle_{\mathbb{R}^q}.$$

Moreover, we assume that for every $i \in \{1,\dots,q\}$, $\sup_{n\in\mathbb{N}^*} \nu_n^\eta(g_i,\omega) < +\infty$, $\mathbb{P}(d\omega) - a.s.$, and that $\widetilde{\Lambda}_{f,i}$ satisfies one of the following two properties:

There exists a measurable function $\underline{\gamma} : (\widetilde{\Omega}, \widetilde{\mathcal{G}}) \to ((0,\bar{\gamma}], \mathcal{B}((0,\bar{\gamma}]))$ such that:

$$\text{I)}\ \widetilde{\mathbb{P}}(d\widetilde{\omega}){-}a.s \qquad \begin{cases} (i) & \forall K \in \mathcal{K}_E, \quad \lim\limits_{\gamma\to 0}\sup\limits_{x\in K} \widetilde{\Lambda}_{f,i}(x,\gamma,\widetilde{\omega}) = 0, \\ (ii) & \lim\limits_{x\to\infty}\sup\limits_{\gamma\in(0,\underline{\gamma}(\widetilde{\omega})]} \widetilde{\Lambda}_{f,i}(x,\gamma,\widetilde{\omega}) = 0, \end{cases} \tag{8}$$

or

$$\text{II)}\ \widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s \qquad \lim_{\gamma\to 0}\sup_{x\in E} \widetilde{\Lambda}_{f,i}(x,\gamma,\widetilde{\omega}) g_i(x) = 0. \tag{9}$$

Remark 2.1. *Notice that assumption $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (i) (see (8)) controls the behavior of the error function $\widetilde{\Lambda}_f$ as the time goes to zero. It is crucial to obtain weak convergence towards the expected target. Hypothesis (7) may indeed be seen as a small time weak approximation that will lead to the result using concatenation techniques. In some specific cases we can obtain the assumption in a uniform way (for the space variable) that is $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ II) (see (9)). However in most cases, it is only possible to obtain this convergence on compact sets (for the space variable) that is $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (i) (see (8)). In this case, it is necessary to introduce the supplementary assumption $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (ii) (see (8)) in order to control the behavior of the error outside those compact sets.*

Remark 2.2. *Let $(F, \mathcal{F}, \lambda)$ be a measurable space. Using the exact same approach, the results we obtain hold when we replace the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{P}})$ by the product measurable space $(\widetilde{\Omega} \times F, \widetilde{\mathcal{G}} \otimes \mathcal{F}, \widetilde{\mathbb{P}} \otimes \lambda)$ in the representation of Λ_f and in (8) and (9) but we restrict to that case for sake of clarity in the writing. This observation can be useful when we study jump process where λ can stand for the jump intensity.*

This representation assumption benefits from the fact that the transition functions $(\mathcal{Q}_\gamma(x,dy))_{\gamma\in(0,\bar{\gamma}]}$, $x \in E$, can be represented using distributions of random variables which are involved in the computation of $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}^*}$. In particular, this approach is well adapted to stochastic approximations associated to a time grid such as numerical schemes for stochastic differential equations with a Brownian part or/and a jump part. We propose in Appendix A an example of checking of condition $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)(I)$ to a (scalar) Brownian diffusion. This case already solved in the original paper [10] can be viewed as a first user guide. In the companion paper [18] devoted to applications of our results to the diffusion processes with switching regimes and to jump diffusions which extend [14] and [20] respectively. However, we can describe a standard approach that can be used to check this assumption. Assume A is local operator of order m (meaning Af involves derivatives of order m of the function f) with form $Af(x)\sum_{l=1}^m T_\ell(x)\partial^l f(x)$. Assume $E = \mathbb{R}$ and $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ follows the same law under $\mathbb{P}$, as the Markov chain defined by the random recurrence

$$\bar{X}_{\Gamma_{n+1}} = F(\bar{X}_{\Gamma_n}, \zeta_{\gamma_{n+1}}), \qquad n \in \mathbb{N}, \quad \widetilde{\mathbb{P}} - a.s.$$

under $\widetilde{\mathbb{P}}$, with F a smooth function (at least continuous) satisfying $F(x,0)=x, x\in E$, and $(\zeta_{\gamma_n})_{n\in\mathbb{N}^*}$ a sequence of $i.i.d$ random variable distributed under the law of ζ_{γ_n} with $(\zeta_\gamma)_{\gamma>0}$ a random process satisfying $\lim_{\gamma\to 0}\zeta_\gamma = 0$ $\widetilde{\mathbb{P}}-a.s.$ with appropriate uniform integrability conditions (for Brownian diffusions one may set $\zeta_\gamma=\sqrt{\gamma}\widetilde{Z}$, $\widetilde{Z}\sim\mathcal{N}(0;1)$). Then we have, for f smooth enough with compact support (e.g. C^2_K for Brownian diffusions corresponding to $m=1$ in what follows),

$$\widetilde{\mathbb{E}}\big[f\circ F(x,\zeta_\gamma)\big] = f(x)+\widetilde{\mathbb{E}}\Big[\sum_{\ell=1}^{m}\frac{(F(x,\zeta_\gamma)-x)^\ell}{\ell!}f^{(\ell)}(x) + \int_0^1\frac{(F(x,\zeta_\gamma)-x)^m}{(m+1)!}f^{(m+1)}\big(x+u(F(x,\zeta_\gamma)-x)\big)du\Big]$$

and then, we have the following writing

$$\begin{aligned}\widetilde{A}_\gamma f(x)-Af(x) &= \widetilde{\mathbb{E}}\Big[\sum_{l=1}^{m}\frac{(F(x,\zeta_\gamma)-x)^\ell}{\gamma\ell!}f^{(\ell)}(x)\\ &\quad+\int_0^1\frac{(F(x,\zeta_\gamma)-x)^m}{\gamma(m+1)!}f^{(m+1)}(x+u(F(x,\zeta_\gamma)-x))du\Big]-Af(x)\\ &=\sum_{l=1}^{m}\Big(\frac{\widetilde{\mathbb{E}}\big[(F(x,\zeta_\gamma)-x)^\ell\big]}{\gamma\ell!}-T_\ell(x)\Big)f^{(\ell)}(x)\\ &\quad+\widetilde{\mathbb{E}}\Big[\int_0^1\frac{(F(x,\zeta_\gamma)-x)^m}{\gamma(m+1)!}f^{(m+1)}(x+u(F(x,\zeta_\gamma)-x))du\Big]\\ &=\sum_{\ell=1}^{m+1}\widetilde{\mathbb{E}}\Big[\Big(\frac{(F(x,\zeta_\gamma)-x)^{l-\mathbb{1}_{\ell\le m+1}}}{\gamma\ell!}-\mathbb{1}_{\ell\le m+1}T_\ell(x)\Big)f^{(\ell)}\big(x+\mathbb{1}_{\ell=m+1}\widetilde{\Theta}(F(x,\zeta_\gamma)-x)\big)\Big]\end{aligned}$$

where $\widetilde{\Theta}$ is uniformly distributed on the interval $[0,1]$ under $\widetilde{\mathbb{P}}$. Now, we have this representation that is well adapted to our framework we check $\mathcal{E}(\widetilde{A},A,\mathcal{D}(A)_0)$ I) (see (8)). To prove $\mathcal{E}(\widetilde{A},A,\mathcal{D}(A)_0)$ I) (i), the idea is to show that $\widetilde{\mathbb{P}}(d\widetilde{\omega})-a.s.$, $\frac{(F(x,\zeta_\gamma)-x)^{\ell-\mathbb{1}_{\ell=m+1}}}{\gamma\ell!}-\mathbb{1}_{\ell=m+1}T_\ell(x)$ (when the expectation of this term is not null) converges to 0 as γ tends to 0 as soon as the approximation is well chosen and using the fact that $\lim_{\gamma\to0}F(x,\zeta_\gamma)=x$, $\widetilde{\mathbb{P}}(d\widetilde{\omega})-a.s.$. The assumption $\mathcal{E}(\widetilde{A},A,\mathcal{D}(A)_0)$ I) (ii) can then be checked studying $f^{(\ell)}(x+\mathbb{1}_{\ell=m+1}\widetilde{\Theta}(F(x,\zeta_\gamma)-x))$. As f is vanishing at infinity, it is trivial when $\ell\in\{1,\dots,m\}$. When $\ell=m+1$ the idea is to introduce $\underline{\gamma}(\widetilde{\omega})$ such that $\widetilde{\mathbb{P}}(d\widetilde{\omega})-a.s.$, $\lim_{x\to\infty}\sup_{\gamma\leqslant\underline{\gamma}(\widetilde{\omega})}x+\widetilde{\Theta}(\widetilde{\omega})(F(x,\zeta_\gamma(\widetilde{\omega}))-x)=+\infty$ and to conclude using the fact that f vanishes at infinity. This is possible as soon as we can find $\bar{\zeta}>0$ such that $\sup_{|\zeta|<\bar{\zeta}}|F(x,\zeta)-x|\leqslant(1-\epsilon)|x|$, for every $x\in E$ and some $\epsilon\in(0,1)$.

Growth control and step-weight assumptions We conclude with hypothesis concerning the control of the martingale part of one step of our approximation. Those will be crucial to prove the convergence of the martingale array terms which appear in the study of the weak error.

Let $\rho\in[1,2]$ and let $\epsilon_{\mathcal{I}}:\mathbb{R}_+\to\mathbb{R}_+$ be an increasing function. For $F\subset\{f, f:(E,\mathcal{B}(E))\to(\mathbb{R},\mathcal{B}(\mathbb{R}))\}$ and $g:E\to\mathbb{R}_+$ a Borel function, we assume that, for every $n\in\mathbb{N}$,

$$\begin{aligned}\mathcal{GC}_Q(F,g,\rho,\epsilon_{\mathcal{I}}) \quad\equiv\quad & \mathbb{P}\text{-}a.s.\ \forall f\in F,\\ & \mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}})-\mathcal{Q}_{\gamma_{n+1}}f(\bar{X}_{\Gamma_n})|^\rho|\bar{X}_{\Gamma_n}]\leqslant C_f\epsilon_{\mathcal{I}}(\gamma_{n+1})g(\bar{X}_{\Gamma_n}),\end{aligned}\tag{10}$$

with $C_f>0$ a finite real constant which may depend on f.

Remark 2.3. *The reader may notice that $\mathcal{GC}_Q(F,g,\rho,\epsilon_{\mathcal{I}})$ holds as soon as (10) is satisfied with an $\mathcal{F}_n^{\bar{X}}:=\sigma(\bar{X}_{\Gamma_k},k\leqslant n)$-adapted sequence $(\mathfrak{X}_n)_{n\in\mathbb{N}^*}$ instead of $\mathcal{Q}_{\gamma_{n+1}}f(\bar{X}_{\Gamma_n})$, since $\mathcal{Q}_{\gamma_{n+1}}f(\bar{X}_{\Gamma_n})=\mathbb{E}[f(\bar{X}_{\Gamma_{n+1}})|\bar{X}_{\Gamma_n}]$ and*

$$\mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}})-\mathcal{Q}_{\gamma_{n+1}}f(\bar{X}_{\Gamma_n})|^\rho|\bar{X}_{\Gamma_n}]\leqslant 2^\rho\mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}})-\mathfrak{X}_n|^\rho|\bar{X}_{\Gamma_n}]\ \ \textit{for every}\ \ \mathfrak{X}_n\in L^2(\mathcal{F}_n^{\bar{X}}).$$

We will combine this assumption with the following step-weight related ones:

$$\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}}) \quad \equiv \quad \mathbb{P}\text{-}a.s. \quad \sum_{n\geq 1}\left|\frac{\eta_n}{H_n\gamma_n}\right|^{\rho}\epsilon_{\mathcal{I}}(\gamma_n)g(\bar{X}_{\Gamma_n}) < +\infty, \tag{11}$$

and

$$\mathcal{SW}_{\mathcal{II},\gamma,\eta}(F) \equiv \quad \mathbb{P}\text{-}a.s. \quad \forall f\in F, \quad \sum_{n\geq 0}\frac{(\eta_{n+1}/\gamma_{n+1}-\eta_n/\gamma_n)_+}{H_{n+1}}|f(\bar{X}_{\Gamma_n})| < +\infty, \tag{12}$$

with the convention $\eta_0/\gamma_0 = 1$. Notice that this last assumption holds as soon as the sequence $(\eta_n/\gamma_n)_{n\in\mathbb{N}^*}$ is non-increasing.

2.2 A.s. weak convergence of the empirical measures

In the beginning of this Section, we present some general results that we use in our approach. Then, we provide convergence results through Theorem 2.3 and Theorem 2.4. The first one establishes almost sure tightness and then the existence of a weak limit for the empirical measures. In Theorem 2.4, we establish the identification of every weak limit with an invariant distribution of the Feller process $(X_t)_{t\geqslant 0}$ with infinitesimal generator A.

2.2.1 Background and preliminary results

In this section, we recall standard general results we employ to study the convergence. Our approach will rely on a specific version of the Martingale problem characterizing the existence of a Feller Markov process which directly provides the existence of a steady regime *i.e.* an invariant distribution. This is the object of the Echeverria-Weiss theorem.

Theorem 2.1. (*a*) ***(Echeverria-Weiss** (see [5], Theorem 9.17)**).*** *Let E be a locally compact and separable metric space and let $A:\mathcal{D}(A)\subset\mathcal{C}_0(E)\to\mathcal{C}_0(E)$ be a linear operator satisfying the positive maximum principle*[1]*, such that $\mathcal{D}(A)$ is dense in $\mathcal{C}_0(E)$ and that there exists a sequence of functions $\varphi_n\in\mathcal{D}(A)$ such that $\lim_n \varphi_n = 1$ and $\lim_n A\varphi_n = 0$ with $\sup_{n\in\mathbb{N}}\|A\varphi_n\|_\infty < +\infty$.*

If $\nu\in\mathcal{P}(E)$ satisfies

$$\forall f\in\mathcal{D}(A), \quad \int_E Af d\nu = 0, \tag{13}$$

then there exists a stationary solution to the martingale problem (A,ν).

(*b*) ***(Hille-Yoshida** (see [22] (Chapter VII, Propositions 1.3 and 1.5) or [5] (Chapter IV, Theorem 2.2))**).*** *The infinitesimal generator of a Feller process satisfies the hypothesis from item (a) except for (13).*

This paper is devoted to the proof of the existence of a measure ν which satisfies (13). Using this result, property (13) is sufficient to prove that ν is an invariant measure for the process with infinitesimal generator A. To be more specific, the measure ν is built as the limit of a sequence of random empirical measures $(\nu_n^\eta)_{n\in\mathbb{N}^*}$. When (13) holds for this limit, we say that the sequence $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ converges towards an invariant distribution of the Feller process with generator A. We begin with some preliminary results.

Lemma 2.1 ((Kronecker)**.** *Let $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}^*}$ be two sequences of real numbers. If $(b_n)_{n\in\mathbb{N}^*}$ is non-decreasing, strictly positive, with $\lim_n b_n = +\infty$ and $\sum_{n\geqslant 1} a_n/b_n$ converges in $\mathbb{R}$, then*

$$\lim_n \frac{1}{b_n}\sum_{k=1}^{n} a_k = 0.$$

[1] $\forall f\in\mathcal{D}(A), f(x_0) = \sup\{f(x), x\in E\}\geqslant 0, x_0\in E \Rightarrow Af(x_0)\leqslant 0.$

Theorem 2.2 (Chow, (see [8], Theorem 2.17)). *Let $(M_n)_{n\in\mathbb{N}^*}$ be a real valued martingale with respect to some filtration $\mathcal{F}=(\mathcal{F}_n)_{n\in\mathbb{N}}$. Then*

$$\lim_n M_n = M_\infty \in \mathbb{R}\ a.s.\ \ on\ the\ event \bigcup_{r\in[0,1]} \Big\{ \sum_{n=1}^{\infty} \mathbb{E}[|M_n - M_{n-1}|^{1+r}|\mathcal{F}_{n-1}] < +\infty \Big\}.$$

2.2.2 Almost sure tightness

From the recursive control assumption, the following Theorem establish the *a.s.* tightness of the sequence $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ and also provides a uniform control of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ on a generic class of test functions.

Theorem 2.3. *Let $s\geqslant 1$, $\rho\in[1,2]$, $v_*>0$, and let us consider the Borel functions $V:E\to[v_*,\infty)$, $g:E\to\mathbb{R}_+$, $\psi:[v_*,\infty)\to\mathbb{R}_+$ and $\epsilon_{\mathcal{I}}:\mathbb{R}_+\to\mathbb{R}_+$ an increasing function. We have the following properties:*

A. Assume that $\widetilde{A}_{\gamma_n}(\psi\circ V)^{1/s}$ exists for every $n\in\mathbb{N}^*$, and that $\mathcal{GC}_Q((\psi\circ V)^{1/s},g,\rho,\epsilon_{\mathcal{I}})$ (see (10)), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}})$ (see (11)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}((\psi\circ V)^{1/s})$ (see (12) hold. Then

$$\mathbb{P}\text{-}a.s. \quad \sup_{n\in\mathbb{N}^*} -\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\widetilde{A}_{\gamma_k}(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}}) < +\infty. \tag{14}$$

B. Let $\alpha>0$ and $\beta\in\mathbb{R}$. Let $\phi:[v_*,\infty)\to\mathbb{R}_+^*$ be a continuous function such that $C_\phi := \sup_{v\in[v_*,\infty)}\phi(v)/v<\infty$. Assume that (14) holds and

(i) $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)$ (see (4)) holds.

(ii) L_V (see (3)).

Then,

$$\mathbb{P}\text{-}a.s. \quad \sup_{n\in\mathbb{N}^*}\nu_n^\eta(\widetilde{V}_{\psi,\phi,s}) < +\infty. \tag{15}$$

with $\widetilde{V}_{\psi,\phi,s}$ defined in (6). Therefore, the sequence $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is $\mathbb{P}$-a.s. tight.

Proof. A. For $n\in\mathbb{N}^*$, we write

$$\begin{aligned}-\sum_{k=1}^{n}\eta_k\widetilde{A}_{\gamma_k}(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}}) =& -\sum_{k=1}^{n}\frac{\eta_k}{\gamma_k}((\psi\circ V)^{1/s}(\bar{X}_{\Gamma_k})-(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}}))\\ &+\sum_{k=1}^{n}\frac{\eta_k}{\gamma_k}((\psi\circ V)^{1/s}(\bar{X}_{\Gamma_k})-\mathcal{Q}_{\gamma_k}(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}})).\end{aligned}$$

We study the first term of the *r.h.s.* First, an Abel transform yields

$$\begin{aligned}-\frac{1}{H_n}\sum_{k=1}^{n}\frac{\eta_k}{\gamma_k}((\psi\circ V)^{1/s}(\bar{X}_{\Gamma_k})-&(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}}))\\ =&\frac{\eta_1}{H_n\gamma_1}(\psi\circ V)^{1/s}(\bar{X}_0)-\frac{\eta_n}{H_n\gamma_n}(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_n})\\ &+\frac{1}{H_n}\sum_{k=2}^{n}\Big(\frac{\eta_k}{\gamma_k}-\frac{\eta_{k-1}}{\gamma_{k-1}}\Big)(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}}).\end{aligned}$$

We recall that $(\psi\circ V)^{1/s}$ is non negative. From $\mathcal{SW}_{\mathcal{II},\gamma,\eta}((\psi\circ V)^{1/s})$ (see (12)), we have

$$\mathbb{E}\Big[\sup_{n\in\mathbb{N}^*}\sum_{k=1}^{n}\frac{1}{H_k}\Big(\frac{\eta_k}{\gamma_k}-\frac{\eta_{k-1}}{\gamma_{k-1}}\Big)_+(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}})\Big] < +\infty,$$

so that

$$\mathbb{P}\text{-}a.s. \quad \sup_{n\in\mathbb{N}^*}\sum_{k=1}^{n}\frac{1}{H_k}\Big(\frac{\eta_k}{\gamma_k}-\frac{\eta_{k-1}}{\gamma_{k-1}}\Big)_+(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}})<+\infty.$$

By Kronecker's lemma, we deduce that

$$\mathbb{P}\text{-}a.s. \quad \lim_n\frac{1}{H_n}\sum_{k=2}^{n}\Big(\frac{\eta_k}{\gamma_k}-\frac{\eta_{k-1}}{\gamma_{k-1}}\Big)_+(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}})=0.$$

This concludes the study of the first term and now we focus on the second one. From Kronecker lemma, it remains to prove the almost sure convergence of the martingale $(M_n)_{n\in\mathbb{N}^*}$ defined by $M_0:=0$ and

$$M_n:=\sum_{k=1}^{n}\frac{\eta_k}{\gamma_k H_k}\big((\psi\circ V)^{1/s}(\bar{X}_{\Gamma_k})-\mathcal{Q}_{\gamma_k}(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{k-1}})\big),\quad n\in\mathbb{N}^*.$$

Using the Chow's theorem (see Theorem 2.2), this $a.s.$ convergence is a direct consequence of the $a.s.$ finiteness of the series

$$\sum_{n=1}^{\infty}\Big(\frac{\eta_n}{\gamma_n H_n}\Big)^{\rho}\mathbb{E}\big[|(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_n})-\mathcal{Q}_{\gamma_n}(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_{n-1}})|^{\rho}\,|\,\bar{X}_{\Gamma_{n-1}}\big],$$

which follows from $\mathcal{GC}_Q((\psi\circ V)^{1/s},g,\rho,\epsilon_{\mathcal{I}})$ (see (10)) and $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}})$ (see (11)).
B. Using $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)(ii)$ (see (4)), there exists $n_0\in\mathbb{N}^*$, such that for every $n\geqslant n_0$, we have

$$\mathbb{E}\Big[\frac{\psi\circ V(\bar{X}_{\Gamma_{n+1}})}{\psi\circ V(\bar{X}_{\Gamma_n})}\Big|\bar{X}_{\Gamma_n}\Big]\leqslant 1+\gamma_{n+1}\frac{\beta-\alpha\phi\circ V(\bar{X}_{\Gamma_n})}{V(\bar{X}_{\Gamma_n})}.$$

Since the function defined on $\mathbb{R}_+^*$ by $v\mapsto v^{1/s}$ is concave and $C_\phi:=\sup_{v\in[v_*,\infty)}\phi(v)/v<+\infty$, for n large enough we use the Jensen's inequality and we derive

$$\begin{aligned}\mathbb{E}\Big[\Big(\frac{\psi\circ V(\bar{X}_{\Gamma_{n+1}})}{\psi\circ V(\bar{X}_{\Gamma_n})}\Big)^{1/s}\Big|\bar{X}_{\Gamma_n}\Big]\leqslant&\Big(1+\gamma_{n+1}\frac{\beta-\alpha\phi\circ V(\bar{X}_{\Gamma_n})}{V(\bar{X}_{\Gamma_n})}\Big)^{1/s}\\ \leqslant&1+\frac{\gamma_{n+1}(\beta-\alpha\phi\circ V(\bar{X}_{\Gamma_n}))}{sV(\bar{X}_{\Gamma_n})}.\end{aligned}$$

Now when $\beta\geqslant 0$, by $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)(iii)$ (see (4)), there exists $\lambda\in(0,1)$ and $v_\lambda\in(0,+\infty)$ such that for every $v>v_\lambda$, then $\phi(v)\geqslant\beta/(\lambda\alpha)$. It follows that the Borel function $C_{\lambda,s}:[v_*,+\infty)\to\mathbb{R}$, $v\mapsto C_{\lambda,s}(v):=v^{-1}\psi(v)^{1/s}(\beta-\lambda\alpha\phi(v))$ is locally bounded on $[v_*,+\infty)$ and non positive on $[v_\lambda,+\infty)$, hence $\bar{C}_{\lambda,s}:=\sup_{v\in[v_*,+\infty)}C_{\lambda,s}(v)<+\infty$. When $\beta<0$, since ϕ and ψ are positive functions, then the function $C_{\lambda,s}$ is non positive and it follows that

$$\begin{aligned}\mathcal{Q}_{\gamma_{n+1}}(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_n})\leqslant&(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_n})\\ &+\frac{\gamma_{n+1}}{s}(C_{\lambda,s}\circ V(\bar{X}_{\Gamma_n})-(1-\lambda)\alpha\widetilde{V}_{\psi,\phi,s}(\bar{X}_{\Gamma_n})),\end{aligned}$$

which yields,

$$\widetilde{V}_{\psi,\phi,s}(\bar{X}_{\Gamma_n})\leqslant-\frac{s}{\alpha(1-\lambda)}\widetilde{A}_{\gamma_{n+1}}(\psi\circ V)^{1/s}(\bar{X}_{\Gamma_n})+\frac{\bar{C}_{\lambda,s}\vee 0}{\alpha(1-\lambda)}.$$

Consequently (15) follows from (14). The tightness of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is a straightforward consequence of (15) and $\lim_{v\to+\infty}\frac{\phi(v)\psi(v)^{1/s}}{v}=+\infty$ (see $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)(iii)$) since we then have $\lim_{x\to\infty}\widetilde{V}_{\psi,\phi,s}(x)=+\infty$. □

2.2.3 Identification of the limit

In Theorem 2.3, we obtained the tightness of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$. It remains to show that every limiting point of this sequence is an invariant distribution of the Feller process with infinitesimal generator A. This is the interest of the following Theorem which relies on the infinitesimal generator approximation.

Theorem 2.4. *Let $\rho\in[1,2]$. We have the following properties:*

A. Let $\mathcal{D}(A)_0 \subset \mathcal{D}(A)$, with $\mathcal{D}(A)_0$ dense in $\mathcal{C}_0(E)$. We assume that $\widetilde{A}_{\gamma_n} f$ exists for every $f \in \mathcal{D}(A)_0$ and every $n \in \mathbb{N}^*$. Also assume that there exists $g : E \to \mathbb{R}_+$ a Borel function and $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function such that $\mathcal{GC}_Q(\mathcal{D}(A)_0, g, \rho, \epsilon_{\mathcal{I}})$ (see (10)) and $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g, \rho, \epsilon_{\mathcal{I}})$ (see (11)) hold and that

$$\lim_n \frac{1}{H_n}\sum_{k=1}^n \left|\eta_{k+1}/\gamma_{k+1} - \eta_k/\gamma_k\right| = 0. \tag{16}$$

Then

$$\mathbb{P}\text{-}a.s. \quad \forall f \in \mathcal{D}(A)_0, \qquad \lim_n \frac{1}{H_n}\sum_{k=1}^n \eta_k \widetilde{A}_{\gamma_k} f(\bar{X}_{\Gamma_{k-1}}) = 0. \tag{17}$$

B. We assume that (17) and $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ (see (7)) hold. Then

$$\mathbb{P}\text{-}a.s. \quad \forall f \in \mathcal{D}(A)_0, \qquad \lim_n \nu_n^\eta(Af) = 0.$$

It follows that, $\mathbb{P}$-a.s., every weak limiting distribution ν_∞^η of the sequence $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ belongs to $\mathcal{V}$, the set of the invariant distributions of $(X_t)_{t\geqslant 0}$. Finally, if the hypothesis from Theorem 2.3 point B. hold and $(X_t)_{t\geqslant 0}$ has a unique invariant distribution, i.e. $\mathcal{V} = \{\nu\}$, then

$$\mathbb{P}\text{-}a.s. \quad \forall f \in \mathcal{C}_{\widetilde{V}_{\psi,\phi,s}}(E), \quad \lim_n \nu_n^\eta(f) = \nu(f), \tag{18}$$

with $\mathcal{C}_{\widetilde{V}_{\psi,\phi,s}}(E)$ defined in (6).

In the particular case where the function ψ is polynomial, (18) also reads as the $a.s.$ convergence of the empirical measures for some L^p-Wasserstein distances, $p > 0$, that we will study further in this paper for some numerical schemes of some diffusion processes. From the liberty granted by the choice of ψ in this abstract framework, where only a recursive control with mean-reverting is required, we will also propose an application for functions ψ with exponential growth.

Proof. A. We write

$$\begin{aligned}-\sum_{k=1}^n \eta_k \widetilde{A}_{\gamma_k} f(\bar{X}_{\Gamma_{k-1}}) = &-\sum_{k=1}^n \frac{\eta_k}{\gamma_k}(f(\bar{X}_{\Gamma_k}) - f(\bar{X}_{\Gamma_{k-1}})) \\ &+ \sum_{k=1}^n \frac{\eta_k}{\gamma_k}(f(\bar{X}_{\Gamma_k}) - \mathcal{Q}_{\gamma_k} f(\bar{X}_{\Gamma_{k-1}})).\end{aligned}$$

We study the first term of the *r.h.s.* We derive by an Abel transform that

$$\begin{aligned}-\frac{1}{H_n}\sum_{k=1}^n \frac{\eta_k}{\gamma_k}(f(\bar{X}_{\Gamma_k}) - f(\bar{X}_{\Gamma_{k-1}})) = &\frac{\eta_1}{H_n\gamma_1} f(\bar{X}_0) - \frac{\eta_n}{H_n\gamma_n} f(\bar{X}_{\Gamma_n}) \\ &+ \frac{1}{H_n}\sum_{k=2}^n \Big(\frac{\eta_k}{\gamma_k} - \frac{\eta_{k-1}}{\gamma_{k-1}}\Big) f(\bar{X}_{\Gamma_{k-1}}).\end{aligned}$$

Since f is bounded and $\lim_n \eta_n/(H_n\gamma_n) = 0$, we deduce that $\lim_n \eta_n f(\bar{X}_{\Gamma_n})/(H_n\gamma_n) \overset{a.s.}{=} 0$ and, on the other hand, we deduce from (16) that

$$\lim_n \frac{1}{H_n}\sum_{k=1}^n \frac{\eta_k}{\gamma_k}(f(\bar{X}_{\Gamma_k}) - f(\bar{X}_{\Gamma_{k-1}})) = 0.$$

This completes the study of the first term. To treat the second term, the approach is quite similar to the one in the proof of Theorem 2.3 point A. using $\mathcal{GC}_Q(\mathcal{D}(A), g, \rho, \epsilon_{\mathcal{I}})$ (see (10)) with $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}})$ (see (11)). Details are left to the reader.
B. First we write

$$\frac{1}{H_n}\sum_{k=1}^n \eta_k \widetilde{A}_{\gamma_k} f(\bar{X}_{\Gamma_{k-1}}) - \nu_n^\eta(Af) = \frac{1}{H_n}\sum_{k=1}^n \eta_k\big(\widetilde{A}_{\gamma_k} f(\bar{X}_{\Gamma_{k-1}}) - Af(\bar{X}_{\Gamma_{k-1}})\big).$$

Now we use the short time approximation $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ (see (7)) and it follows that,

$$\Big|\frac{1}{H_n}\sum_{k=1}^n \eta_k(\widetilde{A}_{\gamma_k} f(\bar{X}_{\Gamma_{k-1}}) - Af(\bar{X}_{\Gamma_{k-1}}))\Big| \leqslant \frac{1}{H_n}\sum_{k=1}^n \eta_k \Lambda_f(\bar{X}_{\Gamma_{k-1}}, \gamma_k).$$

Moreover, we have the following decomposition:

$$\forall f \in \mathcal{D}(A)_0, \forall\, x \in E, \forall \gamma \in [0,\bar{\gamma}], \qquad \Lambda_f(x,\gamma) = \langle g(x), \widetilde{\mathbb{E}}[\widetilde{\Lambda}_f(x,\gamma)]\rangle_{\mathbb{R}^q}$$

with $g : (E, \mathcal{B}(E)) \to \mathbb{R}_+^q$, $q \in \mathbb{N}$, a locally bounded Borel measurable function and $\widetilde{\Lambda}_f : (E \times \mathbb{R}_+ \times \widetilde{\Omega}, \mathcal{B}(E)\otimes\mathcal{B}(\mathbb{R}_+)\otimes\widetilde{\mathcal{G}}) \to \mathbb{R}_+^q$ a measurable function such that $\sup_{i\in\{1,\dots,q\}} \widetilde{\mathbb{E}}\big[\sup_{x\in E}\sup_{\gamma\in(0,\bar{\gamma}]} \widetilde{\Lambda}_{f,i}(x,\gamma)\big] < +\infty$. Since for every $i \in \{1,\dots,q\}$, $\sup_{n\in\mathbb{N}^*} \nu_n^\eta(g_i,\omega) < +\infty$, $\mathbb{P}(d\omega) - a.s.$, the $\mathbb{P}(d\omega) - a.s.$ convergence of $\frac{1}{H_n}\sum_{k=1}^n \eta_k \Lambda_f(\bar{X}_{\Gamma_{k-1}}, \gamma_k)$ towards zero for every $f \in \mathcal{D}(A)_0$, will follow from the following result: Let $(\bar{X}_n)_{n\in\mathbb{N}} \in E^{\otimes\mathbb{N}}$. If

$$\sup_{i\in\{1,\dots,q\}} \sup_{n\in\mathbb{N}^*} \frac{1}{H_n}\sum_{k=1}^n \eta_k g_i(\bar{X}_{k-1}) < +\infty,$$

then, for every $f \in \mathcal{D}(A)_0$, $\lim_n \frac{1}{H_n}\sum_{k=1}^n \eta_k \Lambda_f(\bar{X}_{k-1}, \gamma_k) = 0$. In order to obtain this result, we first show that, for every $f \in \mathcal{D}(A)_0$, every $i \in \{1,\dots,q\}$, and every $(\bar{X}_n)_{n\in\mathbb{N}} \in E^{\otimes\mathbb{N}}$, then

$$\widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s. \qquad \lim_n \frac{1}{H_n}\sum_{k=1}^n \eta_k \widetilde{\Lambda}_{f,i}(\bar{X}_{k-1}, \gamma_k, \widetilde{\omega}) g_i(\bar{X}_{k-1}) = 0,$$

and the result will follow from the Dominated Convergence theorem since, for every $n \in \mathbb{N}^*$,

$$\frac{1}{H_n}\sum_{k=1}^n \eta_k \widetilde{\Lambda}_{f,i}(\bar{X}_{k-1}, \gamma_k, \widetilde{\omega}) g_i(\bar{X}_{k-1}) \leqslant \sup_{x\in E}\sup_{\gamma\in(0,\bar{\gamma}]} \widetilde{\Lambda}_{f,i}(x,\gamma,\widetilde{\omega}) \sup_{n\in\mathbb{N}^*} \frac{1}{H_n}\sum_{k=1}^n \eta_k g_i(\bar{X}_{k-1}) < +\infty.$$

with $\widetilde{\mathbb{E}}[\sup_{x\in E}\sup_{\gamma\in(0,\bar{\gamma}]} \widetilde{\Lambda}_{f,i}(x,\gamma,\widetilde{\omega})] < +\infty$ and $\sup_{n\in\mathbb{N}^*} \frac{1}{H_n}\sum_{k=1}^n \eta_k g_i(\bar{X}_{k-1}) < +\infty$. We fix $f \in \mathcal{D}(A)_0$, $i \in \{1,\dots,q\}$ and $(\bar{X}_n)_{n\in\mathbb{N}} \in E^{\otimes\mathbb{N}}$ and we assume that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (see (8)) holds for $\widetilde{\Lambda}_{f,i}$ and g_i. If instead $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ II) (see (9)) is satisfied, the proof is similar but simpler so we leave it to the reader. By assumption $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I)(ii) (see (9)), $\widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s$, for every $R > 0$, there exists $K_R(\widetilde{\omega}) \in \mathcal{K}_E$ such that $\sup_{x\in K_R^c(\widetilde{\omega})}\sup_{\gamma\in(0,\underline{\gamma}(\widetilde{\omega})]} \widetilde{\Lambda}_{f,i}(x,\gamma,\widetilde{\omega}) < 1/R$. Then from $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (i)(see (8)), we derive that, $\widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s$, for every $R > 0$,

$\lim_n \widetilde{\Lambda}_{f,i}(\bar{X}_{n-1},\gamma_n,\widetilde{\omega})\mathbb{1}_{K_R(\widetilde{\omega})}(\bar{X}_{k-1}) = 0$, Then, since g_i is a locally bounded function, as a straightforward consequence of Césaro's lemma, we obtain

$$\widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s. \quad \forall R > 0,$$
$$\lim_n \frac{1}{H_n}\sum_{k=1}^n \eta_k \widetilde{\Lambda}_{f,i}(\bar{X}_{k-1},\gamma_k,\widetilde{\omega}) g_i(\bar{X}_{k-1})\mathbb{1}_{K_R(\widetilde{\omega})}(\bar{X}_{k-1}) = 0$$

Let $\underline{n}(\widetilde{\omega}) := \inf\{n \in \mathbb{N}^*, \sup_{k\geqslant n}\gamma_k \leqslant \underline{\gamma}(\widetilde{\omega})\}$. By the assumption $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I)(ii) (see (8)), we have, $\widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s$, $\lim_{|x|\to+\infty} \sup_{n\geqslant \underline{n}(\widetilde{\omega})} \widetilde{\Lambda}_{f,i}(x,\gamma_n,\widetilde{\omega}) = 0$, Moreover,

$$\sup_{n\geqslant \underline{n}(\widetilde{\omega})} \frac{1}{H_n}\sum_{k=\underline{n}(\widetilde{\omega})}^n \eta_k \widetilde{\Lambda}_{f,i}(\bar{X}_{k-1},\gamma_k,\widetilde{\omega}) g(\bar{X}_{k-1})\mathbb{1}_{K_R^c(\widetilde{\omega})}(\bar{X}_{k-1})$$
$$\leqslant \sup_{x\in K_R^c(\widetilde{\omega})}\sup_{\gamma\in(0,\underline{\gamma}(\widetilde{\omega})]} \widetilde{\Lambda}_{f,i}(x,\gamma,\widetilde{\omega}) \sup_{n\in\mathbb{N}^*}\frac{1}{H_n}\sum_{k=1}^n \eta_k g_i(\bar{X}_{k-1}).$$

We let R tends to infinity and since $\sup_{n\in\mathbb{N}^*}\frac{1}{H_n}\sum_{k=1}^n \eta_k g_i(\bar{X}_{k-1}) < +\infty$, the $l.h.s.$ of the above equation converges $\widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s.$ to 0. Finally, since $\underline{n}(\widetilde{\omega})$ is $\widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s.$ finite, we also have

$$\widetilde{\mathbb{P}}(d\widetilde{\omega}) - a.s. \quad \forall R > 0,$$
$$\lim_n \frac{1}{H_n}\sum_{k=1}^{\underline{n}(\widetilde{\omega})-1} \eta_k \widetilde{\Lambda}_{f,i}(\bar{X}_{k-1},\gamma_k,\widetilde{\omega}) g(\bar{X}_{k-1})\mathbb{1}_{K_R^c(\widetilde{\omega})}(\bar{X}_{k-1}) = 0.$$

Applying the same approach for every $i \in \{1,\dots,q\}$, the Dominated Convergence Theorem yields:

$$\forall (\bar{X}_n)_{n\in\mathbb{N}} \in E^{\otimes\mathbb{N}}, \forall f \in \mathcal{D}(A)_0, \qquad \lim_n \frac{1}{H_n}\sum_{k=1}^n \eta_k \Lambda_f(\bar{X}_{k-1},\gamma_k) = 0,$$

and since for every $i \in \{1,\dots,q\}$, $\sup_{n\in\mathbb{N}^*}\nu_n^\eta(g_i,\omega) < +\infty$, $\mathbb{P}(d\omega) - a.s.$, then

$$\mathbb{P}(d\omega) - a.s. \qquad \forall f \in \mathcal{D}(A)_0, \quad \frac{1}{H_n}\sum_{k=1}^n \eta_k(\widetilde{A}_{\gamma_k} f(\bar{X}_{\Gamma_{k-1}}) - Af(\bar{X}_{\Gamma_{k-1}})) = 0.$$

It follows that, $\mathbb{P}(d\omega)$-$a.s.$, for every $f \in \mathcal{D}(A)_0$, $\lim_n \nu_n^\eta(Af) = 0$. The conclusion follows from the Echeverria-Weiss theorem (see Theorem 2.1). Simply notice that we maintain the assumptions of this theorem when $\mathcal{D}(A)$ is replaced by $\mathcal{D}(A)_0$, since $\mathcal{D}(A)_0 \subset \mathcal{D}(A)$ and $\mathcal{D}(A)_0$ is dense in $\mathcal{C}_0(E)$. □

2.3 About growth control and step-weight assumptions

The following Lemma presents a L^1-finiteness property that can be obtained under recursive control hypothesis and strongly mean-reverting assumptions ($\phi = I_d$). This result is thus useful to prove $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}})$ (see (11)) or $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(F)$ (see (12)) for well chosen F and g in this specific situation.

Lemma 2.2. *Let $v_* > 0$, $V : E \to [v_*,\infty)$, $\psi : [v_*,\infty) \to \mathbb{R}_+$, such that $\widetilde{A}_{\gamma_n}\psi\circ V$ exists for every $n \in \mathbb{N}^*$. Let $\alpha > 0$, $\beta \in \mathbb{R}$ and $s \geqslant 1$. We assume that $\mathcal{RC}_{Q,V}(\psi, I_d, \alpha, \beta, s)$ (see (4)) holds and that $\mathbb{E}[\psi\circ V(\bar{X}_{\Gamma_{n_0}})] < +\infty$ for every $n_0 \in \mathbb{N}^*$. Then*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\big[\psi\circ V(\bar{X}_{\Gamma_n})\big] < +\infty. \tag{19}$$

In particular, let $\rho \in [1,2]$ *and let* $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \to \mathbb{R}_+$ *be an increasing function. It follows that if* $\sum_{n=1}^{\infty} \left|\frac{\eta_n}{H_n\gamma_n}\right|^{\rho} \epsilon_{\mathcal{I}}(\gamma_n) < +\infty$, *then* $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\psi\circ V,\rho,\epsilon_{\mathcal{I}})$ *holds and if* $\sum_{n\geq 0} \frac{(\eta_{n+1}/\gamma_{n+1}-\eta_n/\gamma_n)_+}{H_{n+1}} < +\infty$, *then* $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(\psi\circ V)$ *is satisfied*

Proof. First, we deduce from $\mathcal{RC}_{Q,V}(\psi, I_d, \alpha, \beta, s)(ii)$ that there exists $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$, we have

$$\mathbb{E}[\psi\circ V(\bar{X}_{\Gamma_{n+1}})|\bar{X}_{\Gamma_n}] \leqslant \psi\circ V(\bar{X}_{\Gamma_n}) + \gamma_{n+1}\frac{\psi\circ V(\bar{X}_{\Gamma_n})}{V(\bar{X}_{\Gamma_n})}(\beta - \alpha V(\bar{X}_{\Gamma_n})).$$

Now, let $\lambda \in (0,1)$ and $v_\lambda = v_* \vee (\beta/(\lambda\alpha))$. It follows that the Borel function $C_\lambda : [v_*,+\infty) \to \mathbb{R}$, $v \mapsto C_\lambda(v) := v^{-1}\psi(v)(\beta - \lambda\alpha v)$ is locally bounded on $[v_*,+\infty)$ and non positive on $[v_\lambda,+\infty)$, hence $\bar{C}_\lambda := \sup_{v\in[v_*,v_\lambda)} C_\lambda(v) < +\infty$ (with the convention $\sup_{v\in\emptyset} C_\lambda(v) = 0$) and

$$\begin{aligned}\mathbb{E}[\psi\circ V(X_{\Gamma_{n+1}})|X_{\Gamma_n}] &\leqslant \psi\circ V(\bar{X}_{\Gamma_n}) + \gamma_{n+1}(C_\lambda\circ V(\bar{X}_{\Gamma_n}) - (1-\lambda)\alpha\psi\circ V(\bar{X}_{\Gamma_n})),\\ &\leqslant \psi\circ V(\bar{X}_{\Gamma_n})(1-\gamma_{n+1}(1-\lambda)\alpha) + \gamma_{n+1}\bar{C}_\lambda.\end{aligned}$$

Applying a simple induction we deduce that $\mathbb{E}[\psi\circ V(X_{\Gamma_n})] \leqslant \mathbb{E}[\psi\circ V(X_{n_0})] \vee \frac{\bar{C}_\lambda}{(1-\lambda)\alpha}$. □

Now, we provide a general way to obtain $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}})$ and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(F)$ for some specific g and F as soon as a recursive control with weakly mean reversion assumption holds.

Lemma 2.3. *Let* $v_* > 0$, $V : E \to [v_*,\infty)$, $\psi,\phi : [v_*,\infty) \to \mathbb{R}_+$, *such that* $\widetilde{A}_{\gamma_n}\psi\circ V$ *exists for every* $n \in \mathbb{N}^*$. *Let* $\alpha > 0$, $\beta \in \mathbb{R}$ *and* $s \geqslant 1$. *We also introduce the non-increasing sequence* $(\theta_n)_{n\in\mathbb{N}^*}$ *such that* $\sum_{n\geqslant 1}\theta_n\gamma_n < +\infty$. *We assume that* $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)$ *(see (4)) holds and that* $\mathbb{E}[\psi\circ V(\bar{X}_{\Gamma_{n_0}})] < +\infty$ *for every* $n_0 \in \mathbb{N}^*$. *Then*

$$\sum_{n=1}^{\infty}\theta_n\gamma_n\mathbb{E}[\widetilde{V}_{\psi,\phi,1}(\bar{X}_{\Gamma_{n-1}})] < +\infty$$

with $\widetilde{V}_{\psi,\phi,1}$ *defined in (6). In particular, let* $\rho \in [1,2]$ *and let* $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \to \mathbb{R}_+$ *be an increasing function. If we also assume*

$$\begin{aligned}\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}}) \quad \equiv \quad & \Big(\gamma_n^{-1}\epsilon_{\mathcal{I}}(\gamma_n)\big(\frac{\eta_n}{H_n\gamma_n}\big)^{\rho}\Big)_{n\in\mathbb{N}^*} \text{ is non-increasing and}\\ & \sum_{n=1}^{\infty}\Big(\frac{\eta_n}{H_n\gamma_n}\Big)^{\rho}\epsilon_{\mathcal{I}}(\gamma_n) < +\infty, \end{aligned} \tag{20}$$

then we have $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\widetilde{V}_{\psi,\phi,1},\rho,\epsilon_{\mathcal{I}})$ *(see (11)). Finally,if*

$$\begin{aligned}\mathcal{SW}_{\mathcal{II},\gamma,\eta} \quad \equiv \quad & \Big(\frac{(\frac{\eta_{n+1}}{\gamma_{n+1}} - \frac{\eta_n}{\gamma_n})_+}{\gamma_n H_n}\Big)_{n\in\mathbb{N}^*} \text{ is non-increasing and}\\ & \sum_{n=1}^{\infty}\frac{(\eta_{n+1}/\gamma_{n+1} - \eta_n/\gamma_n)_+}{H_n} < +\infty, \end{aligned} \tag{21}$$

then we have $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(\widetilde{V}_{\psi,\phi,1})$ *(see (12)).*

Proof. First assume that $\beta \geqslant 0$. By $\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta,s)(iii)$ (see (4)), there exists $\lambda \in (0,1)$ and $v_\lambda \in (0,+\infty)$ such that for every $v > v_\lambda$, then $\phi(v) \geqslant \beta/(\lambda\alpha)$. It follows that the Borel function $C_\lambda : [v_*,+\infty) \to \mathbb{R}$, $v \mapsto C_\lambda(v) := v^{-1}\psi(v)(\beta - \lambda\alpha\phi(v))$ is locally bounded on $[v_*,+\infty)$ and non positive on $[v_\lambda,+\infty)$, hence $\bar{C}_\lambda := \sup_{v\in[v_*,+\infty)} C_\lambda(v) < +\infty$. When $\beta < 0$, since ϕ and ψ are positive functions, then the function C_λ is non positive. Using the same approach as in the proof of

Theorem 2.3 point B., we deduce that there exists $n_0 \in \mathbb{N}$ such that we have the following telescopic decomposition:

$$\begin{aligned}\forall n \geqslant n_0, \quad \theta_{n+1}\gamma_{n+1}\widetilde{V}_{\psi,\phi,1}(\bar{X}_{\Gamma_n}) \leqslant & \theta_{n+1}\frac{\psi\circ V(X_{\Gamma_n}) - \mathbb{E}[\psi\circ V(\bar{X}_{\Gamma_{n+1}})|\bar{X}_{\Gamma_n}]}{\alpha(1-\lambda)} \\ & + \gamma_{n+1}\theta_{n+1}\frac{\bar{C}_\lambda}{\alpha(1-\lambda)} \\ \leqslant & \frac{\theta_n\psi\circ V(\bar{X}_{\Gamma_n}) - \theta_{n+1}\mathbb{E}[\psi\circ V(\bar{X}_{\Gamma_{n+1}})|\bar{X}_{\Gamma_n}]}{\alpha(1-\lambda)} \\ & + \gamma_{n+1}\theta_{n+1}\frac{\bar{C}_\lambda}{\alpha(1-\lambda)},\end{aligned}$$

where the last inequality follows from the fact that the sequence $(\theta_n)_{n\in\mathbb{N}^*}$ is non-increasing. Taking expectancy and summing over n yields the result as ψ takes positive values and $\mathbb{E}[\psi\circ V(X_{n_0})] < +\infty$ for every $n_0 \in \mathbb{N}^*$. □

This result concludes the general approach in a generic framework to prove convergence. The next part of this paper is dedicated to various applications.

2.4 Example: the decreasing step Euler scheme of a diffusion

Using this abstract approach, we can recover the results obtained in [10] and [11] for the Euler scheme of a d-dimensional Brownian diffusion. We do not provide proof in this example, but it is actually similar, though less technical, to the Markov Switching case (see Section 3.2. Notice however, that the approach to obtain infinitesimal approximation assumption is explained in Appendix A for the one-dimensional case. In this part of the paper, our aim is to illustrate how the generic assumptions fit in our general framework.

We consider an N-dimensional Brownian motion $(W_t)_{t\geqslant 0}$. We are interested in the strong solution - assumed to exist and to be unique - of the d-dimensional stochastic equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \tag{22}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times N}$. Let $V : \mathbb{R} \to [1, +\infty)$, the Lyapunov function of this system such that $\mathcal{L}_V$ (see (3)) holds with $E = \mathbb{R}^d$, and

$$|\nabla V|^2 \leqslant C_V V, \qquad \|D^2 V\|_\infty < +\infty.$$

Moreover, we assume that for every $x \in \mathbb{R}$, $|b(x)|^2 + \mathrm{Tr}[\sigma\sigma^*(x)] \leqslant V^a(x)$ for some $a \in (0, 1]$. Finally, for $p \geqslant 1$, we introduce the following L^p-mean-reverting property of V,

$$\exists\, \alpha > 0,\ \beta \in \mathbb{R},\ \forall x \in \mathbb{R}, \quad \langle \nabla V(x), b(x)\rangle + \frac{1}{2}\|\lambda_p\|_\infty 2^{(2p-3)_+}\mathrm{Tr}[\sigma\sigma^*(x)] \leqslant \beta - \alpha V^a(x)$$

with for every $x \in \mathbb{R}^d$, $\lambda_p(x) := \sup\{\lambda_{p,1}(x), \ldots, \lambda_{p,d}(x), 0\}$, with $\lambda_{p,i}(x)$ the i-th eigenvalue of the matrix $D^2V(x) + 2(p-1)\nabla V(x)^{\otimes 2}/V(x)$. We now introduce the Euler scheme of $(X_t)_{t\geqslant 0}$. Let $\rho \in [1, 2]$ and $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho/2}$ and assume that (16), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho, \epsilon_{\mathcal{I}})$ (see (20)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)) hold. Let $(U_n)_n$ be a sequence of $\mathbb{R}^N$-valued centered independent and identically distributed random variables with covariance identity and bounded moments of order $2p$. We define the Euler scheme with decreasing steps $(\gamma_n)_{n\in\mathbb{N}^*}$, $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ of $(X_t)_{t\geqslant 0}$ (22) by

$$\forall n \in \mathbb{N}, \qquad \bar{X}_{\Gamma_{n+1}} = \bar{X}_{\Gamma_n} + \gamma_{n+1}b(\bar{X}_{\Gamma_n}) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_{\Gamma_n})U_{n+1}, \quad \bar{X}_0 = x,$$

where $\Gamma_n = \sum_{k=1}^n \gamma_k$, $n \in \mathbb{N}$. We consider $(\nu_n^\eta(dx, \omega))_{n\in\mathbb{N}^*}$ defined as in (2) with $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ defined above. Now, we specify the measurable functions $\psi, \phi : [1, +\infty) \to [1, +\infty)$ as $\psi(v) = v^p$ and

$\phi(v) = v^a$. Moreover, let $s \geqslant 1$ such that $a\,p\rho/s \leqslant p+a-1, p/s+a-1 > 0$ and $\mathrm{Tr}[\sigma\sigma^*] \leqslant CV^{p/s+a-1}$. Then, we can show that $\mathcal{E}(\widetilde{A}, A, \mathcal{C}_K^2(\mathbb{R}^d))$ (see (7)) holds (the proof is similar but simpler than its counterpart (see Proposition 3.3 for the one state case in the Markov Switching framework) with $\widetilde{A}_\gamma$ and A denoting respectively the infinitesimal generator of the diffusion (22) and the pseudo-generators of the Euler scheme. Then, it follows from Theorem 2.4 that there exists an invariant distribution ν for $(X_t)_{t\geqslant 0}$. Moreover, $(\nu_n^\eta(dx,\omega))_{n\in\mathbb{N}^*}$ *a.s.* weakly converges toward $\mathcal{V}$, the set of invariant distributions of $(X_t)_{t\geqslant 0}$ and when it is unique *i.e.* $\mathcal{V} = \{\nu\}$, we have

$$\lim_n \nu_n^\eta(f) = \nu(f),$$

for every ν-*a.s.* continuous function $f \in \mathcal{C}_{\widetilde{V}_{\psi,\phi,s}}(\mathbb{R}^d)$ defined in (6). Notice that this result was initially obtained in [10] when $a = 1$ and in [11] when $a \in (0,1]$ and in both cases $s = \rho = 2$. Afterwards, the study was extended in the case function ψ with polynomial growth in [13]. We do not recall this result. However, in the sequel we prove the convergence of the empirical measures for both polynomial growth and exponential growth of ψ for the Euler scheme of a Brownian Markov switching diffusions and those mentioned results can be recovered from a simplified version of our approach.

3 Applications

In this section, we propose some concrete applications which follow from the results presented in Section 2. We first give Wasserstein convergence results concerning the Milstein scheme of a weakly mean-reverting Brownian diffusion. Then, we propose a detailed application for the Euler scheme of a Markov Switching diffusion for test functions with polynomial or exponential growth. As a preliminary, we give some standard notations and properties that will be used extensively in the sequel.

NOTATIONS.
• For $\alpha \in (0,1]$ and f an α-Hölder function we denote $[f]_\alpha = \sup_{x\neq y} |f(y)-f(x)|/|y-x|^\alpha$.
• Let $d \in \mathbb{N}$. For any $\mathbb{R}^{d\times d}$-valued symmetric matrix S, we define $\lambda_S := \sup\{\lambda_{S,1}, \ldots, \lambda_{S,d}, 0\}$, with $\lambda_{S,i}$ the i-th eigenvalue of S.

3.1 Wasserstein convergence for the Milstein scheme

In this section, we provide Wasserstein convergence results for the empirical measures (2) built with the Milstein approximation scheme of a one-dimensional weakly mean-reverting Brownian diffusion. The framework presented in Section 2 is well suited this scheme and we present the result that we obtain in this case.

The Milstein scheme has not been investigated until now but the convergence results are similar to the Euler case that is why, even if the proofs are more technical, we simply state them. Moreover, looking at $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ (see (7)), the approximation of A seems to rely on the weak order of the scheme. As a consequence, even from a rate of convergence viewpoint, intuitively, it is not possible to achieve a better rate of convergence of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ with the Milstein scheme than with the Euler scheme. We will give the proof of this result in a further paper.

We consider a one dimensional Brownian motion $(W_t)_{t\geqslant 0}$. We are interested in the strong solution - assumed to exist and to be unique - of the one dimensional stochastic equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \tag{23}$$

where $b, \sigma, \partial_x\sigma : \mathbb{R} \to \mathbb{R}$. Moreover, we assume that for every $x \in \mathbb{R}$, $|b(x)|^2 + |\sigma(x)|^2 + |\sigma\sigma'(x)|^2 \leqslant C(1+|x|^{2a})$ for some $a \in (0,1]$. Finally, for $p \geqslant 1$, we introduce the following L^p-mean-reverting

property:

$$\exists\, \alpha > 0, \beta \in \mathbb{R}, \qquad \forall\, x \in \mathbb{R}, \quad 2xb(x) + (4p-3)2^{(2p-3)_+}\sigma^2(x) \leqslant \beta - \alpha|x|^{2a}.$$

We now introduce the Milstein scheme for $(X_t)_{t\geqslant 0}$. Let $\rho \in [1,2]$ and $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho/2}$ and assume that (16), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}})$ (see (20)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)) hold. Let $(U_n)_n$ be a sequence of centered independent and identically distributed random variables with variance one and bounded moments of order $2p$. We define the Milstein scheme with decreasing steps $(\gamma_n)_{n\in\mathbb{N}^*}$, $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ of $(X_t)_{t\geqslant 0}$ (23) by: $\bar{X}_0 = x$, $\forall n \in \mathbb{N}$,

$$\bar{X}_{\Gamma_{n+1}} = \bar{X}_{\Gamma_n} + \gamma_{n+1} b(\bar{X}_{\Gamma_n}) + \sqrt{\gamma_{n+1}}\sigma(\bar{X}_{\Gamma_n})U_{n+1} + \gamma_{n+1}\sigma\sigma'(\bar{X}_{\Gamma_n})(|U_{n+1}|^2 - 1).$$

Then $V : \mathbb{R} \to [1,+\infty)$, $x \mapsto 1 + x^2$ is a Lyapunov function for this scheme. We consider $(\nu_n^\eta(dx,\omega))_{n\in\mathbb{N}^*}$ defined as in (2) with $(\bar{X}_{\Gamma_n})_{n\in\mathbb{N}}$ defined above. Now, we specify the measurable functions $\psi, \phi : [1,+\infty) \to [1,+\infty)$ as $\psi(v) = v^p$ and $\phi(v) = v^a$. Moreover, let $s \geqslant 1$ such that $ap\rho/s \leqslant p + a - 1$ and $p/s + a - 1 > 0$. Then, it follows from Theorem 2.4 that there exists an invariant distribution ν for $(X_t)_{t\geqslant 0}$. Moreover, $(\nu_n^\eta(dx,\omega))_{n\in\mathbb{N}^*}$ *a.s.* weakly converges toward $\mathcal{V}$, the set of invariant distributions of $(X_t)_{t\geqslant 0}$ and when it is unique *i.e.* $\mathcal{V} = \{\nu\}$, we have

$$\lim_n \nu_n^\eta(f) = \nu(f),$$

for every ν-*a.s.* continuous function $f : \mathbb{R} \to \mathbb{R}$ such that, for every $x \in \mathbb{R}$, $|f(x)| \leqslant C(1 + |x|^{\bar{p}})$, with $\bar{p} < p/s + a - 1$. In other words $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ converges towards ν (as n tends to infinity) for the $\mathrm{L}_{\bar{p}}$ Wasserstein distances.

3.2 The Euler scheme of a Markov switching diffusion

In this part of the paper, we study invariant distributions for Markov switching Brownian diffusions. The framework presented in Section 2 is well suited to this case. Our results extend the convergence results obtained in [14] and inspired by [10]. More particularly, in [14], the convergence of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is established under a strongly mean-reverting assumption that is $\phi = I_d$. In this paper, we do not restrict to that case and consider a weakly mean-reverting setting, namely $\phi(v) = v^a$, $a \in (0,1]$ for every $v \in [v_*,\infty)$. As a first step, we consider polynomial test functions that is $\psi(v) = v^p$, $p \geqslant 1$ for every $v \in [v_*,\infty)$ like in [14] (where $p \geqslant 4$ is required). As a second step, still under a weakly mean-reverting setting (but where ϕ is not explicitly specified), we extend those results to functions ψ with exponential growth which enables to obtain convergence of the empirical measures for much wider class of test functions.

Now, we present the Markov switching model, its decreasing step Euler approximation and the hypothesis necessary to obtain the convergence of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$. We consider a d-dimensional Brownian motion $(W_t)_{t\geqslant 0}$ and $(\zeta_t)_{t\geqslant 0}$ a continuous time Markov chain taking values in the finite state space $\{1,\ldots,M_0\}$, $M_0 \in \mathbb{N}^*$ with generator $\mathfrak{Q} = (q_{z,w})_{z,w\in\{1,\ldots,M_0\}}$ and independent from W. We are interested in the strong solution - assumed to exist and to be unique - of the d-dimensional stochastic equation

$$X_t = x + \int_0^t b(X_s,\zeta_s)ds + \int_0^t \sigma(X_s,\zeta_s)dW_s$$

where for every $z \in \{1,\ldots,M_0\}$, $b(.,z) : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(.,z) \to \mathbb{R}^{d\times d}$ are locally bounded functions. We recall that $q_{z,w} \geqslant 0$ for $z \neq w$, $z,w \in \{1,\ldots,M_0\}$ and $\sum_{w=1}^{M_0} q_{z,w} = 0$ for every $z \in \{1,\ldots,M_0\}$.

The infinitesimal generator of this process reads

$$Af(x,z) = \langle b(x,z), \nabla_x f(x,z)\rangle + \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{i,j}(x,z)\frac{\partial^2 f}{\partial x_i \partial x_j}(x,z) + \sum_{w=1}^{M_0} q_{z,w} f(x,w),$$

for every $(x,z) \in E := \mathbb{R}^d \times \{1,\dots,M_0\}$. Moreover, the domain $\mathcal{D}(A)$ of A contains $\mathcal{D}(A)_0 = \{f \text{ defined on } E, \forall\, z \in \{1,\dots,M_0\}, f(.,z) \in\}$. Notice that $\mathcal{D}(A)_0$ is dense in $\mathcal{C}_0(E)$. The reader may refer to [25] for more details concerning Markov switching diffusion processes where properties such as recurrence, ergodicity and stability are established. We consider the Euler genuine scheme of this process for every $n \in \mathbb{N}$ and every $t \in [\Gamma_n, \Gamma_{n+1}]$, defined by

$$\bar{X}_t = \bar{X}_{\Gamma_n} + (t-\Gamma_n) b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) + \sigma(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})(W_t - W_{\Gamma_n}). \tag{24}$$

We will also denote $\Delta\bar{X}_{n+1} = \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}$ and

$$\Delta\bar{X}^1_{n+1} = \gamma_{n+1} b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}), \quad \Delta\bar{X}^2_{n+1} = \sigma(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})(W_{\Gamma_{n+1}} - W_{\Gamma_n}), \tag{25}$$

and $\bar{X}^i_{\Gamma_{n+1}} = \bar{X}_{\Gamma_n} + \sum_{j=1}^{i} \Delta\bar{X}^i_{n+1}$. In the sequel we will use the notation $U_{n+1} = \gamma_{n+1}^{-1/2}(W_{\Gamma_{n+1}} - W_{\Gamma_n})$. Finally, we consider a Lyapunov function $V : \mathbb{R}^d \times \{1,\dots,M_0\} \to [v_*, \infty)$, $v_* > 0$, which satisfies $\mathcal{L}_V$ (see (3)) with $E = \mathbb{R}^d \times \{1,\dots,M_0\}$, and

$$|\nabla_x V|^2 \leqslant C_V V, \qquad \sup_{(x,z)\in E} \|D_x^2 V(x,z)\| < +\infty. \tag{26}$$

Its mean-reverting properties will be defined further depending on the set of 'test functions' f. We also define

$$\forall\, x \in \mathbb{R}^d, z \in \{1,\dots,M_0\}, \quad \lambda_\psi(x,z) := \lambda_{D_x^2 V(x,z) + 2\nabla_x V(x,z)^{\otimes 2}\psi''\circ V(x,z)\psi'\circ V(x,z)^{-1}}. \tag{27}$$

When $\psi(v) = \psi_p(v) = v^p$, $p > 0$, we will also use the notation λ_p instead of λ_ψ. We suppose that there exists $C > 0$ such that b and σ satisfy

$$\mathfrak{B}(\phi) \quad \equiv \quad \forall\, x \in \mathbb{R}^d, \forall\, z \in \{1,\dots,M_0\}, \quad |b(x,z)|^2 + \mathrm{Tr}[\sigma\sigma^*(x,z)] \leqslant C\phi \circ V(x,z) \tag{28}$$

Test functions with polynomial growth.

Having in mind Wasserstein convergence, we introduce a weaker assumption on the sequence $(U_n)_{n\in\mathbb{N}^*}$ than Gaussian distribution. Let $q \in \mathbb{N}^*$, $p \geqslant 0$. We suppose that $(U_n)_{n\in\mathbb{N}^*}$ is a sequence of i.i.d. random variables such that

$$M_{\mathcal{N},q}(U) \quad \equiv \quad \forall n \in \mathbb{N}^*, \forall \widetilde{q} \in \{1,\dots,q\}, \quad \mathbb{E}[(U_n)^{\otimes\widetilde{q}}] = \mathbb{E}[(\mathcal{N}(0, I_d))^{\otimes\widetilde{q}}] \tag{29}$$

$$M_p(U) \quad \equiv \quad \sup_{n\in\mathbb{N}^*} \mathbb{E}[|U_n|^{2p}] < +\infty. \tag{30}$$

We assume that

$$\exists\, c_V \geqslant 1, \forall\, x \in \mathbb{R}^d, \qquad \sup_{z\in\{1,\dots,M_0\}} V(x,z) \leqslant c_V \inf_{z\in\{1,\dots,M_0\}} V(x,z). \tag{31}$$

Let $\alpha > 0$ and $\beta \in \mathbb{R}$. We introduce the mean-reverting property of the scheme for the Lyapunov function V. We assume that $\liminf_{v\to+\infty} \phi(v) > \beta/\alpha$ and that there exists $\epsilon_0 > 0$, such that we have

$$\mathcal{R}_p(\alpha,\beta,\phi,V) \quad \equiv \quad \forall\, x \in \mathbb{R}^d, \forall\, z \in \{1,\dots,M_0\}, \quad \langle \nabla V(x,z), b(x,z)\rangle + \frac{1}{2}\chi_p(x,z) \leqslant \beta - \alpha\phi \circ V(x,z), \tag{32}$$

with

$$\chi_p(x,z) = \|\lambda_p\|_\infty 2^{(2p-3)_+}\mathrm{Tr}[\sigma\sigma^*(x,z)] + V^{1-p}(x,z)\sum_{w=1}^{M_0}(q_{z,w}+\epsilon_0)V^p(x,w). \tag{33}$$

Theorem 3.1. *Let $p \geqslant 1$,$a \in (0,1]$, $s \geqslant 1$, $\rho \in [1,2]$, $\psi_p(v) = v^p$, $\phi(v) = v^a$ and $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho/2}$. Let $\alpha > 0$ and $\beta \in \mathbb{R}$.*

Assume that $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_{\mathcal{N},2}(U)$ (see (29)) and $M_p(U)$ (see (30)).
Also assume that (26), $\mathfrak{B}(\phi)$ (see (28)), $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (32)), L_V (see (3)), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}})$ (see (20)), $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)), (16) and (31) hold and that $p\rho/s \leqslant p+a-1$.

Then, if $p/s + a - 1 > 0$, $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ (built with $(\bar{X}_t)_{t\geqslant 0}$ defined in (24)) is $\mathbb{P}$-a.s. tight and

$$\mathbb{P}\text{-a.s.} \quad \sup_{n\in\mathbb{N}^*}\nu_n^\eta(V^{p/s+a-1}) < +\infty. \tag{34}$$

Assume also that for every $z \in \{1,\dots,M_0\}$, $b(.,z)$ and $\sigma(.,z)$ have sublinear growth and $\mathrm{Tr}[\sigma\sigma^*(x,z)] \leqslant CV^{p/s+a-1}(x,z)$. Then every weak limiting distribution ν of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is an invariant distribution of $(X_t)_{t\geqslant 0}$ and when ν is unique, we have

$$\mathbb{P}\text{-a.s.} \quad \forall f \in \mathcal{C}_{\widetilde{V}_{\psi_p,\phi,s}}(E), \quad \lim_n \nu_n^\eta(f) = \nu(f), \tag{35}$$

with $\mathcal{C}_{\widetilde{V}_{\psi_p,\phi,s}}(E)$ defined in (6).

Test functions with exponential growth.

We modify the hypothesis concerning the Lyapunov function V in the following way. First, we assume that

$$\forall z \in \{1,\dots,M_0\}, \forall x \in \mathbb{R}^d \quad V(x,z) = V(x,1), \tag{36}$$

and we will use the notation $V(x) := V(x,1)$. We assume that

$$\forall x \in \mathbb{R}^d, \forall z \in \{1,\dots,M_0\}, \\ \mathrm{Tr}[\sigma\sigma^*(x,z)]|b(x)|\big(|\nabla V(x)| + |b(x,z)|\big) \leqslant CV^{1-p}(x)\phi\circ V(x). \tag{37}$$

Now let $p \leqslant 1$ and let $\alpha > 0$ and $\beta \in \mathbb{R}$. We assume that $\liminf_{v\to+\infty}\phi(v) > \beta_+/\alpha$, $\beta_+ = 0 \vee \beta$, and

$$\mathcal{R}_{p,\lambda}(\alpha,\beta,\phi,V) \quad \equiv \quad \forall x \in \mathbb{R}^d, \forall z \in \{1,\dots,M_0\}, \\ \langle \nabla V(x), b(x,z) + \kappa_p(x,z)\rangle + \frac{1}{2}\chi_p(x,z) \leqslant \beta - \alpha\phi\circ V(x), \tag{38}$$

with

$$\kappa_p(x,z) = \lambda p\frac{V^{p-1}(x)}{\phi\circ V(x)}\sigma\sigma^*(x,z)\nabla V(x)$$

and

$$\chi_p(x,z) = -\frac{V^{1-p}(x)}{\phi\circ V(x)C_\sigma(x,z)}\ln(\det(\Sigma(x,z)))$$

with $\Sigma : \mathbb{R}^d \times \{1,\dots,M_0\}, \to \mathcal{S}^d_{+,*}$, $\mathcal{S}^d_{+,*}$ being the set of a positive definite matrix, defined by $(x,z) \mapsto \Sigma(x,z) := I_d - \|D^2V\|_\infty C_\sigma(x,z)V^{p-1}(x)\sigma^*\sigma(x,z)$, where $C_\sigma : \mathbb{R}^d \times \{1,\dots,M_0\} \to \mathbb{R}^*_+$ satisfies $\inf_{x\in\mathbb{R}^d}\inf_{z\in\{1,\dots,M_0\}} C_\sigma(x,z) > 0$.

Theorem 3.2. *Let $p \in [0,1], \lambda \geqslant 0$, $s \geqslant 1$, $\rho \in [1,2]$, let $\phi : [v_*, \infty) \to \mathbb{R}_+$ be a continuous function such that $C_\phi := \sup_{v\in[v_*,+\infty)} \phi(v)/v < +\infty$ and $\liminf_{v\to+\infty} \phi(v) = +\infty$, let $\psi(v) = \exp(\lambda v^p)$, $y \in \mathbb{R}_+$ and let $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho/2}$ and $\widetilde{\epsilon}_{\mathcal{I}}(\gamma) = \gamma^{\rho(p\wedge 1/2)}$. Let $\alpha > 0$ and $\beta \in \mathbb{R}$.*

Assume that $\rho < s$, (36), (26), $\mathfrak{B}(\phi)$ (see (28)), $\mathcal{R}_{p,\lambda}(\alpha,\beta,\phi,V)$ (see (38)) and L_V (see (3)) hold. Also suppose that $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}})$, $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\widetilde{\epsilon}_{\mathcal{I}})$ (see (20)), $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)), (16) and (37) hold.

Then $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ (built with $(\bar{X}_t)_{t\geqslant 0}$ defined in (24)) is $\mathbb{P}$-a.s. tight and

$$\mathbb{P}\text{-}a.s. \quad \sup_{n\in\mathbb{N}^*} \nu_n^\eta\Big(\frac{\phi\circ V}{V}\exp\big(\lambda/sV^p\big)\Big) < +\infty. \tag{39}$$

Assume also that for every $z \in \{1,\dots,M_0\}$, $b(.,z)$ and $\sigma(.,z)$ have sub-linear growth. Then, every weak limiting distribution ν of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is an invariant distribution of $(X_t)_{t\geqslant 0}$ and if ν is unique,

$$\mathbb{P}\text{-}a.s. \quad \forall f \in \mathcal{C}_{\widetilde{V}_{\psi,\phi,s}}(E), \quad \lim_n \nu_n^\eta(f) = \nu(f), \tag{40}$$

with $\mathcal{C}_{\widetilde{V}_{\psi,\phi,s}}(E)$ defined in (6).

3.2.1 Proof of the recursive mean-reverting control

Test functions with polynomial growth

Proposition 3.1. *Let $v_* > 0$, and let $\phi : [v_*,\infty) \to \mathbb{R}_+^*$ be a continuous function such that $C_\phi := \sup_{v\in[v_*,\infty)} \phi(v)/v < +\infty$. Now let $p \geqslant 1$ and define $\psi_p(v) = v^p$, $v \in \mathbb{R}_+$.*

Assume that the sequence $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_{\mathcal{N},2}(U)$ (see (29)) and $M_p(U)$ (see (30)).

Also suppose that (26), (31), $\mathfrak{B}(\phi)$ (see (28)), $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (32)) for some $\alpha > 0$ and $\beta \in \mathbb{R}$, are satisfied.

Then, for every $\widetilde{\alpha} \in (0,\alpha)$, there exists $n_0 \in \mathbb{N}^*$, such that

$$\forall n \geqslant n_0,\ \forall x \in \mathbb{R}^d, \forall z \in \{1,\dots,M_0\},$$
$$\widetilde{A}_{\gamma_n}\psi_p\circ V(x,z) \leqslant \frac{\psi_p\circ V(x,z)}{V(x,z)} p\big(\beta - \widetilde{\alpha}\phi\circ V(x,z)\big). \tag{41}$$

Then $\mathcal{RC}_{Q,V}(\psi_p,\phi,p\widetilde{\alpha},p\beta,s)$ (see (4)) holds for every $\widetilde{\alpha} \in (0,\alpha)$ and $s \geqslant 1$ such that $\liminf_{v\to+\infty}\phi(v) > \beta/\widetilde{\alpha}$ and $\lim_{v\to+\infty} v^{-1}\phi(v)\psi_p(v)^{1/s} = +\infty$. Moreover, when $\phi = Id$, we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\psi_p\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})] < +\infty. \tag{42}$$

Proof. First we write

$$\begin{aligned} V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}) =& V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n}) - V^p(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}) \\ &+ V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n}). \end{aligned} \tag{43}$$

We study the first term of the *r.h.s.* of the above equality. From the second order Taylor expansion and the definition of $\lambda_{\psi_p} = \lambda_p$ (see (27)), we derive

$$\begin{aligned} \psi_p\circ V(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n}) &= \psi_p\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}) + \langle \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}, \nabla_x V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\rangle \psi_p'\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}) \\ &\quad + \frac{1}{2} D_x^2 V(\Upsilon_{n+1},\zeta_{\Gamma_n})\psi_p'\circ V(\Upsilon_{n+1},\zeta_{\Gamma_n})(\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n})^{\otimes 2} \\ &\quad + \frac{1}{2}\nabla_x V(\Upsilon_{n+1},\zeta_{\Gamma_n})^{\otimes 2}\psi_p''\circ V(\Upsilon_{n+1},\zeta_{\Gamma_n})(\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n})^{\otimes 2} \\ &\leqslant \psi_p\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}) + \langle \bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}, \nabla_x V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\rangle \psi_p'\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}) \\ &\quad + \frac{1}{2}\lambda_p(\Upsilon_{n+1},\zeta_{\Gamma_n})\psi_p'\circ V(\Upsilon_{n+1},\zeta_{\Gamma_n})|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^2. \end{aligned} \tag{44}$$

with $\Upsilon_{n+1} \in (\bar{X}_{\Gamma_n}, \bar{X}_{\Gamma_{n+1}})$. First, from (26), we have $\sup_{z\in\{1,\dots,M_0\}} \sup_{x\in\mathbb{R}^d} \lambda_p(x,z) < +\infty$. Now, since $(U_n)_{n\in\mathbb{N}^*}$ is *i.i.d.* and satisfies $M_{\mathcal{N},1}(U)$ (see (29)), we compute

$$\mathbb{E}\big[\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n} | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big] = \gamma_{n+1} b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})$$
$$\mathbb{E}\left[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^2 | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\right] = \gamma_{n+1} \mathrm{Tr}[\sigma\sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})] + \gamma_{n+1}^2 |b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^2.$$

We focus on the study of the last term of the *r.h.s* of (44), also called the "remainder".

Case $p = 1$. Assume first that $p = 1$. Using $\mathfrak{B}(\phi)$ (see (28)), for every $\widetilde{\alpha} \in (0, \alpha)$, there exists $n_0(\widetilde{\alpha})$ such that, for every $n \geqslant n_0(\widetilde{\alpha})$,

$$\frac{1}{2}\|\lambda_1\|_\infty \gamma_{n+1}^2 |b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^2 \leqslant \gamma_{n+1}(\alpha - \widetilde{\alpha}) \phi \circ V(\bar{X}_{\Gamma_n}, , \zeta_{\Gamma_n}). \tag{45}$$

From assumption $\mathcal{R}_p(\alpha, \beta, \phi, V)$ (see (32) and (33)), we gather all the terms of (44) together and we conclude that

$$\gamma_{n+1}^{-1} \mathbb{E}[V(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] + \sum_{z=1}^{M_0} (q_{\zeta_{\Gamma_n}, z} + \epsilon_0) V(\bar{X}_{\Gamma_n}, z)$$
$$\leqslant \beta - \widetilde{\alpha} \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}).$$

Case $p > 1$. Assume now that $p > 1$ so that $\psi_p'(v) = p v^{p-1}$. Since $|\nabla V|^2 \leqslant C_V V$ (see (26)), then $\sqrt{V}$ is Lipschitz. Now, we use the following inequality: Let $l \in \mathbb{N}^*$. We have

$$\forall \alpha > 0, \forall u_i \in \mathbb{R}^d, i = 1, \dots, l, \qquad \Big|\sum_{i=1}^{l} u_i\Big|^\alpha \leqslant l^{(\alpha-1)_+} \sum_{i=1}^{l} |u_i|^\alpha. \tag{46}$$

It follows that

$$V^{p-1}(\Upsilon_{n+1}, \zeta_{\Gamma_n}) \leqslant \big(\sqrt{V}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) + [\sqrt{V}]_1 |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|\big)^{2p-2}$$
$$\leqslant 2^{(2p-3)_+} (V^{p-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) + [\sqrt{V}]_1^{2p-2} |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p-2}).$$

To study the 'remainder' of (44), we multiply the above inequality by $|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^2$. First, we study the second term which appears in the *r.h.s.* and using $\mathfrak{B}(\phi)$ (see (28)), for any $p \geqslant 1$,

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p} \leqslant C \gamma_{n+1}^p \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})^p (1 + |U_{n+1}|^{2p}).$$

Let $\hat{\alpha} \in (0, \alpha)$. Therefore, we deduce from $M_p(U)$ (see (30)) that there exists $n_0(\hat{\alpha}) \in \mathbb{N}$ such that for any $n \geqslant n_0(\hat{\alpha})$, we have

$$\mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p} | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}]$$
$$\leqslant \gamma_{n+1} \phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})^p \frac{\alpha - \hat{\alpha}}{C_\phi^{p-1} \|\lambda_p\|_\infty 2^{(2p-3)_+} [\sqrt{V}]_1^{2p-2}}.$$

To treat the other term of the 'remainder' of (44), we proceed as in (45) with $\|\lambda_1\|_\infty$ replaced by $\|\lambda_p\|_\infty 2^{2p-3} [\sqrt{V}]_1^{2p-2}$, α replaced by $\hat{\alpha}$ and $\widetilde{\alpha} \in (0, \hat{\alpha})$. We gather all the terms of (44) together and

using $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (32) and (33)), for every $n \geqslant n_0(\widetilde{\alpha}) \vee n_0(\hat{\alpha})$, we obtain

$$\begin{aligned}\mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n}) - V^p(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}] \\ + V^{1-p}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\sum_{z=1}^{M_0}(q_{\zeta_{\Gamma_n},z}+\epsilon)V^p(\bar{X}_{\Gamma_n},z) \\ \leqslant \gamma_{n+1}pV^{p-1}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})(\beta-\alpha\phi\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})) \\ + \gamma_{n+1}pV^{p-1}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\Big(\phi\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})(\hat{\alpha}-\widetilde{\alpha}) \\ + (\alpha-\hat{\alpha})\frac{V^{1-p}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\phi\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})^p}{C_\phi^{p-1}}\Big) \\ \leqslant \gamma_{n+1}V^{p-1}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\big(\beta p-\widetilde{\alpha}p\phi\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\big).\end{aligned}$$

Now, we focus on the second term of the *r.h.s.* of (43). First, since ζ and W are independent, it follows, with notations (25), that

$$\begin{aligned}\mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n})|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n},\Delta\bar{X}_{n+1}] \\ = \gamma_{n+1}\sum_{z=1}^{M_0}(q_{\zeta_{\Gamma_n},z} + \underset{n\to+\infty}{o}(\gamma_{n+1}))V^p(\bar{X}_{\Gamma_{n+1}},z).\end{aligned}$$

Now, using the same reasoning as for the first term of the *r.h.s.* of (43) and (31), since $p \geqslant 1$, we derive, for every $z \in \{1,\dots,M_0\}$,

$$\begin{aligned}|\mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}},z) - V^p(\bar{X}_{\Gamma_n},z)|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}]| \\ \leqslant C(\gamma_{n+1}^{1/2}V^{p-1}(\bar{X}_{\Gamma_n},z)\phi\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}) + \gamma_{n+1}^p\phi\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})^p \\ + \gamma_{n+1}V^{p-1/2}(\bar{X}_{\Gamma_n},z)\sqrt{\phi\circ V(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})}) \\ \leqslant C\gamma_{n+1}^{1/2}V^p(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\end{aligned}$$

where $C > 0$ is a constant which may change from line to line. We deduce that there exists $\varepsilon : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{\gamma\to 0}\varepsilon(\gamma) = 0$, such that we have

$$\begin{aligned}\mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_{n+1}}) - V^p(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n})|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}] \\ = \gamma_{n+1}\sum_{z=1}^{M_0}\big(q_{\zeta_{\Gamma_n},z}+o(\gamma_{n+1})\big)\mathbb{E}[V^p(\bar{X}_{\Gamma_{n+1}},z)|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}] \\ \leqslant \gamma_{n+1}\sum_{z=1}^{M_0}(q_{\zeta_{\Gamma_n},z}+\varepsilon(\gamma_{n+1}))V^p(\bar{X}_{\Gamma_n},z).\end{aligned}$$

This yields (41) as a direct consequence of $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (32) and (33)). The proof of (42) is a straightforward application of Lemma 2.2 as soon as we notice that the increments of the Euler scheme (for Markov Switching diffusions) have finite polynomial moments which implies (19). □

Test functions with exponential growth

In this section we do not relax the assumption on the Gaussian structure of the increment as we do in the polynomial case with hypothesis (29) and (30). In particular, it leads the following result:

Lemma 3.1. *Let $\Lambda \in \mathbb{R}^{d\times d}$ and $U \sim \mathcal{N}(0,I_d)$. We define $\Sigma \in \mathbb{R}^{d\times d}$ by $\Sigma = I_d - 2\Lambda^*\Lambda$. Assume that $\Sigma \in \mathcal{S}_{+,*}^d$. Then, for every $h \in (0,1)$,*

$$\forall v \in \mathbb{R}^d, \qquad \mathbb{E}\Big[\exp\Big(\sqrt{h}\langle v,U\rangle + h|\Lambda U|^2\Big)\Big] \leqslant \exp\Big(\frac{h}{2(1-h)}|v|^2\Big)\det(\Sigma)^{-h/2}. \tag{47}$$

Proof. A direct computation yields

$$\mathbb{E}[\exp(|\Lambda U|^2)] = \int_{\mathbb{R}^d} (2\pi)^{-d/2} \exp\Big(-\frac{1}{2}\langle -2\Lambda^*\Lambda u + u, u\rangle\Big) du = \det(\Sigma)^{-1/2}.$$

Now, (47) follows from the Hölder inequality since

$$\begin{aligned}\mathbb{E}\big[\exp(\sqrt{h}\langle v, U\rangle + h|\Lambda U|^2)\big] &\leqslant \mathbb{E}\Big[\exp\Big(\frac{\sqrt{h}}{1-h}\langle v, U\rangle\Big)\Big]^{1-h}\mathbb{E}[\exp(|\Lambda U|^2)]^h \\ &= \exp\Big(\frac{h}{2(1-h)}|v|^2\Big)\det(\Sigma)^{-h/2}.\end{aligned}$$

□

Using those results, we deduce the recursive control for exponential test functions.

Proposition 3.2. *Let $v_* > 0$, and let $\phi : [v_*, \infty) \to \mathbb{R}_+$ be a continuous function such that $C_\phi := \sup_{v\in[v_*,\infty)} \phi(v)/v < +\infty$. Now let $p \in [0,1]$, $\lambda \geqslant 0$ and define $\psi(v) = \exp(\lambda v^p)$, $y \in \mathbb{R}_+$.*

Suppose that (26), (36), $\mathfrak{B}(\phi)$ (see (28)) and $\mathcal{R}_{p,\lambda}(\alpha, \beta, \phi, V)$ (see (38)) are satisfied.

Then, for every $\widetilde{\alpha} \in (0, \alpha)$, there exists $\widetilde{\beta} \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}^*$, such that

$$\forall n \geqslant n_0, \forall x \in \mathbb{R}^d, \quad \widetilde{A}_{\gamma_n}\psi \circ V(x) \leqslant \frac{\psi\circ V(x)}{V(x)} p\big(\widetilde{\beta} - \widetilde{\alpha}\phi\circ V(x)\big). \tag{48}$$

Then, $\mathcal{RC}_{Q,V}(\psi, \phi, p\widetilde{\alpha}, p\widetilde{\beta}, s)$ (see (4)) holds for every $s \geqslant 0$, as soon as $\liminf_{v\to+\infty} \phi(v) = +\infty$. Moreover, when $\phi = Id$ we have

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\psi\circ V(\bar{X}_{\Gamma_n})] < +\infty. \tag{49}$$

Proof. When $p = 0$, the result is straightforward. Since $p \leqslant 1$, the function defined on $\mathbb{R}_+$ by $y \mapsto v^p$ is concave. Using then the Taylor expansion at order 2 of the function V, we have, for every $x, y \in \mathbb{R}^d$,

$$\begin{aligned}V^p(y) - V^p(x) \leqslant& pV^{p-1}(x)\big(V(y) - V(x)\big) \\ \leqslant& pV^{p-1}(x)\big(\langle\nabla V(x), y-x\rangle + \frac{1}{2}\|D^2V\|_\infty|y-x|^2\big).\end{aligned}$$

Using this inequality with $x = \bar{X}_{\Gamma_n}$ and $y = \bar{X}_{\Gamma_{n+1}} = \bar{X}_{\Gamma_n} + \Delta\bar{X}^1_{n+1} + \Delta\bar{X}^2_{n+1}$, with notations (25), we derive

$$\begin{aligned}&V^p(\bar{X}_{\Gamma_n} + \Delta\bar{X}_{n+1}) - V^p(\bar{X}_{\Gamma_n}) \\ &\quad\leqslant pV^{p-1}(\bar{X}_{\Gamma_n})\langle\nabla V(\bar{X}_{\Gamma_n}), \Delta\bar{X}^1_{n+1} + \Delta\bar{X}^2_{n+1}\rangle \\ &\qquad + \frac{1}{2}pV^{p-1}(\bar{X}_{\Gamma_n})\|D^2V\|_\infty\big(|\Delta\bar{X}^1_{n+1}|^2 + |\Delta\bar{X}^2_{n+1}|^2 + 2\langle\Delta\bar{X}^1_{n+1}, \Delta\bar{X}^2_{n+1}\rangle\big).\end{aligned}$$

It follows that

$$\mathbb{E}[\exp(\lambda V^p(\bar{X}_{\Gamma_{n+1}}))|\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \leqslant H_{\gamma_{n+1}}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})L_{\gamma_{n+1}}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})$$

with, for every $x \in \mathbb{R}^d$, every $z \in \{1, \dots, M_0\}$ and every $\gamma \in \mathbb{R}^*_+$,

$$\begin{aligned}H_\gamma(x, z) =& \exp(\lambda V^p(x) + \gamma\lambda pV^{p-1}(x)\langle\nabla V(x), b(x, z)\rangle \\ &+ \gamma^2\frac{1}{2}\lambda p\|D^2V\|_\infty V^{p-1}(x)|b(x, z)|^2)\end{aligned}$$

and

$$L_\gamma(x,z) = \mathbb{E}[\exp(\sqrt{\gamma}\lambda p V^{p-1}(x)\langle \nabla V(x) + \gamma \|D^2 V\|_\infty b(x,z), \sigma(x,z)U\rangle + \frac{1}{2}\gamma\lambda p \|D^2 V\|_\infty V^{p-1}(x)|\sigma(x,z)U|^2)]$$

where $U \sim \mathcal{N}(0, I_d)$. In order to compute $L_\gamma(x,z)$, we use Lemma 3.1 (see (47)) with parameters $h = C_\sigma(x,z)^{-1}\gamma\lambda p$, $v = \sqrt{C_\sigma(x)\lambda p}V^{p-1}(x)\sigma^*(x,z)(\nabla V(x) + \gamma\|D^2V\|_\infty b(x,z))$ and the matrix

$$\Sigma(x,z) = I_d - \|D^2V\|_\infty C_\sigma(x,z)V^{p-1}(x)\sigma^*\sigma(x,z)$$

where $\inf_{x\in\mathbb{R}^d}\inf_{z\in\{1,\dots,M_0\}} C_\sigma(x,z) > 0$ and $\Sigma(x,z) \in\in \mathcal{S}^d_{+,*}$. It follows from (47) and $h/(2-2h) \leqslant h$ for $h \in (0,1/2]$, that for every $\gamma \leqslant \inf_{x\in\mathbb{R}^d}\inf_{z\in\{1,\dots,M_0\}} C_\sigma(x,z)/(2\lambda p)$,

$$\begin{aligned} L_\gamma(x,z) &\leqslant \exp\Big(\frac{\gamma\lambda p C_\sigma(x,z)^{-1}}{2(1-\gamma\lambda p C_\sigma(x,z)^{-1})}|v|^2 - \frac{1}{2}\gamma\lambda p C_\sigma(x,z)^{-1}\ln(\det(\Sigma(x,z)))\Big) \\ &\leqslant \exp\Big(\gamma\lambda p C_\sigma(x,z)^{-1}|v|^2 - \frac{1}{2}\gamma\lambda p C_\sigma(x,z)^{-1}\ln(\det(\Sigma(x,z)))\Big). \end{aligned}$$

At this point, we focus on the first term inside the exponential. We have

$$\begin{aligned} |v|^2 \leqslant & C_\sigma(x,z)\lambda p V^{2p-2}(x)\big(\langle\sigma\sigma^*(x,z)\nabla V(x), \nabla V(x)\rangle \\ & + \mathrm{Tr}[\sigma\sigma^*(x,z)](\gamma\|D^2V\|_\infty 2\langle\nabla V(x), b(x,z)\rangle + \gamma^2\|D^2V\|^2_\infty |b(x,z)|^2)\big) \end{aligned}$$

Using $\mathfrak{B}(\phi)$ (see (28)), (37) and $\mathcal{R}_{p,\lambda}(\alpha,\beta,\phi,V)$ (see (38)), it follows that there exists $\bar{C} > 0$ such that

$$H_\gamma(x,z)L_\gamma(x,z) \leqslant \exp\big(\lambda V^p(x) + \gamma\lambda p V^{p-1}(x)(\beta - \alpha\phi\circ V(x)) + \bar{C}\gamma^2 V^{p-1}(x)\phi\circ V(x)\big)$$

which can be rewritten

$$\begin{aligned} H_\gamma(x,z)L_\gamma(x,z) \leqslant \exp\Big(\Big(1 - \gamma p\alpha\frac{\phi\circ V(x)}{V(x)}\Big)\lambda V^p(x) \\ + \gamma p\alpha\frac{\phi\circ V(x)}{V(x)}V^p(x)\Big(\frac{\lambda\beta}{\alpha\phi\circ V(x)} + \gamma\bar{C}/(\alpha p)\Big)\Big). \end{aligned}$$

Using the convexity of the exponential function, we have for every $\gamma p\alpha C_\phi < 1$,

$$\begin{aligned} H_\gamma(x,z)L_\gamma(x,z) \leqslant \exp\big(\lambda V^p(x)\big) - \gamma p\alpha\frac{\phi\circ V(x)}{V(x)}\exp\big(\lambda V^p(x)\big) \\ + \gamma p\alpha\frac{\phi\circ V(x)}{V(x)}\exp\Big(V^p(x)\Big(\frac{\lambda\beta}{\alpha\phi\circ V(x)} + \gamma\bar{C}/(\alpha p)\Big)\Big). \end{aligned}$$

It remains to study the last term of the *r.h.s* of the above inequality. The function defined on $[v_*, +\infty)$ by $v \mapsto \exp(v^p(\frac{\lambda\beta}{\alpha\phi(v)} + \gamma\bar{C}/(\alpha p)))$ is continuous and locally bounded. Moreover, by $\mathcal{R}_{p,\lambda}(\alpha,\beta,\phi,V)$ (see (38)), we have $\liminf_{v\to+\infty}\phi(v) > \beta_+/\alpha$. Hence, there exists $\theta \in (0,1)$ and $v_\theta \geqslant v_*$ such that $\phi(v) \geqslant \beta_+/(\alpha\theta)$ for every $v \geqslant v_\theta$. Consequently, as soon as $\gamma < \theta\lambda\alpha p/\bar{C}$, for every $\widetilde{\alpha} \in (0,\alpha)$ there exists $\widetilde{\beta} \geqslant 0$ such that

$$\begin{aligned} \frac{\phi\circ V(x)}{V(x)}\exp\Big(V^p(x)\Big(\frac{\lambda\beta}{\alpha\phi\circ V(x)} + \gamma\bar{C}/(\alpha p)\Big)\Big) \leqslant & \frac{\widetilde{\beta}}{\alpha}\frac{\exp(\lambda V^p(x))}{V(x)} \\ & + \frac{\alpha - \widetilde{\alpha}}{\alpha}\frac{\phi\circ V(x)}{V(x)}\exp(\lambda V^p(x)) \end{aligned}$$

and the proof of the recursive control (48) is completed. Finally (49) follows from (19), which follow from the equation above, and Lemma 2.2. □

3.2.2 Infinitesimal control

Proposition 3.3. *Suppose that the sequence $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_{\mathcal{N},2}(U)$ (see (29)). Also assume that for every $z \in \{1,\dots,M_0\}$, $b(.,z)$ and $\sigma(.,z)$ have sublinear growth and that $\sup_{n\in\mathbb{N}^*} \nu_n^\eta(\mathrm{Tr}[\sigma\sigma^*]) < +\infty$, a.s.*

Then, $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ (see (7)) is fulfilled.

Proof. First we recall that $\mathcal{D}(A)_0 = \{f : \mathbb{R}^d \times \{1,\dots,M_0\}, \forall\, z \in \{1,\dots,M_0\}, f(.,z) \in \mathcal{C}_K^2(\mathbb{R}^d)\}$ and we write, for $f \in \mathcal{D}(A)_0$,

$$\begin{aligned} f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}) =& f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) \\ &+ f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - f(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}). \end{aligned}$$

We study the first term of the *r.h.s.* of the above equation. Since U and ζ are independent, we have, with notation (25),

$$\mathbb{E}\big[f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}, \Delta\bar{X}_{n+1}\big] = \gamma_{n+1} \sum_{z=1}^{M_0} \big(q_{\zeta_{\Gamma_n},z} + o(\gamma_{n+1})\big) f(\bar{X}_{\Gamma_{n+1}}, z).$$

Using Taylor expansions at order one and two, for every $z \in \{1,\dots,M_0\}$ and the fact that the sequence $(U_n)_{n\in\mathbb{N}^*}$ is *i.i.d.*, we obtain

$$\begin{aligned} &\mathbb{E}\big[f(\bar{X}_{\Gamma_{n+1}}, z) - f(\bar{X}_{\Gamma_n}, z) | \bar{X}_{\Gamma_n} = x, \zeta_{\Gamma_n}\big] \\ &\quad = \mathbb{E}\big[f(\bar{X}_{\Gamma_n} + \Delta\bar{X}^1_{n+1}, z) - f(\bar{X}_{\Gamma_n}, z) | \bar{X}_{\Gamma_n} = x, \zeta_{\Gamma_n}\big] \\ &\qquad + \mathbb{E}\big[f(\bar{X}_{\Gamma_{n+1}}, z) - f(\bar{X}_{\Gamma_n} + \Delta\bar{X}^1_{n+1}, z) | \bar{X}_{\Gamma_n} = x, \zeta_{\Gamma_n}\big] \\ &\quad \leqslant \int_0^1 |\nabla_x f(x + \theta b(x,\zeta_{\Gamma_n})\gamma_{n+1}, z)||b(x,\zeta_{\Gamma_n})\gamma_{n+1}| d\theta \\ &\qquad + \int_0^1 \|D_x^2 f(x + b(x,\zeta_{\Gamma_n})\gamma_{n+1} + \theta\sigma(x,\zeta_{\Gamma_n})\sqrt{\gamma_{n+1}}u, z)\| |\sqrt{\gamma_{n+1}}\sigma(x,\zeta_{\Gamma_n})u|^2 d\theta \widetilde{\mathbb{P}}_U(du) \end{aligned}$$

where $\widetilde{\mathbb{P}}_U$ denotes the distribution of U_1. Combining the two last inequalities, we derive

$$\begin{aligned} &\gamma_{n+1}^{-1}\mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ &\qquad \leqslant \sum_{z=1}^{M_0} q_{\zeta_{\Gamma_n},z} f(\bar{X}_{\Gamma_n}, z) + o(\gamma_{n+1})\|f\|_\infty \\ &\qquad\quad + \sum_{z=1}^{M_0} \big(|q_{\zeta_{\Gamma_n},z}| + o(\gamma_{n+1})\big)\big(\Lambda_{f,1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}, \gamma_{n+1})|b(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})| \\ &\qquad\qquad\qquad + \Lambda_{f,2}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}, \gamma_{n+1})\,\mathrm{Tr}[\sigma\sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})]\big). \end{aligned}$$

We study each term in the *r.h.s.* of the above inequality. First, we have $\Lambda_{f,1}(x,z,\gamma) = |b(x,z)|\widetilde{\mathbb{E}}[\widetilde{\Lambda}_{f,1}(x,z,\gamma)]$ where $\widetilde{\Lambda}_{f,1}(x,z,\gamma) = \widetilde{\mathcal{R}}_{f,1}(x,z,\gamma,\Theta)$ with $\Theta \sim \mathcal{U}_{[0,1]}$ under $\widetilde{\mathbb{P}}$, and

$$\begin{aligned} \widetilde{\mathcal{R}}_{f,1} \quad : \mathbb{R}^d \times \{1,\dots,M_0\} \times \mathbb{R}_+ \times [0,1] \quad &\to \quad \mathbb{R}_+ \\ (x,z,\gamma,\theta) \quad &\mapsto \quad \gamma \sum_{w=1}^{M_0} |\nabla_x f(x + \theta b(x,z)\gamma, w)|. \end{aligned}$$

We are going to prove that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (see (8)) holds.

Since b has sublinear growth w.r.t. its first variable, there exists $C_b \geqslant 0$ such that $|b(x,z)| \leqslant C_b(1+|x|)$ for every $x \in \mathbb{R}^d$ and $z \in \{1,\dots,M_0\}$. Therefore, since f has a compact support, it follows

that there exists $\gamma_0 > 0$ and $R > 0$ such that we have $\sup_{|x|>R,z\in\{1,\dots,M_0\}} \sup_{\gamma\leqslant\underline{\gamma}} \widetilde{\mathcal{R}}_{f,1}(x,z,\gamma,\theta) = 0$ for every $\theta \in [0,1]$ which implies $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I)(ii).
Since $\nabla_x f$ is bounded, it is straightforward that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (i) holds.
Finally, b is locally bounded and defining and $g_1(x,z) = \mathbb{1}_{x\leqslant R}|b(x,z)|$, the couple $(\widetilde{\Lambda}_{f,1}, g_1)$ satisfies $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I).

Now, we have $\Lambda_{f,2}(x,z,\gamma) = g_2(x,z)\widetilde{\mathbb{E}}[\widetilde{\Lambda}_{f,2}(x,z,\gamma)]$ where $\widetilde{\Lambda}_{f,2}(x,z,\gamma) = \widetilde{\mathcal{R}}_{f,2}(x,z,\gamma,U,\Theta)$ with $U \sim \mathbb{P}_U$, $\Theta \sim \mathcal{U}_{[0,1]}$ under $\widetilde{\mathbb{P}}$ and $g_2(x,z) = \mathrm{Tr}[\sigma\sigma^*(x,z)]$ and

$$\begin{aligned}\widetilde{\mathcal{R}}_{f,2} \quad &: \mathbb{R}^d \times \{1,\dots,M_0\} \times \mathbb{R}_+ \times \mathbb{R}^d \times [0,1] \;\to\; \mathbb{R}_+ \\ &\qquad\qquad (x,z,\gamma,u,\theta) \;\mapsto\; \widetilde{\mathcal{R}}_{f,2}(x,z,\gamma,u,\theta),\end{aligned}$$

with
$$\widetilde{\mathcal{R}}_{f,2}(x,z,\gamma,u,\theta) = |\sqrt{\gamma}u|^2 \sum_{w=1}^{M_0} \|D_x^2 f(x + b(x,z)\gamma + \theta\sigma(x,z)\sqrt{\gamma}u, w)\|.$$

We are going to prove that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (see (8)) holds for the couple $(\widetilde{\Lambda}_{f,2}, g_2)$. We fix $u \in \mathbb{R}^N$ and $\theta \in [0,1]$.

Since the functions b and σ have sublinear growth, there exists $C_{b,\sigma} \geqslant 0$ such that $|b(x,z)| + |\sigma(x,z)| \leqslant C_{b,\sigma}(1+|x|)$ for every $x \in \mathbb{R}^d$ and $z \in \{1,\dots,M_0\}$. Therefore, since f has compact support, there exists $\underline{\gamma}(u,\theta) > 0$ and $R > 0$ such that

$$\sup_{|x|>R,z\in\{1,\dots,M_0\}} \sup_{\gamma\leqslant\underline{\gamma}(u,\theta)} |\widetilde{\mathcal{R}}_{f,2}(x,z,\gamma,u,\theta)| = 0.$$

It follows that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I)(ii) holds.
Moreover since $D_x^2 f$ is bounded, it is straightforward that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) (i) is also satisfied.
Finally, we recall that $\sup_{n\in\mathbb{N}^*} \nu_n^\eta(\mathrm{Tr}[\sigma\sigma^*]) < +\infty$, $a.s.$ and U is bounded in L^2 and then $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) holds for $(\widetilde{\Lambda}_{f,2}, g_2)$.

Moreover, it is straightforward to show that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ II) (see 9)) holds for every couple of functions of the form $(\underset{n\to+\infty}{o}(\gamma_{n+1})\|f\|_\infty, 1)$ which concludes the study of the first term.

It remains to study $\mathbb{E}[f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - f(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}]$. Using once again Taylor expansions at order one and two, we derive

$$\begin{aligned}\gamma_{n+1}^{-1}\Big(&\mathbb{E}\big[f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - f(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|\bar{X}_{\Gamma_n} = x, \zeta_{\Gamma_n} = z\big] \\ &\qquad - \langle\nabla_x f(x,z), b(x,z)\rangle - \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^*)_{i,j}(x,z)\frac{\partial^2 f}{\partial x_i \partial x_j}(x,z)\Big) \\ &\leqslant \int_0^1 |\nabla_x f(x + \theta b(x,z)\gamma_{n+1}, z) - \nabla_x f(x)||b(x,z)|d\theta \\ &\quad + \int_0^1 \|D_x^2 f(x + b(x,z)\gamma_{n+1} + \theta\sigma(x,z)\sqrt{\gamma_{n+1}}u, z) \\ &\qquad\qquad - D_x^2 f(x)\||\sigma(x,z)u|^2 d\theta p_U(du).\end{aligned}$$

Using a similar reasoning as before, one can show that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I) holds for $(\widetilde{\Lambda}_{f,3}, g_1)$ and $(\widetilde{\Lambda}_{f,4}, g_2)$ where $\widetilde{\Lambda}_{f,3}(x,z,\gamma) = \widetilde{\mathcal{R}}_{f,3}(x,z,\gamma,\Theta)$ and $\widetilde{\Lambda}_{f,4}(x,z,\gamma) = \widetilde{\mathcal{R}}_{f,4}(x,z,\gamma,U,\Theta)$ with $U \sim p_U$ and $\Theta \sim \mathcal{U}_{[0,1]}$ under $\widetilde{\mathbb{P}}$,

$$\begin{aligned}\widetilde{\mathcal{R}}_{f,3} \quad &: \mathbb{R}^d \times \{1,\dots,M_0\} \times \mathbb{R}_+ \times [0,1] \;\to\; \mathbb{R}_+ \\ &\qquad\qquad (x,z,\gamma,\theta) \;\mapsto\; |\nabla_x f(x + \theta b(x,z)\gamma, z) - \nabla_x f(x,z)|,\end{aligned}$$

and

$$\begin{aligned}\widetilde{\mathcal{R}}_{f,4} \quad &: \mathbb{R}^d \times \{1,\dots,M_0\} \times \mathbb{R}_+ \times \mathbb{R}^d \times [0,1] \quad \to \quad \mathbb{R}_+ \\ &\qquad\qquad\qquad\qquad\qquad\qquad (x,z,\gamma,u,\theta) \quad \mapsto \quad \widetilde{\mathcal{R}}_{f,4}(x,z,\gamma,u,\theta),\end{aligned}$$

with

$$\widetilde{\mathcal{R}}_{f,4}(x,z,\gamma,u,\theta) = \|D_x^2 f(x + b(x,z)\gamma + \theta\sigma(x,z)\sqrt{\gamma}u, z) - D_x^2 f(x)\||u|^2.$$

We gather all the terms together noticing that $\widetilde{\Lambda}_{f,q} = \widetilde{\Lambda}_{-f,q}$, $q \in \{1,\dots,4\}$, and the proof is completed. □

3.2.3 Proof of Growth control and step-weight assumptions

Test functions with polynomial growth.

Lemma 3.2. *Let $p \geqslant 1$, $a \in (0,1]$, $\rho \in [1,2]$, $s \geqslant 1$ and let $\psi_p(v) = v^p$ and $\phi(v) = v^a$. We suppose that the sequence $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_{(\rho/2)\vee(p\rho/s)}(U)$ (see (30)). Then, for every $n \in \mathbb{N}$, we have*

$$\begin{aligned}\forall f \in \mathcal{D}(A)_0, \quad \mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}^1_{\Gamma_n}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ \leqslant C_f \gamma_{n+1}^{\rho/2} 1 \vee \mathrm{Tr}[\sigma\sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})]^{\rho/2},\end{aligned} \tag{50}$$

with notations (25). In other words, we have $\mathcal{GC}_Q(\mathcal{D}(A)_0, 1 \vee \mathrm{Tr}[\sigma\sigma^*]^{\rho/2}, \rho, \epsilon_{\mathcal{I}})$ (see (10)) with $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho/2}$ for every $\gamma \in \mathbb{R}_+$.
Moreover, if (26), (31) and $\mathfrak{B}(\phi)$ (see (28)) hold and $p\rho/s \leqslant p + a - 1$, then, for every $n \in \mathbb{N}$, we have

$$\mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - V^{p/s}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \leqslant C\gamma_{n+1}^{\rho/2} V^{p+a-1}(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}), \tag{51}$$

In other words, we have $\mathcal{GC}_Q(V^{p/s}, V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$ (see (10)) with $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho/2}$ for every $\gamma \in \mathbb{R}_+$.

Proof. We begin by noticing that, with notations (25),

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}^1_{\Gamma_{n+1}}| \leqslant C\gamma_{n+1}^{1/2} \mathrm{Tr}[\sigma\sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})]^{1/2} |U_{n+1}|$$

Let $f \in \mathcal{D}(A)_0$. We employ this estimation and since for $f \in \mathcal{D}(A)_0$ then $f(.,z)$ is uniformly Lipschitz in $z \in \{1,\dots,M_0\}$, it follows that

$$\mathbb{E}\big[|f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n}) - f(\bar{X}^1_{\Gamma_n}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big] \leqslant C\gamma_{n+1}^{\rho/2} |\sigma\sigma^*(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})|^{\rho/2}.$$

Moreover,

$$\begin{aligned}&\mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_{n+1}}) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ &= \gamma_{n+1} \sum_{z=1}^{M_0} (q_{\zeta_{\Gamma_n}, z} + \underset{n\to+\infty}{o}(\gamma_{n+1})) \mathbb{E}[|f(\bar{X}_{\Gamma_{n+1}}, z) - f(\bar{X}_{\Gamma_{n+1}}, \zeta_{\Gamma_n})|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \\ &\leqslant C\gamma_{n+1} \|f\|_\infty^\rho.\end{aligned}$$

Gathering both terms concludes the study for $f \in \mathcal{D}(A)_0$.
We focus now on the case $f = V^{p/s}$. We notice that $\mathfrak{B}(\phi)$ (see (28)) implies that for any $n \in \mathbb{N}$,

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}| \leqslant C\gamma_{n+1}^{1/2} \sqrt{\phi \circ V(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})}(1 + |U_{n+1}|).$$

We rewrite the term that we study as follows

$$V^{p/s}(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_{n+1}})-V^{p/s}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})=V^{p/s}(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n})-V^{p/s}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})+V^{p/s}(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_{n+1}})-V^{p/s}(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n}).$$

We study the first term of the *r.h.s.* of the equality above. Using the following inequality

$$\forall u,v\in\mathbb{R}_+,\forall\alpha\geqslant 1,\qquad |u^\alpha-v^\alpha|\leqslant\alpha 2^{\alpha-1}(v^{\alpha-1}|u-v|+|u-v|^\alpha),\tag{52}$$

with $\alpha=2p/s$, it follows from (26) that $\sqrt{V(.,z)}$ is Lipschitz uniformly in $z\in\{1,\dots,M_0\}$ and

$$\begin{aligned}\big|V^{p/s}(\bar{X}_{\Gamma_{n+1}},z)-&V^{p/s}(\bar{X}_{\Gamma_n},z)\big|\\&\leqslant 2^{2p/s}p/s\big(V^{p/s-1/2}(\bar{X}_{\Gamma_n},z)\big|\sqrt{V}(\bar{X}_{\Gamma_{n+1}},z)-\sqrt{V}(\bar{X}_{\Gamma_n},z)\big|\\&\quad+|\sqrt{V}(\bar{X}_{\Gamma_{n+1}},z)-\sqrt{V}(\bar{X}_{\Gamma_n},z)|^{2p/s}\big)\\&\leqslant 2^{2p/s}p/s\big([\sqrt{V}]_1V^{p/s-1/2}(\bar{X}_{\Gamma_n},z)|\bar{X}_{\Gamma_{n+1}}-\bar{X}_{\Gamma_n}|\\&\quad+[\sqrt{V}]_1^{2p/s}|\bar{X}_{\Gamma_{n+1}}-\bar{X}_{\Gamma_n}|^{2p/s}\big).\end{aligned}$$

We use the assumption $p\rho/s\leqslant p+a-1$, $a\in(0,1]$, $p\geqslant 1$ and it follows from $\mathfrak{B}(\phi)$ (see (28)) and (31) when $z\neq\zeta_{\Gamma_n}$, that

$$\mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}},z)-V^{p/s}(\bar{X}_{\Gamma_n},z)|^\rho|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}]\leqslant C\gamma_{n+1}^{\rho/2}V^{p+a-1}(\bar{X}_{\Gamma_n},z).$$

In order to treat the first term, we put $z=\zeta_{\Gamma_n}$ in this estimation. It remains to study the second term. We notice that since $p\rho/s\leqslant p+a-1$, it is straightforward from the previous inequality that for every $z\in\{1,\dots,M_0\}$, we have

$$\mathbb{E}\big[V^{p\rho/s}(\bar{X}_{\Gamma_{n+1}},z)|\bar{X}_{\Gamma_n},z\big]\leqslant CV^{p+a-1}(\bar{X}_{\Gamma_n},z).$$

We focus on the term to estimate and using this inequality, we obtain

$$\begin{aligned}\mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}},&\zeta_{\Gamma_{n+1}})-V^{p/s}(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n})|^\rho|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}]\\&=\gamma_{n+1}\sum_{z=1}^{M_0}\big(q_{\zeta_{\Gamma_n},z}+o(\gamma_{n+1})\big)\times\mathbb{E}[|V^{p/s}(\bar{X}_{\Gamma_{n+1}},z)-V^{p/s}(\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n})|^\rho|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}]\\&\leqslant C\gamma_{n+1}\sum_{z=1}^{M_0}\big(|q_{\zeta_{\Gamma_n},z}|+\gamma_{n+1}\big)\big(V^{p+a-1}(\bar{X}_{\Gamma_n},z)+V^{p+a-1}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n})\big)\\&\leqslant C\gamma_{n+1}V^{p+a-1}(\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}),\end{aligned}$$

where the last inequality follows from (31). We rearrange the terms and the proof of (51) is completed. □

Test functions with exponential growth.

Lemma 3.3. *Let $p\in[0,1]$, $\lambda\geqslant 0$, $s\geqslant 1$, $\rho\in[1,2]$ and let $\phi:[v_*,\infty)\to\mathbb{R}_+$ be a continuous function such that $C_\phi:=\sup_{v\in[v_*,\infty)}\phi(v)/v<+\infty$ and let $\psi(v)=\exp(\lambda v^p)$. We assume that $\rho<s$, (26), (36) and $\mathfrak{B}(\phi)$ (see (28)) hold, and that*

$$\forall\widetilde{\lambda}\leqslant\lambda,\exists C\geqslant 0,\forall n\in\mathbb{N},\qquad\mathbb{E}[\exp(\widetilde{\lambda}V^p(\bar{X}_{\Gamma_{n+1}}))|\bar{X}_{\Gamma_n},\zeta_{\Gamma_n}]\leqslant C\exp(\widetilde{\lambda}V^p(\bar{X}_{\Gamma_n})).\tag{53}$$

Then, for every $n \in \mathbb{N}$, we have

$$\mathbb{E}[|\exp(\lambda/sV^p(\bar{X}_{\Gamma_{n+1}})) - \exp(\lambda/sV^p(\bar{X}_{\Gamma_n}))|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}] \leqslant C\gamma_{n+1}^{\rho(p\wedge 1/2)} \frac{\phi \circ V(\bar{X}_{\Gamma_n})}{V(\bar{X}_{\Gamma_n})} \exp(\lambda V^p(\bar{X}_{\Gamma_n})). \tag{54}$$

In other words, we have $\mathcal{GC}_Q(\exp(\lambda/sV^p), V^{-1}.\phi\circ V.\exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ (see (10)) and $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho(p\wedge 1/2)}$ for every $\gamma \in \mathbb{R}_+$.

Proof. When $p = 0$ the result is straightforward. We begin by noticing that $\mathfrak{B}(\phi)$ (see (28)) implies that for every $n \in \mathbb{N}$,

$$|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}| \leqslant C\gamma_n^{1/2}\sqrt{\phi \circ V(\bar{X}_{\Gamma_n})}(1 + |U_{n+1}|^2).$$

Let $x, y \in \mathbb{R}^d$. From Taylor expansion at order one, we derive,

$$\begin{aligned}&\big|\exp(\lambda/sV^p(y)) - \exp(\lambda/sV^p(x))\big| \\ &\quad\leqslant \frac{\lambda}{s}\big(\exp(\lambda/sV^p(y)) + \exp(\lambda/sV^p(x))\big)\big|V^p(y) - V^p(x)\big|.\end{aligned} \tag{55}$$

First, let $p \in [1/2, 1]$ we use (52) with $\alpha = 2p$ and since $\sqrt{V}$ is Lipschitz, we obtain

$$\begin{aligned}|V^p(y) - V^p(x)| &\leqslant 2^{2p}p(V^{p-1/2}(x)|\sqrt{V}(y) - \sqrt{V}(x)| + |\sqrt{V}(y) - \sqrt{V}(x)|^{2p}) \\ &\leqslant 2^{2p}p(V^{p-1/2}(x)[\sqrt{V}]_1|y - x| + [\sqrt{V}]_1^{2p}|y - x|^{2p}).\end{aligned}$$

When $p \in [0, 1/2]$. We notice that from (26), the function V^p is α-Hölder for every $\alpha \in [2p, 1]$ (see Lemma 3. in [20]) and then V^p is $2p$-Hölder that is

$$|V^p(y) - V^p(x)| \leqslant [\sqrt{V}]_{2p}|y - x|^{2p}.$$

We focus on the case $p \in [1/2, 1]$. When $p \leqslant 1/2$ the proof is similar and left to the reader. Using (55), we derive from the Hölder inequality that

$$\begin{aligned}&\mathbb{E}\big[|\exp(\lambda/sV^p(\bar{X}_{\Gamma_{n+1}})) - \exp(\lambda/sV^p(\bar{X}_{\Gamma_n}))|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big] \\ &\quad\leqslant C\exp(\lambda\rho/sV^p(\bar{X}_{\Gamma_n}))\Big(V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n})\mathbb{E}\big[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big] \\ &\qquad\qquad + \mathbb{E}\big[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p\rho} | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big]\Big) \\ &\qquad + C\mathbb{E}\Big[\exp(\lambda\rho/sV^p(\bar{X}_{\Gamma_{n+1}}))\big(V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n})|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^\rho \\ &\qquad\qquad + |\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p\rho}\big)\Big|\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\Big] \\ &\quad\leqslant C\exp(\lambda\rho/sV^p(\bar{X}_{\Gamma_n}))\Big(V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n})\mathbb{E}\big[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^\rho | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big] \\ &\qquad\qquad + \mathbb{E}\big[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p\rho} | \bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big]\Big) \\ &\qquad + CV^{p\rho-\rho/2}(\bar{X}_{\Gamma_n})\mathbb{E}\big[\exp(\lambda\rho\theta/sV^p(\bar{X}_{\Gamma_{n+1}}))|\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big]^{1/\theta} \\ &\qquad\qquad \times \mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{\rho\theta/(\theta-1)}|\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big]^{(\theta-1)/\theta} \\ &\qquad + C\mathbb{E}\big[\exp(\lambda\rho\theta/sV^p(\bar{X}_{\Gamma_{n+1}}))|\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big]^{1/\theta} \\ &\qquad\qquad \times \mathbb{E}[|\bar{X}_{\Gamma_{n+1}} - \bar{X}_{\Gamma_n}|^{2p\rho\theta/(\theta-1)}|\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big]^{(\theta-1)/\theta},\end{aligned}$$

for every $\theta > 1$. From (53) and since $\rho < s$, we take $\theta \in (1, \rho/s]$ and we get

$$\mathbb{E}\big[\exp(\lambda\rho\theta/sV^p(\bar{X}_{\Gamma_{n+1}})|\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n}\big] \leqslant C\exp(\lambda\theta\rho/sV^p(\bar{X}_{\Gamma_n}, \zeta_{\Gamma_n})).$$

Rearranging the terms and since $\rho < s$, we conclude from $\mathfrak{B}(\phi)$ (see (28)) that

$$\begin{aligned}\mathbb{E}[|\exp(\lambda/sV^p(\bar{X}_{\Gamma_{n+1}}))-\exp(\lambda/sV^p(\bar{X}_{\Gamma_{n+1}}))|^\rho|\bar{X}_{\Gamma_{n+1}},\zeta_{\Gamma_n}]&\\ \leqslant C\exp(\lambda\rho/sV^p(\bar{X}_{\Gamma_n}))\big(\gamma_n^{\rho/2}V^{p\rho-\rho/2}(\bar{X}_{\Gamma_n})|\phi\circ V(\bar{X}_{\Gamma_n})|^{\rho/2}+\gamma_n^{p\rho}|\phi\circ V(\bar{X}_{\Gamma_n})|^{p\rho}\big)&\\ \leqslant C\gamma_n^{\rho/2}\frac{\phi\circ V(\bar{X}_{\Gamma_n})}{V(\bar{X}_{\Gamma_n})}\exp(\lambda V^p(\bar{X}_{\Gamma_n})),&\end{aligned}$$

and the proof of (55) is completed. □

3.2.4 Proof of Theorem 3.1

This result follows from Theorem 2.3 and Theorem 2.4. The proof consists in showing that the assumptions from those theorems are satisfied.

Step 1. Mean-reverting recursive control First, we show that $\mathcal{RC}_{Q,V}(\psi_p,\phi,p\widetilde{\alpha},p\beta,s)$ (see (4)) is satisfied for every $\widetilde{\alpha}\in(0,\alpha)$ and $s\geqslant 1$ such that $p/s+a-1>0$.

Since (26), $\mathfrak{B}(\phi)$ (see (28)) and $\mathcal{R}_p(\alpha,\beta,\phi,V,s)$ (see (32)) hold, it follows from Proposition 3.1 that $\mathcal{RC}_{Q,V}(\psi_p,\phi,p\widetilde{\alpha},p\beta,s)$ (see (4)) is satisfied for every $\widetilde{\alpha}\in(0,\alpha)$ and $s\geqslant 1$ such that $p/s+a-1>0$ since we have then $\liminf_{v\to+\infty}\phi(v)>\beta/\widetilde{\alpha}$ and $\lim_{v\to+\infty}v^{-1}\phi(v)\psi_p(v)^{1/s}=+\infty$.

Step 2. Step-weight assumption Now, we show that $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (11)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{p+a-1})$ (see (12)) hold.

First we recall that $\mathcal{RC}_{Q,V}(\psi_p,\phi,p\widetilde{\alpha},p\beta,s)$ (see (4)) is satisfied for every $\widetilde{\alpha}\in(0,\alpha)$ and $s\geqslant 1$ such that $p/s+a-1>0$. Then, using $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}})$ (see (20)) with Lemma 2.3 gives $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (11)). Similarly, $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{p+a-1})$ (see (12) follows from $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)) and Lemma 2.3.

Step 3. Growth control assumption Now, we prove $\mathcal{GC}_Q(F,V^{a+p-1},\rho,\epsilon_{\mathcal{I}})$ (see (10)) for $F=\mathcal{D}(A)_0$ and $F=\{V^{p/s}\}$.

This is a consequence of Lemma 3.2. We notice that $\rho\leqslant 2p$ and $\rho/s\leqslant 1$. Consequently $M_{(\rho/2)\vee(p\rho/s)}(U)$ (see (30)) follows from $M_p(U)$. Now, we notice that Lemma 3.2 and the fact that under $\mathfrak{B}(\phi)$ (see (28)) and $p\geqslant 1$, we have $\mathrm{Tr}[\sigma\sigma^*]\leqslant CV^{p+a-1}$, imply that for $F=\mathcal{D}(A)_0$ and $F=\{V^{p/s}\}$, then $\mathcal{GC}_Q(F,V^{a+p-1},\rho,\epsilon_{\mathcal{I}})$ (see (10)) holds.

Step 4. Conclusion

i. The first part of Theorem 3.1 (see (34)) is a consequence of Theorem 2.3. Let us observe that assumptions from Theorem 2.3 indeed hold.

   On the one hand, we observe that from Step 2. and Step 3. the assumptions $\mathcal{GC}_Q(V^{p/s},V^{a+p-1},\rho,\epsilon_{\mathcal{I}})$ (see (10)), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (11)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{p+a-1})$ (see (12)) hold which are the hypothesis from Theorem 2.3 point A. with $g=V^{p+a-1}$.

   On the other hand, from Step 1. the assumption$\mathcal{RC}_{Q,V}(\psi_p,\phi,p\widetilde{\alpha},p\beta,s)$ (see (4)) is satisfied for every $\widetilde{\alpha}\in(0,\alpha)$ and $s\geqslant 1$ such that $p/s+a-1>0$. Moreover, since L_V (see (3)) holds, then the hypothesis from Theorem 2.3 point B. are satisfied.

   We thus conclude from Theorem 2.3 that $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ (built with $(\bar{X}_t)_{t\geqslant 0}$ defined in (24)) is $\mathbb{P}$-*a.s.* tight and (34) holds which concludes the proof of the first part of Theorem 3.1.

ii. Let us now prove the second part of Theorem 3.1 (see (35)) which is a consequence of Theorem 2.4.

On the one hand,we observe that from Step 2. and Step 3. the assumptions $\mathcal{GC}_Q(\mathcal{D}(A)_0, V^{a+p-1}, \rho, \epsilon_{\mathcal{I}})$ (see (10)) and $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p+a-1}, \rho, \epsilon_{\mathcal{I}})$ (see (11)) hold which are the hypothesis from Theorem 2.4 point A. with $g = V^{p+a-1}$.

On the other hand, since $z \in \{1, \dots, M_0\}$, $b(.,z)$ and $\sigma(.,z)$ have sublinear growth and $\mathrm{Tr}[\sigma\sigma^*] \leqslant CV^{p/s+a-1}$, so that $\mathbb{P}$-a.s. $\sup_{n\in\mathbb{N}^*} \nu_n^\eta(\mathrm{Tr}[\sigma\sigma^*]) < +\infty$, it follows from Proposition 3.3 that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ (see (7)) is satisfied. Then, the hypothesis from Theorem 2.4 point B. hold and (35) follows from (18).

3.2.5 Proof of Theorem 3.2

This result follows from Theorem 2.3 and Theorem 2.4. The proof consists in showing that the assumptions from those theorems are satisfied.

Step 1. Mean-reverting recursive control First, we show that for every $\widetilde{\alpha} \in (0, \alpha)$, there exists $\widetilde{\beta} \in \mathbb{R}_+$ such that $\mathcal{RC}_{Q,V}(\widetilde{\psi}, \phi, p\widetilde{\alpha}, p\widetilde{\beta}, s)$ (see (4)) is satisfied for every function $\widetilde{\psi} : [v_*, \infty) \to \mathbb{R}_+$ such that $\widetilde{\psi}(v) = \exp(\widetilde{\lambda}v^p)$ with $\widetilde{\lambda} \leqslant \lambda$ and every $s \geqslant 1$. Notice that this property and the fact that ϕ has sublinear growth imply (53).

We begin by noticing that $\mathcal{R}_{p,\lambda}(\alpha, \beta, \phi, V)$ (see (38)) implies $\mathcal{R}_{p,\widetilde{\lambda}}(\alpha, \beta, \phi, V)$ for every $\widetilde{\lambda} \leqslant \lambda$. Since (26), $\mathfrak{B}(\phi)$ (see (28)), $\mathcal{R}_{p,\lambda}(\alpha, \beta, \phi, V)$ (see (38)) and (37) hold, it follows from Proposition 3.2 with $\lim_{v\to+\infty} \phi(v) = +\infty$, that that for every $\widetilde{\alpha} \in (0, \alpha)$, there exists $\widetilde{\beta} \in \mathbb{R}_+$ such that $\mathcal{RC}_{Q,V}(\widetilde{\psi}, \phi, p\widetilde{\alpha}, p\widetilde{\beta}, s)$ (see (4)) is satisfied for every function $\widetilde{\psi} : [v_*, \infty) \to \mathbb{R}_+$ such that $\widetilde{\psi}(v) = \exp(\widetilde{\lambda}v^p)$ with $\widetilde{\lambda} \leqslant \lambda$ and every $s \geqslant 1$.

Step 2. Step-weight assumption Now, we show that $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{-1}.\phi \circ V.\exp(\lambda V^p), \rho, \widetilde{\epsilon_{\mathcal{I}}})$, $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{-1}.\phi \circ V.\exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ (see (11)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(\exp(\lambda/sV^p))$ (see (12)) hold.

First we recall that that there exists $\widetilde{\alpha} \in (0, \alpha)$ and $\widetilde{\beta} \in \mathbb{R}_+$ such that $\mathcal{RC}_{Q,V}(\psi, \phi, \widetilde{\alpha}, \widetilde{\beta}, s)$ (see (4)) is satisfied for every $s \geqslant 1$. Then, using $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho, \widetilde{\epsilon_{\mathcal{I}}})$ and $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho, \epsilon_{\mathcal{I}})$ (see (20)) with Lemma 2.3 gives $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{-1}.\phi \circ V.\exp(\lambda V^p), \rho, \widetilde{\epsilon_{\mathcal{I}}})$ and $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{-1}.\phi \circ V.\exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ (see (11)). Similarly, $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{-1}.\phi \circ V.\exp(\lambda V^p))$ (see (12) follows from $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)) and Lemma 2.3.

Step 3. Growth control assumption Now, we prove $\mathcal{GC}_Q(F, V^{-1}.\phi \circ V.\exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ (see (10)) for $F = \mathcal{D}(A)_0$ and $F = \{\exp(\lambda/sV^p)\}$.

This is a consequence of Lemma 3.2 and Lemma 3.3. We notice indeed that $\mathfrak{B}(\phi)$ (see (28)) gives $\mathrm{Tr}[\sigma\sigma^*]^{\rho/2} \leqslant (\phi \circ V)^\rho$. Moreover, we have already shown that (53) is satisfied in Step 1. These observations combined with (54) imply that $\mathcal{GC}_Q(\mathcal{D}(A)_0, V^{-1}\phi \circ V\exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ and $\mathcal{GC}_Q(\exp(\lambda/sV^p), V^{-1}.\phi \circ V.\exp(\lambda V^p), \rho, \widetilde{\epsilon_{\mathcal{I}}})$ (see (10)) hold.

Step 4. Conclusion

i. The first part of Theorem 3.2 (see (39)) is a consequence of Theorem 2.3. Let us observe that assumptions from Theorem 2.3 indeed hold.

On the one hand, we observe that from Step 2. and Step 3. the assumptions $\mathcal{GC}_Q(\exp(\lambda/sV^p), V^{-1}\phi\circ V\exp(\lambda V^p), \rho, \widetilde{\epsilon_{\mathcal{I}}})$ (see (10)), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{-1}\phi\circ V\exp(\lambda V^p), \rho, \widetilde{\epsilon_{\mathcal{I}}})$ (see (11)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{-1}\phi\circ V\exp(\lambda V^p))$ (see (12)) hold which are the hypothesis from Theorem 2.3 point A. with $g = V^{-1}\phi \circ V\exp(\lambda V^p)$.

On the other hand, from Step 1. for every $\widetilde{\alpha} \in (0, \alpha)$, there exists $\widetilde{\beta} \in \mathbb{R}_+$ such that $\mathcal{RC}_{Q,V}(\psi, \phi, p\widetilde{\alpha}, p\widetilde{\beta}, s)$ (see (4)) is satisfied for every $s \geqslant 1$. Moreover, since L_V (see (3)) holds, then the hypothesis from Theorem 2.3 point B. are satisfied.

We thus conclude from Theorem 2.3 that $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ (built with $(\bar{X}_t)_{t\geqslant 0}$ defined in (24)) is $\mathbb{P}$-$a.s.$ tight and (39) holds which concludes the proof of the first part of Theorem 3.2.

ii. Let us now prove the second part of Theorem 3.2 (see (40)) which is a consequence of Theorem 2.4.

On the one hand,we observe that from Step 2. and Step 3. the assumptions $\mathcal{GC}_Q(\mathcal{D}(A)_0, V^{-1}\phi\circ V\exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ (see (10)) and $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{-1}\phi\circ V\exp(\lambda V^p), \rho, \epsilon_{\mathcal{I}})$ (see (11)) hold which are the hypothesis from Theorem 2.4 point A. with $g = V^{-1}\phi\circ V\exp(\lambda V^p)$.

On the other hand, since $z\in\{1,\dots,M_0\}$, $b(.,z)$ and $\sigma(.,z)$ have sublinear growth and $\mathrm{Tr}[\sigma\sigma^*]\leqslant CV^{-1}\phi\circ V\exp(\lambda/sV^p)$, so that $\mathbb{P}$-a.s. $\sup_{n\in\mathbb{N}^*}\nu_n^\eta(\mathrm{Tr}[\sigma\sigma^*])<+\infty$, it follows from Proposition 3.3 that $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ (see (7)) is satisfied. Then, the hypothesis from Theorem 2.4 point B. hold and (40) follows from (18).

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A Infinitesimal approximation: Example of a Brownian diffusion and user guide

We consider a scalar Brownian diffusion ($d = q = 1$) with Lipschitz continuous drifts b and diffusion coefficient σ. Hence $E = \mathbb{R}$ and there exists a real constant $C \geqslant 1$ such that $|b(x)| \vee |\sigma(x)| \leqslant C(1+|x|)$ (set e.g. $C = [b]_{\mathrm{Lip}} \vee [\sigma]_{\mathrm{Lip}} \vee |b(0)| \vee |\sigma(0)| \vee 1$). We will show how to check Assumption $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ (see (7)). We define the Euler operator with step $\gamma > 0$ by $\mathfrak{E}_\gamma(x,z) = x + \gamma b(x) + \sqrt{\gamma}\sigma(x)z$ and we consider the Euler scheme with decreasing steps and and Brownian increments. We set $\mathcal{D}(A)_0 = \mathcal{C}^2_K(\mathbb{R},\mathbb{R})$ (twice differentiable functions with compact support). Let $\widetilde{Z} : (\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{P}}) \to \mathbb{R}$ and $\mathcal{N}(0,1)$-distributed random variable so that $\mathcal{Q}_\gamma(f)(x) = \mathbb{E}\big[f\big(\mathfrak{E}_\gamma(x,\widetilde{Z})\big)\big]$. Elementary computations based on a second order Taylor expansion with integral remainder show that, for every $f \in \mathcal{C}^2_K(\mathbb{R},\mathbb{R})$,

$$\widetilde{A}_\gamma f(x) - Af(x) = \widetilde{\mathbb{E}}_{\widetilde{\omega}}\big[\widetilde{\Lambda}_f(x,\gamma,\widetilde{\omega})\big]$$

with (note that $g \equiv 1$)

$$\widetilde{\Lambda}_f(x,\gamma,\widetilde{\omega}) = \frac{(\mathfrak{E}_\gamma(x,\widetilde{Z}) - x)^2}{\gamma} \int_0^1 \Big[f''\big(x + (\mathfrak{E}_\gamma(x,\widetilde{Z}(\widetilde{\omega})) - x)u\big) - f''(x)\Big]\, du.$$

We now check $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ I).
(i). Let K be a compact set of $\mathbb{R}$. For every $\gamma \in (0,\bar{\gamma}]$, $u \in [0,1]$ and $x \in K$,

$$\big|\widetilde{\Lambda}_f(x,\gamma,\widetilde{\omega})\big| \leq 2\big(\bar{\gamma} b_K^2 + \sigma_K^2 \widetilde{Z}(\widetilde{\omega})^2\big) \int_0^1 (2\|f''\|_{\sup}) \wedge w\big(f'', \delta(x,u,\gamma,\widetilde{Z}(\widetilde{\omega}))\big) du$$

where $g_K = \sup_{x\in K} |g(x)|$ and $w(g,\delta)$, $\delta > 0$, denotes the uniform continuity modulus of a function $g : \mathbb{R} \to \mathbb{R}$. One easily checks that, for every $u \in [0,1]$ and $x \in K$

$$0 < \delta(x,u,\gamma,\widetilde{Z}(\widetilde{\omega})) \leqslant \sqrt{\gamma}\, C_{b_K,\sigma_K,\bar{\gamma}}\big(1 + |\widetilde{Z}|\big).$$

Consequently by Lebesgue's dominated convergence theorem

$$\lim_{\gamma\to 0} \widetilde{\mathbb{E}}\Big[\sup_{x\in K} \big|\widetilde{\Lambda}_f(x,\gamma,\widetilde{\omega})\big|\Big] = 0.$$

(ii). Set $\underline{\gamma}(\widetilde{\omega}) = \frac{1}{9(1+|\widetilde{Z}(\widetilde{\omega})|^2)C^2}$. Then, one checks that for every $x \in \mathbb{R}$ and $u \in [0,1]$,

$$\big|x + u\big(\mathfrak{E}_\gamma(x,\widetilde{Z}(\widetilde{\omega})) - x\big)\big| \geqslant |x| - \frac{2}{3}\big(1+|x|\big) = \frac{|x|}{3} - \frac{2}{3}.$$

Hence, for every $\widetilde{\omega}$ such that $\widetilde{Z}(\widetilde{\omega})$ is finite,

$$\lim_{|x|\to+\infty} \sup_{\gamma\in(0,\underline{\gamma}],u\in[0,1]} \big|x + u\big(\mathfrak{E}_\gamma(x,\widetilde{Z}(\widetilde{\omega})) - x\big)\big| = +\infty.$$

Finally, as f'' has a compact support,

$$\sup_{\gamma\in(0,\underline{\gamma}]} \left|\widetilde{\Lambda}_f(x,\gamma,\widetilde{\omega})\right| = 0 \quad \text{for } |x| \text{ large enough.}$$