



**HAL**  
open science

## Some new results related to Lorentz $G\Gamma$ -spaces and interpolation

Irshaad Ahmed, Alberto Fiorenza, Maria Rosaria Formica, Amiran Gogatishvili,  
Jean-Michel Rakotoson

### ► To cite this version:

Irshaad Ahmed, Alberto Fiorenza, Maria Rosaria Formica, Amiran Gogatishvili, Jean-Michel Rakotoson. Some new results related to Lorentz  $G\Gamma$ -spaces and interpolation. *Journal of Mathematical Analysis and Applications*, 2020, 483, pp.123623 -. [10.1016/j.jmaa.2019.123623](https://doi.org/10.1016/j.jmaa.2019.123623). [hal-03488463](https://hal.science/hal-03488463)

**HAL Id: hal-03488463**

**<https://hal.science/hal-03488463v1>**

Submitted on 21 Dec 2021

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY-NC 4.0 - Attribution - Non-commercial use - International License

## SOME NEW RESULTS RELATED TO LORENTZ $G\Gamma$ -SPACES AND INTERPOLATION

IRSHAAD AHMED<sup>1</sup>, ALBERTO FIORENZA<sup>2</sup>, MARIA ROSARIA FORMICA<sup>3</sup>,  
AMIRAN GOGATISHVILI<sup>4</sup> and JEAN MICHEL RAKOTOSON<sup>5\*</sup>

**ABSTRACT.** We compute the  $K$ -functional related to some couple of spaces as small or classical Lebesgue space or Lorentz-Marcinkiewicz spaces completing the results of [12]. This computation allows to determine the interpolation space in the sense of Peetre for such couple. It happens that the result is always a  $G\Gamma$ -space, since this last space covers many spaces. The motivations of such study are various, among them we wish to obtain a regularity estimate for the so called very weak solution of a linear equation in a domain  $\Omega$  with data in the space of the integrable function with respect to the distance function to the boundary of  $\Omega$ .

### 1. Introduction

The present work finds its motivation in the recent results in [13, 11, 23]. The original question comes from an unpublished manuscript by H. Brezis (see comments in [11]) and later presented in [6] (see also the mention made in [27]) concerning the following problem : let  $f$  be given in  $L^1(\Omega, \text{dist}(x, \partial\Omega))$  ( $\Omega$  bounded smooth open set of  $\mathbb{R}^n$ ), then H. Brezis shows the existence and uniqueness of a function  $u \in L^1(\Omega)$  satisfying

$$|u|_{L^1(\Omega)} \leq c|f|_{L^1(\Omega, \text{dist}(x, \partial\Omega))}$$

with

$$GD(\Omega) = \left\{ \begin{array}{l} - \int_{\Omega} u \Delta \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \forall \varphi \in C_0^2(\bar{\Omega}), \\ \text{with } C_0^2(\bar{\Omega}) = \left\{ \varphi \in C^2(\bar{\Omega}), \varphi = 0 \text{ on } \partial\Omega \right\}. \end{array} \right.$$

Therefore, the question of the integrability of the generalized derivative of  $u$  arises in a natural way and was raised already in the note by H. Brezis and developed in [11], [23], [24]. More generally, the question of the regularity of  $u$  is arised, according to  $f$ .

In [10, 13], we have shown the following theorem:

#### **Theorem 1.1.**

Let  $\Omega$  be a bounded open set of class  $C^2$  of  $\mathbb{R}^n$ ,  $|\Omega| = 1$  and  $\alpha \geq \frac{1}{n'}$  where  $n' = \frac{n}{n-1}$ ,

---

*Date:* Received: xxxxxx; Revised: yyyyyy; Accepted: zzzzzz.

\*Corresponding author.

2010 *Mathematics Subject Classification.* Primary 46E30,46B70, Secondary 35J65.

*Key words and phrases.* Grand and Small Lebesgue spaces, classical Lorentz-spaces, Interpolation, very weak solution.

$f \in L^1(\Omega; \delta)$ , with  $\delta(x) = \text{dist}(x; \partial\Omega)$ .

Consider  $u \in L^{n', \infty}(\Omega)$ , the very weak solution (v.w.s.) of

$$-\int_{\Omega} u \Delta \varphi dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in C^2(\bar{\Omega}), \varphi = 0 \text{ on } \partial\Omega. \quad (1.1)$$

Then,

(1) if  $f \in L^1(\Omega; \delta(1 + |\text{Log } \delta|)^\alpha)$  and  $\alpha > \frac{1}{n'}$ :

$$u \in L^{(n', n\alpha - n + 1)}(\Omega) = G\Gamma(n', 1; w_\alpha), \quad w_\alpha(t) = t^{-1}(1 - \text{Log } t)^{\alpha - 1 - \frac{1}{n'}}$$

and

$$\|u\|_{G\Gamma(n', 1; w_\alpha)} \leq K_0 \|f\|_{L^1(\Omega; \delta(1 + |\text{Log } \delta|)^\alpha)} \quad (1.2)$$

(2) if  $f \in L^1(\Omega; \delta(1 + |\text{Log } \delta|)^{\frac{1}{n'}})$  then

$$u \in L^{n'}(\Omega) \text{ and similar estimate as (1.2) holds.}$$

Note that the assumption on the regularity of  $\Omega$ , needed in the proof of Theorem 1.1, is necessary for the development of the theory of very weak solutions; we stress that the estimates in this paper will be obtained following arguments valid regardless of the regularity of  $\Omega$ , which will be definitively dropped in our statements.

The Lorentz  $G\Gamma$ -space is defined as follows :

**Definition 1.2. of Generalized Gamma space with double weights (Lorentz- $G\Gamma$ )**

Let  $w_1, w_2$  be two weights on  $(0, 1)$ ,  $m \in [1, +\infty]$ ,  $1 \leq p < +\infty$ . We assume the following conditions:

- c1) There exists  $K_{12} > 0$  such that  $w_2(2t) \leq K_{12} w_2(t) \forall t \in (0, 1/2)$ . The space  $L^p(0, 1; w_2)$  is continuously embedded in  $L^1(0, 1)$ .
- c2) The function  $\int_0^t w_2(\sigma) d\sigma$  belongs to  $L^{\frac{m}{p}}(0, 1; w_1)$ .

A generalized Gamma space with double weights is the set :

$$G\Gamma(p, m; w_1, w_2) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable } \int_0^t v_*^p(\sigma) w_2(\sigma) d\sigma \text{ is in } L^{\frac{m}{p}}(0, 1; w_1) \right\}.$$

A similar definition has been considered in [17]. They were interested in the embeddings between  $G\Gamma$ -spaces.

The following property has been proved in [12], p.424.

**Property 1.3.**

Let  $G\Gamma(p, m; w_1, w_2)$  be a Generalized Gamma space with double weights and let us define for  $v \in G\Gamma(p, m; w_1, w_2)$

$$\rho(v) = \left[ \int_0^1 w_1(t) \left( \int_0^t v_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{m}}$$

with the obvious change for  $m = +\infty$ .

Then,

- (1)  $\rho$  is a quasinorm.
- (2)  $G\Gamma(p, m; w_1, w_2)$  endowed with  $\rho$  is a quasi-Banach function space.
- (3) If  $w_2 = 1$

$$G\Gamma(p, m; w_1, 1) = G\Gamma(p, m; w_1).$$

Generalized Gamma spaces with double weights originate from the  $G\Gamma(p, m, w)$  spaces introduced and studied in [16]

**Example 1.1. of weights**

Let  $w_1(t) = (1 - \text{Log } t)^\gamma$ ,  $w_2(t) = (1 - \text{Log } t)^\beta$  with  $(\gamma, \beta) \in \mathbb{R}^2$ . Then

$$w_2 \text{ satisfies condition c1) and } w_1 \text{ and } w_2 \text{ are in } L_{exp}^{\max(\gamma; \beta)}(]0, 1[).$$

**Question 1** The natural question is *how to extend of Theorem 1.1 for  $\alpha < \frac{1}{n'}$  and how to improve the estimate when  $\alpha = \frac{1}{n'}$ ?*

We note that the solution  $u$  of (1.1) satisfies also

$$|u|_{L^{n', \infty}(\Omega)} \leq K_1 |f|_{L^1(\Omega; \delta)}, \quad (1.3)$$

therefore, since (1.1) defines implicitly the linear operator

$$\text{datum } f \rightarrow \text{solution } u,$$

a natural idea is to obtain an estimate through the real interpolation method of Marcinkiewicz (see [4, 5, 7]) to derive

$$|u|_{(L^{n', \infty}, L^{(n')})_{\alpha, 1}} \leq K_2 |f|_{L^1(\Omega; \delta(1 + |\text{Log } \delta|)^\alpha)} \quad \text{for } 0 < \alpha \leq 1. \quad (1.4)$$

Note that  $L^{(n', 1)} = L^{n'}$  (see below for a full definition.)

**Question 2** *How to characterize the space  $(L^{n', \infty}(\Omega), L^{(n')})_{\alpha, 1}$ ?*

We still have not an answer to this question. Therefore, we will provide a lower estimate for the norm of  $u$  in relation (1.4), a particular overbound can be obtained from our work made in

[12] :

Since  $L^{n',\infty}(\Omega) \subset L^{n'}$ , then we have

$$\left( L^{n',\infty}(\Omega), L^{n'}(\Omega) \right)_{\alpha,1} \subset \left( L^{n'}(\Omega), L^{n'}(\Omega) \right)_{\alpha,1}$$

and we have shown in [12] the following

**Theorem 1.4. (characterization of the interpolation between Grand and Small Lebesgue space)**

$$\left( L^{n'}(\Omega), L^{n'}(\Omega) \right)_{\alpha,1} = G\Gamma(n'; 1; w_1; w_2) \text{ with } w_1(t) = \frac{(1 - \text{Log } t)^{\alpha-1}}{t}, \quad w_2(t) = \frac{1}{1 - \text{Log } t}.$$

(see next section for the definition of  $G\Gamma$ ).

Therefore, we have the following **non optimal result** but valid for all  $\alpha$ .

**Proposition 1.5.**

Let  $u$  be the solution of (1.1). Then,

$$\|u\|_{G\Gamma(n';1;w_1;w_2)} = \int_0^1 (1 - \text{Log } t)^\alpha \left( \int_0^t \frac{u_*^{n'}(x) dx}{1 - \text{Log } x} \right)^{\frac{1}{n'}} \frac{dt}{(1 - \text{Log } t)t} \leq K_4 \|f\|_{L^1(\Omega; \delta(1+|\text{Log } \delta|)^\alpha)}$$

whenever  $0 < \alpha < 1$ .

To give a new improved statement for Proposition 1.5 namely, we will show the

**Theorem 1.6.**

Let  $1 < p < \infty$ ,  $0 < \theta < 1$  and  $1 \leq r < \infty$ . Then

$$\|f\|_{G\Gamma(p,r;w_1,w_2) \cap G\Gamma(\infty,r;v_1,v_2)} \lesssim \|f\|_{(L^{p,\infty}, L^{(p)_{\theta,1}})},$$

where  $w_1(t) = t^{-1}(1 - \text{Log } t)^{r\theta-1}$ ,  $w_2(t) = (1 - \text{Log } t)^{-1}$ ,  $v_1(t) = t^{-1}(1 - \text{Log } t)^{r\theta(1-1/p)-1}$  and  $v_2(t) = t^{1/p}$ .

Thanks to this last theorem, we deduce from relation (1.4) a new estimate of the solution  $u$  valid also for  $\alpha < \frac{1}{n'}$  and better than Proposition 1.5 in that case. To complete the results of [12], we shall introduce different results on the interpolation spaces namely, between  $(L^{n'}, L^{(n')_{\theta,r}}$ ,  $(L^{p,\alpha}, L^{(p),\beta})_{\theta,r}$ ,  $(L^p, L^p)_{\theta,r}$ . It happens all of these spaces are Lorentz G-gamma spaces. We state few of those results.

**Theorem 1.7.**

For  $0 < \theta < 1$ ,  $r \in [1, +\infty[$

$$(L^{n'}, L^{(n')_{\theta,r}}) = G(n', r; w_1; 1) \quad \text{with } w_1(t) = t^{-1}(1 - \text{Log } t)^{r\frac{\theta}{n'}-1}.$$

**Corollary 1.8. of Theorem 1.7**

For  $0 < \theta < 1$ , one has

$$(L^{n'}, L^{(n')})_{\theta,1} = L^{(n',\theta)}.$$

As in [12, 10], the proofs of the above results rely on the computation of the  $K$ -functional; similarly, in order to treat the couple  $(L^{n',\infty}, L^{n'})$ , we will show the following

**Theorem 1.9.**

The  $K$ -functional for  $(L^{n',\infty}, L^{n'})$  is given by, for  $t \in ]0, 1[$ ,  $f \geq 0$  in  $L^{n',\infty}$

$$K_0(f; t) = t \sup \left\{ \left( \int_E f_*^{n'}(\sigma) d\sigma \right)^{\frac{1}{n'}} ; t^{-n'} = \int_E \frac{dx}{x} \right\}$$

**Remark 1.10.**

Setting  $d\nu = \frac{dx}{x}$ ,  $|E|_\nu = \int_E d\nu$ ,  $f_{*,\nu}$  the decreasing rearrangement of a nonnegative function  $f$  with respect to the measure  $\nu$ , then we can write the preceding theorem as :

**Theorem 1.11.**

The  $K$ -functional for the couple  $(L^{p,\infty}, L^p)$  is given, for  $f \geq 0$  in  $L^{p,\infty}$ ,  $t > 0$

$$K_0(f; t) = t \left( \int_0^{t^{-p}} \left( \psi(s) \right)_{*,\nu}^p(x) dx \right)^{\frac{1}{p}}.$$

Here  $1 \leq p < +\infty$ ,  $\psi(s) = s^{\frac{1}{p}} f_*(s)$ ,  $s \in (0, 1)$ .

Applying Theorem 1.9 with real interpolation method of Marcinkiewicz, we then deduce the following partial answer for very weak solution :

**Proposition 1.12.**

For  $0 < \alpha \leq 1$ , let  $u$  be the solution of (1.1). Then one has a constant  $c > 0$  such that

$$\int_0^1 t^{-\alpha} \sup_{\{E:|E|_\nu=t^{-n'}\}} \left( \int_E u_*^{n'}(x) dx \right)^{\frac{1}{n'}} dt \leq c |f|_{L^1(\Omega; \delta(1+|\text{Log } \delta|)^{\frac{\alpha}{n'}})}.$$

Other consequences of the above interpolation results are the interpolation inequalities, we state few of them.

**Property 1.13. (Interpolation inequalities for small and grand Lebesgue spaces)** If

$1 \geq \alpha > \frac{1}{n'}$ , then  $\forall v \in L^{(n')}$

$$\|v\|_{L^{n',\infty}} \leq c \|v\|_{L^{n'}}^{1-\alpha} \|v\|_{L^{(n')}}^\alpha.$$

## 2. Notation and Primary results

For a measurable function  $f : \Omega \rightarrow \mathbb{R}$ , we set for  $t \geq 0$

$$D_f(t) = \text{measure} \left\{ x \in \Omega : |f(x)| > t \right\}$$

and  $f_*$  the decreasing rearrangement of  $|f|$ ,

$$f_*(s) = \inf \left\{ t : D_f(t) \leq s \right\} \text{ with } s \in (0, |\Omega|), \quad |\Omega| \text{ is the measure of } \Omega,$$

that we shall assume to be equal to 1 for simplicity.

If  $A_1$  and  $A_2$  are two quantities depending on some parameters, we shall write

$$A_1 \lesssim A_2 \text{ if there exists } c > 0 \text{ (independent of the parameters) such that } A_1 \leq cA_2$$

$$A_1 \simeq A_2 \text{ if and only if } A_1 \lesssim A_2 \text{ and } A_2 \lesssim A_1.$$

We recall also the following definition of interpolation spaces.

Let  $(X_0, \|\cdot\|_0)$ ,  $(X_1, \|\cdot\|_1)$  two Banach spaces contained continuously in a Hausdorff topological vector space (that is  $(X_0, X_1)$  is a compatible couple).

For  $g \in X_0 + X_1$ ,  $t > 0$  one defines the so called  $K$  functional  $K(g, t; X_0, X_1) \doteq K(g, t)$  by setting

$$K(g, t) = \inf_{g=g_0+g_1} (\|g_0\|_0 + t\|g_1\|_1). \quad (2.1)$$

For  $0 \leq \theta \leq 1$ ,  $1 \leq p \leq +\infty$ ,  $\alpha \in \mathbb{R}$  we shall consider

$$(X_0, X_1)_{\theta, p; \alpha} = \left\{ g \in X_0 + X_1, \|g\|_{\theta, p; \alpha} = \|t^{-\theta - \frac{1}{p}} (1 - \text{Log } t)^\alpha K(g, t)\|_{L^p(0,1)} \text{ is finite} \right\}.$$

Here  $\|\cdot\|_V$  denotes the norm in a Banach space  $V$ . The weighted Lebesgue space  $L^p(0, 1; \omega)$ ,  $0 < p \leq +\infty$ , is endowed with the usual norm or quasi norm defined through  $\|g\|_{L^p(0,1;\omega)} = \left( \int_0^1 |g(t)|^p \omega(t) dt \right)^{1/p}$ , where  $\omega$  is a weight function on  $(0, 1)$ .

Our definition of the interpolation space is different from the usual one (see [4, 26]) since we restrict the norms on the interval  $(0, 1)$ . We recall that these modified interpolation spaces have been already used by other authors, see e.g. [21, 8] and the references given there.

If we consider ordered couple, i.e.  $X_1 \hookrightarrow X_0$  and  $\alpha = 0$ , the following equality holds with equivalence of norms

$$(X_0, X_1)_{\theta, p; 0} = (X_0, X_1)_{\theta, p},$$

where the space on the right hand side is the interpolation space as defined by J. Peetre (see [4, 26, 5]).

### 2.1. Some remarkable $G\Gamma$ -spaces.

In this paragraph, we want to prove among other that  $G\Gamma$ -spaces cover many well-known spaces.

#### Proposition 2.1.

Let  $1 \leq p < +\infty$ . If  $w_1$  and  $w_2$  are integrable weights on  $(0, 1)$  and  $w_2$  satisfies c1), then

$$G\Gamma(p, m; w_1, w_2) = \Lambda^p(w_2),$$

where  $\Lambda^p(w_2)$  denotes the classical Lorentz space defined through

$$\left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable} : \left( \int_0^1 f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} = \|f\|_{\Lambda^p(w_2)} < +\infty \right\}.$$

#### Proof

If  $v \in \Lambda^p(w_2)$  then  $\rho(v) \leq \|v\|_{\Lambda^p(w_2)} \left[ \int_0^1 w_1(t) dt \right]^{\frac{1}{m}} < +\infty$ .

Conversely, let  $v$  be such that  $\rho(v) < +\infty$ . We have for some  $a > 0$ ,  $\int_a^1 w_1(t) dt > 0$ . Then for all  $t \geq a$

$$\left( \int_0^a f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{m}{p}} \leq \left( \int_0^t f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{m}{p}},$$

from which we derive after multiplying by  $w_1(t)$  and integrating from  $a$  to 1,

$$\left( \int_0^a f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} \leq \frac{\rho(f)}{\left[ \int_a^1 w_1(t) dt \right]^{\frac{1}{m}}} \lesssim \rho(f) < +\infty. \quad (2.2)$$

Between  $(a, 1)$ , we have :

$$\left( \int_a^1 f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} \leq f_*(a) \|w_2\|_{L^1}^{\frac{1}{p}} \lesssim \int_0^a f_*(\sigma) d\sigma = \int_0^1 f_*(\sigma) \chi_{[0,a]}(\sigma) d\sigma. \quad (2.3)$$

The condition c1) implies

$$\int_0^a f_*(\sigma) d\sigma \lesssim \left( \int_0^1 (f_* \chi_{[0,a]})^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}}. \quad (2.4)$$

So that relations (2.2) to (2.4) imply

$$\left( \int_a^1 f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} \lesssim \left( \int_0^a f_*^p(\sigma) w_2(\sigma) d\sigma \right)^{\frac{1}{p}} \lesssim \rho(v) < +\infty. \quad (2.5)$$

This shows

$$\|f\|_{\Lambda^p(w_2)} \lesssim \rho(v).$$

◇

Next we want to focus in a special case :

**Proposition 2.2.**

Assume that  $w_1(t) = t^{-1}(1 - \text{Log } t)^\gamma$ ,  $w_2(t) = (1 - \text{Log } t)^\beta$ ,  $(\gamma, \beta) \in \mathbb{R}^2$ ,  $m \in [1, +\infty[$ ,  $p \in [1, +\infty[$ .

- (1) If  $\gamma < -1$  then  $G\Gamma(p, m; w_1, w_2) = \Lambda^p(w_2)$ .
- (2) If  $\gamma > -1$  and  $\gamma + \beta \frac{m}{p} + 1 \geq 0$  then

$$G\Gamma(p, m; w_1, w_2) = G\Gamma(p, m; \bar{w}_1, 1), \quad \bar{w}_1(t) = t^{-1}(1 - \text{Log } t)^{\gamma + \beta \frac{m}{p}}.$$

**Proof**

For the first statement, we observe that if  $\gamma + 1 < 0$ ,  $\int_0^1 (1 - \text{Log } t)^\gamma \frac{dt}{t}$  is finite. Then applying Proposition 2.1 we derive the first result.

For the case  $\gamma + 1 > 0$ , we shall need the following lemma whose proof is in [12], p. 447:

**Lemma 2.3.**

Let  $t_k = 2^{1-2^k}$ ,  $k \in \mathbb{N}$ ,  $\lambda > 0$ ,  $q > 0$ ,  $H$  a nonnegative locally integrable function on  $(0, 1)$  satisfying

$$\int_0^1 H(x) dx \lesssim \int_0^{\frac{1}{2}} H(x) dx.$$

Then

- (1)  $2^k \approx 1 - \text{Log } x$ ,  $x \in [t_{k+1}, t_k]$ .
- (2)

$$\begin{aligned} \int_0^1 \left[ (1 - \text{Log } t)^\lambda \int_0^t H(x) dx \right]^q \frac{dt}{(1 - \text{Log } t)t} &\approx \sum_{k \in \mathbb{N}} \left( \int_0^{t_k} H(x) dx \right)^q 2^{\lambda k q} \\ &\approx \sum_{k \in \mathbb{N}} \left( 2^{\lambda k} \int_{t_{k+1}}^{t_k} H(x) dx \right)^q. \end{aligned}$$

We shall apply this Lemma with  $H(x) = f_*^p(x)(1 - \text{Log } x)^\beta$ . We have  $\int_0^1 H(x) dx \lesssim \int_0^{\frac{1}{2}} H(x) dx$  since  $f_*^p$  is decreasing and  $\int_0^1 (1 - \text{Log } t)^\gamma dt < +\infty \quad \forall \gamma \in \mathbb{R}$ . Indeed

$$\begin{aligned} \left( \int_{\frac{1}{2}}^1 H(x) dx \right)^{\frac{1}{p}} &\lesssim f_* \left( \frac{1}{2} \right) \lesssim \int_0^{\frac{1}{2}} f_*(t) dt \\ &\leq \left( \int_0^{\frac{1}{2}} f_*^p(t) (1 - \text{Log } t)^\beta dt \right)^{\frac{1}{p}} \cdot \left( \int_0^{\frac{1}{2}} (1 - \text{Log } t)^{-\beta \frac{p'}{p}} dt \right)^{\frac{1}{p'}} \\ &\lesssim \left( \int_0^{\frac{1}{2}} H(x) dx \right)^{\frac{1}{p}}. \end{aligned}$$

Applying statement 2. of this Lemma 2.3, we derive

$$\begin{aligned}
\rho^m(f) &= \int_0^1 (1 - \text{Log } t)^\gamma \left( \int_0^t H(x) dx \right)^{\frac{m}{p}} \frac{dt}{t} \\
&= \int_0^1 \left[ (1 - \text{Log } t)^{(\gamma+1)\frac{p}{m}} \int_0^t H(x) dx \right]^{\frac{m}{p}} \frac{dt}{(1 - \text{Log } t)t} \quad \text{if } \gamma + 1 > 0 \\
&\approx \sum_{k \in \mathbb{N}} \left( 2^{\lambda k} \int_{t_{k+1}}^{t_k} H(x) dx \right)^q \quad \text{with } \lambda = (\gamma + 1)\frac{p}{m}, \quad q = \frac{m}{p}
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\approx \sum_{k \in \mathbb{N}} \left( 2^{(\lambda+\beta)k} \int_{t_{k+1}}^{t_k} f_*^p(x) dx \right)^q \\
&\approx \int_0^1 \left[ (1 - \text{Log } t)^{\lambda+\beta} \int_0^t f_*^p(x) dx \right]^q \frac{dt}{(1 - \text{Log } t)t} \\
&\approx \int_0^1 (1 - \text{Log } t)^{(\lambda+\beta)q-1} \left( \int_0^t f_*^p(x) dx \right)^q \frac{dt}{t}.
\end{aligned} \tag{2.7}$$

If  $\gamma > -1$ ,  $\gamma + \beta \frac{m}{p} + 1 \geq 0$ , then the equality comes from the definition of  $G\Gamma(p, m; \bar{w}_1)$ .

This ends of the proof of Proposition 2.2 ◇

**Lemma 2.4.**

Assume that  $w_1(t) = t^{-1}(1 - \text{Log } t)^\gamma$ ,  $w_2(t) = (1 - \text{Log } t)^\beta$ ,  $(\gamma, \beta) \in \mathbb{R}^2$ ,  $m \in [1, \infty[$ ,  $p \in [1, \infty[$ .  
If  $\gamma > -1$  and  $\gamma + \beta \frac{m}{p} + 1 < 0$ , then

$$\|f\|_{G\Gamma(p, m; w_1, w_2)}^m \approx \int_0^1 (1 - \text{Log } t)^{\gamma + \beta \frac{m}{p}} \left( \int_t^1 f_*^p(x) dx \right)^{m/p} \frac{dt}{t}.$$

*Proof.* Put

$$I = \int_0^1 (1 - \text{Log } t)^{\gamma + \beta \frac{m}{p}} \left( \int_t^1 f_*^p(x) dx \right)^{m/p} \frac{dt}{t}.$$

Let  $t_k = 2^{1-2^k}$ ,  $k \in \mathbb{N}$ . Since  $\gamma + \beta \frac{m}{p} + 1 < 0$ , we can apply the second assertion of Lemma 6.3 in [12] to obtain

$$I \approx \sum_{k \in \mathbb{N}} \left( \int_{t_{k+1}}^1 f_*^p(x) dx \right)^{m/p} 2^{(\gamma + \beta \frac{m}{p} + 1)k},$$

then using the second assertion of Lemma 6.1 in [12] gives

$$I \approx \sum_{k \in \mathbb{N}} \left( \int_{t_{k+1}}^{t_k} f_*^p(x) dx \right)^{m/p} 2^{(\gamma + \beta \frac{m}{p} + 1)k},$$

by first assertion in Lemma 2.3, we get

$$I \approx \sum_{k \in \mathbb{N}} \left( \int_{t_{k+1}}^{t_k} (1 - \text{Log } x)^\beta f_*^p(x) dx \right)^{m/p} 2^{(\gamma+1)k},$$

since  $\gamma > -1$ , we can apply the first assertion of Lemma 6.1 in [12] to obtain

$$I \approx \sum_{k \in \mathbb{N}} \left( \int_0^{t_k} (1 - \text{Log } x)^\beta f_*^p(x) dx \right)^{m/p} 2^{(\gamma+1)k},$$

finally an application of the second assertion in Lemma 2.3 yields

$$I \approx \|f\|_{G\Gamma(p,m;w_1,w_2)}^m,$$

which completes the proof.  $\square$

We shall need in particular the Corollary 2.7, consequence of relation (2.7) and the following

**Definition 2.5. of the small Lebesgue space [22, 9]**

The small Lebesgue space associated to the parameters  $p \in ]1, +\infty[$  and  $\theta > 0$  is the set  $L^{(p,\theta)}(\Omega) =$

$$\left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable} : \|f\|_{(p,\theta)} = \int_0^1 (1 - \text{Log } t)^{-\frac{\theta}{p} + \theta - 1} \left( \int_0^t f_*^p(\sigma) d\sigma \right)^{1/p} \frac{dt}{t} < +\infty \right\}.$$

**Definition 2.6. of the grand Lebesgue space [22, 9]**

The grand Lebesgue space is the associate space of the small Lebesgue space, with the parameters  $p \in ]1, +\infty[$  and  $\theta > 0$  is the set  $L^{p),\theta}(\Omega) =$

$$\left\{ f : \Omega \rightarrow \mathbb{R} \text{ measurable} : \|f\|_{p),\theta} = \sup_{0 < t < 1} (1 - \text{Log } t)^{-\frac{\theta}{p}} \left( \int_t^1 f_*^p(\sigma) d\sigma \right)^{1/p} < +\infty \right\}.$$

**Corollary 2.7. of Proposition 2.2**

If  $m = 1, \gamma + 1 + \frac{\beta}{p} > 0, \gamma > -1$  and  $\beta \in \mathbb{R}$ , the functions  $w_i, i = 1, 2$  as in Proposition 2.2 then

$$G\Gamma(p, 1; w_1, w_2) = L^{(p,\theta)}, \quad \theta = p' \left( \gamma + 1 + \frac{\beta}{p} \right).$$

### 3. Some $K$ -functional computations and the associated interpolation spaces

#### 3.1. The case of the couple $(L^{n'}, L^{(n')})$ .

**Theorem 3.1.**

Let  $1 < n < +\infty$ , and let  $\varphi(t) = e^{1-\frac{1}{t^n}}$ ,  $0 < t \leq 1$ . Then

$$K(f, t; L^{n'}, L^{(n')}) \approx t \int_{\varphi(t)}^1 (1 - \text{Log } \sigma)^{-\frac{1}{n'}} \left( \int_0^\sigma f_*^{n'}(x) dx \right)^{\frac{1}{n'}} \frac{d\sigma}{\sigma} \doteq K_2(t)$$

for all  $f \in L^{n'}$ .

**Proof:**

First, let us show:

$$K_2(t) \lesssim K(f, t; L^{n'}, L^{(n')}).$$

Let  $f = g + h$  with  $g \in L^{n'}$  and  $h \in L^{(n')}$ . Then, for all  $x$ ,  $f_*(x) \leq g_*\left(\frac{x}{2}\right) + h_*\left(\frac{x}{2}\right)$ . Therefore, we have

$$\begin{aligned} K_2(t) &\leq \|g\|_{L^{n'}} t \int_{\varphi(t)}^1 (1 - \text{Log } \sigma)^{-\frac{1}{n'}} \frac{d\sigma}{\sigma} + t \|h\|_{L^{(n')}} \\ &\lesssim \|g\|_{L^{n'}} + t \|h\|_{L^{(n')}} \end{aligned}$$

Taking the infimum, one derives

$$K_2(t) \lesssim K(f, t; L^{n'}, L^{(n')}). \quad (3.1)$$

For the converse, we adopt the same decomposition as in [12]

$$g = (|f| - f_*(\varphi(t)))_+, \quad h = f - g. \quad (3.2)$$

Then

$$f_* = h_* + g_*, \quad g_* = (f_* - f_*(\varphi(t)))_+,$$

$$h_* = f_*(\varphi(t))\chi_{[0, \varphi(t)]} + f_*(s)\chi_{[\varphi(t), 1]}$$

$$\begin{aligned} \|g\|_{L^{n'}} &\lesssim \left[ \int_0^{\frac{\varphi(t)}{2}} (f_*(s) - f_*(\varphi(t)))_+^{n'} ds \right]^{\frac{1}{n'}} \lesssim \left[ \int_0^{\frac{\varphi(t)}{2}} f_*^{n'}(s) ds \right]^{\frac{1}{n'}} \\ &= \|f\|_{L^{n'}(0, \frac{\varphi(t)}{2})} \lesssim \frac{K_2(t)}{\int_{\varphi(t)/2}^1 (1 - \text{Log } \sigma)^{-\frac{1}{n'}} \frac{d\sigma}{\sigma}} \lesssim K^2(t). \end{aligned} \quad (3.3)$$

As in [12], we have

$$\begin{aligned}
t\|h\|_{L^{(n')}} &\leq t \left( \int_0^{\varphi(t)} (1 - \text{Log } s)^{-1/n'} s^{1/n'} \frac{ds}{s} \right) f_*(\varphi(t)) \\
&\quad + t \left( \int_{\varphi(t)}^1 (1 - \text{Log } s)^{-1/n'} \frac{ds}{s} \right) \varphi(t)^{1/n'} f_*(\varphi(t)) \\
&\quad + t \int_{\varphi(t)}^1 (1 - \text{Log } s)^{-1/n'} \left( \int_{\varphi(t)}^s f_*^{n'}(x) dx \right)^{1/n'} \frac{ds}{s} \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{3.4}$$

Since

$$\int_0^{\varphi(t)} s^{1/n'} (1 - \text{Log } s)^{-1/n'} \frac{ds}{s} \lesssim \varphi(t)^{1/n'} (1 - \text{Log } \varphi(t))^{-1/n'},$$

we obtain

$$I_1 \lesssim t \left( 1 - \text{Log } \varphi(t) \right)^{-1/n'} \varphi(t)^{1/n'} f_*(\varphi(t)) \lesssim K_2(t) \tag{3.5}$$

$$I_2 \lesssim \varphi(t)^{1/n'} f_*(\varphi(t)) \lesssim \sup_{0 < s < \varphi(t)} s^{1/n'} f_*(s) \lesssim K_2(t) \tag{3.6}$$

and

$$I_3 \lesssim K_2(t), \tag{3.7}$$

and using these relations (3.5) to (3.7), we derive

$$t\|h\|_{L^{(n')}} \leq I_1 + I_2 + I_3 \lesssim K_2(t). \tag{3.8}$$

From relations (3.3) and (3.8) we infer:

$$\|g\|_{L^{n'}} + t\|h\|_{L^{(n')}} \lesssim K_2(t). \tag{3.9}$$

and therefore

$$K(f, t; L^{n'}, L^{(n')}) \lesssim K_2(t). \tag{3.10}$$

The combination of the above relations (3.10), (3.1) gives Theorem 3.1.  $\diamond$

**Corollary 3.2.** (of Theorem 3.1)

If  $n \in ]1, +\infty[$ ,  $r \in [1, +\infty[$ ,  $0 < \theta < 1$ , then

$$\|f\|_{(L^{n'}, L^{(n')})_{\theta, r}}^r \approx \int_0^1 (1 - \text{Log } x)^{\frac{\theta r}{n}} \left( \int_0^x f_*^{n'}(s) ds \right)^{\frac{r}{n'}} \frac{dx}{x(1 - \text{Log } x)}.$$

**Proof:**

One has for  $f \in L^{n'} + L^{(n')}$ ,  $1 \leq r < \infty$

$$\|f\|_{(L^{n'}, L^{(n')})_{\theta, r}}^r \approx \int_0^1 [t^{-\theta} K(f, t)]^r \frac{dt}{t} \tag{3.11}$$

Using Theorem 3.1 and making a change of variable  $x = \varphi(t)$  that is  $t = (1 - \text{Log } x)^{-\frac{1}{n}}$ , one derives from relation (3.11)

$$\|f\|_{(L^{n'}, L^{(n')})_{\theta, r}}^r \approx J_f$$

$$J_f = \int_0^1 \left[ (1 - \text{Log } x)^{\frac{\theta-1}{n}} \int_x^1 (1 - \text{Log } \sigma)^{-\frac{1}{n'}} \left( \int_0^\sigma f_*^{n'}(s) ds \right)^{\frac{1}{n'}} \frac{d\sigma}{\sigma} \right]^r \frac{dx}{x(1 - \text{Log } x)}.$$

Applying Hardy's inequality (we take into account that  $\theta < 1$  and apply [3] Theorem 6.5, (6.14)), we have

$$J_f \lesssim \int_0^1 \left[ (1 - \text{Log } x)^{\frac{\theta}{n}} \left( \int_0^x f_*^{n'}(\sigma) d\sigma \right)^{\frac{1}{n'}} \right]^r \frac{dx}{x(1 - \text{Log } x)} = \tilde{J}_f.$$

For the converse, since we have for all  $x > 0$

$$\int_x^1 (1 - \text{Log } \sigma)^{-\frac{1}{n'}} \left( \int_0^\sigma f_*^{n'}(s) ds \right)^{\frac{1}{n'}} \frac{d\sigma}{\sigma} \geq \left( \int_0^x f_*^{n'}(s) ds \right)^{\frac{1}{n'}} (1 - \text{Log } x)^{-\frac{1}{n'}} |\text{Log } x|,$$

we then have

$$J_f \geq \int_0^1 \left[ (1 - \text{Log } x)^{\frac{\theta}{n} - 1 - \frac{1}{r}} |\text{Log } x| \left( \int_0^x f_*^{n'}(s) ds \right)^{\frac{1}{n'}} \right]^r \frac{dx}{x}. \quad (3.12)$$

From this relation we deduce

$$\tilde{J}_f \lesssim J_f + \int_0^1 \left[ (1 - \text{Log } x)^{\frac{\theta}{n} - 1 - \frac{1}{r}} \left( \int_0^x f_*^{n'}(s) ds \right)^{\frac{1}{n'}} \right]^r \frac{dx}{x} \doteq J_f + I_r, \quad (3.13)$$

while to estimate the last integral, one has

$$I_r \leq \|f\|_{L^{n'}}^r \int_0^1 (1 - \text{Log } x)^{\left(\frac{\theta}{n} - 1\right)r - 1} \frac{dx}{x} \leq c \|f\|_{L^{n'}}^r.$$

Since  $(L^{n'}, L^{(n')})_{\theta, r}$  is continuously embedded in  $L^{n'}$ , we then have

$$I_r \leq c \|f\|_{(L^{n'}, L^{(n')})_{\theta, r}}^r. \quad (3.14)$$

Thus, we derive

$$\tilde{J}_f \lesssim J_f + I_r \lesssim \|f\|_{(L^{n'}, L^{(n')})_{\theta, r}}^r.$$

◇

### Proof of Theorem 1.7

We derive it from Corollary 3.2 of Theorem 3.1.

◇

### 3.2. Interpolation between grand and classical Lebesgue spaces in the critical case.

#### Lemma 3.3.

Let  $1 < p < \infty$ , and let  $f \in L^p$ . Then, for all  $0 < t < 1$ ,

$$K(f, t; L^p, L^p) \approx \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-1/p} \left( \int_s^1 f_*^p(x) dx \right)^{1/p},$$

where  $\varphi(t) = e^{1 - \frac{1}{t^p}}$ .

*Proof.* Fix  $f \in L^p$  and  $0 < t < 1$ . Set

$$K_1(t) = \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-1/p} \left( \int_s^1 f_*^p(x) dx \right)^{1/p}.$$

First we show that

$$K_1(t) \lesssim K(f, t; L^p, L^p). \quad (3.15)$$

Let  $f = g + h$  be an arbitrary decomposition with  $g \in L^p$  and  $h \in L^p$ . Using the elementary inequality  $f_*(x) \leq g_*(x/2) + h_*(x/2)$ , we derive

$$\begin{aligned} K_1(t) &\lesssim \|g\|_{L^p} + \|h\|_{L^p} \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-1/p} \\ &= \|g\|_{L^p} + t\|h\|_{L^p}, \end{aligned}$$

from which (3.15) follows. Next we establish the converse estimate

$$K(f, t; L^p, L^p) \lesssim K_1(t). \quad (3.16)$$

To this end, we take the same particular decomposition  $f = g + h$  as in Theorem 3.1 relation (3.2). Clearly,

$$\|g\|_{L^p} \leq K_1(t). \quad (3.17)$$

Next we note that

$$\begin{aligned} t\|h\|_{L^p} &= t f_*(\varphi(t)) [\varphi(t)]^{1/p} + t \left( \int_{\varphi(t)}^1 f_*^p(x) dx \right)^{1/p} \\ &\approx \left( 1 - \text{Log } \frac{\varphi(t)}{2} \right)^{-1/p} f_*(\varphi(t)) [\varphi(t)]^{1/p} \\ &\quad + \left( \sup_{0 < s < \phi(t)} (1 - \text{Log } s)^{-1/p} \right) \left( \int_{\varphi(t)}^1 f_*^p(x) dx \right)^{1/p}, \\ &\lesssim \left( 1 - \text{Log } \frac{\varphi(t)}{2} \right)^{-1/p} \left( \int_{\frac{\varphi(t)}{2}}^{\varphi(t)} f_*(x) dx \right)^{1/p} \\ &\quad + \sup_{0 < s < \phi(t)} (1 - \text{Log } s)^{-1/p} \left( \int_s^1 f_*^p(x) dx \right)^{1/p}, \end{aligned}$$

which gives

$$t\|h\|_{L^p} \leq K_1(t). \quad (3.18)$$

Now (3.16) follows from (3.17) and (3.18). The proof is complete.  $\square$

**Theorem 3.4.**

Let  $1 < p < \infty$ ,  $0 < \theta < 1$ , and  $1 \leq r < \infty$ . Then

$$(L^p), L^p)_{\theta,r} = GF(p, r; w_1, w_2),$$

where  $w_1(t) = t^{-1}(1 - \text{Log } t)^{r\theta/p-1}$  and  $w_2(t) = (1 - \text{Log } t)^{-1}$ .

*Proof.*

Let  $f \in L^p$ . Then, using Lemma 3.3, we get at

$$\|f\|_{(L^p), L^p)_{\theta,r}}^r \approx \int_0^1 (1 - \text{Log } t)^{\frac{r\theta}{p}-1} \left[ \sup_{0 < s < t} \psi(s) \right]^r \frac{dt}{t}, \quad (3.19)$$

where

$$\psi(s) = (1 - \text{Log } s)^{-1/p} \left( \int_s^1 f_*^p(x) dx \right)^{1/p}.$$

Now, in view of Lemma 2.4 (applied with  $\gamma = \frac{r\theta}{p} - 1$ ,  $\beta = -1$ ,  $m = r$ ), it is sufficient to establish that

$$\|f\|_{(L^p), L^p)_{\theta,r}}^r \approx \int_0^1 (1 - \text{Log } t)^{r(\frac{\theta-1}{p})-1} \left( \int_t^1 f_*^p(x) dx \right)^{r/p} \frac{dt}{t}. \quad (3.20)$$

The estimate “ $\gtrsim$ ” in (3.20) follows trivially from (3.19), while for the converse estimate we infer from Bennet-Rudnick Lemma ([3] Lemma 6.1) that

$$\sup_{0 < s < t} \psi(s) \lesssim \int_0^t (1 - \text{Log } s)^{-1} \psi(s) \frac{ds}{s},$$

which, combined with (3.19), gives

$$\|f\|_{(L^p), L^p)_{\theta,r}}^r \lesssim \int_0^1 \left[ (1 - \text{Log } t)^{\frac{\theta}{p}-\frac{1}{r}} \int_0^t (1 - \text{Log } s)^{-1} \psi(s) \frac{ds}{s} \right]^r \frac{dt}{t},$$

from which follows the desired estimate “ $\lesssim$ ” in (3.20) by Hardy inequality [3] Theorem 6.5. The proof is complete.  $\square$

### 3.3. Interpolation between grand Lebesgue spaces in the critical case.

**Lemma 3.5.**

Let  $1 < p < \infty$  and  $0 < \beta < \alpha$ . Let  $f \in L^{p,\alpha}$ . Then, for all  $0 < t < 1$ ,

$$\begin{aligned} K(f, t; L^{p,\alpha}, L^{p,\beta}) &\approx \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-\frac{\alpha}{p}} \left( \int_s^{\varphi(t)} f_*^p(x) dx \right)^{1/p} \\ &\quad + t \sup_{\varphi(t) < s < 1} (1 - \text{Log } s)^{-\frac{\beta}{p}} \left( \int_s^1 f_*^p(x) dx \right)^{1/p}, \end{aligned}$$

where  $\varphi(t) = e^{1-t\frac{p}{\beta-\alpha}}$ .

*Proof.*

Fix  $f \in L^{p,\alpha}$  and  $0 < t < 1$ . Set

$$K_1(t) = \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-\frac{\alpha}{p}} \left( \int_s^{\varphi(t)} f_*^p(x) dx \right)^{1/p},$$

and

$$K_2(t) = t \sup_{\varphi(t) < s < 1} (1 - \text{Log } s)^{-\frac{\beta}{p}} \left( \int_s^1 f_*^p(x) dx \right)^{1/p}.$$

First we show that

$$K_1(t) + K_2(t) \lesssim K(f, t; L^{p,\alpha}, L^{p,\beta}). \quad (3.21)$$

Let  $f = g + h$  be an arbitrary decomposition with  $g \in L^{\alpha,p}$  and  $h \in L^{\beta,p}$ . Using the elementary inequality  $f_*(x) \leq g_*(x/2) + h_*(x/2)$ , we derive

$$\begin{aligned} K_1(t) &\lesssim \|g\|_{L^{p,\alpha}} + \|h\|_{L^{p,\beta}} \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{\frac{\beta-\alpha}{p}} \\ &= \|g\|_{L^{p,\alpha}} + t \|h\|_{L^{p,\beta}}, \end{aligned}$$

and

$$\begin{aligned} K_2(t) &\lesssim t \|g\|_{L^{p,\alpha}} \sup_{\varphi(t) < s < 1} (1 - \text{Log } s)^{\frac{\alpha-\beta}{p}} + t \|h\|_{L^{p,\beta}} \\ &= \|g\|_{L^{p,\alpha}} + t \|h\|_{L^{p,\beta}}. \end{aligned}$$

Thus, we get

$$K_1(t) + K_2(t) \lesssim \|g\|_{L^{p,\alpha}} + t \|h\|_{L^{p,\beta}},$$

from which (3.21) follows.

It remains to establish the converse estimate

$$K(f, t; L^{p,\alpha}, L^{p,\beta}) \lesssim K_1(t) + K_2(t). \quad (3.22)$$

We again take the same particular decomposition  $f = g + h$  as in Theorem 3.1. (relation (3.2)).

It is easy to check that

$$\|g\|_{L^{p,\alpha}} \lesssim K_1(t). \quad (3.23)$$

Next we observe that

$$t \|h\|_{L^{p,\beta}} = J_1(t) + J_2(t) + K_2(t), \quad (3.24)$$

where

$$J_1(t) = t f_*(\varphi(t)) \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-\frac{\beta}{p}} \left( \int_s^{\varphi(t)} dx \right)^{1/p},$$

and

$$J_2(t) = t \left( \int_{\varphi(t)}^1 f_*^p(x) dx \right)^{1/p} \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-\frac{\beta}{p}}.$$

Since

$$\sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-\frac{\beta}{p}} = (1 - \text{Log } \varphi(t))^{-\frac{\beta}{p}},$$

we have

$$J_2(t) \leq K_2(t). \quad (3.25)$$

Next we show that

$$J_1(t) \lesssim K_1(t). \quad (3.26)$$

We have

$$\begin{aligned} J_1(t) &\leq t f_*(\varphi(t)) \left( \int_0^{\varphi(t)} dx \right)^{1/p} \sup_{0 < s < \varphi(t)} (1 - \text{Log } s)^{-\frac{\beta}{p}} \\ &= t^{\frac{\alpha}{\alpha-\beta}} f_*(\varphi(t)) [\varphi(t)]^{1/p} \\ &\lesssim (1 - \text{Log } \varphi(t))^{-\frac{\alpha}{p}} \left( \int_{\frac{\varphi(t)}{2}}^{\varphi(t)} f_*^p(x) dx \right)^{1/p} \end{aligned}$$

from which follows (3.26). Altogether from the relations (3.23)-(3.26), we get (3.22). The proof is complete.  $\square$

### Theorem 3.6.

Let  $1 < p < \infty$ ,  $0 < \beta < \alpha$ ,  $0 < \theta < 1$ , and  $1 \leq r < \infty$ . Then

$$(L^p)^{\alpha}, (L^p)^{\beta})_{\theta, r} = GF(p, r; w_1, w_2),$$

where  $w_1(t) = t^{-1}(1 - \text{Log } t)^{\frac{r\theta}{p}(\alpha-\beta)-1}$  and  $w_2(t) = (1 - \text{Log } t)^{-\alpha}$ .

*Proof.*

Let  $X = (L^p)^{\alpha}, (L^p)^{\beta})_{\theta, r}$  and take  $f \in L^p)^{\alpha}$ . Then

$$\|f\|_X^r \approx I_1 + I_2,$$

where

$$I_1 = \int_0^1 (1 - \text{Log } t)^{\frac{r\theta(\alpha-\beta)}{p}-1} \left[ \sup_{0 < s < t} (1 - \text{Log } s)^{-\frac{\alpha}{p}} \left( \int_s^t f_*^p(x) dx \right)^{1/p} \right]^r \frac{dt}{t},$$

and

$$I_2 = \int_0^1 (1 - \text{Log } t)^{\frac{r(\theta-1)(\alpha-\beta)}{p}-1} \left[ \sup_{t < s < 1} (1 - \text{Log } s)^{-\frac{\beta}{p}} \left( \int_s^1 f_*^p(x) dx \right)^{1/p} \right]^r \frac{dt}{t}.$$

Put

$$I_3 = \int_0^1 (1 - \text{Log } t)^{\frac{r\theta(\alpha-\beta)}{p} - \frac{r\alpha}{p} - 1} \left( \int_t^1 f_*^p(x) dx \right)^{r/p} \frac{dt}{t}.$$

In view of Lemma 2.4 (applied with  $\gamma = r\theta(\alpha - \beta)/p - 1$ ,  $\beta = -\alpha$ ,  $m = r$ ), it is sufficient to show that

$$I_1 + I_2 \approx I_3.$$

Clearly,  $I_3 \lesssim I_2$ . Thus, it remains to establish that  $I_1 \lesssim I_3$  and  $I_2 \lesssim I_3$ . Now

$$\begin{aligned} I_1 &\leq \int_0^1 (1 - \text{Log } t)^{\frac{r\theta(\alpha-\beta)}{p}-1} \left[ \sup_{0 < s < t} (1 - \text{Log } s)^{-\frac{r\alpha}{p}} \left( \int_s^1 f_*^p(x) dx \right)^{1/p} \right]^r \frac{dt}{t} \\ &= \int_0^1 (1 - \text{Log } t)^{\frac{r\theta(\alpha-\beta)}{p}-1} \left[ \sup_{0 < s < t} \frac{p}{r\alpha} \int_0^s (1 - \text{Log } y)^{-\frac{r\alpha}{p}-1} \frac{dy}{y} \left( \int_s^1 f_*^p(x) dx \right)^{r/p} \right] \frac{dt}{t} \\ &\lesssim \int_0^1 (1 - \text{Log } t)^{\frac{r\theta(\alpha-\beta)}{p}-1} \int_0^t (1 - \text{Log } y)^{-\frac{r\alpha}{p}-1} \left( \int_y^1 f_*^p(x) dx \right)^{r/p} \frac{dy}{y} \frac{dt}{t}, \end{aligned}$$

and applying Fubini's Theorem, we obtain  $I_1 \lesssim I_3$ . Finally,

$$\begin{aligned} I_2 &= \int_0^1 (1 - \text{Log } t)^{\frac{r(\theta-1)(\alpha-\beta)}{p}-1} \left[ \sup_{t < s < 1} \left( (1 - \text{Log } s)^{-\frac{r\beta}{p}} - (1 - \text{Log } t)^{-\frac{r\beta}{p}} \right) \left( \int_s^1 f_*^p(x) dx \right)^{r/p} \right] \frac{dt}{t} \\ &+ \int_0^1 (1 - \text{Log } t)^{\frac{r(\theta-1)(\alpha-\beta)}{p}-1} (1 - \text{Log } t)^{-\frac{r\beta}{p}} \left( \int_t^1 f_*^p(x) dx \right)^{r/p} \frac{dt}{t} \\ &= \int_0^1 (1 - \text{Log } t)^{\frac{r(\theta-1)(\alpha-\beta)}{p}-1} \left[ \sup_{t < s < 1} \frac{p}{r\beta} \left( \int_t^s (1 - \text{Log } y)^{-\frac{r\beta}{p}-1} \frac{dy}{y} \right) \left( \int_s^1 f_*^p(x) dx \right)^{r/p} \right] \frac{dt}{t} \\ &+ \int_0^1 (1 - \text{Log } t)^{\frac{r(\theta-1)(\alpha-\beta)}{p}-\frac{r\beta}{p}-1} \left( \int_t^1 f_*^p(x) dx \right)^{r/p} \frac{dt}{t} \\ &\lesssim \int_0^1 (1 - \text{Log } t)^{\frac{r(\theta-1)(\alpha-\beta)}{p}-1} \left[ \int_t^1 (1 - \text{Log } s)^{-\frac{r\beta}{p}-1} \left( \int_s^1 f_*^p(x) dx \right)^{r/p} \frac{ds}{s} \right] \frac{dt}{t} + I_3, \end{aligned}$$

applying again Fubini's Theorem, we obtain  $I_2 \lesssim I_3$ . The proof is complete.  $\square$

### 3.4. The $K$ -functional for the couple $(L^{p,\infty}, L^p)$ , $1 < p < +\infty$ .

#### Theorem 3.7.

For a measurable set  $E \subset [0, 1]$ , we denote  $|E|_\nu = \int_E \frac{dx}{x}$  and for  $f \in L^{p,\infty}$ ,  $1 < p < +\infty$ , we define

$$K_p(f, t) = t \sup \left\{ \left( \int_E f_*^p(\sigma) d\sigma \right)^{\frac{1}{p}} : |E|_\nu = t^{-p} \right\} \quad t \in ]0, 1].$$

Then

$$K(f, t; L^{p,\infty}, L^p) \approx K_p(f, t)$$

and

$$K_p(f, t) = t \left[ \int_0^{t^{-p}} \psi_{*,\nu}(x)^p dx \right]^{\frac{1}{p}}$$

where  $\psi(s) = s^{\frac{1}{p}} f_*(s)$ ,  $\psi_{*,\nu}$  its decreasing rearrangement with respect to the measure  $\nu$ .

**Proof:**

Let  $f = g + h$  with  $g \in L^{p,\infty}$  and  $h \in L^p$ . Then,  $f_*(s) \leq g_*\left(\frac{s}{2}\right) + h_*\left(\frac{s}{2}\right)$ , for  $s \in ]0, 1]$

$$K_p(f, t) \leq t \sup_{|E|_\nu=t^{-p}} \left( \int_E g_*^p\left(\frac{s}{2}\right) ds \right)^{\frac{1}{p}} + t \sup_{|E|_\nu=t^{-p}} \left( \int_E h_*^p\left(\frac{s}{2}\right) ds \right)^{\frac{1}{p}}. \quad (3.27)$$

The first term can be bound as follows :

$$t \sup_{|E|_\nu=t^{-p}} \left( \int_E \left[ s^{\frac{1}{p}} g_*\left(\frac{s}{2}\right) \right]^p \frac{ds}{s} \right)^{\frac{1}{p}} \lesssim t \|g\|_{L^{p,\infty}} \sup_{|E|_\nu=t^{-p}} |E|_\nu^{\frac{1}{p}} = \|g\|_{L^{p,\infty}} \quad (3.28)$$

While the second term satisfies

$$t \sup_{|E|_\nu=t^{-p}} \left( \int_E h_*^p\left(\frac{s}{2}\right) ds \right)^{\frac{1}{p}} \leq t \|h\|_{L^p}. \quad (3.29)$$

From the three last relations, we have

$$K_p(f, t) \lesssim \|g\|_{L^{p,\infty}} + t \|h\|_{L^p}. \quad (3.30)$$

From which we derive

$$K_p(f, t) \lesssim K(f, t; L^{p,\infty}, L^p). \quad (3.31)$$

For the converse, let  $t$  be fixed and set  $A_t = \{s : \psi(s) > \psi_{*,\nu}(t^{-p})\}$ , where  $\psi(s) = s^{\frac{1}{p}} f_*(s)$ ,  $s \in [0, 1]$ , and  $\psi_{*,\nu}$  denotes its decreasing rearrangement with respect to  $\nu$ .

By equimesurability, we have

$$|A_t|_\nu = t^{-p}.$$

Let us consider the measure preserving mapping  $\sigma : \mathbb{R} \rightarrow (0, +\infty)$  such that  $f = f_* \circ \sigma$  and set  $f_i = g_i \circ \sigma$ ,  $i = 1, 2$  where, for  $s \in (0, 1)$

$$\begin{aligned} g_1(s) &= s^{-\frac{1}{p}} \psi_{*,\nu}(t^{-p}) \chi_{A_t}(s) + f_*(s) \chi_{A_t^c}(s) \\ g_2(s) &= s^{-\frac{1}{p}} \left( \psi(s) - \psi_{*,\nu}(t^{-p}) \right) \chi_{A_t} \end{aligned}$$

and  $A_t^c$  is the complement of  $A_t$  in  $(0, 1)$ , say  $A_t^c = \left\{ s : \psi(s) \leq \psi_{*,\nu}(t^{-p}) \right\}$ .

Since  $\sigma$  is measure preserving we have

$$\|f_2\|_{L^p}^p = \|g_2\|_{L^p}^p = \int_0^{|A_t|_\nu} \left( \psi_{*,\nu}(x) - \psi_{*,\nu}(t^{-p}) \right)^p dx.$$

From which we derive

$$\|f_2\|_{L^p}^p \leq \int_0^{t^{-p}} \psi_{*,\nu}(x)^p dx. \quad (3.32)$$

While for  $f_1$ , we have

$$\begin{aligned} \|f_1\|_{L^{p,\infty}} &= \|g_1\|_{L^{p,\infty}} \leq \sup_s \left[ \psi_{*,\nu}(t^{-p})\chi_{A_t}(s) + s^{\frac{1}{p}} f_*(s)\chi_{A_t^c}(s) \right] \\ &\leq \sup_s \left[ \psi_{*,\nu}(t^{-p})\chi_{A_t}(s) + \psi(s)\chi_{A_t^c}(s) \right] \\ &\leq \psi_{*,\nu}(t^{-p}) \text{ (by definition of } A_t^c \text{)}. \end{aligned} \quad (3.33)$$

Since  $f = f_1 + f_2 \in L^{p,\infty} + L^p$ , we derive from relation (3.32) and (3.33) that

$$K(f, t; L^{p,\infty}, L^p) \leq \|f_1\|_{L^{p,\infty}} + t\|f_2\|_{L^p} \leq \psi_{*,\nu}(t^{-p}) + t \left[ \int_0^{t^{-p}} \psi_{*,\nu}^p(x) dx \right]^{\frac{1}{p}}. \quad (3.34)$$

Since the function  $x \rightarrow \psi_{*,\nu}(x)$  is decreasing one has

$$\psi_{*,\nu}(t^{-p}) \leq t \left[ \int_0^{t^{-p}} \psi_{*,\nu}^p(x) dx \right]^{\frac{1}{p}}. \quad (3.35)$$

Thus, we derive from (3.34) and (3.35)

$$K(f, t; L^{p,\infty}, L^p) \lesssim t \left[ \int_0^{t^{-p}} \psi_{*,\nu}^p(x) dx \right]^{\frac{1}{p}}. \quad (3.36)$$

Making use of a corollary of the the Hardy Littlewood inequality (see Cor. 1.2.1 p. 12 in [25]), we have

$$\int_0^{t^{-p}} \psi_{*,\nu}^p(x) dx = \text{Max}_{|E|_\nu=t^{-p}} \int_E \psi^p(s) \frac{ds}{s} = \text{Max}_{|E|_\nu=t^{-p}} \int_E f_*^p(s) ds. \quad (3.37)$$

Thus

$$K_p(f, t) = t \left[ \int_0^{t^{-p}} \psi_{*,\nu}(x)^p dx \right]^{\frac{1}{p}}. \quad (3.38)$$

This equality with relation (3.36) leads to

$$K(f, t; L^{p,\infty}, L^p) \lesssim K_p(f, t).$$

◇

We end this section by proving Theorem 1.6 so we start with the following lemma :

**Lemma 3.8.**

Let  $1 < p < \infty$ . Then for any  $f \in L^{p,\infty}$  and all  $0 < t < 1$ ,

$$\sup_{0 < s < t} s^{\frac{1}{p}} f_*(s) \lesssim K(\rho(t), f; L^{p,\infty}, L^p),$$

where  $\rho(t) = (1 - \text{Log } t)^{-1 + \frac{1}{p}}$ .

*Proof.* Fix  $f \in L^{p,\infty}$  and  $0 < t < 1$ . Set

$$I(t, f) = \sup_{0 < s < t} s^{\frac{1}{p}} f_*(s).$$

It is sufficient to show that the following estimate

$$I(t, f) \lesssim \|f_0\|_{L^{p,\infty}} + \rho(t) \|f_1\|_{L^{(p)}},$$

holds for an arbitrary decomposition  $f = f_0 + f_1$  with  $f_0 \in L^{p,\infty}$  and  $f_1 \in L^{(p)}$ . In view of the elementary inequality  $f_*(t) \leq f_{0*}(t/2) + f_{1*}(t/2)$ , it follows that

$$I(t, f) \lesssim I(t/2, f_0) + I(t/2, f_1).$$

Clearly,  $I(t/2, f_0) \leq \|f_0\|_{L^{p,\infty}}$ . Therefore, it remains to show that

$$I(t/2, f_1) \lesssim \rho(t) \|f_1\|_{L^{(p)}}. \quad (3.39)$$

Note that  $\rho(t) \approx 1$ ,  $1/2 < t < 1$ . Therefore, (3.39) holds for all  $1/2 < t < 1$  in view of the fact that  $L^{(p)} \hookrightarrow L^{p,\infty}$ . Next let  $0 < t < 1/2$  and take  $0 < v < t/2$ . Then

$$\begin{aligned} \|f_1\|_{L^{(p)}} &\geq \int_v^1 (1 - \text{Log } s)^{-1/p} \left( \int_0^s f_{1*}^p(u) du \right)^{\frac{1}{p}} \frac{ds}{s} \\ &\geq \left( \int_0^v f_{1*}^p(u) du \right)^{\frac{1}{p}} \int_v^1 (1 - \text{Log } s)^{-1/p} \frac{ds}{s} \\ &\geq v^{1/p} f_{1*}(v) \int_t^1 (1 - \text{Log } s)^{-1/p} \frac{ds}{s} \\ &\approx (1 - \text{Log } t)^{1-1/p} v^{1/p} f_{1*}(v), \end{aligned}$$

whence we obtain (3.39) since  $v$  was arbitrarily taken to be between 0 and  $t/2$ . The proof is complete.  $\square$

**Theorem 3.9.** *Let  $1 < p < \infty$ ,  $0 < \theta < 1$ , and  $1 \leq r < \infty$ . Then, for any  $f \in (L^{p,\infty}, L^{(p)})_{\theta,r}$ , one has*

$$\|f\|_{G\Gamma(\infty,r;v_1,v_2)} \lesssim \|f\|_{(L^{p,\infty}, L^{(p)})_{\theta,r}},$$

where  $v_1(t) = t^{-1}(1 - \text{Log } t)^{r\theta(1-1/p)-1}$  and  $v_2(t) = t^{1/p}$ .

*Proof.* Put  $\rho(t) = (1 - \text{Log } t)^{-1+\frac{1}{p}}$ ,  $0 < t < 1$ . It immediately follows from Lemma 3.8 that

$$\left( \int_0^1 [\rho(t)]^{-\theta r} \left[ \sup_{0 < s < t} s^{1/p} f_*(s) \right]^r \frac{\rho'(t)}{\rho(t)} dt \right)^{1/r} \lesssim \|f\|_{(L^{p,\infty}, L^{(p)})_{\theta,r}}.$$

The simple observation

$$\frac{\rho'(t)}{\rho(t)} \approx t^{-1}(1 - \text{Log } t)^{-1}$$

completes the proof.  $\square$

Now the estimate resulting from Theorem 1.3 in [12] is:

**Theorem 3.10.**

Let  $1 < p < \infty$ ,  $0 < \theta < 1$ , and  $1 \leq r < \infty$ . Then

$$\|f\|_{G\Gamma(p,r;w_1,w_2)} \lesssim \|f\|_{(L^p,\infty,L^p)_{\theta,r}},$$

where  $w_1(t) = t^{-1}(1 - \text{Log } t)^{r\theta-1}$  and  $w_2(t) = (1 - \text{Log } t)^{-1}$ .

The  $G\Gamma$  spaces in Theorems 3.9 say  $G\Gamma(\infty, r; v_1, v_2)$  and 3.10 say  $G\Gamma(p, r, w_1, w_2)$  are not comparable. Thus we get Theorem 1.6.

#### 4. Some interpolation inequalities for Small and Grand Lebesgue spaces

One may combine the above results with some standard results on interpolation spaces to deduce few inequalities as Property 1.13.

We recall the following result that can be found in [26].

**Theorem 4.1.**

Let  $E_0$  and  $E_1$  be two Banach spaces continuously embedded into some topological vector space. For  $0 \leq \theta \leq 1$ , one has

$$(E_0, E_1)_{\theta,1} \subset E \subset E_0 + E_1$$

if and only if

$$\text{there exists } c > 0 : \forall a \in E_0 \cap E_1 \quad \|a\|_E \leq c \|a\|_0^{1-\theta} \|a\|_1^\theta$$

where  $\|\cdot\|_i$  denotes the norm in  $E_i$ ,  $i = 0, 1$ .

**Proof of Property 1.13**

We apply the above Theorem 4.1 with  $E_0 = L^{n'}$ ,  $E_1 = L^{(n')}$ .

Then from Theorem 1.4 and Corollary 2.7 of Proposition 2.2 one has

$$(L^{n'}, L^{(n')})_{\alpha,1} = L^{(n',\beta)}$$

with  $\beta = n\alpha - n + 1$  and  $\alpha > \frac{1}{n'}$ .

Since  $L^{(n',\beta)} \subset L^{n',\infty} \subset L^{n'}$ , we deduce the result from Theorem 4.1 with  $E = L^{n',\infty}$ ,  $\theta = \alpha$ .

◇

### 5. A remark on the associate space of $G\Gamma(p, r; \omega)$

#### Theorem 5.1.

Let  $1 < p < \infty$ ,  $1 < r < \infty$  and  $\delta > 0$ . Put  $w(t) = t^{-1}(1 - \text{Log } t)^{\delta-1}$ . Then

$$[G\Gamma(p, r; w)]' = G\Gamma(p', r'; w_1, w_2),$$

where  $w_1(t) = t^{-1}(1 - \text{Log } t)^{\frac{r'\delta}{r}-1}$  and  $w_2(t) = (1 - \text{Log } t)^{-\frac{2p'\delta}{r}}$ .

*Proof.*

Put  $\eta = p'\delta/r$ , and take  $\alpha = 2\eta$ ,  $\beta = \eta/2$  and  $\theta = 1/3$ . Then by [1] Theorem 7, we have

$$G\Gamma(p, r; w) = (L^{(p,\beta)}, L^{(p,\alpha)})_{\theta,r}.$$

Since  $L^{(p,\beta)} \cap L^{(p,\alpha)} = L^{(p,\max\{\alpha,\beta\})}$  contains  $L^\infty$  and therefore is dense in both  $L^{(p,\beta)}$ ,  $L^{(p,\alpha)}$  (see [15]), and since in this setting the associate space and the dual space coincide (see [4]), we may use the duality relation for real interpolation spaces (see, for instance, [5] Theorem 3.7.1), and we get

$$\begin{aligned} [G\Gamma(p, r; w)]' &= (L^{p'}, \beta, L^{p'}, \alpha)_{\theta, r'} \\ &= (L^{p'}, \alpha, L^{p'}, \beta)_{1-\theta, r'}. \end{aligned}$$

Finally, an application of Theorem 3.6 completes the proof.  $\square$

**Remark 5.2.** In view of Lemma 2.4, we get the following equivalent norm on  $[G\Gamma(p, r; w)]'$ :

$$\|f\|_{[G\Gamma(p,r;w)]'} \approx \left( \int_0^1 (1 - \text{Log } t)^{-\frac{r'\delta}{r}-1} \left( \int_t^1 f_*^{p'}(x) dx \right)^{\frac{r'}{p'}} \frac{dt}{t} \right)^{1/r'},$$

which is apparently simpler than the one which follows from [19] Theorem 1.1 (vi).

#### Acknowledgment :

The third author, M.R. Formica, has been partially supported by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM) and by Università degli Studi di Napoli Parthenope through the project "sostegno alla Ricerca individuale".

The work of fourth author, A.Gogatishvili, has been partially supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) [grand number FR17-589], by Grant 18-00580S of the Czech Science Foundation and RVO:67985840.

We address a grateful thanks to Professor Teresa Signes to point out an error in the first version of this work.

Finally, we are grateful to the anonymous referee for his/her careful reading of the paper and for the suggestions which improved the original version of this paper.

## REFERENCES

- [1] **I. Ahmed, A. Hafeez and G. Murtaza**, Real interpolation of small Lebesgue spaces in a critical case, *J. Funct. Spaces*, <https://doi.org/10.1155/2018/3298582>, (2018), Art. ID 3298582, 5pp.
- [2] **J. Bastero, F. Ruiz**, Elementary reverse Hölder type inequalities with application to operator interpolation theory, *Proc. A.M.S.* **124** (10) (1996), 3183-3192.
- [3] **C. Bennett, K. Rudnick**, *On Lorentz-Zygmund spaces*, *Dissertationes Math. (Rozrrawy Mat.)* 175 (1980) 67 pp.
- [4] **C. Bennett, R. Sharpley**, *Interpolation of Operators*, Academic Press, 1988.
- [5] **J. Bergh and J. Löfström**, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.
- [6] **H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa**, Blow up for  $u_t - \mathbb{D}u = g(u)$  revisited. *Advance in Diff. Eq.* **1**, (1996) 73-90.
- [7] **J. Cerdà, H. Coll**, Interpolation of classical Lorentz Spaces, *Positivity* **7**, 3, (2003) 225-234.
- [8] **F. Cobos, O. Domínguez**, On Besov spaces of logarithmic smoothness and Lipschitz spaces, *J. Math. Anal. Appl.* **425**, (2015), 71-84.
- [9] **G. Di Fratta, A. Fiorenza**, A direct approach in the duality of grand and small Lebesgue spaces, *Nonlinear Anal. TMA* **70** (2009) 2582-2592.
- [10] **J. I. Díaz, D. Gómez-Castro, J. M. Rakotoson, R. Temam**, Linear diffusion with singular absorption potential and/or unbounded convective flow: the weighted space approach, *Discrete and Continuous Dynamical Systems* **38**, 2, (2018) 509-546
- [11] **J.I. Díaz, J.M. Rakotoson**, On the differentiability of the very weak solution with right-hand side data integrable with respect to the distance to the boundary *J. Functional Analysis* **257** (2009) 807-831.
- [12] **A. Fiorenza, M.R. Formica, A. Gogatishvili, T. Kopaliani, J.M. Rakotoson**, Characterization of interpolation between Grand, small or classical Lebesgue spaces, *Non Linear Analysis* **177** (2018) 422-453.
- [13] **A. Fiorenza, M.R. Formica, J.M. Rakotoson**, Pointwise estimates for  $GT$ -functions and applications, *Differential Integral Equations* **30**, 11-12 (2017) 809-824.
- [14] **A. Fiorenza, M. Krbec**, On an optimal decomposition in Zygmund spaces, *Georgian Math. J.* **9**, (2) (2002), 271-286.
- [15] **A. Fiorenza, J.M. Rakotoson**, New properties of small Lebesgue spaces and their applications, *Math. Ann.* **326**, (2003), 543-561.

- [16] **A. Fiorenza, J.M. Rakotoson**, Some estimates in  $GF(p,m,w)$  spaces, *J. Math. Anal. Appl.* **340**, (2008), 793–805.
- [17] **A. Gogatishvili, M. Krepela, L. Pick, F. Soudsky**, Embeddings of Lorentz-type spaces involving weighted integral means, *J.F.A.* **273**, 9 (2017) 2939-2980.
- [18] **A. Gogatishvili, L. Pick**, Discretization and anti-discretization of rearrangement-invariant norms, *Publ. Mat.* **47** (2003) 311-358.
- [19] **A. Gogatishvili, L. Pick and F. Soudský**, Characterization of associate spaces of weighted Lorentz spaces with applications. *Studia Math.* 224 (2014), no. 1, 1-23.
- [20] **M.L. Gol'dman, H.P. Heinig, V.D. Stepanov**, On the principle of duality in Lorentz spaces, *Canad. J. Math.* **48** 5, (1996) 959-979.
- [21] **M.E. Gomez and M. Milman**, Extrapolation spaces and almost-everywhere convergence of singular integrals, *J. London Math. Soc.* **34**, (1986), 305–316.
- [22] **T. Iwaniec, C. Sbordone**, On the integrability of the jacobian under minimal, *Arch. Rat. Mech. Anal.* **119** (1992), 129-143.
- [23] **J.M. Rakotoson**, New hardy inequalities and behaviour of linear elliptic equations *J. Functional Analysis* **263** 9 (2012) 2893-2920.
- [24] **J.M. Rakotoson**, A sufficient condition for a blow-up on the space. Absolutely conditions functions for the very weak solution *AMO* **73**(2016) 153-163.
- [25] **J.M. Rakotoson**, *Rearrangement Relatif Un instrument d'estimation pour les problemes aux limites*, Springer (2008).
- [26] **L. Tartar**, *An introduction to Sobolev spaces and Interpolation spaces*, Springer-Verlag Berlin (2007)
- [27] **L. Veron** *Singularities of solutions of second order quasilinear equations*, Longman, Edinburgh Gate, Harlow,(1995) p 176-180.

<sup>1</sup>DEPARTMENT OF MATHEMATICS, SUKKUR IBA UNIVERSITY, SUKKUR, **Pakistan**  
 Email address: [quantized84@yahoo.com](mailto:quantized84@yahoo.com), [irshaad.ahmed@iba-suk.edu.pk](mailto:irshaad.ahmed@iba-suk.edu.pk)

<sup>2</sup> UNIVERSITÀ DI NAPOLI "FEDERICO II", VIA MONTEOLIVETO, 3, I-80134 NAPOLI, **Italy** AND ISTITUTO PER LE APPLICAZIONI DEL CALCOLO "MAURO PICONE" CONSIGLIO NAZIONALE DELLE RICERCHE VIA PIETRO CASTELLINO, 111 I-80131 NAPOLI, **Italy**  
 Email address: [fiorenza@unina.it](mailto:fiorenza@unina.it)

<sup>3</sup>UNIVERSITÀ DEGLI STUDI DI NAPOLI "PARTHENOPE", VIA GENERALE PARISI 13, 80132, NAPOLI,  
**Italy**

*Email address:* [mara.formica@uniparthenope.it](mailto:mara.formica@uniparthenope.it)

<sup>4</sup>INSTITUTE OF MATHEMATICS OF THE CZECH ACADEMY OF SCIENCES - ŽITNÁ, 115 67 PRAGUE  
1, **Czech Republic**

*Email address:* [gogatish@math.cas.cz](mailto:gogatish@math.cas.cz)

<sup>5</sup> LABORATOIRE DE MATHÉMATIQUES ET APPLICATIONS - UNIVERSITÉ DE POITIERS, 11 BD MARIE  
ET PIERRE CURIE, TÉLÉPORT 2, 86073 POITIERS CEDEX 9, **France**

*Email address:* [rako@math.univ-poitiers.fr](mailto:rako@math.univ-poitiers.fr), [jean.michel.rakotoson@univ-poitiers.fr](mailto:jean.michel.rakotoson@univ-poitiers.fr)