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Abstract

This work deals with the propagation of chaos without cut-off for some aggregation-diffusion models known as fractional Keller Segel equations. The diffusion considered here is given by the fractional Laplacian operator $-(-\Delta)^{\frac{a}{2}}$ with $a \in (1,2)$ and the singularity of the aggregation kernel behaves like $|x|^{1-\alpha}$ with $\alpha \in]1, a]$. In the Diffusion Dominated case $\alpha \in (1, a)$, we give a propagation of chaos result, thanks to the Γ lower semi continuity of the fractional Fisher information, already known for the classical Fisher information, using a result of [20]. In the fair competition case $a = \alpha$, we only prove a convergence/consistency result in a sub-critical sensitivity regime, similarly as the result obtained for the classical Keller-Segel equation in [16].

Résumé français. Ce travail concerne la propagation du chos moléculaire pour une classe de modèles d'agrégation-diffusion appelés équations de Keller-Segel fractionnaires. La diffusion en jeu est modélisée par l'opérateur de Laplacien fractionnaire $-(-\Delta)^{\frac{a}{2}}$ pour $a \in (1,2)$ et la singularité du noyau d'agrégation se comporte comme $|x|^{1-\alpha}$ avec $\alpha \in]1, a]$. Dans le cas où la diffusion est dominante $\alpha \in (1, a)$, on donne un résultat de propagation du chaos, grâce la Γ semi continuité inférieure de l'information de Fisher fractionnaire, déjà connue dans le cas classique, en utilisant un résultat de [20]. Dans le cas où la compétition entre la diffusion et l'agrgation est équitable $a = \alpha$, on donne seulement un résultat de compacité/convergence dans un régime de chimiosensibilité sous critique, comparablement aux résultats obtenus pour l'équation de Keller-Segel classique dans [16].

Keywords: fractional Laplacian, aggregation-diffusion, tightness, Fisher infromation
2010 MSC: 35Q70, 35Q92, 35R11, 60G52

1. Introduction

The parabolic-elliptic Keller Segel equation has received a large attention from the kinetic community lately. This model deals with the chemotaxis of cells or bacteria evolving in a environment, which they are able to modify in order to communicate with each other. More precisely the evolution of the density of bacteria ρ_t and the concentration of chemoattractant c_t is given by the following system of coupled equations

$$\begin{aligned} \partial_t \rho_t + \chi \nabla \cdot (\rho_t \nabla c_t) &= \Delta \rho_t, \\ -\Delta c_t &= \rho_t, \end{aligned} \tag{1.1}$$

where $\chi > 0$ is a sensitivity parameter encoding the intensity of the aggregation. The parabolic-elliptic from of the system follows form the assumption that the diffusion coefficient of the chemoattractant is much larger than the one of the bacteria (see [23]). We refer to [4] for a proper biological and mathematical motivation. This model has been extensively studied, especially in dimension 2 which is the best understood and which makes particular biological sense in the context of bacteria motion. Some blow up phenomena are known to arise if the initial mass is too large

[4, Corollary 2.2], and global well posedness holds when the mass is small enough [11]. However the question of propagation of chaos for this model remains open.
Some bacteria are known for their "run and tumble" motion, therefore their trajectories are better described by Lévy flights than Brownian motion (see for instance [5]). This consideration motivates to replace the classical diffusion in the evolution equation of the density of bacteria with a fractional diffusion. We define the fractional Laplacian on $\mathbb{R}^d$ of exponent $a/2 \in (0,1)$ for smooth function $u \in C_c^\infty(\mathbb{R}^d)$ as

$$\begin{aligned} -(-\Delta)^{a/2}u(x) &= c_{d,a}\text{v.p.}\int_{\mathbb{R}^d}\frac{u(x)-u(y)}{|x-y|^{d+a}}dy \\ &= c_{d,a}\text{v.p.}\int_{\mathbb{R}^d}\frac{u(x)-u(y)-(x-y)\cdot\nabla u(x)}{|x-y|^{d+a}}dy, \end{aligned} \tag{1.2}$$

where $c_{d,a}$ is a normalization constant defined as

$$c_{d,a} = -\frac{2^a\Gamma\left(\frac{d+a}{2}\right)}{\pi^{d/2}\Gamma\left(-\frac{a}{2}\right)}.$$

The equality between the two lines of (1.2) comes from the fact that for any $x \in \mathbb{R}^d$ it holds

$$\text{v.p.}\int_{\mathbb{R}^d}\frac{x-y}{|x-y|^{d+a}}dy = 0.$$

Moreover the fractional Laplacian can be equivalently defined in terms of Fourier multipliers as

$$\mathcal{F}(-(-\Delta)^{a/2}u)(\xi) = -|2\pi\xi|^a\widehat{u}(\xi), \tag{1.3}$$

where $\mathcal{F}(\cdot)$ and $\widehat{\cdot}$ denote the Fourier transform, defined as $\mathcal{F}(h)(\xi) = \hat{h}(\xi) = \int_{\mathbb{R}^d} h(x)e^{-2i\pi x\cdot\xi}dx$ (see for instance [25] for equivalent definitions of the fractional Laplacian). Not only for the purpose of modeling, but also because of the recent popularity of fractional diffusion equation, the problem

$$\begin{aligned} &\partial_t\rho_t + \chi\nabla\cdot(\rho_t\nabla c_t) + (-\Delta)^{\frac{a}{2}}\rho_t = 0, \\ &-\Delta c_t = \rho_t, \end{aligned} \tag{1.4}$$

has been studied under various perspectives by different authors. In [22], Huang and Liu obtained local in time existence for L^2 initial condition when $a \in (1,2)$ in dimension 2. Escudero obtained global existence for a similar system in dimension 1 in [13]. In dimension 1, equation (1.4) has also been studied by Clavez and Bournaveas in [5] who prove global existence in case $a \in (0,1]$ for some initial condition in $L^p(\mathbb{R})$ for some $p \geq \frac{1}{a}$, and in case $a \in (1,2]$. They also show blow-up in case $a \in (0,1)$ if the initial condition has a small first order moment compared to its mass. More recently, in 2 dimensional settings and for $a \in (0,2)$, Biler et al obtained a blow-up condition for the solution (1.4) in [2, Theorem 2.1] for large $M^{\frac{2}{a}}$-Morrey norm of the initial condition.
In this paper we address the question of propagation of chaos, (we refer to [31] and next section for details) for a similar equation as (1.4), where we replace the Newtonian attraction force with a less singular interaction kernel. Let $\alpha \in (0,2)$, and on $\mathbb{R}^d$ define

$$W_\alpha(x) = \frac{|x|^{2-\alpha}}{2-\alpha}, \quad \text{and} \quad K_\alpha(x) = -\nabla W_a(x) = -\frac{x}{|x|^\alpha},$$

(with the convention "$\frac{|x|^0}{0} = \ln|x|$"). For $(a,\alpha) \in (0,2)\times(0,d)$ and $N \geq 1$ let $(\mathcal{Z}_t^i)_{i=1,\cdots,N,t\geq 0}$ be N independent a-stable Lévy flights on $\mathbb{R}^d$ (more precisions will be given about a-stable process in the next session), $(X_0^{N,1},\cdots,X_0^{N,N})$ a random variable on $\mathbb{R}^{dN}$ independent of the N Lévy flights and consider the particle system evolving on $\mathbb{R}^d$ defined as

$$X_t^{N,i} = X_0^{N,i} + \frac{\chi}{N}\int_0^t\sum_{j\neq i}^N K_\alpha(X_s^{N,i} - X_s^{N,j})ds + \mathcal{Z}_t^i. \tag{1.5}$$

We expect that when the number of particle goes to infinity, and the family of initial condition is assumed to be ρ_0-chaotic (see [31, Definition 2.1]), the above particle system approaches the following nonlinear PDE

$$\begin{aligned} &\partial_t \rho_t + \chi \nabla \cdot (\rho_t (K_\alpha * \rho_t)) + (-\Delta)^{\frac{a}{2}} \rho_t = 0 \\ &\rho_{t=0} = \rho_0. \end{aligned} \tag{1.6}$$

This nonlinear conservation equation, which will be referred as fractional Keller-Segel equation, is solved by the time marginals of the process solution to the nonlinear SDE

$$\mathcal{X}_t = \mathcal{X}_0 + \chi \int_0^t \int_{\mathbb{R}^d} K_\alpha(\mathcal{X}_s - y)\rho_s(dy)\, ds + \mathcal{Z}_t, \quad \rho_s = \mathcal{L}(\mathcal{X}_s), \tag{1.7}$$

with $(\mathcal{Z}_t)_{t\geq 0}$ an a-stable Lévy flight on $\mathbb{R}^d$ independent of $\mathcal{X}_0$.

Notation :

- Lebesgue and Sobolev's norm: In the rest of the paper (and sometimes above as well), v.p. stands for the Cauchy principal value of a singular integral, $\|\cdot\|_{L^p}$ is the L^p norm on $\mathbb{R}^d$, $\|\cdot\|_{L^p\cap L^q} := \|\cdot\|_{L^p} + \|\cdot\|_{L^q}$ and $|\cdot|_{H^s(\mathbb{R}^d)}$ is the fractional Sobolev semi-norm of exponent $s \in (0,1)$ defined as

$$|u|^2_{H^s(\mathbb{R}^d)} := \int\int \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}} dxdy.$$

- Norms and inner product on the Euclidean space : the Euclidean inner product on $\mathbb{R}^d$ will be denoted either $x\cdot y$ or $\langle x, y\rangle$, and sometimes the latter notation will be used for the inner product on other spaces when it will make sense. We will also use the notation $\langle x\rangle = \sqrt{1+|x|^2}$, and for $\kappa \in \mathbb{R}$,

$$m_\kappa(x) = \langle x\rangle^\kappa.$$

  The unit ball on $\mathbb{R}^d$ will be denoted $\mathbb{B}$, its complementary will be denoted $\mathbb{B}^c$, and the ball of radius $r > 0$, $\mathbb{B}_r$.

- Probability measures : For a functional F on $\mathbb{R}^{dN}$, and $i = 1, \cdots, N$ we note $\nabla_i F = (\partial_{di-(d-1)}F, \cdots, \partial_{di}F) \in \mathbb{R}^d$, and the notation X^x_k will stand for the integration variable $(x_1, \cdots, x_{k-1}, x, x_{k+1}, \cdots, x_N)$ in $\mathbb{R}^{dN}$.
  The notation $\mathcal{P}(E)$ stands for the set of probability measures on E, $\mathcal{P}_{\mathrm{sym}}(E^k)$ for the set of symmetric probability measures on E^k, and $\mathcal{P}_{\mathrm{sym}}(E^N)$ for the set of sequences of symmetric probabilities on E^N, i.e. invariant by permutation. W_p is the Wasserstein metric on $\mathcal{P}(\mathbb{R}^d)$ of order $p \geq 1$. For $T > 0$, the notation $\mathcal{D}([0,T];\mathbb{R}^d)$ stands for the *càdlàg* (right continuous with left limits) paths on $\mathbb{R}^d$, or equivalently the Skorokhod space from $[0,T]$ to $\mathbb{R}^d$. We define $\mathbf{e}_t : \gamma \in \mathcal{D}(0,T;\mathbb{R}^d) \mapsto \gamma(t) \in \mathbb{R}^d$ the evolution map at time t, and for $\rho \in \mathcal{P}\left(\mathcal{D}(0,T;\mathbb{R}^d)\right)$ we implicitly associate the family of probability measures $(\rho_t \in \mathcal{P}(\mathbb{R}^d))_{t\in[0,T]}$ defined as $\rho_t = \mathbf{e}_t \# \rho$.
  All the probability measure at stake in the paper are assumed to admit a density with respect to the Lebesgue measure, and the confusion will be abused between measures and their densities.

- Finally, the dependence of some generic constant C on the parameters of the problem will always be expressed in its indexation. We define the functional Φ as

$$\Phi(x,y) = (x-y)\ln\left(\frac{x}{y}\right),\ \forall (x,y) \in (0,\infty)\times(0,\infty).$$

  Recall that for any $x, y \geq 0$ it holds (see for instance the inequality below equation (6) of [34])

$$\Phi(x,y) \geq 4\left(\sqrt{x} - \sqrt{y}\right)^2. \tag{1.8}$$

The rest of the paper is organized as follows. In Section 2, we make some luminary discussions about propagation of chaos and a-stable processes, and state the main results of this paper after recalling the state of the literature on

similar problems. In section 3 we obtain a priori bound on some quantities of interests for the underlying particle system, by entropy dissipation method or stochastic calculus. In section 4 we adapt some classical coupling method for interacting diffusion which enables to conclude the compactness argument in the diffusion dominated case. In Appendix A, we pursue an informal discussion aiming at enlightening some technical limitations of the main result in the fair competition case. Appendix B discusses the case of non singular interactions $\alpha \leq 1$, and Appendix C contains a delayed proof.

2. Preliminaries and main results

2.1. Propagation of chaos

In this paper, we address the question of propagation of chaos for the particle system (1.5). For the sake of completeness we recall some basic notions on this topic, and refer to [31] for some further explanations. We begin with the

Definition 2.1 (Definition 2.1 in [31])**.** *Let $(u_N)_{N\geq 1}$ be a sequence of symmetric probabilities on E^N ($u_N \in \mathcal{P}_{sym}(E^N)$), with E some polish space. We say that u_N is u-chaotic, with $u \in \mathcal{P}(E)$, if for any $k \geq 2$ and $\phi_1, \cdots, \phi_k \in C_b(E)$ it holds*

$$\lim_{N\to+\infty} \int_{E^N} \bigotimes_{j=1}^{k} \phi_j u_N = \prod_{j=1}^{k} \left(\int_E \phi_j u \right).$$

Let us give a short comment about this definition. Consider $(u_N)_{N\geq 1}$ a sequence of symmetric probabilities, u-chaotic in the sense of the above Definition for some $u \in \mathcal{P}(E)$ and let $(X_N)_{N\geq 1}$ be a sequence of random vectors on E^N of law u_N for each $N \geq 1$. The chaos assumption then implies that asymptotically in N, the correlations between two different components of X_N is small, and each of its components follows the law u . In other words that u_N behaves like $u^{\otimes N}$ as N goes to infinity. Then we need the important

Proposition 2.1 (Proposition 2.2 of [31])**.** *Let be $(u_N)_{N\geq 1}$ be a sequence of symmetric probabilities on E^N (E a polish space), $(X^1, \cdots, X^N)_{N\geq 1}$ a sequence of random vector of law u_N, and $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^i}$ the emprical measure associated to this vector. Then*

(*i*) *u_N is u chaotic if and only if $(\mu_N)_{N\geq 1}$ converges in law (weakly in $\mathcal{P}(E)$) toward $u \in \mathcal{P}(E)$.*

(*ii*) *The sequence of random variables $(\mu_N)_{N\geq 1}$ is tight if and only if the sequence of law of X^1 under u_N is tight, or equivalently if and only if the sequence of the first marginal of the u_N is tight.*

Our aim is therefore to prove that the dynamic (1.5) propagates chaos i.e. that if one starts this dynamic from some initial condition which law is ρ_0-chaotic, the law of the solution at time $t > 0$ to (1.5) is ρ_t-chaotic, with ρ_t the solution at time $t > 0$ to (1.6). Or equivalently, due to the above Proposition, to prove that

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_0^i} \overset{*,(\mathcal{L})}{\underset{N\to+\infty}{\rightharpoonup}} \rho_0 \;\Rightarrow\; \forall t > 0,\ \mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} \overset{*,(\mathcal{L})}{\underset{N\to+\infty}{\rightharpoonup}} \rho_t.$$

Implicitly, such a statement requires a good knowledge of the well posedness of the limit problem (1.6). When this knowledge is not available (typically when the singularity of the kernel K_α is too high), one can not expect such a strong result as it does not make sense to talk about "the" solution at time $t > 0$, ρ_t. However one can look at a weaker result of the type, if

$$\mu_0^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_0^i} \overset{*,(\mathcal{L})}{\underset{N\to+\infty}{\rightharpoonup}} \rho_0,$$

then there exists a subsequence of the $\left((\mu_t^N)_{t\in[0,T]}\right)_{N\geq 1}$ converging in law toward some (possibly random) $(\mu_t)_{t\in[0,T]} \in \mathcal{P}\left(\mathcal{D}(0,T;\mathbb{R}^d)\right)$, which solves to (1.6) and such that $\mu_0 = \rho_0$. in this case we say that we derive a convergence/consistency result, rather than propagation of chaos result.

2.2. a-stable processes with index $a \in (1,2)$

Let M be a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^d$ of intensity $Leb \times \lambda$ where λ is a σ-finite measure on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\lambda(dx) < \infty,$$

(see for instance [9, Definition 2.3, Chapter V]). Denote $\bar{M}$ its compensated measure, i.e.

$$\bar{M}(ds, dx) = M(ds, dx) - ds\lambda(dx), \tag{2.9}$$

, and denote $(Z_t)_{t\geq 0}$ the following Lévy process

$$Z_t = \int_{[0,t]\times\mathbb{B}} x\overline{M}(ds, dx) + \int_{[0,t]\times\mathbb{B}^c} xM(ds, dx). \tag{2.10}$$

The first stochastic integral in the r.h.s. converges in the sense of the Cauchy principal value i.e.

$$\int_{[0,t]\times\mathbb{B}} x\overline{M}(ds, dx) = \lim_{\varepsilon\to 0}\left(\int_{[0,t]\times\mathbb{B}\setminus\mathbb{B}_\varepsilon} xM(ds, dx) - t\int_{\mathbb{B}\setminus\mathbb{B}_\varepsilon} x\lambda dx\right) \quad \text{a.s.}$$

Due to Ito's rule [1, Theorem 4.4.7, p 226] we have for a test function ϕ smooth enough

$$\begin{aligned}\phi(Z_t) =& \phi(Z_s) + \int_{[s,t]\times\mathbb{B}} x\cdot\nabla\phi(Z_{u_-})\overline{M}(du, dx) + \int_{[s,t]\times\mathbb{B}^c} x\cdot\nabla\phi(Z_{u_-})M(du, dx)\\ &+ \int_{[s,t]\times\mathbb{R}^d} \left(\phi(Z_{u_-} + x) - \phi(Z_{u_-}) - x\cdot\nabla\phi(Z_{u_-})\right) M(du, dx).\end{aligned}$$

In the particular case

$$\lambda(dx) = c_{d,a}\frac{dx}{|x|^{d+a}},$$

we can rewrite

$$Z_t = \int_{[0,t]\times\mathbb{R}^d} x\bar{M}(ds, dx),$$

since $\int_{\mathbb{B}^c} x\lambda(dx) = 0$, and then

$$\begin{aligned}\phi(Z_t) =& \phi(Z_s) + \int_{[s,t]\times\mathbb{R}^d} \left(\phi(Z_{u_-} + x) - \phi(Z_{u_-})\right)\bar{M}(du, dx)\\ &+ c_{d,a}\int_s^t \text{v.p.}\int_{\mathbb{R}^d} \frac{\phi(Z_u + x) - \phi(Z_u) - x\cdot\nabla\phi(Z_u)}{|x|^{d+a}}dx\,du.\end{aligned} \tag{2.11}$$

This particular choice of intensity makes of $(Z_t)_{t\geq 0}$ defined in (2.10) an a-stable Lévy process, i.e. $(Z_t)_{t\geq 0}$ has the same law as $(u^{-1/a}Z_{ut})_{t\geq 0}$ for any $u > 0$. Necessarily, such a process can only exists for $a \in [0, 2]$ [9, Exercice 2.34, Chapter VI], the case $a = 0$ corresponding to the null process, and the case $a = 2$, to the standard Brownian motion. It is well known, but we also see from (2.11), that the infinitesimal generator of the a-stable Lévy process is the fractional Laplacian of exponent $a/2$ defined in (1.2). It follows also from (2.11) and classical properties of Poisson random measures that for any smooth function ϕ the process $(\mathcal{M}_t)_{t\geq 0}$ defined as

$$\mathcal{M}_t = \phi(Z_t) - \phi(0) - c_{d,a}\int_0^t \text{v.p.}\int_{\mathbb{R}^d} \frac{\phi(Z_u + x) - \phi(Z_u) - z\cdot\nabla\phi(Z_u)}{|z|^{d+a}}dx\,du,$$

is a martingale, and the a-stable Lévy flight $(Z_t)_{t\geq 0}$ is the only process such that $(\mathcal{M}_t)_{t\geq 0}$ defined as above is a martingale.

2.3. Main result

We now give some comments on propagation of chaos results for similar systems as the one studied here, already existing in the literature after which we will give the main result of this paper. We emphasize that there are only a few results of propagation of chaos for particle system with singular interaction and additive diffusion, beside the ones we recall here. They rely essentially on the fact that the diffusion is non degenerate, and in particular the strategy would not apply for second order system with a diffusion only in velocity.
We introduce three different cases. When $a > \alpha$ we say we are in *Diffusion Dominated* case, when $a = \alpha$, in *Fair competition* case and $a < \alpha$ in *Aggregation Dominated* case. This terminology has been introduced by Carrillo et al (see for instance [6, Definition 2.6]) and is based on the homogeneity analysis of the free energy for which the system they study is a gradient flow (in Wasserstein metric). However, the presence of fractional diffusion here makes difficult to write equation (1.6) in a gradient flow form (in usual Wasserstein metric see for instance [12]), and in some sense we abuse their terminology. Nevertheless note that the classical 2 dimensional Keller-Segel (which is $a = \alpha = 2$) falls in Fair competition case, as they define it, as the sub-critical Keller Segel equation (i.e. $2 = a > \alpha > 1$) studied in [18] falls in the Diffusion Dominated case. So that the extension we give here of their nomenclature is not completely senseless.

We begin with the *Aggregation Dominated* case, as it is the less understood of all. To the best of the author's knowledge, there is no result without cut-off, except for the case $a = 0$ that is without any diffusion at all. In [8], the authors consider the case $a = 0$ and $\alpha \in (0, 2)$, as the absence of diffusion makes possible here to control the minimal inter-particle distance, thus the singularity of the interaction, but under some very restrictive assumptions on the initial distribution of the particles. As for cut-off result, Huang and Liu treated the case $\alpha = 2$ and $a \in (0, 1)$ with logarithmic cut-off of order $(\ln N)^{-1/2}$ in [22]. More recently Garcia and Pickl treated in [17] the classical Keller Segel case $a = \alpha = 2$ in the sub-critical sensitivity regime $\chi < 4$, with polynomial cut-off of order $N^{-\delta}$ with $\delta \in (0, 1/2)$, but it is likely that the coupling techniques they used, can be extended to the full Aggregation Dominated case, provided that one controls the L^∞ norm of the limit solution.
For the *Fair competition* case, the only existing result to the best of the author's knowledge, is the one of Fournier and Jourdain [16, Theorem 6] for the classical Keller Segel equation.
Finally the *Diffusion Dominated* case is the easiest as far as propagation of chaos is concerned. Godinho and Quininao treated the case $a = 2, \alpha \in (1, 2)$ in [18]. Let also be mentioned that when $a = 2$ and $\alpha \in (0, 1)$, the interaction K_α is $(1 - \alpha)$-Holder, and the propagation of chaos follows from a recent and more general result by Holding [21, Theorem 2.1]. This result provides convergence rate in Wasserstein metric, and is very different from the other results quoted in this section. Nevertheless, this result is obtained by taking advantage of the diffusion, and could not be stated in the deterministic case.
To summarize, the Newtonian interaction is critically controlled by a classical diffusion, and a less singular than Newtonian interaction is perfectly controlled without any diffusion at all or with a classical diffusion. As these two cases correspond to the two extremal exponent for stable Lévy process ($a = 0$ and $a = 2$), a natural question to investigate is, which type of singularity can be controlled by a fractional diffusion. It is the object of the main result of this paper given in the

Theorem 2.1. *Let be $T > 0$.*

Diffusion Dominated *Let be $2 > a > \alpha > 1$ and $\left((X_t^{N,1}, \cdots, X_t^{N,N})_{t\in[0,T]}\right)_{N\geq 1}$ be a sequence of solutions to equation (1.5) with initial condition of law $\left(\rho_0^{\otimes N}\right)_{N\geq 1}$ with $\rho_0 \in L\log L(\mathbb{R}^d) \cap \mathcal{P}_\kappa(\mathbb{R}^d)$ for some $\kappa \in (1, a)$. Then the sequence $\left(\left(\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}\right)_{t\in[0,T]}\right)_{N\geq 1}$ converges in law $\mathcal{P}\left(\mathcal{D}(0, T; \mathbb{R}^d)\right)$ to $(\rho_t)_{t\in[0,T]}$ the unique solution to equation (1.6) in $L^1(0, T; L^1 \cap L^p(\mathbb{R}^d))$ for any $p \in \left(\frac{d}{d-\alpha}, \frac{d}{d-a}\right)$ starting from ρ_0.*

Fair Competition *Assume that $a \in (a^*, 2)$, $\alpha = a$ and $\chi \in (0, \chi(a))$ for some given real $a^* \in (1, 2)$ and function $\chi : (a^*, 2) \mapsto (0, \infty)$. Let $\left((X_t^{N,1}, \cdots, X_t^{N,N})_{t\in[0,T]}\right)_{N\geq 1}$ be a sequence of solutions to equation (1.5) with initial condition of laws $\left(F_0^N\right)_{N\geq 1} \in \mathcal{P}_{sym}(\mathbb{R}^{dN})$ being ρ_0-chaotic in the sense of Definition 2.1 and satisfying*

$$\sup_{N\geq 1} \int_{\mathbb{R}^{dN}} \langle x_1\rangle^\kappa F_0^N(dx_1, \cdots, dx_N) < \infty,$$

for some $\kappa \in (1, a)$. *Then there exists a subsequence of* $\left(\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}\right)_{t\in[0,T]}\right)_{N\geq 1}$ *which converges in law in* $\mathcal{P}(\mathcal{D}(0,T;\mathbb{R}^d))$ *to* $(\rho_t)_{t\in[0,T]}$, *a solution to equation* (1.6) *starting from* $\rho_0 \in \mathcal{P}_\kappa(\mathbb{R}^d)$ *which satisfies for any* $\varepsilon \in (0,1)$

$$\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^{\varepsilon-a}\rho_s(dx)\rho_s(dy)\,ds < \infty.$$

Remark 2.1. *In the Diffusion Dominated case, the result could be extended to the case where the initial condition is not necessarily tensorized. Actually, we could perform the proof with a sequence of initial condition* $\left(F_0^N\right)_{N\geq 1} \in \mathcal{P}_{sym}(\mathbb{R}^{dN})$ *which is* ρ_0*-chaotic in the sense of Definition 2.1, and satisfying*

$$\sup_{N\geq 1} N^{-1}\int_{\mathbb{R}^{dN}} F_0^N \ln F_0^N < \infty,\ \sup_{N\geq 1}\int_{\mathbb{R}^{dN}} \langle x_1\rangle^\kappa F_0^N < \infty.$$

But as it is customary in propagation of chaos result, to start from i.i.d. initial conditions, we will do so for the sake of simplicity.
In the Fair Competition case, the limitations $a \in (a^*, 2)$ *is mainly technical. In practice,* a^* *is equal to* 1, *and* $\chi(a)$ *can be defined explicitly as*

$$\chi(a) = 2^{a-1}\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{a-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+a-1}{2}\right)},$$

see Appendix A for further considerations on this topic.

We emphasize that this result is, to the best of the author's knowledge, the first propagation of chaos result with singular kernel and anomalous diffusion.
Let us briefly sketch the proof of this theorem. In both Diffusion Dominated and Fair Competition cases, the key argument is given in the

Lemma 2.1. *Let* $(((X_t^{N,i})_{t\geq 0}))_{i=1,\cdots,N})_{N\geq 1}$ *be a family of processes such that for each* $t \geq 0$ *and* $N \geq 1$ *the law of the random vector* $(X_t^{N,i})_{i=1,\cdots,N}$ *is symmetric, and for some* $p > 1$ *and* $T > 0$ *it holds*

$$\sup_{N\geq 1}\mathbb{E}\left[\int_0^T |X_s^{N,1} - X_s^{N,2}|^{(1-\alpha)p} ds\right] < \infty. \tag{2.12}$$

Then the family of laws of the processes $((J_t^N)_{t\in[0,T]})_{N\geq 1}$*defined by*

$$J_t^N = \int_0^t \frac{1}{N}\sum_{k>1}^{N} K_\alpha(X_s^{N,1} - X_s^{N,k})ds,$$

is tight under $\mathbb{P}$.

This Lemma can be seen as a particular application of the Kolmogorov's tightness criteria in a mean field settings and has been much used in similar problems (see for instance [16, Lemma 11], [15, Lemma 5.2] or [18, Lemma 4.1]). For the sake of completeness we detail the proof below

Proof. For any $0 \leq s < t \leq T$ we have

$$\begin{aligned}
\left|J_t^N - J_s^N\right| &\leq \frac{1}{N}\sum_{j>1}\int_s^t \left|X_u^{N,1} - X_u^{N,j}\right|^{1-\alpha} du \\
&\leq |t-s|^{p'}\frac{1}{N}\sum_{j>1}\left(\int_0^T |X_u^{N,1} - X_u^{N,j}|^{(1-\alpha)p} du\right)^{1/p} \\
&\leq |t-s|^{p'}\left(1 + \frac{1}{N}\sum_{j>1}\int_0^T |X_u^{N,1} - X_u^{N,j}|^{(1-\alpha)p}\,du\right) := |t-s|^{p'} Z_{N,p}^T.
\end{aligned}$$

Then for $R > 0$ let us denote

$$\mathcal{K}^R = \{h \in C([0,T],\mathbb{R}^d),\ h(0) = 0,\ \sup_{0\le s<t\le T} \frac{|h(s)-h(t)|}{|s-t|^{p'}} \le R\},$$

which is compact by Ascoli-Azerla's Theorem. Then using Markov's inequality and the symmetry in law of the $X_t^{N,i}$ yields

$$\begin{aligned}\sup_{N\ge 1}\mathbb{P}\left((J_t^N)_{t\in[0,T]} \notin \mathcal{K}^R\right) &= \sup_{N\ge 1}\mathbb{P}\left(Z_{N,p}^T \ge R\right)\\ &\le R^{-1}\sup_{N\ge 1}\mathbb{E}\left[Z_{N,p}^T\right]\\ &= R^{-1}\left(1+\sup_{N\ge 1}\mathbb{E}\left[\int_0^T |X_s^{N,1}-X_s^{N,2}|^{(1-\alpha)p}ds\right]\right),\end{aligned}$$

and the sequence of laws of $\left((J_t^N)_{t\in[0,T]}\right)_{N\ge1}$ is tight (see Definition before Theorem 5.1 of [3]), since for any $\varepsilon > 0$, we can choose R large enough such that it holds

$$\sup_{N\ge 1}\mathbb{P}\left((J_t^N)_{t\in[0,T]} \notin \mathcal{K}^R\right) \le \varepsilon.$$

□

There now remains to obtain the bound (2.12) from the particle dynamic (1.5) but this estimate is obtained with different techniques in the Diffusion Dominated and the Fair Competition case.

For the Fair Competition case $a = \alpha$, we follow in Proposition 3.4 the strategy of [16] (see also [28]). Namely, the strategy relies on the following formal Ito's computation. If for $\varepsilon \in (0,1)$ we denote $\phi(x) = |x|^\varepsilon$ and $Z_t^{1,2} = X_t^{N,1} - X_t^{N,2}$, since $M_1 - M_2 \overset{(\mathcal{L})}{=} 2^{1/a}M_1$, stochastic calculus for jump processes formally yields

$$\begin{aligned}\phi\left(Z_t^{1,2}\right) = \phi\left(Z_0^{1,2}\right) &- \int_0^t \frac{\chi\varepsilon}{N}|Z_s^{1,2}|^{\varepsilon-2}Z_s^{1,2}\cdot\left(\sum_{k>2}^N\left(K_\alpha(Z_s^{1,k})-K_\alpha(Z_s^{2,k})\right)\right)ds - \int_0^t \frac{2\chi\varepsilon}{N}\left|Z_s^{1,2}\right|^{\varepsilon-a}ds\\ &+ \int_{[0,t]\times\mathbb{R}^d}\left(\phi\left(Z_{s_-}^{1,2}+2^{1/a}x\right)-\phi\left(Z_{s_-}^{1,2}\right)-2^{1/a}x\cdot\nabla\phi\left(Z_{s_-}^{1,2}\right)\right)M_1(ds,dx)\\ &+ \int_{[0,t]\times\mathbb{R}^d} 2^{1/a}x\cdot\nabla\phi\left(Z_{s_-}^{1,2}\right)\bar{M}_1(ds,dx).\end{aligned} \tag{2.13}$$

Using the symmetry of the roles played by the particles yields

$$\mathbb{E}\left[|Z_s^{1,2}|^{\varepsilon-2}Z_s^{1,2}\cdot\left(\sum_{k>1}^N\left(K_a(Z_s^{1,k})-K_a(Z_s^{2,k})\right)\right)\right] \le 2(N-2)\mathbb{E}\left[|Z_s^{i,j}|^{\varepsilon-a}\right].$$

Now since $M_1(ds,dx)$ has the intensity $dsc_{d,a}\frac{dx}{|x|^{d+a}}$, and by definition of the fractional Laplacian we have

$$\begin{aligned}&\mathbb{E}\left[\int_{[0,t]\times\mathbb{R}^d}\left(\phi\left(Z_{s_-}^{1,2}+2^{1/a}x\right)-\phi\left(Z_{s_-}^{1,2}\right)-2^{1/a}x\cdot\nabla\phi\left(Z_{s_-}^{1,2}\right)\right)M_1(ds,dx)\right]\\ &\quad= \mathbb{E}\left[\int_0^t c_{d,a}\int_{\mathbb{R}^d}\frac{\left(\phi\left(Z_{s_-}^{1,2}+2^{1/a}x\right)-\phi\left(Z_{s_-}^{1,2}\right)-2^{1/a}x\cdot\nabla\phi\left(Z_{s_-}^{1,2}\right)\right)}{|x|^{d+a}}dxds\right]\\ &\quad= 2\int_0^t\mathbb{E}\left[(-\Delta)^{a/2}\phi(Z_s^{1,2})\right]ds = 2\varepsilon c_{\varepsilon,a}\int_0^t\mathbb{E}\left[|Z_s^{1,2}|^{\varepsilon-a}\right]ds,\end{aligned}$$

where we used the definition of the fractional Laplacian by Fourier multipliers in the last line. Finally since the last term in the r.h.s. (2.13) is a martingale, in the end we obtain

$$\sup_{N\geq 1}\mathbb{E}\left[|X_T^{N,1}-X_T^{N,2}|^{\varepsilon}\right]\geq 2\varepsilon(c_{\varepsilon,a}-\chi)\sup_{N\geq 1}\int_0^T\mathbb{E}\left[|X_s^{N,1}-X_s^{N,2}|^{\varepsilon-a}\right]ds,$$

which yields the desired bound (2.12) with $p=\frac{a-\varepsilon}{a-1}$ for some $\varepsilon\in(0,1)$, provided that $\chi\in\left(0,\sup_{\varepsilon\in(0,1)}c_{\varepsilon,a}\right)$. Of course this scheme of proof is only formal as we can not apply Ito's rule with the non smooth function $\phi(x)=|x|^{\varepsilon}$ and we have to replace it with a mollification $\phi_\eta(x)=(|x|^2+\eta^2)^{\varepsilon/2}$ for any $\eta>0$. But the fractional Laplacian of this mollified function can not be explicitly computed, and have to be estimated by below (see Lemma 3.2). In Appendix A we develop the formal computation (2.13) in order to give a sharper idea of the critical value of the sensitivity χ

For the Diffusion Dominated case, we follow in Proposition 3.3 the strategy of [15]. Namely the law of solutions to (1.5), $F_t^N=\mathcal{L}((X_t^{N,i})_{i=1,\cdots,N})\in\mathcal{P}_{sym}(\mathbb{R}^{dN})$ solves the Liouville equation

$$\partial_t F_t^N+\sum_{i=1}^N\nabla_i\cdot\left(\frac{1}{N}\sum_{j\neq i}K_\alpha(x_i-x_j)F_t^N\right)+\sum_{i=1}^N c_{d,a}\text{p.v.}\int_{\mathbb{R}^d}\frac{F_t^N(x_1,\cdot,x_i+z,\cdot,x_N)-F_t^N(x_1,\cdots,x_N)}{|z|^{d+a}}dz=0,\qquad(2.14)$$

for the initial condition $F^N_{|t=0}=F_0^N$, the law of initial conditions of (1.5). Since we assume that these initial conditions are i.i.d. of law $\rho_0\in L\ln L(\mathbb{R}^d)$ i.e. $F_0^N=\rho_0^{\otimes N}$, estimating the entropy dissipation along the Liouville equation (2.14) yields the following uniform control

$$\sup_{N\geq 1}\int_0^T\mathcal{I}_a^N(F_t^N)dt<\infty.$$

where $\mathcal{I}_a^N$ is the normalized fractional Fisher information defined on $\mathcal{P}(\mathbb{R}^{dN})$ as the dissipation of entropy along the fractional many-body heat flow

$$\mathcal{I}_a^N(F_N)=\frac{1}{2}\frac{1}{N}\sum_{k=1}^N\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\Phi\left(F_N(X_k^x),F_N(X_k^y)\right)}{|x-y|^{d+a}}dxdy\,dX^{N-1}.\qquad(2.15)$$

Thanks to natural properties of this Fisher information (Proposition 3.1) and Sobolev's embeddings (Lemma (3.5)), this control implies the desired bound.

In both cases, once the bound 2.12 is obtained, by Lemma 2.1 and point (ii) of Proposition 2.1, the family of the laws of the sequence of empirical measures $\left((\frac{1}{N}\sum_{i=1}^N\delta_{X_t^{N,i}})_{t\in[0,T]}\right)_{N\geq 1}$ is tight. Thus by Prokhorov's Theorem (see for instance [3, Theorem 5.1]), there exits a subsequence converging in law to some limit point $(\rho_t)_{t\in[0,T]}$ as N goes to infinity. In Theorem 3.1, we show that this limit point almost surely solves (1.6) for the initial condition ρ_0.

In the Fair Competition case, this solution almost surely satisfies for any $\varepsilon\in(0,1)$

$$\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^{\varepsilon-a}\rho_s(dx)\rho_s(dy)ds<\infty.$$

This information is not sufficient to conclude to the uniqueness of this cluster point, and we obtain only some convergence/consistency result in this case as in [16].

In the Diffusion Dominated case, this solution satisfies almost surely, thanks to Corollary 3.1

$$\int_0^T\mathcal{I}_a^1(\rho_s)ds=\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\Phi\left(\rho_s(x),\rho_s(y)\right)}{|x-y|^{d+a}}dxdy<\infty,$$

so that thanks to inequality (1.8), it holds almost surely $\rho\in L^2((0,T),H^{a/2}(\mathbb{R}^d))$, and thus by Sobolev's embeddings $\rho\in L^1((0,T),L^p(\mathbb{R}^d))$ for any $p\in\left(1,\frac{d}{d-a}\right)$. In Proposition 4.1, we use some rather classical coupling method to prove

that there is at most one solution to (1.6) lying in $L^1((0,T),L^p(\mathbb{R}^d))$ for some $p \in \left(\frac{d}{d-\alpha},\infty\right)$. In other words, there is one and only one accumulation point (for the convergence in law) of the sequence $\left((\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}})_{t\in[0,T]}\right)_{N\geq 1}$, which is the unique solution to (1.6) lying in $\rho \in L^1((0,T),L^p(\mathbb{R}^d))$ for some $p \in \left(\frac{d}{d-\alpha},\frac{d}{d-a}\right)$. Which yields to the convergence in law of the full sequence, and concludes the proof.

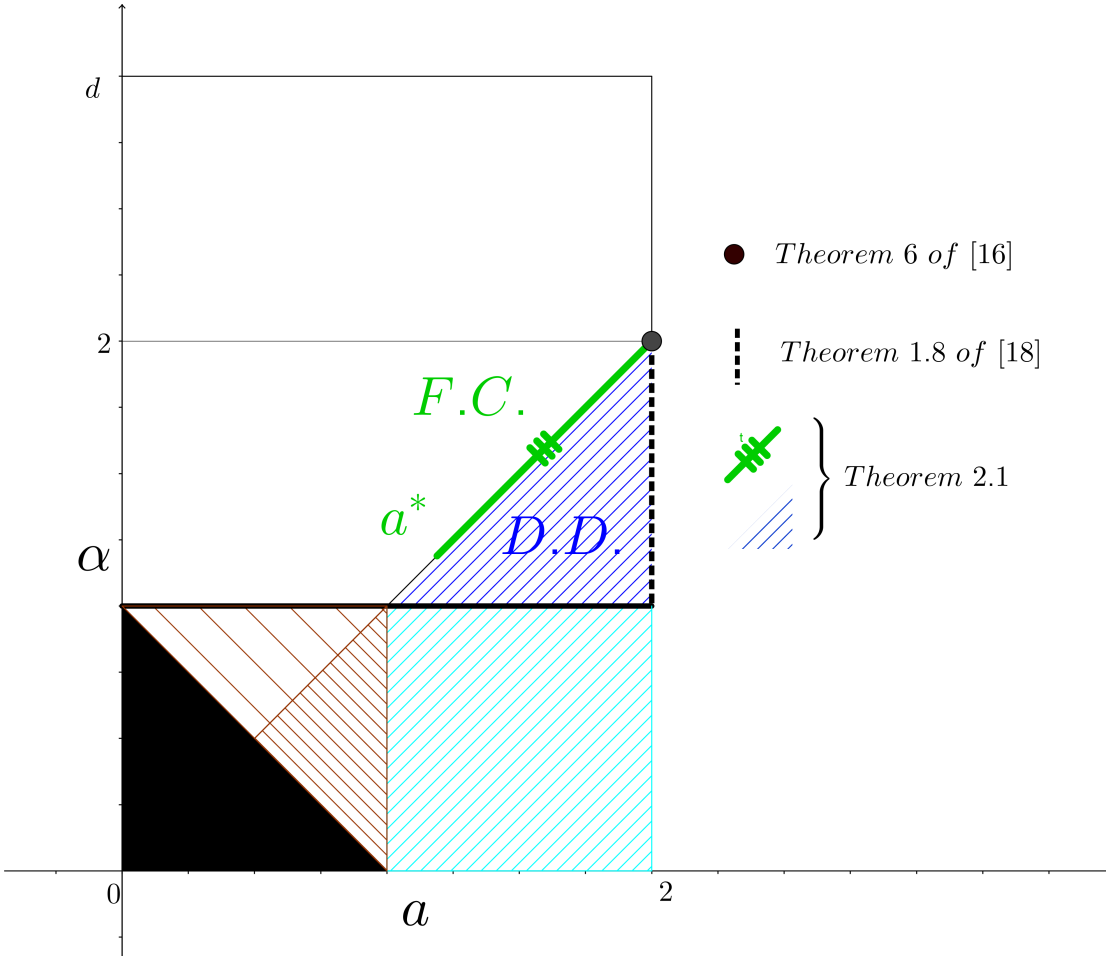

Figure 1. Topography of propagation of chaos results without cut-off for equation (1.6) in the plane (a,α). It seems delicate to make sense out of equation (1.6) in the region $\alpha + a < 1$, since moments of order at least $1-\alpha$ on ρ are needed to make sense of the term $(K_\alpha * \rho)$, and $-(-\Delta)^{a/2}$ propagates moments of order at most a (see Lemma 3.3). See also Appendix B for some results in the case $\alpha \leq 1$ depending on whether $\alpha \geq a$, $1 \geq a > \alpha$ or $\alpha \leq 1 < a \leq 2$.

Remark 2.2. *Note that we have not said anything so far regarding the existence of solutions to the particles system equation* (1.5)*. The difficulty comes from the non smoothness of the drift coefficient. However this can be solved by mollifying the interaction kernel, and showing that the family of the (unique due to classical SDE theory) solution with such a mollified drift is tight in the mollification parameter for fixed N. But as the computations done in the proofs of Proposition 3.3 and 3.4 show the tightness uniformly in N with the not mollified interaction kernel, they a fortiori show the tightness in the regularization parameter for the regularized system with fixed number of particles. Hence with a similar strategy as described above (see also [16, Theorem 5]) it is possible to build a solution to the particle system thanks to the tightness argument. We leave the reader check that the less singular kernel or the a-stable Lévy noise considered here instead of the Newtonian force and Brownian motion considered in [16], do not change the argument used by Fournier and Jourdain. However, this argument does not provide uniqueness, but it is not required in order to obtain Theorem 2.1.*

3. Tightness estimate

The key point of the proof of Theorem 2.1, is the tightness of the law of the particle system (1.5). Such a result follows from getting an estimation of the expectation of some singular function of the distance between the first and second particle (by ex changeability), but this estimate is obtained with very different techniques in the Diffusion Dominated and the Fair Competition case, as explained in the sketch of proof.

3.1. Useful estimates

In this section we provide some basic estimates which will be used later in the paper. We begin with the

Lemma 3.1. *The following properties hold.*

(*i*) *For all* $a, b \geq 0$ *and* $\alpha, \beta \geq 0$ *we have*

$$(\alpha a - \beta b)\,(\ln a - \ln b) \geq (a - b)(\alpha - \beta),$$

with equality if and only if $a = b$.

(*ii*) *The functional* Φ *defined on* $(0, \infty) \times (0, \infty)$ *as* $\Phi(x, y) = (x - y)(\ln x - \ln y)$ *is jointly convex.*

(*iii*) *For any* $\kappa \geq 1$ *and* $x, y \in \mathbb{R}^d$ *we have*

$$\left(\langle x\rangle^{\kappa-2} x - \langle y\rangle^{\kappa-2} y\right) \cdot (x - y) \geq 0.$$

Proof. (*i*) By symmetry of the role of a, b, α, β we only treat the cases $a \geq b, \alpha \geq \beta$ and $a \geq b, \alpha \leq \beta$. In the first case we easily obtain, since $x \mapsto \ln x$ is increasing

$$\begin{aligned}(a\alpha - b\beta)(\ln a - \ln b) &= (\alpha - \beta)a(\ln a - \ln b) + \beta(a - b)(\ln a - \ln b)\\ &\geq (\alpha - \beta)(a - b),\end{aligned}$$

because since $x \mapsto \ln x$ is concave, it holds

$$(\ln a - \ln b) \geq \frac{(a - b)}{a},$$

with equality if and only if $a = b$. On the other case we get

$$\begin{aligned}(a\alpha - b\beta)(\ln a - \ln b) &= \alpha(a - b)(\ln a - \ln b) + b(\ln a - \ln b)(\alpha - \beta)\\ &\geq (\alpha - \beta)(a - b),\end{aligned}$$

because it holds

$$(\ln a - \ln b) \leq \frac{(a - b)}{b},$$

which concludes the proof.
(*ii*) direct computations yields

$$\nabla^2\Phi(x, y) = (x + y)\begin{pmatrix} \frac{1}{x^2} & -\frac{1}{xy} \\ -\frac{1}{xy} & \frac{1}{y^2} \end{pmatrix},$$

which is nonnegative. (*iii*) The functional $m_\kappa : x \in \mathbb{R}^d \mapsto \langle x\rangle^\kappa$ is convex for $\kappa \geq 1$. And then for any $x, y \in \mathbb{R}^d$

$$(\nabla m_\kappa(x) - \nabla m_\kappa(y)) \cdot (x - y) \geq 0.$$

□

3.2. Fractional Laplacian and fractional Fisher information

In this section we provide the key arguments from which the tightness of the law of the particle system will follow. We start with the tool needed in the Fair Competition case. It consists roughly in giving a bound from below of the Ito's correction of the process $(|Z_t|^\varepsilon)_{t\geq 0}$ with $\varepsilon \in (0,1)$ and $(Z_t)_{t\geq 0}$ some d-dimensional and a-stable Lévy process. Precisely we have the

Lemma 3.2. *Let $\eta > 0$, $\varepsilon \in (0,1)$ and ϕ_η defined as*

$$\phi_\eta : x \in \mathbb{R}^d \mapsto \left(|x|^2+\eta^2\right)^{\frac{\varepsilon}{2}} = \eta^\varepsilon \langle \frac{x}{\eta}\rangle^\varepsilon.$$

Then for any $a \in (1,2)$ there exist some constants $C_{\varepsilon,a}, c_d > 0$ such that for any $x \in \mathbb{R}^d$ it holds

$$\int_{\mathbb{R}^d} \frac{\phi_\eta(x+z)-\phi_\eta(x)-z\cdot\nabla\phi_\eta(x)}{|z|^{d+a}}dz \geq |\mathbb{S}^{d-1}|\left(\frac{\varepsilon^2 c_d}{2(2-a)}(|x|^2+\eta^2)^{\frac{\varepsilon-4}{2}}|x|^{4-a} - \varepsilon C_{\varepsilon,a}|x|^{\varepsilon-a}\right),$$

with

$$C_{\varepsilon,a} = \left(\frac{2-\varepsilon}{\sqrt{4-\varepsilon}(3-a)\varepsilon} + \frac{1}{a\varepsilon}\right),$$
$$c_d = \begin{cases} \frac{1}{2} \text{ if } d=2 \\ \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}B\left(\frac{d-1}{2},\frac{3}{2}\right) \text{ if } d>2, \end{cases}.$$

where B stands for the beta Euler function.

Proof. Let be $x, z \in \mathbb{R}^d \times \mathbb{R}^d$, $\eta > 0$ and define $\phi_\eta^{z,x}$ as

$$\phi_\eta^{z,x}(t) = \left(|x+tz|^2+\eta^2\right)^{\frac{\varepsilon}{2}}.$$

Then straightforward computations yield

$$(\phi_\eta^{z,x})'(t) = \varepsilon\langle z, x+tz\rangle\left(|x+tz|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}},$$
$$(\phi_\eta^{z,x})''(t) = \varepsilon\left(|x+tz|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}|z|^2\left((\varepsilon-2)X_{x,t,z}^2+1\right),$$
$$(\phi_\eta^{z,x})^{(3)}(t) = \varepsilon(\varepsilon-2)\left(|x+tz|^2+\eta^2\right)^{\frac{\varepsilon-3}{2}}|z|^3\left((\varepsilon-4)X_{x,t,z}^3+3X_{x,t,z}\right),$$

with $X_{x,t,z} = \left\langle \frac{x+tz}{|x+tz|}, \frac{z}{|z|}\right\rangle\left(\frac{|x+tz|^2}{|x+tz|^2+\eta^2}\right)^{\frac{1}{2}}$.

Then Taylor's formula yields

$$\begin{aligned}\int_{\mathbb{R}^d}\frac{\phi_\eta^{z,x}(1)-\phi_\eta^{z,x}(0)-(\phi_\eta^{z,x})'(0)}{|z|^{d+a}}dz &= \int_{|z|\leq|x|}\frac{\phi_\eta^{z,x}(1)-\phi_\eta^{z,x}(0)-(\phi_\eta^{z,x})'(0)}{|z|^{d+a}} \\ &+ \int_{|z|\geq|x|}\frac{\phi_\eta^{z,x}(1)-\phi_\eta^{z,x}(0)}{|z|^{d+a}}dz \\ &= \int_{|z|\leq|x|}\frac{(\phi_\eta^{z,x})''(0)}{2|z|^{d+a}}dz \\ &+ \int_{|z|\leq|x|}\left(\int_0^1\frac{(\phi_\eta^{z,x})^{(3)}(t)}{2!}(1-t)^2dt\right)\frac{dz}{|z|^{d+a}} \\ &+ \int_{|z|\geq|x|}\frac{\phi_\eta^{z,x}(1)-\phi_\eta^{z,x}(0)}{|z|^{d+a}}dz \\ &:= I_1+I_2+I_3.\end{aligned}$$

⋄ Estimate of I_1

It is direct to obtain, rewriting $x = |x|\sigma_0$

$$\begin{aligned}
I_1 &= \frac{\varepsilon}{2}\left(|x|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\int_{|z|\le|x|}|z|^{2-d-a}\left((\varepsilon-2)\left\langle\frac{z}{|z|},\frac{x}{|x|}\right\rangle^2\frac{|x|^2}{|x|^2+\eta^2}+1\right)dz\\
&= \frac{\varepsilon}{2}\left(|x|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\int_0^{|x|}\int_{\mathbb{S}^{d-1}} r^{2-d-a}\left(1+(\varepsilon-2)\langle\sigma,\sigma_0\rangle^2\frac{|x|^2}{|x|^2+\eta^2}\right)r^{d-1}drd\sigma\\
&= \frac{\varepsilon}{2}\left(|x|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\left(\int_0^{|x|}r^{1-a}dr\right)\int_{\mathbb{S}^{d-1}}\left(1+(\varepsilon-2)\langle\sigma,\sigma_0\rangle^2\frac{|x|^2}{|x|^2+\eta^2}\right)d\sigma\\
&= \frac{\varepsilon}{2}\left(|x|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\frac{|x|^{2-a}}{2-a}|\mathbb{S}^{d-1}|\left(1+\frac{\int_{\mathbb{S}^{d-1}}\langle\sigma,\sigma_0\rangle^2\,d\sigma}{|\mathbb{S}^{d-1}|}(\varepsilon-2)\frac{|x|^2}{|x|^2+\eta^2}\right).
\end{aligned}$$

When $d = 2$, we write $\sigma_0 = (\cos(\theta_0), \sin(\theta_0))$ and then we have

$$\frac{\int_{\mathbb{S}^{d-1}}\langle\sigma,\sigma_0\rangle^2\,d\sigma}{|\mathbb{S}^{d-1}|} = \frac{\int_0^{2\pi}\cos^2(\theta_0-\theta)d\theta}{2\pi} = \frac{1}{2}.$$

When $d > 2$ we find by changes of of variables (see for instance equation (61) of [33])

$$\frac{\int_{\mathbb{S}^{d-1}}\langle\sigma,\sigma_0\rangle^2\,d\sigma}{|\mathbb{S}^{d-1}|} = \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}\int_0^\pi \sin^{d-2}(\theta)\cos^2(\theta)d\theta = \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|}B\left(\frac{d-1}{2},\frac{3}{2}\right) = \sqrt{\pi}^{-1}\frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-2}{2}\right)}B\left(\frac{d-1}{2},\frac{3}{2}\right).$$

It can be numerically checked that $c_d \le 1/2$ for any $d \ge 2$, so that we obtain

$$\begin{aligned}
I_1 &= \frac{\varepsilon^2 c_d|\mathbb{S}^{d-1}|}{2(2-a)}\left(|x|^2+\eta^2\right)^{\frac{\varepsilon-4}{2}}|x|^{4-a} + \frac{\varepsilon|\mathbb{S}^{d-1}|}{2(2-a)}\left(|x|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}|x|^{2-a}\frac{(1-2c_d)|x|^2+\eta^2}{|x|^2+\eta^2}\\
&\ge \frac{\varepsilon^2 c_d|\mathbb{S}^{d-1}|}{2(2-a)}\left(|x|^2+\eta^2\right)^{\frac{\varepsilon-4}{2}}|x|^{4-a}.
\end{aligned}$$

⋄ Estimate of I_2

In the case $|z| \le |x|$ we easily get

$$|x+tz| \ge |x| - t|z| \ge (1-t)|x|,$$

and since

$$\left|(\varepsilon-4)X^3+3X\right| \le \frac{2}{\sqrt{4-\varepsilon}},$$

for $X \in (-1,1)$, and we deduce

$$|(\phi_\eta^{z,x})^{(3)}(t)| \le \varepsilon(2-\varepsilon)|x|^{\varepsilon-3}|z|^3(1-t)^{\varepsilon-3}\frac{2}{\sqrt{4-\varepsilon}},$$

therefore

$$\begin{aligned}
I_2 &\ge -\frac{\varepsilon(2-\varepsilon)}{2}\frac{2}{\sqrt{4-\varepsilon}}\left(\int_0^1(1-t)^{\varepsilon-1}dt\right)|x|^{\varepsilon-3}\int_{|z|\le|x|}\frac{|z|^3}{|z|^{d+a}}dz\\
&\ge -\frac{\varepsilon(2-\varepsilon)|\mathbb{S}^{d-1}|}{\varepsilon(3-a)\sqrt{4-\varepsilon}}|x|^{\varepsilon-a}.
\end{aligned}$$

⋄ Estimate of I_3

Note that if $|z| \ge |x|$ then

$$-(|z|^2+\eta^2)^{\frac{\varepsilon}{2}} = -\phi_\eta^{x,z}(0) \le -\phi_\eta^{z,x}(0) = -(|x|^2+\eta^2)^{\frac{\varepsilon}{2}},$$

then since $\phi_\eta^{z,x}(1) = \phi_\eta^{x,z}(1)$, we have

$$\begin{aligned}
I_3 &= \int_{|z|\geq|x|} \frac{\phi_\eta^{z,x}(1) - \phi_\eta^{z,x}(0)}{|z|^{d+a}} dz \\
&\geq \int_{|z|\geq|x|} \frac{\phi_\eta^{x,z}(1) - \phi_\eta^{x,z}(0)}{|z|^{d+a}} dz \\
&= \int_{|z|\geq|x|} \frac{(\phi_\eta^{x,z})'(0) + \int_0^1 (\phi_\eta^{x,z})^{(2)}(t)(1-t)dt}{|z|^{d+a}} dz \\
&= \int_{|z|\geq|x|} \frac{\varepsilon(|z|^2+\eta^2)^{\frac{\varepsilon-2}{2}} \langle x, z\rangle}{|z|^{d+a}} dz + \int_{|z|\geq|x|} \int_0^1 (\phi_\eta^{x,z})^{(2)}(t)(1-t)|z|^{-(d+a)} dtdz \\
&= \int_{|z|\geq|x|} \int_0^1 (\phi_\eta^{x,z})^{(2)}(t)(1-t)|z|^{-(d+a)} dtdz,
\end{aligned}$$

but recall that

$$(\phi_\eta^{x,z})''(t) = \varepsilon\left(|z+tx|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}} |x|^2 \left((\varepsilon-2)\left\langle \frac{x}{|x|}, \frac{z+tx}{|z+tx|}\right\rangle^2 \frac{|z+tx|^2}{|z+tx|^2+\eta^2} + 1\right),$$

and since $|z| \geq |x|$ we easily get

$$|z+tx| \geq |z| - t|x| \geq (1-t)|x|.$$

Moreover since for any $X \in (0,1)$

$$\left|(\varepsilon-2)X^2 + 1\right| \leq 1,$$

it follows

$$|(\phi_\eta^{z,x})''(t)| \leq \varepsilon|x|^\varepsilon (1-t)^{\varepsilon-2}.$$

This yields

$$I_3 \geq -\frac{|\mathbb{S}^{d-1}|\varepsilon}{\varepsilon a}|x|^{\varepsilon-a},$$

and the result holds with the desired constant. □

Remark 3.1. *Also recall the bound for any $\varepsilon \in (0,a)$*

$$-(-\Delta)^{a/2}(m_\varepsilon)(x) \leq C_{a,\varepsilon} m_{\varepsilon-a}(x), \tag{3.16}$$

see for instance [26, Proposition 2.2, (18)].

Next we need some tools for the Diffusion Dominated case $\alpha < a$. As this case is close to the one studied in [18] which relies on properties of the classical Fisher information, we need to extend those properties to the anomalous diffusion case. Such a fractional Fisher information has not been very studied in the literature. The main results in this domain, to the best of the author's knowledge, have been obtained by Toscani [32], also note the contribution of [19] where the author also consider such a fractional Fisher information for probability measures on the $1d$ torus or the real line. The work by Erbar in [12] should also be quoted, where the author establishes some new metric on the probability measure space, with respect to which the Boltzmann's entropy is a gradient flow functional for the fractional heat equation. In this purpose the author introduces a (relative) fractional Fisher information.
By definition, considering N independent d-dimensional Brownian motion and one dN-dimensional Brownian motion is the same. Therefore establishing the Liouville equation associated to a particle system with independent Brownian diffusions, falls by using an Ito's formula in $\mathbb{R}^{dN}$ (see for instance [15, Proof of Proposition 5.1, Step 1]). However, for any $a \in (0,2)$, a vector valued process whose components are N independent a-stable Lévy flights on $\mathbb{R}^d$, is Lévy processes on $\mathbb{R}^{dN}$ which is not a-stable.
Let $\mu_1, \cdots, \mu_N$ be N independent Poisson random measures on $\mathbb{R}^+ \times \mathbb{R}^d$ with intensity $ds \times c_{d,a}\frac{dx}{|x|^{d+a}}$, and denote $\tilde{\mathcal{Z}}_t^N$ the $\mathbb{R}^{dN}$-valued process defined by

$$\tilde{\mathcal{Z}}_t^N = \left(\int_{[0,t]\times\mathbb{R}^d} x\bar{\mu}_1(ds,dx), \cdots, \int_{[0,t]\times\mathbb{R}^d} x\bar{\mu}_N(ds,dx)\right).$$

It is classical (see [9, Chapter VII, 3)]), since the $(\mu_i)_{i=1,\cdots,N}$ are independent, to obtain for any $r^N = (r_1, \cdots, r_N) \in \mathbb{R}^{dN}$

$$\mathbb{E}\left[e^{ir^N\cdot\tilde{\mathcal{Z}}_t^N}\right] = e^{t\sum_{k=1}^N c_{d,a}|r_k|^a} = e^{t\sum_{k=1}^N \psi(r_k)},$$

where $\psi(r) = c_{d,a}|r|^a$ is the characteristic exponent of the d-dimensional a-stable Lévy process. Let now be $\mathcal{M}^N$ be a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^d \times \{1\cdots, N\}$ with intensity $Nds \times c_{d,a}|x|^{-d+a)}dx \times \mathcal{U}(\{1, \cdots, N\})$ and denote

$$\bar{\mathcal{Z}}_t^N = \int_{[0,t]\times\mathbb{R}^d\times\{1\cdots,N\}} (\delta_{l=1}x, \cdots, \delta_{l=N}x)\,\bar{\mathcal{M}}^N(ds, dx, dl).$$

Hence it holds due to classical properties of Poisson random measure (see [9, Theorem 2.9, p 252] for instance)

$$\begin{aligned}\mathbb{E}\left[e^{ir^N\cdot\bar{\mathcal{Z}}_t^N}\right] &= e^{-\int_0^t c_{d,a}\mathrm{v.p.}\int_{\mathbb{R}^d}\frac{1}{N}\sum_{l=1}^N\left(1-e^{i\sum_{k=1}^N\delta_{l,k}x\cdot r_k}\right)N\frac{dx}{|x|^{d+a}}ds}\\ &= e^{t\sum_{k=1}^N c_{d,a}\mathrm{v.p.}\int_{\mathbb{R}^d}\left(e^{ix\cdot r_k}-1\right)\frac{dx}{|x|^{d+a}}} = e^{t\sum_{k=1}^N\psi(r_k)}.\end{aligned}$$

Then it follows that the law $F_t^N = \mathcal{L}(\tilde{\mathcal{Z}}_t^N) = \mathcal{L}(\bar{\mathcal{Z}}_t^N)$ solves the many-body fractional heat equation

$$\partial_t F_t^N(X^N) + \sum_{k=1}^N\int_{\mathbb{R}^{dN}}\frac{F_t^N(X^N + z\mathbf{a}_\mathbf{k}) - F_t^N(X^N))}{|z|^{d+a}}dz = 0, \tag{3.17}$$

with $z\mathbf{a}_k = \left(\underbrace{0,\cdots 0}_{2(k-1)}, z_1, z_2, \underbrace{0\cdots,0}_{2(N-k)}\right) \in \mathbb{R}^{dN}$ and $z = (z_1, z_2)$. Next we look at the dissipation of entropy along this equation. For a probability measure $F^N \in \mathcal{P}(\mathbb{R}^{dN})$ introduce the *normalized Boltzmann's entropy*

$$\mathcal{H}^N(F^N) = \frac{1}{N}\int_{\mathbb{R}^{dN}} F^N(x)\ln F^N(x)dx,$$

then it holds

$$\begin{aligned}\frac{d}{dt}\mathcal{H}^N(F_t^N) &= -\frac{1}{N}\int_{\mathbb{R}^{dN}}(1+\ln F_t^N(X^N))\left(\sum_{k=1}^N \mathrm{p.v.}\int_{\mathbb{R}^d}\frac{F_t^N(X^N+z\mathbf{a}_k) - F_t^N(X^N)}{|z|^{d+a}}dz\right)dX^N\\ &= -\frac{1}{N}\sum_{k=1}^N\int_{\mathbb{R}^{dN}}\mathrm{p.v.}\int_{\mathbb{R}^d}\frac{\left(F_t^N(X^N+z\mathbf{a}_k) - F_t^N(X^N)\right)\ln F_t^N(X^N)}{|z|^{d+a}}dzdX^N\\ &= \frac{1}{N}\sum_{k=1}^N\int_{\mathbb{R}^{dN}}\mathrm{p.v.}\int_{\mathbb{R}^d}\frac{\left(F_t^N(X^N) - F_t^N(X^N+z\mathbf{a}_k)\right)\ln F_t^N(X^N+z\mathbf{a}_k)}{|z|^{d+a}}dzdX^N\\ &= -\frac{1}{2}\frac{1}{N}\sum_{k=1}^N\int_{\mathbb{R}^{dN}}\mathrm{p.v.}\int_{\mathbb{R}^d}\frac{\left(F_t^N(X^N+z\mathbf{a}_k) - F_t^N(X^N)\right)\left(\ln F_t^N(X^N+z\mathbf{a}_k) - \ln F_t^N(X^N)\right)}{|z|^{d+a}}dzdX^N\\ &= -\frac{1}{2}\frac{1}{N}\sum_{k=1}^N\int_{\mathbb{R}^{dN}}\mathrm{p.v.}\int_{\mathbb{R}^d}\frac{\Phi\left(F_t^N(X^N+z\mathbf{a}_k), F_t^N(X^N)\right)}{|z|^{d+a}}dzdX^N.\end{aligned}$$

Henceforth we consider the functional defined in 2.15 on $\mathcal{P}(\mathbb{R}^{dN})$ as

$$\begin{aligned}
\mathcal{I}_a^N(G_N) &= \frac{1}{2}\frac{1}{N}\sum_{k=1}^N \int_{\mathbb{R}^{d(N-1)}} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\Phi\left(G_N(X_k^x), G_N(X_k^y)\right)}{|x-y|^{d+a}} dxdy\, dx_1, \cdots, dx_{k-1}, dx_{k+1}, \cdots, dx_N,\\
&= \frac{1}{2}\frac{1}{N}\sum_{k=1}^N \int_{\mathbb{R}^{d(N-1)}} \int_{\mathbb{R}^4} \frac{\left(G_N(X_k^x) - G_N(X_k^y)\right)\left(\ln G_N(X_k^x) - \ln G_N(X_k^y)\right)}{|x-y|^{d+a}} dxdy\, dx_1, \cdots, dx_{k-1}, dx_{k+1}, \cdots, dx_N,\\
&= \frac{1}{2}\frac{1}{N}\sum_{k=1}^N \int_{\mathbb{R}^{dN}} \int_{\mathbb{R}^d} \frac{(G_N(x+z\mathbf{a}_k) - G_N(x))\,(\ln G_N(x+z\mathbf{a}_k) - \ln G_N(x))}{|z|^{d+a}} dzdx\\
&\text{with} \quad X_k^x = (x_1, \cdots, x_{k-1}, x, x_{k+1}, \cdots, x_N),
\end{aligned}$$

as the pendent for fractional diffusion of the normalized Fisher information (with the convention $\mathcal{I}_a^1 = I_a$).

Remark 3.2. *In the classical case the Fisher information can be rewrtiten as*

$$\begin{aligned}
\mathcal{I}^N(F^N) &= \frac{1}{N}\int_{\mathbb{R}^{dN}} \frac{\left|\nabla F^N\right|^2}{F^N} dX^N = \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{dN}} \frac{\left|\nabla_i F^N\right|^2}{F^N} dX^N\\
&= \frac{1}{N}\sum_{i=1}^N \int_{\mathbb{R}^{dN}} \nabla_i F^N \cdot \nabla_i \ln F^N dX^N,
\end{aligned}$$

which has the same the form of the one defined in (2.15)*, except that the H^1 inner product between F^N and $\ln F^N$ w.r.t. the i-th component is replaced in the fractional case with the $H^{a/2}$ inner product.*

This quantity is so far an entropy dissipation, but not an *information* yet. In oder to properly qualify it as such, we have the

Proposition 3.1. *The fractional Fisher information defined in* (2.15)

(*i*) *is proper, convex,*

(*ii*) *is lower semi continuous w.r.t. the weak convergence in $\mathcal{P}(\mathbb{R}^{dN})$,*

(*iii*) *is super-additive in the sense that for $G^N \in \mathcal{P}(\mathbb{R}^{dN})$ and $G^i \in \mathcal{P}(\mathbb{R}^{2i}), G^{N-i} \in \mathcal{P}(\mathbb{R}^{2(N-i)})$ its marginal on $\mathbb{R}^{2i}$ (resp. $\mathbb{R}^{2(N-i)}$ for $i = 1, \cdots, N-1$ it holds*

$$\mathcal{I}_a^N(G^N) \geq \frac{i}{N}\mathcal{I}_a^i(G^i) + \frac{N-i}{N}\mathcal{I}_a^{N-i}(G^{N-i}).$$

Moreover equality holds if and only if $G^N = G^i \otimes G^{N-i}$.

(*iv*) *satisfies, for any $G^N \in \mathcal{P}_{sym}(\mathbb{R}^{dN})$, and $G^k \in \mathcal{P}_{sym}(\mathbb{R}^{2k})$ its marginal on $\mathbb{R}^{2k}$*

$$\mathcal{I}_a^N(G^N) \geq \mathcal{I}_a^k(G^k),$$

and for any $g \in \mathcal{P}(\mathbb{R}^d)$

$$\mathcal{I}_a^N(g^{\otimes N}) = I_a(g).$$

Proof. Proof of point (*i*)
Convexity holds form point (*ii*) of Lemma 3.1. We delay the proof of the fact that $\mathcal{I}_a^N$ is proper after the proof of point (*iv*).
Proof of point (*ii*)
Let be $(u_n^k)_{n\in\mathbb{N}}$, such that $u_n^k \overset{*}{\rightharpoonup} u^k \in \mathcal{P}(\mathbb{R}^{2k})$ and for $\varepsilon > 0$ set $\rho_\varepsilon(x_1, \cdots, x_k) = \frac{1}{(2\pi\varepsilon)^{\frac{k}{2}}} e^{-\frac{\sum_{k=1}^N |x_k|^2}{2\varepsilon}}$. Then it holds that

$u_n^k * \rho_\varepsilon =: u_n^{k,\varepsilon} \overset{*}{\rightharpoonup} u^{k,\varepsilon} := u^k * \rho_\varepsilon \in \mathcal{P}(\mathbb{R}^{2k})$ for any $\varepsilon > 0$, and that $u^{k,\varepsilon}$ is a smooth function which is always strictly larger than 0. Note that due to point (ii) of Lemma 3.1 and Jensen's inequality it holds

$$\begin{aligned}\mathcal{I}_a^k(u_n^{k,\varepsilon}) &= \frac{1}{2k}\sum_{j=1}^{k}\int_{\mathbb{R}^{2(k-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\Phi\left(\int u_n^k(X_j^x+z)\rho_\varepsilon(z)dz, \int u_n^k(X_j^y+z)\rho_\varepsilon(z)dz\right)}{|x-y|^{d+a}}dxdy\, d\hat{X}^j,\\ &\leq \frac{1}{2k}\int\sum_{j=1}^{k}\int_{\mathbb{R}^{2(k-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\Phi\left(u_n^k(X_j^x+z), u_n^k(X_j^y+z)\right)}{|x-y|^{d+a}}dxdy\, d\hat{X}_N^j\rho_\varepsilon(z)dz = \mathcal{I}_a^k(u_n^k).\end{aligned}$$

Moreover it holds (dealing only with the case $j = 1$ by symmetry)

$$\begin{aligned}\mathcal{I}_a^k(u_n^{k,\varepsilon}) &= \int_{\mathbb{R}^{2(k-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(u_n^{k,\varepsilon}(x,X^k)-u_n^{k,\varepsilon}(y,X^k)\right)\left(\ln u_n^{k,\varepsilon}(x,X^k)-\ln u_n^{k,\varepsilon}(y,X^k)\right)}{|x-y|^{d+a}}dxdydX^k\\ &= \int_{\mathbb{R}^{2(k-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(u_n^{k,\varepsilon}(x,X^k)-u_n^{k,\varepsilon}(y,X^k)\right)\left(\ln u^{k,\varepsilon}(x,X^k)-\ln u^{k,\varepsilon}(y,X^k)\right)}{|x-y|^{d+a}}dxdydX^k\\ &+ \int_{\mathbb{R}^{2(k-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(u_n^{k,\varepsilon}(x,X^k)-u_n^{k,\varepsilon}(y,X^k)\right)\left(\ln\left(\frac{u_n^{k,\varepsilon}(x,X^k)}{u^{k,\varepsilon}(x,X^k)}\right)-\ln\left(\frac{u_n^{k,\varepsilon}(y,X^k)}{u^{k,\varepsilon}(y,X^k)}\right)\right)}{|x-y|^{d+a}}dxdydX^k\\ &:= \mathcal{E}_1^n+\mathcal{E}_2^n.\end{aligned}$$

$\diamond$ Estimate of $\mathcal{E}_1^n$
We rewrite

$$\mathcal{E}_1^n = 2\int_{\mathbb{R}^{2k}}u_n^{k,\varepsilon}(x,X^k)\int_{\mathbb{R}^d}\frac{\left(\ln u^{k,\varepsilon}(x,X^k)-\ln u^{k,\varepsilon}(y,X^k)\right)}{|x-y|^{d+a}}dydxdX^k,$$

by symmetry
$\diamond$ Estimate of $\mathcal{E}_2^n$
Using point (i) of Lemma 3.1 and also by symmetry we find that

$$\begin{aligned}\mathcal{E}_2^n &\geq \int_{\mathbb{R}^{2(k-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(u^{k,\varepsilon}(x,X^k)-u^{k,\varepsilon}(y,X^k)\right)\left(\frac{u_n^{k,\varepsilon}(x,X^k)}{u^{k,\varepsilon}(x,X^k)}-\frac{u_n^{k,\varepsilon}(y,X^k)}{u^{k,\varepsilon}(y,X^k)}\right)}{|x-y|^{d+a}}dxdydX^k\\ &= 2\int_{\mathbb{R}^{2k}}u_n^{k,\varepsilon}(x,X^k)\int_{\mathbb{R}^d}\frac{\left(u^{k,\varepsilon}(x,X^k)-u^{k,\varepsilon}(y,X^k)\right)}{u^{k,\varepsilon}(x,X^k)|x-y|^{d+a}}dydxdX^k.\end{aligned}$$

Now since the functions

$$(x,X_k)\mapsto \text{v.p.}\int_{\mathbb{R}^d}\frac{\left(\ln u^{k,\varepsilon}(x,X^k)-\ln u^{k,\varepsilon}(y,X^k)\right)}{|x-y|^{d+a}}dy,$$

and

$$(x,X_k)\mapsto \text{v.p.}\int_{\mathbb{R}^d}\frac{\left(u^{k,\varepsilon}(x,X^k)-u^{k,\varepsilon}(y,X^k)\right)}{u^{k,\varepsilon}(x,X^k)|x-y|^{d+a}}dy,$$

are continuousfor any $\varepsilon > 0$, we deduce by weak convergence of $u_n^{k,\varepsilon}$ to $u^{k,\varepsilon}$ as n goes to infinity, that

$$\lim_{n\to\infty}\mathcal{E}_1^n = 2\int_{\mathbb{R}^{2k}}u^{k,\varepsilon}(x,X^k)\int_{\mathbb{R}^d}\frac{\left(\ln u^{k,\varepsilon}(x,X^k)-\ln u^{k,\varepsilon}(y,X^k)\right)}{|x-y|^{d+a}}dydxdX^k = \mathcal{I}_a^k(u^{k,\varepsilon}),$$

and

$$\lim_{n\to\infty}\mathcal{E}_2^n = 2\int_{\mathbb{R}^{2k}}u^{k,\varepsilon}(x,X^k)\int_{\mathbb{R}^d}\frac{\left(u^{k,\varepsilon}(x,X^k)-u^{k,\varepsilon}(y,X^k)\right)}{u^{k,\varepsilon}(x,X^k)|x-y|^{d+a}}dydxdX^k = 0.$$

Hence it follows that

$$\liminf_n \mathcal{I}_a^k(u_n^k) \geq \liminf_n \mathcal{I}_a^k(u_n^{k,\varepsilon}) = \mathcal{I}_a^k(u^{k,\varepsilon}).$$

Then for almost all $x, y, x_2, \cdots, x_k \in \mathbb{R}^{2(k+1)}$ it holds

$$\lim_{\varepsilon\to 0} \frac{\Phi(u^{k,\varepsilon}(x, X^k), u^{k,\varepsilon}(y, X^k))}{|x-y|^{d+a}} = \frac{\Phi(u^k(x, X^k), u^k(y, X^k))}{|x-y|^{d+a}},$$

up to taking a sequence $(\varepsilon_n)_{n\geq 1}$ converging to 0. Hence by Fatou's Lemma it holds

$$\begin{aligned}\mathcal{I}_a^k(u^k) &= \int_{\mathbb{R}^{2(k-1)}} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\Phi(u^k(x, X^y), u^k(y, X^k))}{|x-y|^{d+a}} dxdydx_2\cdots dx_k \\ &\leq \liminf_{\varepsilon>0} \int_{\mathbb{R}^{2(k-1)}} \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\Phi(u^{k,\varepsilon}(x, X^k), u^{k,\varepsilon}(y, X^k))}{|x-y|^{d+a}} dxdydx_2\cdots dx_k = \liminf_{\varepsilon>0} \mathcal{I}_a^k(u^{k,\varepsilon}),\end{aligned}$$

and therefore

$$\liminf_n \mathcal{I}_a^k(u_n^k) \geq \mathcal{I}_a^k(u^k).$$

.

Proof of point (*iii*)
Let be $G_N \in \mathcal{P}(\mathbb{R}^{dN})$ and $(X_1, \cdots, X_N)$ a random vector of law G_N, fix $i = 1, \cdots, N$ and denote

$$g_i(x_1, \cdots, x_i | x_{i+1}, \cdots, x_N) = \mathcal{L}(X_1, \cdots, X_i \,|\, X_{i+1}, \cdots, X_N),$$

then

$$G_N(x_1, \cdots, x_N) = g_i(x_1, \cdots, x_i | x_{i+1}, \cdots, x_N) G_{N-i}(x_{i+1}, \cdots, x_N) = g_{N-i}(x_{i+1}, \cdots, x_N | x_1, \cdots, x_i) G_i(x_1, \cdots, x_i).$$

Next observe that for $k \leq i$

$$\begin{aligned}\Phi\left(G_N(X_k^x), G_N(X_k^y)\right) &= \left(G_N(X_k^x) - G_N(X_k^y)\right)\left(\ln G_i(X_{k,i}^x) - \ln G_i(X_{k,i}^y)\right) \\ &\quad + \left(G_N(X_k^x) - G_N(X_k^y)\right)\left(\ln g_{N-i}(X^{N-i}|X_{k,i}^x) - \ln g_{N-i}(X^{N-i}|X_{k,i}^y)\right),\end{aligned}$$

with the notations

$$\begin{aligned}&X_k^x = (x_1, \cdots, x_{k-1}, x, x_{k+1}, \cdots, x_N), \\ &X_{k,i}^x = (x_1, \cdots, x_{k-1}, x, x_{k+1}, \cdots, x_i), \quad \text{if} \quad k = i \text{ the last component is } x \\ &X^{N-i} = (x_{i+1}, \cdots, x_N).\end{aligned}$$

Similarly if $k > i$

$$\begin{aligned}\Phi\left(G_N(X_k^x), G_N(X_k^y)\right) &= \left(G_N(X_k^x) - G_N(X_k^y)\right)\left(\ln G_{N-i}(X_{k,N-i}^x) - \ln G_{N-i}(X_{k,N-i}^y)\right) \\ &\quad + \left(G_N(X_k^x) - G_N(X_k^y)\right)\left(\ln g_i(X^i|X_{k,N-i}^x) - \ln g_i(X^i|X_{k,N-i}^y)\right),\end{aligned}$$

with the similar notations

$$\begin{aligned}&X_{k,N-i}^x = (x_{i+1}, \cdots, x_{k-1}, x, x_{k+1}, \cdots, x_N), \quad \text{if} \quad k = i+1 \text{ the first component is } x \\ &X^i = (x_1, \cdots, x_i).\end{aligned}$$

Using all this yields

$$\begin{aligned}
\mathcal{I}_a^k(G_N) &= \frac{1}{2}\frac{1}{N}\sum_{k=1}^{N}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\Phi\left(G_N(X_k^x),G_N(X_k^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_N^k,\\
&\geq \frac{1}{2}\frac{1}{N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(G_N(X_k^x)-G_N(X_k^y)\right)\left(\ln G_i(X_k^x)-\ln G_i(X_k^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_N^k\\
&+\frac{1}{2}\frac{1}{N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(G_N(X_k^x)-G_N(X_k^y)\right)\left(\ln g_{N-i}(X^{N_i}|X_{k,i}^x)-\ln g_{N-i}(X^{N-i}|X_{k,i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_N^k\\
&+\frac{1}{2}\frac{1}{N}\int\sum_{k=i+1}^{N}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(G_N(X_k^x)-G_N(X_k^y)\right)\left(\ln G_{N-i}(X_{k,N-i}^x)-\ln G_{N-i}(X_{k,N-i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_N^k\\
&+\frac{1}{2}\frac{1}{N}\int\sum_{k=i+1}^{N}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(G_N(X_k^x)-G_N(X_k^y)\right)\left(\ln g_i(X^i|X_{k,N-i}^x)-\ln g_i(X^i|X_{k,N-i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_N^k\\
&:=\mathcal{J}_1^i+\mathcal{J}_2^i+\mathcal{J}_1^{N-i}+\mathcal{J}_2^{N-i}.
\end{aligned}$$

$\diamond$ Estimate of $\mathcal{J}_1$:
Both terms are treated equally, so we will focus only on the i term. Using Fubini's Theorem yields

$$\begin{aligned}
\mathcal{J}_1^1 &= \frac{1}{2}\frac{1}{N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{G_N(X_{k,i}^x,X^{N-i})\left(\ln G_i(X_{k,i}^x)-\ln G_i(X_{k,i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_N^k\\
&-\frac{1}{2}\frac{1}{N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{G_N(X_{k,i}^y,X^{N-i})\left(\ln G_i(X_{k,i}^x)-\ln G_i(X_{k,i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_N^k\\
&=\frac{1}{2}\frac{1}{N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{2(i-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\int G_N(X_{k,i}^x,X^{N-i})dX^{N-i}\left(\ln G_i(X_{k,i}^x)-\ln G_i(X_{k,i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_i^k\\
&-\frac{1}{2}\frac{1}{N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{2(i-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\int G_N(X_{k,i}^y,X^{N-i})dX^{N-i}\left(\ln G_i(X_{k,i}^x)-\ln G_i(X_{k,i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_i^k\\
&=\frac{1}{2}\frac{1}{N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{2(i-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(G_i(X_{k,i}^x)-G_i(X_{k,i}^y)\right)\left(\ln G_i(X_{k,i}^x)-\ln G_i(X_{k,i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_i^k=\frac{i}{N}\mathcal{I}_a^i(G_i).
\end{aligned}$$

$\diamond$ Estimate of $\mathcal{J}_2$:
Similarly we only treat $\mathcal{J}_2^i$. Using point (i) of Lemma 3.1 and once again Fubini's Theorem we get

$$\begin{aligned}
\mathcal{J}_2^i &\geq \frac{1}{2}\frac{1}{N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(G_i(X_{k,i}^x)-G_i(X_{k,i}^y)\right)\left(g_{N-i}(X^{N-i}|X_{k,i}^x)-g_{N-i}(X^{N-i}|X_{k,i}^y)\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_N^k\\
&=\frac{1}{2N}\int\sum_{k=1}^{i}\int_{\mathbb{R}^{2(i-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(G_i(X_{k,i}^x)-G_i(X_{k,i}^y)\right)\left(\int g_{N-i}(X^{N-i}|X_{k,i}^x)dX^{N-i}-\int g_{N-i}(X^{N-i}|X_{k,i}^y)dX^{N-i}\right)}{|x-y|^{d+a}}dxdy\,d\hat{X}_i^k\\
&=0
\end{aligned}$$

Moreover due to point (i) of Lemma 3.1, $\mathcal{J}_2=0$ only if for any $k=1,\cdots,i$ for almost every $x,y\in\mathbb{R}^d\times\mathbb{R}^d$ and $x_1,\cdots,x_{k-1},x_{k+1},x_i\in\mathbb{R}^{2(i-1)}$ it holds

$$g_{N-i}(X^{N-i}|x_1,\cdots,x_{k-1},x,x_{k+1},x_i)=g_{N-i}(X^{N-i}|x_1,\cdots,x_{k-1},y,x_{k+1},x_i)=\mu(X^{N-i}),$$

for some $\mu\in\mathcal{P}(\mathbb{R}^{2(N-i)})$. But necessarily $\mu=G_{N-i}$ and we deduce that

$$G_N=G_i\otimes G_{N-i}.$$

Proof of point (*iv*)
Note that the symmetry of G_N yields

$$\mathcal{I}_a^N(G_N) = \frac{1}{2}\int_{\mathbb{R}^{d(N-1)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\Phi\left(G_N(X_1^x), G_N(X_1^y)\right)}{|x-y|^{d+a}}dxdy\, d\hat{X}_N^1.$$

In the tensorised case, Fubini's Theorem yields

$$\begin{aligned}\mathcal{I}_a^N(g^{\otimes N}) &= \int\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\left(g(x)\prod_{k=2}^N g(x_k) - g(y)\prod_{k=2}^N g(x_k)\right)\left(\ln\left(g(x)\prod_{k=2}^N g(x_k)\right) - \ln\left(g(y)\prod_{k=2}^N g(x_k)\right)\right)}{|x-y|^{d+a}}dxdy\hat{X}_N^1\\ &= \int\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{(g(x)-g(y))\left(\ln(g(x)) - \ln(g(y))\right)}{|x-y|^{d+a}}dxdy\right)\prod_{k=2}^N g(x_k)d\hat{X}_N^1 = I_a(g).\end{aligned}$$

On the other hand, with similar notations as in point (*iii*) write

$$G_N(X_1^x) = G_k(X_1^x)g_{N-k}(X^{N-k}|X_1^x),$$

and then

$$\begin{aligned}\Phi\left(G_N(X_1^x), G_N(X_1^y)\right) &= \left(G_N(X_1^x) - G_N(X_1^y)\right)\left(\ln G_k(X_1^x) - \ln G_k(X_1^y)\right)\\ &\quad + \left(G_N(X_1^x) - G_N(X_1^y)\right)\left(\ln g_{N-k}(X^{N-k}|X_1^x) - \ln g_{N-k}(X^{N-k}|X_1^y)\right)\\ &\geq \left(G_N(X_1^x) - G_N(X_1^y)\right)\left(\ln G_k(X_1^x) - \ln G_k(X_1^y)\right)\\ &\quad + \left(G_k(X_1^x) - G_k(X_1^y)\right)\left(g_{N-k}(X^{N-k}|X_1^x) - g_{N-k}(X^{N-k}|X_1^y)\right),\end{aligned}$$

dividing the above inequality by $|x-y|^{d+a}$ and integrating over $dxdydx_2,\cdots,dx_N$ yields the desired result thanks to similar computations as the one done in the proof of point (*iii*).
To see that $\mathcal{I}_a^N$ is proper, take $\nu\in\mathcal{P}(\mathbb{R}^d)$ and $\psi_\varepsilon = \varepsilon^{-2}e^{-\sqrt{1+\varepsilon^{-2}|x|^2}}$, and define $\nu^{N,\varepsilon} := (\nu_\varepsilon)^{\otimes} := (\nu*\psi_\varepsilon)^{\otimes N}\in\mathcal{P}(\mathbb{R}^{dN})$. Then we have

$$\begin{aligned}\mathcal{I}_a^N(\nu^{N,\varepsilon}) = I_a(\nu*\psi_\varepsilon) &= \int_{\mathbb{R}^4}\frac{\Phi(\nu_\varepsilon(x),\nu_\varepsilon(y))}{|x-y|^{d+a}}dxdy\\ &\leq \int_{|x-y|\leq 1}\frac{\nu_\varepsilon(x)-\nu_\varepsilon(y)}{\ln(\nu_\varepsilon(x)) - \ln(\nu_\varepsilon(y))}\frac{|\ln(\nu_\varepsilon(x)) - \ln(\nu_\varepsilon(y))|^2}{|x-y|^{d+a}}dxdy\\ &+ \int_{|x-y|\geq 1}(\nu_\varepsilon(x)+\nu_\varepsilon(y))\frac{|\ln(\nu_\varepsilon(x)) - \ln(\nu_\varepsilon(y))|}{|x-y|^{d+a}}dxdy\\ &\leq \int_{|x-y|\leq 1}(\nu_\varepsilon(x)+\nu_\varepsilon(y))\frac{|\ln(\nu_\varepsilon(x)) - \ln(\nu_\varepsilon(y))|^2}{|x-y|^{d+a}}dxdy\\ &+ \int_{|x-y|\geq 1}(\nu_\varepsilon(x)+\nu_\varepsilon(y))\frac{|\ln(\nu_\varepsilon(x)) - \ln(\nu_\varepsilon(y))|}{|x-y|^{d+a}}dxdy,\end{aligned}$$

Using [20, Lemma 5.8] (see also Lemma Appendix C.1 below) we find that $\|\nabla\ln\nu_\varepsilon\|_{L^\infty(\mathbb{R}^d)}\leq\varepsilon^{-1}$. Therefore

$$\mathcal{I}_a^N(\nu^{N,\varepsilon}) \leq 2\int_{\mathbb{R}^d}\nu_\varepsilon(x)\left(\int_{\mathbb{R}^d}(|x-y|^{-(d-2+a)}\wedge|x-y|^{-(d-1+a)})dy\right)dx < \infty.$$

So that $\mathcal{I}_a^N$ is non identically equal to $+\infty$, and it is proper. □

Remark 3.3. *All the properties established on the fractional Fisher information can be proved with the same techniques for the classical Fisher information*

$$\mathcal{I}^N(F^N) = \frac{1}{N}\sum_{i=1}^N\int_{\mathbb{R}^{dN}}\nabla_i F^N\cdot\nabla_i\ln F^N dX^N,$$

using this particular form, and a slight modification of point (i) *of Lemma 3.1 which reads*

$$\nabla(fg)\cdot\nabla\ln g=\nabla f\cdot\nabla g+\frac{f}{g}\,|\nabla g|^2\geq\nabla f\cdot\nabla g,$$

for f,g *two nonnegative functions. Usually, those properties are obtained with the duality form*

$$\mathcal{I}(F^N)=\sup_{\varphi\in C_b^1(\mathbb{R}^{dN})}\left\langle F^N,-\frac{\psi^2}{4}-\nabla\cdot\psi\right\rangle,$$

see [20, Lemma 3.7] for instance. See also [7, Theorem 3] for the first proof of the superadditivity of the Fisher information, i.e. point (iii) *of the above Proposition, where the form*

$$\mathcal{I}^N(F^N)=\frac{4}{N}\sum_{i=1}^N\int_{\mathbb{R}^{dN}}\left|\nabla_i\sqrt{F^N}\right|^2=dX^N=\frac{4}{N}|\sqrt{F^N}|^2_{H^1(\mathbb{R}^{dN})},$$

is used.

According to [20, Definition 5.2], define for $\kappa\geq 1$ the sets $\mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$ as

$$\begin{aligned}\mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d)):&=\{\pi\in\mathcal{P}(\mathcal{P}(\mathbb{R}^d))\,|\int_{\mathcal{P}(\mathbb{R}^d)}\int_{\mathbb{R}^d}\langle x\rangle^\kappa\rho(dx)\pi(d\rho)<\infty\}\\&:=\{\pi\in\mathcal{P}(\mathcal{P}(\mathbb{R}^d))\,|\int_{\mathcal{P}(\mathbb{R}^d)}M_\kappa(\rho)\pi(d\rho)<\infty\}\\&:=\{\pi\in\mathcal{P}(\mathcal{P}(\mathbb{R}^d))\,|\,\mathcal{M}_\kappa(\pi)<\infty\},\end{aligned}$$

and

$$\mathcal{P}_\kappa(\mathbb{R}^{dN}):=\{(F^N)_{N\geq1}\,|F^N\in\mathcal{P}_{\rm sym}(\mathbb{R}^{dN})\text{ and }\sup_{N\geq1}\int_{\mathbb{R}^{dN}}\langle x_1\rangle^\kappa F^N(dx_1,\cdots,dx_N)<\infty\}.$$

In some sense we abuse the N in the notation, so we emphasize that $\mathcal{P}_\kappa(\mathbb{R}^{dN})$ is a set of sequences. For $\pi\in\mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$, define $(\pi_N)_{N\geq1}$ its Hewitt and Savage (see for instance [20, Theorem 5.1]) projection on $\mathcal{P}_\kappa(\mathbb{R}^{dN})$ as

$$\pi_N:=\int_{\mathcal{P}(\mathbb{R}^d)}\rho^{\otimes N}\pi(d\rho).$$

Next define on $\mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$ the mean fractional Fisher information $\tilde{\mathcal{I}}_a$ as

$$\tilde{\mathcal{I}}_a(\pi)=\sup_{N\geq1}\mathcal{I}_a^N(\pi_N)=\lim_{N\geq1}\mathcal{I}_a^N(\pi_N).\tag{3.18}$$

The fact that the lim equals the sup comes from the fact that the sequence $\left(\mathcal{I}_a^N(\pi_N)\right)_{N\geq1}$ is nondecreasing. Indeed the sequence of symmetric probability measures $(\pi_N)_{\geq1}$ is compatible, i.e. for any $k<N$ the marginal on $\mathbb{R}^{d(N-k)}$ of π_N is π_{N-k}. So that π_{N-1} is the marginal on $\mathbb{R}^{d(N-1)}$ of π_N and by point (iv) of Proposition 3.1 we have $\mathcal{I}_a^N(\pi_N)\geq\mathcal{I}_a^{N-1}(\pi_{N-1})$.

We now give the last technical result, which proof is delayed in appendix, and which enables to conclude to the desired Γ-l.s.c. property in the

Proposition 3.2. *The functional* $\tilde{\mathcal{I}}_a$ *defined in* (3.18) *is affine in the following sense. For any* $\pi\in\mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$ *and any partition of* $\mathcal{P}_\kappa(\mathbb{R}^d)$ *by some sets* $(\omega_i)_{i=1,\cdots,M}$*, such that* ω_i *is an open set in* $\mathcal{P}_\kappa(\mathbb{R}^d)\setminus\left(\bigcap_{j=1}^{i-1}\omega_j\right)$ *for all* $1\leq i\leq M-1$ *and* $\pi(\omega_i)>0$ *for all* $1\leq i\leq M$*, defining*

$$\begin{aligned}&\alpha_i:=\pi(\omega_i)\quad\textit{and}\quad\gamma_i=(\alpha_i)^{-1}\mathbf{1}_{\omega_i}\pi\in\mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))\\&\gamma_i=0\quad\textit{if}\quad\alpha_i=0,\end{aligned}$$

so that

$$\pi = \sum_{i=1}^{M} \alpha_i \gamma_i, \quad \textit{and} \quad \sum_{i=1}^{M} \alpha_i = 1,$$

it holds

$$\tilde{\mathcal{I}}_a(\pi) = \sum_{i=1}^{M} \alpha_i \tilde{\mathcal{I}}_a(\gamma_i).$$

Remark 3.4. *If one considers the classical normalized Fisher information $\mathcal{I}^N$ (cf Remark 3.3), that is the derivative of the Boltzmann's entropy along the classical many-body heat flow (i.e. equation* (3.17) *with fractional Laplacian replaced with standard Laplacian), a similar result as the one of the above Proposition can be found in [24, Proposition 3]. It based on the fact that the mean Fisher information $\tilde{\mathcal{I}}$ defined thanks to $\mathcal{I}^N$ through a similar relation as* (3.18)*, is the time derivative of the mean entropy along the infinite dimensional heat flow. And then the mean Fisher information is affine by linearity of the time derivative, the heat flow, and the mean entropy (see for instance [29]). Another proof can be found in [20], which strategy will be followed in the proof of Proposition 3.2 in Appendix C. But it is very likely that the lines of the proof of [24, Proposition 3] can also be followed in the fractional case.*
Another proof of the affinity of some mean information closely related to the one studied in this paper can be found in [30, Lemma A.6]. More precisely it is shown that the mean information $\tilde{\mathcal{T}}_a$ associated to the family of functionals

$$\mathcal{T}_a^N(F_N) = \frac{1}{2}\frac{1}{N}\sum_{k=1}^{N} \int_{\mathbb{R}^{d(N-1)}} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\left(\sqrt{F_N(X_k^x)} - \sqrt{F_N(X_k^y)}\right)^2}{|x-y|^{d+a}} dxdy\, dX^{N-1},$$

through a similar relation as (3.18)*, is affine. Due to* (1.8)*, we have $\mathcal{I}_a^N \geq 4\mathcal{T}_a^N$. Furthermore, we claim that the functionals $\mathcal{T}_a^N$ satisfies the properties listed in Proposition 3.1. So that the same result as the one of Corollary 3.1 below applies to $\tilde{\mathcal{T}}_a$. Moreover since*

$$\mathcal{T}_a^N(f^{\otimes N}) = |f|^2_{H^{a/2}(\mathbb{R}^d)},$$

the information $\tilde{\mathcal{T}}_a$ provides the required regularity at the limit. So that it is very likely that all our analysis could be rewritten thanks to these functionals.

We now can state the key argument which enables in the Diffusion Dominated case to go beyond a convergence/consistency result, and provide a complete propagation of chaos result.

Corollary 3.1. *For any $\pi \in \mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$ it holds*

$$\tilde{\mathcal{I}}_a(\pi) = \int_{\mathcal{P}(\mathbb{R}^d)} I_a(\rho)\pi(d\rho),$$

Moreover the functional $\tilde{\mathcal{I}}_a$ is affine, proper and l.s.c. w.r.t. the weak convergence in $\mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$ and satisfies the Γ-lower semi continuous property, i.e. for any sequence $(F_N)_{N\geq 1} \in \mathcal{P}_\kappa(\mathbb{R}^{dN})$ converging toward $\pi \in \mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$ in the sense that

$$\forall\, j \geq 2, F_N^j \overset{*}{\rightharpoonup} \pi^j \, in \, \mathcal{P}(\mathbb{R}^{dj}), \; as \; N \to +\infty,$$

where F_N^j denotes the marginal on $\mathbb{R}^{dj}$ of F_N, and π^j the π Hewitt and Savage projection on $\mathbb{R}^{dj}$, then it holds

$$\liminf_N \mathcal{I}_a^N(F_N) \geq \tilde{\mathcal{I}}_a(\pi) = \int_{\mathcal{P}(\mathbb{R}^d)} I_a(\rho)\pi(d\rho).$$

Proof. This result is an immediate consequence of [20, Lemma 5.6], and Propositions 3.1 and 3.2. We leave the reader check that the last two propositions consist in checking that the family of functionals $(\mathcal{I}_a^N)_{N\geq 1}$ satisfies the assumptions of [20, Lemma 5.6]. □

This result also provides the so-called 3-level representation of the fractional Fisher information in 2d defined in (3.18) i.e.

$$\tilde{\mathcal{I}}_a(\pi) = \int_{\mathcal{P}(\mathbb{R}^d)} I_a(\rho)\pi(d\rho) = \sup_{N\geq 1} \mathcal{I}_a^N(\pi^N) = \lim_{N\geq 1} \mathcal{I}_a^N(\pi^N),$$

with $\pi^N = \int_{\mathcal{P}(\mathbb{R}^d)} \rho^{\otimes N}\pi(d\rho)$.

3.3. Convergence/consistency of particle system (1.5)

In this section we establish the tightness of the law of the particle system (1.5) in both case $\alpha < a$ and $\alpha = a$. Note that in both case we have that

$$X_t^{N,1} = X_0^{N,1} + \int_0^t \frac{1}{N}\sum_{j>1}^{N} K_\alpha(X_s^{N,1} - X_s^{N,j})\, ds + \mathcal{Z}_t^1 := X_0^{N,1} + J_t^{N,1} + \mathcal{Z}_t^1,$$

so that it is enough to show the tightness of the $\left((J_t^{N,1})_{t\in[0,T]}\right)_{N\geq 1}$ to deduce the tightness of the law of $\left((\frac{1}{N}\sum_{i=1}^N \delta_{X_t^i})_{t\in[0,T]}\right)_{N\geq 1}$, due to point (*ii*) of Proposition 2.1. First we need some moments estimates given in the

Lemma 3.3. *Let $(X_t^{N,1}, \cdots, X_t^{N,N})$ be a solution to* (1.5) *with $\alpha \in (1, a]$ (with law F_t^N). Then for any $\kappa \in (1, a)$ and $t > 0$ there is a constant $C_{a,\kappa}$ such that*

$$\sup_{i=1,\cdots,N} \mathbb{E}\left[\langle X_t^i\rangle^\kappa\right] = \int_{\mathbb{R}^{dN}} \langle x_1\rangle^\kappa F_t^N \leq C_{a,\kappa} t + \int_{\mathbb{R}^{dN}} \langle x_1\rangle^\kappa F_0^N.$$

Proof. Using Ito's formula with the C^2 functional m_κ yields

$$\begin{aligned} m_\kappa(X_t^1) = m_\kappa(X_0^1) &- \int_0^t \frac{\chi}{N} a\left(1 + \left|X_s^1\right|^2\right)^{\frac{\kappa-2}{2}} X_s^1 \cdot \left(\sum_{k>1} \frac{Z_s^{1,k}}{|Z_s^{1,k}|^\alpha}\right) ds \\ &+ \int_{[0,t]\times\mathbb{R}^d} \left(m_\kappa(X_{s_-}^1 + x) - m_\kappa(X_{s_-}^1) - x\cdot\nabla m_\kappa(X_{s_-}^1)\right) M_1(ds, dx) \\ &+ \int_{[0,t]\times\mathbb{R}^d} x\cdot\nabla m_\kappa(X_{s_-}^1)\bar{M}_1(ds, dx). \end{aligned}$$

Taking the expectation yields

$$\begin{aligned} \mathbb{E}\left[m_\kappa(X_t^1)\right] = \mathbb{E}\left[m_\kappa(X_0^1)\right] &- \int_0^t \frac{\chi}{N} a\mathbb{E}\left[\left(1 + \left|X_s^1\right|^2\right)^{\frac{\kappa-2}{2}} X_s^1 \cdot \left(\sum_{k>1} \frac{Z_s^{1,k}}{|Z_s^{1,k}|^\alpha}\right)\right] ds \\ &+ \int_0^t \mathbb{E}\left[c_{2,a}\int_{\mathbb{R}^d} \frac{m_\kappa(X_s^1 + x) - m_\kappa(X_s^1) - x\cdot\nabla m_\kappa(X_s^1)}{|x|^{d+a}} dx\right] ds. \end{aligned}$$

By ex-changeability and point (*iii*) of Lemma 3.1 we get

$$\mathbb{E}\left[\left(1 + \left|X_s^1\right|^2\right)^{\frac{\kappa-2}{2}} X_s^1 \cdot \frac{Z_s^{1,k}}{|Z_s^{1,k}|^\alpha}\right] = \frac{1}{2}\mathbb{E}\left[\left(\left(1 + \left|X_s^1\right|^2\right)^{\frac{\kappa-2}{2}} X_s^1 - \left(1 + \left|X_s^k\right|^2\right)^{\frac{\kappa-2}{2}} X_s^k\right) \cdot \frac{Z_s^{1,k}}{|Z_s^{1,k}|^\alpha}\right] \geq 0,$$

and then

$$\mathbb{E}\left[m_\kappa(X_t^1)\right] \leq \mathbb{E}\left[m_\kappa(X_0^1)\right] + \int_0^t \mathbb{E}\left[-(-\Delta)^{a/2} m_\kappa(X_s^1)\right] ds.$$

Using (3.16) then yields

$$\begin{aligned} \mathbb{E}\left[m_\kappa(X_t^{N,1})\right] &\leq \mathbb{E}\left[m_\kappa(X_0^{N,1})\right] + \int_0^t C_{a,\kappa}\mathbb{E}\left[m_{\kappa-a}(X_s^{N,1})\right] ds \\ &\leq \mathbb{E}\left[m_\kappa(X_0^1)\right] + tC_{a,\kappa}, \end{aligned}$$

since $\kappa < a$ and therefore $m_{\kappa-a}(x) \leq 1$ for any $x \in \mathbb{R}^d$. And the result is proved since by ex-changeability of the particles it holds for any $i = 1, \cdots, N$

$$\mathbb{E}\left[\langle X_t^i\rangle^\kappa\right] = \mathbb{E}\left[\langle X_t^1\rangle^\kappa\right] = \int_{\mathbb{R}^{dN}} \langle x_1\rangle^\kappa F_t^N.$$

□

We now give the two main results of this section. The first is the

Proposition 3.3. *Let $1 < \alpha < a < 2$, and $(X_t^{N,1}, \cdots, X_t^{N,N})$ be a solution to equation (1.5) for an initial condition with law $(F_0^N)_{N\geq 1} \in \mathcal{P}_\kappa(\mathbb{R}^{dN})$ for some $\kappa \in (1, a)$ and such that*

$$\mathcal{H}^N(F_0^N) = \frac{1}{N}\int_{\mathbb{R}^{dN}} F_0^N \ln F_0^N dX^N < \infty.$$

Then for any $t > 0$ it holds

$$\int_0^t \mathcal{I}_a^N(F_s^N)ds \leq C\left(\mathcal{H}^N(F_0^N) + \int_{\mathbb{R}^{dN}} \langle x_1\rangle^\kappa F_0^N + t\right),$$

where $C > 0$ is a constant independent of F_0^N. In particular for any $\gamma \in (0, a)$ it holds

$$\sup_{N\geq 1}\int_0^T \sup_{1\leq i\neq j\leq N} \mathbb{E}\left[|X_u^{N,i} - X_u^{N,j}|^{-\gamma}\right] du < \infty.$$

The proof of this proposition is based on [18] itself inspired by [15]. It relies on a control of the Fisher information. The second result proved in this section is given in the

Proposition 3.4. *Let $1 < \alpha = a < 2$ and $(X_t^{N,1}, \cdots, X_t^{N,N})$ be a solution to equation (1.5) for an initial condition with law $(F_0^N)_{N\geq 1} \in \mathcal{P}_\kappa(\mathbb{R}^{dN})$ for some $\kappa \in (1, a)$. There exists $a^* \in (1, 2)$, and $\chi : a \in (a^*, 2) \mapsto (0, \infty)$, such that if $a \in (a^*, 2)$ and $\chi \in (0, \chi(a))$ then it holds for any $T > 0$ and $\varepsilon \in (0, 1)$*

$$\sup_{N\geq 1}\int_0^T \sup_{1\leq i\neq j\leq N} \mathbb{E}\left[|X_u^{N,i} - X_u^{N,j}|^{\varepsilon - a}\right] du < \infty.$$

The proof of this proposition is based on [16] itself inspired by [28]. Unfortunately, the technical limitation $a^* \in (1, 2)$ is not sharp, and it seems not clear how to improve them. The proofs of both these propositions are given later in this section.

3.3.1. Proof of Proposition 3.3

We begin the proof of this proposition with some fractional logarithmic Gagliardano-Nirenberg-Sobolev inequality. More precisely we have the

Lemma 3.4. *Let be $d \geq 2$, for any $p \in (1, \frac{d}{d-a}]$, there is a constant $C_{p,d,a} > 0$ s.t. $\forall u \in \mathcal{P}(\mathbb{R}^d)$ it holds*

$$||u||_{L^p(\mathbb{R}^d)} \leq C_{p,d,a} I_a(u)^{1-\frac{d}{a}\left(\frac{1}{p}-\frac{d-a}{d}\right)}.$$

Remark 3.5. *This Lemma can be seen as a generalization in the fractional case of [15, Lemma 3.2], in the case $\alpha < 2$. However in the case $d = 2$, the critical exponent $\frac{2}{2-a}$ can be reached when $a < 2$, whereas the exponent ∞ has to be excluded case $a = 2$.*

Proof. First recall that by (1.8) for any $x, y \geq 0$ it holds

$$\Phi(x, y) \geq 4\left(\sqrt{x} - \sqrt{y}\right)^2,$$

so that

$$\begin{aligned} I_a(u) &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\Phi(u(x), u(y))}{|x-y|^{d+a}} \\ &\geq 4\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\left(\sqrt{u(x)} - \sqrt{u(y)}\right)^2}{|x-y|^{d+a}} = 4|\sqrt{u}|^2_{H^{a/2}(\mathbb{R}^d)}. \end{aligned}$$

By fractional Sobolev's embeddings (see for instance [10, Theorem 6.5]) there is $C_{a,d} > 0$ such that

$$|\sqrt{u}|^2_{H^{a/2}(\mathbb{R}^d)} \geq C_{a,d} \|\sqrt{u}\|^2_{L^{\frac{2d}{d-a}}(\mathbb{R}^d)} = C_{a,d} \|u\|_{L^{\frac{d}{d-a}}(\mathbb{R}^d)},$$

we conclude to the desired result since for any $p \in (1, \frac{d}{d-a}]$ by interpolation inequality we have

$$\|u\|_{L^p(\mathbb{R}^d)} \leq \|u\|^{\frac{d}{a}\left(\frac{1}{p} - \frac{d-a}{d}\right)}_{L^1(\mathbb{R}^d)} \|u\|^{1-\frac{d}{a}\left(\frac{1}{p} - \frac{d-a}{d}\right)}_{L^{\frac{d}{d-a}}(\mathbb{R}^d)}.$$

□

Lemma 3.5. *Let be $\gamma \in (0, a)$ and $p \in \left(\frac{d}{d-\gamma}, \frac{d}{d-a}\right)$. There exists a constant $C_{\gamma,p,a} > 0$ such that for any $N \geq 2$ and $F^N \in \mathcal{P}_{sym}(\mathbb{R}^{dN})$ it holds*

$$\int_{\mathbb{R}^{dN}} |x_1 - x_2|^{-\gamma} F^N(dx_1 \cdots dx_N) \leq C_{\gamma,p,a} \left(1 + \mathcal{I}^N_a(F^N)^{1-\frac{2}{a}\left(\frac{1}{p} - \frac{2-a}{2}\right)}\right).$$

Proof. We introduce the unitary linear transformation

$$\forall (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \Psi(x_1, x_2) = (x_1 - x_2, x_2).$$

Denote $F^2 \in \mathcal{P}_{\mathrm{sym}}(\mathbb{R}^{2d})$ the marginal of F^N on $\mathbb{R}^{2d}$ and $\tilde{F}^2 = F^2 \circ \Psi^{-1}$. A simple substitution shows that $\mathcal{I}^2_a(F^2) = \mathcal{I}^2_a(\tilde{F}^2)$. Indeed

$$\begin{aligned}
\mathcal{I}^2_a(\tilde{F}^2) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\Phi(F^2 \circ \Psi^{-1}(x,z), F^2 \circ \Psi^{-1}(y,z))}{|x-y|^{d+a}} dxdydz \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\Phi(F^2(x+z,z), F^2(y+z,z))}{|(x+z)-(y+z)|^{d+a}} dxdydz \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\Phi(F^2(x,z), F^2(y,z))}{|x-y|^{d+a}} dxdydz = \mathcal{I}^2_a(F^2).
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^{dN}} |x_1 - x_2|^{-\gamma} F^N(dx_1 \cdots dx_N) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^{-\gamma} F^2(x_1, x_2) dx_1 dx_2 \\
&= \int_{\mathbb{R}^2 \times \mathbb{R}^d} |y_1|^{-\gamma} \tilde{F}^2(y_1, y_2) dy_1 dy_2 \\
&\leq 1 + \int_{\mathbb{R}^2 \times \mathbb{R}^d} \mathbf{1}_{|y_1| \leq 1} |y_1|^{-\gamma} \tilde{F}^2(y_1, y_2) dy_1 dy_2 \\
&\leq 1 + \int_{|y_1| \leq 1} |y_1|^{-\gamma} \tilde{f}^2_1(y_1) dy_1
\end{aligned}$$

where we used the unitary change of variables $(y_1, y_2) = \Psi(x_1, x_2)$, $\tilde{f}^2_1$ denotes the first marginal of $\tilde{F}^2$. Then for any $p > \frac{d}{d-\gamma}$

$$\begin{aligned}
\int_{\mathbb{R}^2 \times \mathbb{R}^d} |x_1 - x_2|^{-\gamma} F^2(x_1, x_2) dx_1 dx_2 &\leq 1 + \left(\int_{|y_1| \leq 1} |y_1|^{-\gamma p'} dy_1\right)^{1/p'} \|\tilde{f}^2_1\|_{L^p(\mathbb{R}^d)} \\
&\leq C_{\gamma,p} \left(1 + I_a^{1-\frac{2}{a}\left(\frac{1}{p} - \frac{2-a}{2}\right)}(\tilde{f}^2_1)\right),
\end{aligned}$$

where we used Lemma 3.4. The proof is then concluded since

$$I_a(\tilde{f}^2_1) \leq \mathcal{I}^2_a(\tilde{F}^2) = \mathcal{I}^2_a(F^2) \leq \mathcal{I}^N_a(F^N),$$

thanks to point (*iv*) Proposition 3.1. □

Remark 3.6. *Note that in the tensorized case $F^N = f^{\otimes N}$ with $f \in \mathcal{P}(\mathbb{R}^d)$ (in which we are clearly not), Hardy-Littlewood-Sobolev's inequality (see for instance [27, Theorem 4.3]) yields*

$$\int_{\mathbb{R}^{dN}} |x_1 - x_2|^{-\gamma} F^N(dx_1 \cdots dx_N) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|^{-\gamma} f(x_1) f(x_2) dx_1 dx_2 \leq C \|f\|^2_{L^{\frac{2d}{2d-\gamma}}}.$$

But by Lemma 3.4

$$\|f\|^2_{L^{\frac{2d}{2d-\gamma}}} \leq C(I_a(f))^{\gamma/a},$$

so that using point (iv) of Proposition 3.1 yields

$$\int_{\mathbb{R}^{dN}} |x_1 - x_2|^{-\gamma} F^N(dx_1 \cdots dx_N) \leq C \left(\mathcal{I}^2_a(F^N)\right)^{\gamma/a},$$

which holds even in the critical case $\gamma = a$, (provided that $a \neq 2$ when $d = 2$). The latter condition excludes the classical Keller Segel case $a = d = 2$.

We now have all the ingredient to mimic the entropy dissipation estimate of [15]. Precisely we have the

Lemma 3.6. *Let be $\alpha \in (1, a)$, $\chi > 0$, and $(X^{N,1}_t, \cdots, X^{N,N}_t)_{t\geq 0}$ a solution to (1.5) with initial law $F^N_0 \in \mathcal{P}_\kappa(\mathbb{R}^{dN})$ for some $\kappa \in (1, a)$, and denote $F^N_t \in \mathcal{P}_\kappa(\mathbb{R}^{dN})$ its law. Then for any $p \in \left(\frac{d}{d-\alpha}, \frac{d}{d-a}\right)$ there is a constant $C_{\alpha,p,a,\chi,\kappa} > 0$ such that*

$$\int_0^t \mathcal{I}^N_a(F^N_s)\, ds \leq C_{\alpha,p,a,\chi,\kappa} \left(\mathcal{H}^N(F^N_0) + \int_{\mathbb{R}^{dN}} \langle x_1 \rangle^\kappa F^N_0 + t \right).$$

Proof. We first prove that F^N_t is a weak solution at time t to the Liouville equation (2.14). Let be $\phi \in C^\infty(\mathbb{R}^{dN})$. Due to Ito's rule we get

$$\begin{aligned}
\phi(X^{N,1}_t, \cdots, X^{N,N}_t) &= \phi(X^{N,1}_0, \cdots, X^{N,N}_0) + \int_0^t \sum_{i=1}^N \nabla_i \phi(X^{N,1}_s, \cdots, X^{N,N}_s) \cdot \frac{1}{N} \sum_{j \neq i} K_\alpha(X^{N,i}_s - X^{N,j}_s)\, ds \\
&+ \sum_{i=1}^N \int_{[0,t]\times\mathbb{R}^d} \left(\phi(X^{N,1}_{s_-}, \cdot, X^{N,i}_{s_-} + z, \cdot, X^{N,N}_{s_-}) - \phi(X^{N,1}_{s_-}, \cdots, X^{N,N}_{s_-}) - z \cdot \nabla_i \phi(X^{N,1}_{s_-}, \cdots, X^{N,N}_{s_-}) \right) M_i(ds, dz) \\
&+ \sum_{i=1}^N \int_{[0,t]\times\mathbb{R}^d} z \cdot \nabla_i \phi(X^{N,1}_{s_-}, \cdots, X^{N,N}_{s_-}) \bar{M}_i(ds, dz),
\end{aligned}$$

and since $F^N_t = \mathcal{L}(X^{N,1}_t, \cdots, X^{N,N}_t)$ taking the expectation yields

$$\begin{aligned}
\int_{\mathbb{R}^{dN}} \phi(x_1, \cdots, x_N) F^N_t &= \int_{\mathbb{R}^{dN}} \phi(x_1, \cdots, x_N) F^N_0 + \int_0^t \int_{\mathbb{R}^{dN}} \sum_{i=1}^N \nabla_i \phi(x_1, \cdots, x_N) \cdot \frac{1}{N} \sum_{j \neq i} K_\alpha(x_i - x_j) F^N_s\, ds \\
&+ \int_0^t \sum_{i=1}^N \int_{\mathbb{R}^{dN}} c_{d,a} \int_{\mathbb{R}^d} \frac{\phi(x_1, \cdot, x_i + z, \cdot, x_N) - \phi(x_1, \cdots, x_N) - z \cdot \nabla_i \phi(x_1, \cdots, x_N)}{|z|^{d+a}} dz F^N_s ds,
\end{aligned}$$

which is nothing but the weak formulation of (2.14). Hence we deduce, dropping the t in the notation for the sake of

simplicity

$$\begin{aligned}\frac{d}{dt}\mathcal{H}^N(F^N) &= \frac{1}{N}\int_{\mathbb{R}^{dN}} \partial_t F^N(1+\ln F^N)\\ &= -\frac{\chi}{N^2}\sum_{j\neq i}\int_{\mathbb{R}^{dN}} \nabla_i\cdot\Big(K_\alpha(x_i-x_j)F^N\Big)(1+\ln F^N)\\ &\quad -\frac{1}{N}\sum_{i=1}^N\int_{\mathbb{R}^{dN}}\text{p.v.}\int_{\mathbb{R}^d}\frac{F^N(x_1,\cdot,x_i+z,\cdot,x_N)-F^N(x_1,\cdots,x_N)}{|z|^{d+a}}(1+\ln F^N(x_1,\cdots,x_N))dz dx_1\cdots dx_N\\ &= \frac{\chi}{N^2}\sum_{j\neq i}\int_{\mathbb{R}^{dN}}\nabla_i\cdot\Big(K_\alpha(x_i-x_j)\Big)F^N(x_1,\cdots,x_N)dx_1,\cdots,dx_N\\ &\quad -\frac{1}{N}\sum_{i=1}^N\int_{\mathbb{R}^{dN}}\int_{\mathbb{R}^d}\frac{\Big(F^N(x_1,\cdot,x,\cdot,x_N)-F^N(x_1,\cdot,y,\cdot,x_N)\Big)\Big(\ln F^N(x_1,\cdot,x,\cdot,x_N)-\ln F^N(x_1,\cdot,y,\cdot,x_N)\Big)}{|x-y|^{d+a}}dxdyd\tilde{X}^i_N\\ &= \frac{\chi(N-1)(\alpha-1)}{N}\int_{\mathbb{R}^d\times\mathbb{R}^d}|x_1-x_2|^{-\alpha}F^2(x_1,x_2)dx_1dx_2-\mathcal{I}^N_a(F^N),\end{aligned}$$

where $\tilde{X}^i_N = dx_1,\cdot,dx_{i-1},dx_{i+1},\cdot,dx_N$ and $F^2\in\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)$ stands for the two particles marginal of F^N. But using Lemma 3.5, we find for any $p\in\left(\frac{d}{d-\alpha},\frac{d}{d-a}\right)$ that

$$\begin{aligned}\frac{d}{dt}\mathcal{H}^N(F^N_t) &\le \chi(\alpha-1)C_{\alpha,p,a}\left(1+\mathcal{I}^N_a(F^N_t)^{1-\frac{2}{a}\left(\frac{1}{p}-\frac{2-a}{2}\right)}\right)-\mathcal{I}^N_a(F^N_t)\\ &\le \chi(\alpha-1)C_{\alpha,p,a}-\left(\mathcal{I}^N_a(F^N_t)-\chi(\alpha-1)C_{\alpha,p,a}\mathcal{I}^N_a(F^N_t)^{1-\frac{2}{a}\left(\frac{1}{p}-\frac{2-a}{2}\right)}\right)\\ &\le C_{\alpha,p,a,\chi}-\frac{1}{2}\mathcal{I}^N_a(F^N_t),\end{aligned}$$

so that

$$\mathcal{H}^N(F^N_t)+\int_0^t\frac{1}{2}\mathcal{I}^N_a(F^N_s)ds\le\mathcal{H}^N(F^N_0)+C_{\alpha,p,a,\chi}t. \tag{3.19}$$

Then define $G^N_\kappa = e^{-\sum_{i=1}^N\lambda_\kappa|x_i|^\kappa}$, with $\lambda_\kappa>0$ being such that

$$\int_{\mathbb{R}^d}e^{-\lambda_\kappa|x|^\kappa}dx=1.$$

By positivity of the relative entropy

$$\mathcal{H}^N(F^N_t)=\int_{\mathbb{R}^{dN}}F^N_t\ln\left(\frac{F^N_t}{G^N_\kappa}\right)+\int_{\mathbb{R}^{dN}}F^t_t\ln\left(G^N_\kappa\right)\ge-\lambda_\kappa\int_{\mathbb{R}^{dN}}\langle x_1\rangle^\kappa F^N_t$$

and summing $\lambda_\kappa\int_{\mathbb{R}^{dN}}\langle x_1\rangle^\kappa F^N_t$ to (3.19), combined to Lemma 3.3 yields

$$\left(\mathcal{H}^N(F^N_t)+\lambda_\kappa\int_{\mathbb{R}^{dN}}\langle x_1\rangle^\kappa F^N_t\right)+\int_0^t\frac{1}{2}\mathcal{I}^N_a(F^N_s)ds\le\mathcal{H}^N(F^N_0)+\lambda_\kappa\int_{\mathbb{R}^{dN}}\langle x_1\rangle^\kappa F^N_0+\Big(C_{\alpha,p,a,\chi}+\lambda_\kappa C_{a,\kappa}\Big)t$$

which concludes the desired result, since the l.h.s. of the above inequality is the sum of two nonnegative term. □

Combining Lemmas 3.6 and 3.5 concludes the proof of Proposition 3.3.

3.3.2. Proof of Proposition 3.4

In this section we now set $\alpha=a$. In this case we extend the method used in [16]. In this case let $(X^{N,1}_t,\cdots,X^{N,N}_t)_{t\in[0,T]}$ be a solution to (1.5) and denote $Z^{i,j}_s := X^{N,i}_s - X^{N,j}_s$ note that it solves

$$Z^{i,j}_t = Z^{i,j}_0-\frac{\chi}{N}\int_0^t\sum_{k\neq i,j}\left(\frac{Z^{i,k}_s}{|Z^{i,k}_s|^a}-\frac{Z^{j,k}_s}{|Z^{j,k}_s|^a}\right)ds-\frac{2\chi}{N}\int_0^t\frac{Z^{i,j}_s}{|Z^{i,j}_s|^a}ds+\int_{[0,t]\times\mathbb{R}^d}x\left(\bar{M}_i-\bar{M}_j\right)(ds,dx).$$

Recall that $(M_i)_{i=1,\cdots,N}$ are independent Poisson random measures with intensity $dsc_{d,a}|x|^{-(d+a)}dx$, and $\bar{M}_i$ are their compensated measures (see (2.9)). Denote

$$H_t^{i,j} := \int_{[0,t]\times\mathbb{R}^d} x\left(\bar{M}_i - \bar{M}_j\right)(ds,dx).$$

It holds (see [9]) for any $r \in \mathbb{R}^d$, since M_i is independent of M_j

$$\begin{aligned}\mathbb{E}\left[e^{ir\cdot H_t^{i,j}}\right] &= \mathbb{E}\left[e^{ir\cdot\int_{[0,t]\times\mathbb{R}^d} x\bar{M}_i(ds,dx)}\right]\mathbb{E}\left[e^{-ir\cdot\int_{[0,t]\times\mathbb{R}^d} x\bar{M}_j(ds,dx)}\right]\\ &= e^{-t2c_{a,d}|r|^a} = e^{-tc_{a,d}|2^{1/a}r|^a} = \mathbb{E}\left[e^{i2^{1/a}r\cdot\int_{[0,t]\times\mathbb{R}^d} x\bar{M}_i(ds,dx)}\right],\end{aligned}$$

hence by equality of the Fourier transform, it follows that

$$H_t^{i,j} \overset{(\mathcal{L})}{=} \int_{[0,t]\times\mathbb{R}^d} 2^{1/a}x\bar{M}_i(ds,dx).$$

Let be $\varepsilon \in (0,1)$, $\eta > 0$ and similarly as in Lemma 3.2 define

$$\phi_\eta(x) = \eta^\varepsilon\langle\frac{x}{\eta}\rangle^\varepsilon = \left(|x|^2+\eta^2\right)^{\frac{\varepsilon}{2}},$$

using Ito's rule yields

$$\begin{aligned}\phi_\eta\left(Z_t^{i,j}\right) = \phi_\eta\left(Z_0^{i,j}\right) &- \int_0^t \frac{\chi}{N}\varepsilon\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}} Z_s^{i,j}\cdot\left(\sum_{k\neq i,j}\left(\frac{Z_s^{i,k}}{|Z_s^{i,k}|^a} - \frac{Z_s^{j,k}}{|Z_s^{j,k}|^a}\right)\right)ds\\ &- \int_0^t \frac{2\chi\varepsilon}{N}\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\left|Z_s^{i,j}\right|^{2-a}ds\\ &+ \int_{[0,t]\times\mathbb{R}^d}\left(\phi_\eta\left(Z_{s_-}^{i,j}+2^{1/a}x\right) - \phi_\eta\left(Z_{s_-}^{i,j}\right) - 2^{1/a}x\cdot\nabla\phi_\eta\left(Z_{s_-}^{i,j}\right)\right)M_i(ds,dx)\\ &+ \int_{[0,t]\times\mathbb{R}^d} 2^{1/a}x\cdot\nabla\phi_\eta\left(Z_{s_-}^{i,j}\right)\bar{M}_i(ds,dx).\end{aligned} \tag{3.20}$$

Taking the expectation yields

$$\begin{aligned}\mathbb{E}\left[\phi_\eta\left(Z_t^{i,j}\right)\right] = \mathbb{E}\left[\phi_\eta\left(Z_0^{i,j}\right)\right] &- \int_0^t \frac{\chi}{N}\varepsilon\mathbb{E}\left[\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}} Z_s^{i,j}\cdot\left(\sum_{k\neq i,j}\left(\frac{Z_s^{i,k}}{|Z_s^{i,k}|^a} - \frac{Z_s^{j,k}}{|Z_s^{j,k}|^a}\right)\right)\right]ds\\ &- \int_0^t \frac{2\chi\varepsilon}{N}\mathbb{E}\left[\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}|Z_s^{i,j}|^{2-a}\right]ds\\ &+ \int_0^t \mathbb{E}\left[c_{d,a}\int_{\mathbb{R}^d}\frac{\phi_\eta\left(Z_s^{i,j}+2^{1/a}x\right) - \phi_\eta\left(Z_s^{i,j}\right) - 2^{1/a}x\cdot\nabla\phi_\eta\left(Z_s^{i,j}\right)}{|x|^{d+a}}dx\right]ds.\end{aligned}$$

Note that the change of variable $x' = 2^{1/a}x$ yields

$$\begin{aligned}c_{d,a}\int_{\mathbb{R}^d}\frac{\phi_\eta\left(Z+2^{1/a}x\right) - \phi_\eta(Z) - 2^{1/a}x\cdot\nabla\phi_\eta(Z)}{|x|^{d+a}}dx &= c_{d,a}\int_{\mathbb{R}^d}\frac{\phi_\eta(Z+x') - \phi_\eta(Z) - x'\cdot\nabla\phi_\eta(Z)}{|2^{-1/a}x'|^{d+a}}2^{-d/a}dx'\\ &= 2(-\Delta)^{a/2}\phi_\eta(Z)\end{aligned}$$

Also note that due to ex changeability we find

$$\begin{aligned}
A_s^{i,j} &:= \mathbb{E}\left[\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\left|Z_s^{i,j}\cdot\sum_{k\neq i,j}\left(\frac{Z_s^{i,k}}{|Z_s^{i,k}|^a}-\frac{Z_s^{j,k}}{|Z_s^{j,k}|^a}\right)\right|\right]\\
&\leq \sum_{k\neq i,j}\left(\mathbb{E}\left[\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\left|Z_s^{i,j}\right|\left|Z_s^{i,k}\right|^{1-a}\right]+\mathbb{E}\left[\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\left|Z_s^{i,j}\right|\left|Z_s^{j,k}\right|^{1-a}\right]\right)\\
&= 2\sum_{k\neq i}\mathbb{E}\left[\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\left|Z_s^{i,j}\right|\left|Z_s^{i,k}\right|^{1-a}\right]\\
&\leq 2\sum_{k\neq i}\mathbb{E}\left[\left(\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-2}{2}}\left|Z_s^{i,j}\right|\right)^p\right]^{1/p}\mathbb{E}\left[\left|Z_s^{i,k}\right|^{(1-a)p/(p-1)}\right]^{(p-1)/p}\\
&\leq 2(N-2)\mathbb{E}\left[\left(\left(|Z_s^{1,2}|^2+\eta^2\right)^{\frac{\varepsilon-1}{2}}\right)^p\right]^{1/p}\mathbb{E}\left[\left|Z_s^{1,2}\right|^{(1-a)p/(p-1)}\right]^{(p-1)/p},
\end{aligned}$$

for any $p>1$. Choosing $p=1+\frac{a-1}{1-\varepsilon}$ yields

$$A_s^{i,j}\leq 2(N-2)\mathbb{E}\left[\left(\left(|Z_s^{1,2}|^2+\eta^2\right)^{\frac{\varepsilon-1}{2}}\right)^{\frac{a-\varepsilon}{1-\varepsilon}}\right]^{\frac{1-\varepsilon}{a-\varepsilon}}\mathbb{E}\left[\left|Z_s^{1,2}\right|^{\varepsilon-a}\right]^{\frac{a-1}{a-\varepsilon}}.$$

Putting all those estimates together and using also Lemma 3.2, we get

$$\begin{aligned}
\mathbb{E}\left[\phi_\eta\left(Z_t^{1,2}\right)\right] \geq &-\frac{2\chi\varepsilon(N-2)}{N}\int_0^t\mathbb{E}\left[\left(|Z_s^{1,2}|^2+\eta^2\right)^{\frac{\varepsilon-a}{2}}\right]^{\frac{1-\varepsilon}{a-\varepsilon}}\mathbb{E}\left[\left|Z_s^{1,2}\right|^{\varepsilon-a}\right]^{\frac{a-1}{a-\varepsilon}}ds-\int_0^t\frac{2\chi\varepsilon}{N}\mathbb{E}\left[\left(|Z_s^{i,j}|^2+\eta^2\right)^{\frac{\varepsilon-a}{2}}\right]ds\\
&+2c_{d,a}|\mathbb{S}^{d-1}|\varepsilon\int_0^t\left(\frac{\varepsilon c_d}{2(2-a)}\mathbb{E}\left[\left(|Z_s^{1,2}|^2+\eta^2\right)^{\frac{\varepsilon-4}{2}}|Z_s^{1,2}|^{4-a}\right]-C_{\varepsilon,a}\mathbb{E}\left[|Z_s^{1,2}|^{\varepsilon-a}\right]\right)ds.
\end{aligned}\tag{3.21}$$

On the other hand, provided that $\eta\in(0,1)$, since $\kappa>1>\varepsilon$ using ex-changeability

$$\begin{aligned}
\mathbb{E}\left[\phi_\eta\left(Z_t^{1,2}\right)\right] &\leq \mathbb{E}\left[\langle Z_t^{1,2}\rangle^\kappa\right]\\
&\leq 2^{\frac{\kappa}{2}+1}\mathbb{E}\left[\langle X_t^1\rangle^\kappa\right]\leq 2^{\frac{\kappa}{2}+1}C_{a,\kappa,t},
\end{aligned}$$

where $C_{a,\kappa,t}$ is the constant exhibited in Lemmma 3.3. This and letting η go to 0 in (3.21) yields

$$\frac{2^{\frac{\kappa}{2}+1}C_{a,\kappa,t}}{2\varepsilon\left(c_{d,a}|\mathbb{S}^{d-1}|(\frac{\varepsilon c_d}{2(2-a)}-C_{\varepsilon,a})-\chi\right)}\geq\int_0^t\mathbb{E}\left[\left|Z_s^{1,2}\right|^{\varepsilon-a}\right]ds,$$

provided that

$$\frac{\varepsilon c_{d,a}|\mathbb{S}^{d-1}|c_d}{2(2-a)}-c_{d,a}|\mathbb{S}^{d-1}|C_{\varepsilon,a}>\chi,$$

where we recall that $C_{\varepsilon,a}=\left(\frac{2-\varepsilon}{\sqrt{4-\varepsilon(3-a)\varepsilon}}+\frac{1}{a\varepsilon}\right)$. Now for fixed $d\geq 2$, we may define a^* as

$$a^*:=\inf\Big\{a\in(1,2),\ s.t.\ \sup_{\varepsilon\in(0,1)}\Big\{\frac{\varepsilon c_d}{2(2-a)}-C_{\varepsilon,a}\Big\}>0\Big\},$$

and for fixed $a\in(a^*,2)$, we define $\chi(a)$ as

$$\chi(a):=c_{d,a}|\mathbb{S}^{d-1}|\sup_{\varepsilon\in(0,1)}\Big\{\frac{\varepsilon c_d}{2(2-a)}-C_{\varepsilon,a}\Big\}.$$

And the proof of Proposition 3.4 is complete.

3.3.3. Martingale method for convergence/consistency

Using now Propositions 3.3 and 3.4, we can give the

Theorem 3.1. *Let be $T > 0$ and either $1 < \alpha < a < 2$ and $\chi > 0$ or $a^* < \alpha = a < 2$ and $\chi \in (0, \chi(a))$. Consider $(X_t^{N,i})_{t\in[0,T],i=1,\cdots,N}$ a sequence of solution to (1.5), and denote $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}$ the associated empirical measure. Assume that $F_0^N = \mathcal{L}\left(X_0^{N,1}, \cdots, X_0^{N,N}\right)$ the law of the initial condition satisfies*

$$F_0^N = \rho_0^{\otimes N}, \text{ with } \rho_0 \in L\log L(\mathbb{R}^d) \cap \mathcal{P}_\kappa(\mathbb{R}^d), \text{ if } a > \alpha,$$

or

$$(F_0^N)_{N\geq 1} \text{ is } \rho_0 - \text{chaotic}, \ (F_0^N)_{N\geq 1} \in \mathcal{P}_\kappa(\mathbb{R}^{dN}), \text{ if } a = \alpha \in (a^*, 2),$$

for some $\kappa \in (1, a)$. Then

(*i*) *$(\mu_\cdot^N)_{N\in\mathbb{N}}$ is tight in $\mathcal{P}(\mathcal{D}([0,T],\mathbb{R}^d))$*

(*ii*) *any accumulation point of $(\mu_\cdot^N)_{N\in\mathbb{N}}$ almost surely belongs to $\mathcal{S}_\alpha^a$ in case $a > \alpha$ or $\mathcal{S}_a$ in case $a = \alpha$ respectfully defined as*

$$\begin{aligned}
&\mathcal{S}_\alpha^a = \{\rho \in \mathcal{P}\left(\mathcal{D}\left(0,T;\mathbb{R}^d\right)\right) \,|\, \rho_\cdot = \mathcal{L}(X_\cdot), \ X_\cdot \text{solution to (1.7), and } \int_0^T I_a(\rho_t)dt < \infty\},\\
&\mathcal{S}_a = \{\rho \in \mathcal{P}\left(\mathcal{D}\left(0,T;\mathbb{R}^d\right)\right) \,|\, \rho_\cdot = \mathcal{L}(X_\cdot), \ X_\cdot \text{solution to (1.7), and } \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^{\varepsilon-a}\rho_t(x)\rho_t(y)dxdydt < \infty\},\\
&\quad \forall \varepsilon \in (0,a).
\end{aligned} \tag{3.22}$$

Remark 3.7. *This Theorem is mostly a retranscription of [16, Step 2 of proof of Theorem 6], [15, Proposition 6.1] or [18, Proposition 4.2]. It is a natural step in propagation of chaos by tightness to characterize the limit point obtained by compactness.*

Proof. Proof of (*i*)
First the sequence $(X_0^{N,1})_{N\geq 1}$ is tight, since the initial condition are assumed to be ρ_0-chaotic. Then, we deduce from Propositions 3.3 or 3.4 and Lemma 2.1 that $\left((X_0^{N,1} + J_t^{N,1})_{t\in[0,T]}\right)_{N\geq 1}$ is tight in $C(0,T;\mathbb{R}^d)$. Hence $\left((X_0^{N,1} + J_t^{N,1} + \mathcal{Z}_t^1)_{t\in[0,T]}\right)_{N\geq 1}$ is tight in $\mathcal{D}([0,T];\mathbb{R}^d)$, the space of cadlag trajectories. But using point (*ii*) of Proposition 2.1 concludes the proof of point (*i*) since $\mathcal{D}([0,T],\mathbb{R}^d)$ is Polish (see [3, Theorem 12.2]).
Proof of point (*ii*)
Due to point (*i*) and Prokhorov's Theorem (see for instance [3, Theorem 5.1]), we know that there is a subsequence of $(\mu_\cdot^N)_{N\geq 1}$ (for which we will use the same notation for the sake of notational simplicity) going in law to some $\mu_\cdot \in \mathcal{P}\left(\mathcal{D}(0,T;\mathbb{R}^d)\right)$. We now define the martingale problem of unknown $Q \in \mathcal{P}\left(\mathcal{D}(0,T;\mathbb{R}^d)\right)$

$$\begin{cases}
(a) \quad \mathbf{e}_0 \# Q = \rho_0,\\
(b) \quad Q_t := \mathbf{e}_t \# Q, (Q_t)_{t\in[0,T]} \quad \text{satisfies the relevant inequality of (3.22) depending on whether } a = \alpha \text{ or } a > \alpha,\\
(c) \quad \forall 0 < t_1 < \cdots < t_k < s < t \leq T, \phi_1, \cdots, \phi_k \in C_b(\mathbb{R}^d), \phi \in C_b^2(\mathbb{R}^d), \quad \mathcal{F}(Q) = 0, \text{with}\\
\qquad \mathcal{F}(Q) :=\\
\int\int \prod_{k=1}^N \phi_k(\gamma_{t_k})\left(\phi(\gamma_t) - \phi(\gamma_s) - \chi\int_s^t K_\alpha(\gamma_u - \tilde{\gamma}_u)\cdot\nabla\phi(\gamma_u) - \text{v.p.}c_{d,a}\int_{\mathbb{R}^d} \frac{\phi(\gamma_u+z)-\phi(\gamma_u)-z\cdot\nabla\phi(\gamma_u)}{|z|^{d+a}}dz\,du\right) Q(d\gamma)Q(d\tilde{\gamma}).
\end{cases} \tag{3.23}$$

Recall that $\mathbf{e}_t$ is the evaluation of cadlag trajectories at time t map defined as

$$\mathbf{e}_t : \gamma \in \mathcal{D}(0,T;\mathbb{R}^d) \mapsto \gamma(t) \in \mathbb{R}^d,$$

so that condition (a) means that the time marginal Q at time $t = 0$ is ρ_0, and (b) means that the family of time marginals $(Q_t)_{t\in[0,T]}$ of Q satisfies

$$\forall \varepsilon \in (0,1),\ \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^{\varepsilon-a} Q_t(dx)Q_t(dy)dt < \infty,\ \text{when } a=\alpha, \text{ and } \int_0^T I_a(Q_t)dt < \infty, \text{when } a > \alpha.$$

We now show that the limit point $\mu_\cdot \in \mathcal{P}\left(\mathcal{D}(0,T;\mathbb{R}^d)\right)$ almost surely solves this martingale problem, and divide it in 3 steps.

$\diamond$ Step 1

It is straightforward that in both cases $a = \alpha$ or $a > \alpha$, $\mu_\cdot$ satisfies point (a), due to the fact F_0^N is ρ_0-chaotic.

$\diamond$ Step 2

• Step 2.a

In the case $a = \alpha$ we use the techniques of [16, *Proof of Theorem 6, step 2.3*] and introduce $m > 0$. Due to Proposition 3.4 we get

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} (m \wedge |x-y|^{\varepsilon-a})\mu_s^N(dx)\mu_s^N(dy)ds\right] &= \frac{1}{N^2}\sum_{i,j}\mathbb{E}\left[\int_0^T \left(m \wedge |X_s^{N,i} - X_s^{N,j}|^{\varepsilon-a}\right) ds\right] \\
&\le \frac{Tm}{N} + \frac{1}{N^2}\sum_{i\neq j}\int_0^T \mathbb{E}\left[|X_s^{N,i} - X_s^{N,j}|^{\varepsilon-a}\right] ds \\
&\le \frac{Tm}{N} + C_{\varepsilon,T}.
\end{aligned}$$

Letting N go to infinity, we find that the l.h.s. converges to $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d}(m \wedge |x-y|^{\varepsilon-a})\mu_s(dx)\mu_s(dy)\, ds\right]$ since $\mu_\cdot^N$ converges in law to $\mu_\cdot$. Letting then m go to infinity yields, thanks to the monotone convergence Theorem

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^{\varepsilon-a}\mu_s(dx)\mu_s(dy)ds\right] ds \le C_{\varepsilon,T},$$

and therefore

$$\int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^{\varepsilon-a}\mu_s(dx)\mu_s(dy)ds < \infty, \text{ a.s..}$$

.

• Step 2.b

In case $a > \alpha$ we use the techniques of [15, Proof of Proposition 6.1, Step 2]. Denote $\pi_t = \mathcal{L}(\mu_t) \in \mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$, $\pi_t^j = \int_{\mathcal{P}(\mathbb{R}^d)} g^{\otimes j}\pi_t(dg)$ its Hewitt and Savage projection, and $F_t^{N,j}$ the marginal of F_t^N on $\mathbb{R}^{2j}$ (recall that $F_t^N = \mathcal{L}(X_t^1,\cdots,X_t^N) \in \mathcal{P}_\kappa(\mathbb{R}^{dN})$). It is classical (see [31]) to deduce from point (i) that $F_t^{N,j}$ converges weakly to π_t^j as N goes to infinity. Hence, using Corollary 3.1, Fatou's Lemma and Proposition 3.3 we find

$$\begin{aligned}
\mathbb{E}\left[\int_0^T I_a(\mu_t)\, dt\right] = \int_0^T \int_{\mathcal{P}(\mathbb{R}^d)} I_a(g)\pi_t(dg)dt &\le \int_0^T \liminf_{N\to+\infty} \mathcal{I}_a^N(F_t^N)dt \\
&\le \liminf_{N\to+\infty}\int_0^T \mathcal{I}_a^N(F_t^N)dt \le 2\liminf_{N\to+\infty}\mathcal{H}^N(F_0^N) + CT \\
&= 2\liminf_{N\to+\infty}\mathcal{H}^N(\rho_0^{\otimes N}) + CT = 2\int_{\mathbb{R}^d}\rho_0\ln\rho_0 + CT,
\end{aligned}$$

and therefore

$$\int_0^T I_a(\mu_t)\, dt < \infty, \text{ a.s.}$$

⋄ Step 3
Regardless of $a = \alpha$ or $a > \alpha$, define

$$\begin{aligned} O_i(t) &:= \phi(X_t^i) - \int_0^t \frac{\chi}{N} \sum_{j\neq i}^N \nabla\phi(X_s^i)\cdot K_\alpha(X_s^i - X_s^j)ds \\ &\quad - \int_0^t c_{d,a} \int_{\mathbb{R}^d} \frac{\phi(X_s^i + z) - \phi(X_s^i) - z\cdot\nabla\phi(X_s^i)}{|z|^{d+a}} dzds \\ &= \phi(X_0^i) + \int_{[0,t]\times\mathbb{R}^d} \left(\phi(X_{s_-}^i + z) - \phi(X_{s_-}^i)\right) \bar{M}_i(ds,dz). \end{aligned}$$

Note that $(O_t^i)_{t\in[0,T]}$ is a martingale with respect to the natural filtration of $\mathcal{F}_t = \sigma(\mathcal{X}_r^N, \mathcal{Z}_r^N)_{r\geq t}$. Hence since the $(M_i)_{i=1,\cdots,N}$ are independent, we have,

$$\begin{aligned} \mathbb{E}\left[\mathcal{F}^2(\mu_\cdot^N)\right] =& \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^N \left(\prod_{l=1}^k \phi_l(X_{t_l}^i)\right)(O_i(t) - O_i(s))\right)^2\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\frac{1}{N^2}\sum_{i,j}\prod_{l=1}^k \phi_l(X_{t_l}^i)\prod_{l=1}^k \phi_l(X_{t_l}^j)\left(O_t^i - O_s^i\right)\left(O_t^j - O_s^j\right) | \mathcal{F}_s\right]\right] \\ &= \frac{1}{N^2}\sum_{i=1}^N \mathbb{E}\left[\prod_{l=1}^k \phi_l(X_{t_l}^i)^2\right] \mathbb{E}\left[\left(\int_{[s,t]\times\mathbb{R}^d} \left(\phi(X_{u_-}^i + z) - \phi(X_{u_-}^i)\right)\bar{M}_i(du,dz)\right)^2\right] \leq \frac{C_{\mathcal{F}}}{N}, \end{aligned} \tag{3.24}$$

since (see for instance [1, point (2) of Theorem 4.2.3 p200])

$$\begin{aligned} \mathbb{E}\left[\left(\int_{[s,t[\times\mathbb{R}^d} \left(\phi(X_{u_-}^i + z) - \phi(X_{u_-}^i)\right)\bar{M}_i(du,dz)\right)^2\right] &= \int_s^t \mathbb{E}\left[c_{d,a}\int_{\mathbb{R}^d} \frac{\left(\phi(X_{u_-}^i + z) - \phi(X_{u_-}^i)\right)^2}{|z|^{d+a}} dz\right] du \\ &\leq c_{d,a}(t-s)\left(\|\nabla\phi\|_{L^\infty}^2 \int_{\mathbb{B}} |z|^{-(d+a)}dz + 4\|\phi\|_{L^\infty}^2 \int_{\mathbb{B}^c} |z|^{-(d-2+a)}dz\right). \end{aligned}$$

After which it is classical to deduce that $\mu_\cdot$ satisfies c) (see [16, *Proof of Theorem 6, Step 2.3.2-3-4*]). Indeed for $\eta > 0$ define $K_{\eta,\alpha}$ as

$$K_{\eta,\alpha}(x) = -\frac{x}{\max(|x|^\alpha, \eta)^\alpha},$$

and define $\mathcal{F}_\eta$ as in (3.23) c) with K_α replaced with $K_{\alpha,\eta}$. Note that since $K_\alpha(0) = 0$ it holds for any $x \in \mathbb{R}^d$ and $\varepsilon \in (0,1)$

$$\left|K_\alpha(x) - K_{\alpha,\eta}(x)\right| \leq \mathbf{1}_{|x|\leq\eta}|x|^{1-\alpha} \leq \eta^{1-\varepsilon}|x|^{\varepsilon-a}.$$

This implies that for any $Q \in \mathcal{P}(\mathcal{D}(0,T;\mathbb{R}^d))$

$$\begin{aligned} |\mathcal{F}(Q) - \mathcal{F}_\varepsilon(Q)| &\leq \eta^{1-\varepsilon}\int\int\int_0^t |\gamma_u - \tilde{\gamma}_u|^{\varepsilon-a} Q(d\gamma)Q(d\tilde{\gamma}) \\ &\leq \eta^{1-\varepsilon}\int_0^T \int\int |z-\tilde{z}|^{\varepsilon-a} Q_u(dz)Q_u(d\tilde{z})\, du. \end{aligned} \tag{3.25}$$

Hence since for any $N \geq 1$ and $\eta > 0$ it holds

$$\mathcal{F}(\mu_\cdot) = \mathcal{F}(\mu_\cdot) - \mathcal{F}_\eta(\mu_\cdot) + \mathcal{F}_\eta(\mu_\cdot) - \mathcal{F}_\eta(\mu_\cdot^N) + \mathcal{F}_\eta(\mu_\cdot^N) - \mathcal{F}(\mu_\cdot^N) + \mathcal{F}(\mu_\cdot^N),$$

using some triangular inequalities and (3.25) yields

$$\begin{aligned}\mathbb{E}\left[|\mathcal{F}(\mu_\cdot)|\right] &\leq \mathbb{E}\left[\left|\mathcal{F}(\mu_\cdot) - \mathcal{F}_\eta(\mu_\cdot)\right|\right] + \mathbb{E}\left[\left|\mathcal{F}_\eta(\mu_\cdot) - \mathcal{F}_\eta(\mu_\cdot^N)\right|\right] + \mathbb{E}\left[\left|\mathcal{F}_\eta(\mu_\cdot^N) - \mathcal{F}(\mu_\cdot^N)\right|\right] + \mathbb{E}\left[\left|\mathcal{F}(\mu_\cdot^N)\right|\right] \\ &\leq \eta^{1-\varepsilon}\int_0^T \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} |z-\tilde{z}|^{\varepsilon-a}\mu_u(dz)\mu_u(d\tilde{z})\right]du + \mathbb{E}\left[\left|\mathcal{F}_\eta(\mu_\cdot) - \mathcal{F}_\eta(\mu_\cdot^N)\right|\right] \\ &+ \eta^{1-\varepsilon}\int_0^T \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} |z-\tilde{z}|^{\varepsilon-a}\mu_u^N(dz)\mu_u^N(d\tilde{z})\right]du + \sqrt{\frac{C_{\mathcal{F}}}{N}}.\end{aligned}$$

We fix $\delta > 0$. Due to Step 2 of this proof and Proposition 3.3 or Proposition 3.4 depending on whether $a = \alpha$ or $a > \alpha$, we have

$$\int_0^T \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} |z-\tilde{z}|^{\varepsilon-a}\mu_u(dz)\mu_u(d\tilde{z})\right]du < \infty,\ \sup_{N\geq 1}\int_0^T \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} |z-\tilde{z}|^{\varepsilon-a}\mu_u^N(dz)\mu_u^N(d\tilde{z})\right]du < \infty.$$

We fix $\eta > 0$ such that

$$\eta^{1-\varepsilon}\int_0^T \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} |z-\tilde{z}|^{\varepsilon-a}\mu_u(dz)\mu_u(d\tilde{z}) + \int_{\mathbb{R}^d\times\mathbb{R}^d} |z-\tilde{z}|^{\varepsilon-a}\mu_u^N(dz)\mu_u^N(d\tilde{z})\right]du < \delta.$$

Now since $\mathcal{F}_\eta$ is a smooth function on $\mathcal{P}\left(\mathcal{D}([0,T];\mathbb{R}^d)\right)$ and $\mu_\cdot^N$ converges in law to $\mu_\cdot$, we may take N large enough such that

$$\mathbb{E}\left[\left|\mathcal{F}_\eta(\mu_\cdot) - \mathcal{F}_\eta(\mu_\cdot^N)\right|\right] \leq \delta,$$

and simultaneously

$$\sqrt{\frac{C_{\mathcal{F}}}{N}} \leq \delta.$$

Finally we deduce that $\forall\delta > 0, \mathbb{E}\left[|\mathcal{F}(\mu_\cdot)|\right] \leq 4\delta$, i.e.

$$\mathbb{E}\left[|\mathcal{F}(\mu_\cdot)|\right] = 0,$$

which implies $\mathcal{F}(\mu_\cdot) = 0$ a.s. and concludes the proof.

□

4. Uniqueness of the limit equation

Now, in order to complete the propagation of chaos result, we need to investigate the uniqueness of the accumulation points of the sequence of the law of solution to equation (1.5), which have been proved to be tight in both case $a > \alpha$ and $a = \alpha$ in the previous section. However this uniqueness can not be obtained in the Fair Competition case in the class where lie the accumulation points. However in the Diffusion Dominated we are able to conclude to the well posedness of equation (1.6) for an initial condition in $L\log L(\mathbb{R}^d)$.

From Lemma 3.4, we know that any solution to (1.6) starting from an $L\log L$ initial condition is $L^1(0,T;L^p(\mathbb{R}^d))$ for any $p \in \left(1, \frac{d}{d-a}\right)$. There remains to prove the uniqueness of solution to equation (1.6) in this class. Note that this is known in the case $a = 2$ and $\alpha \in (1,2)$ (see [18]). Precisely we have strong-strong stability estimate

Lemma 4.1. *Let be $p > \frac{d}{d-\alpha}, q \geq 1$. There is a constant $C_{\alpha,q,p} > 0$ such that for any X, Y random variables of respective laws ρ_1, ρ_2 with $\rho_1, \rho_2 \in L^p(\mathbb{R}^d) \cap \mathcal{P}_q(\mathbb{R}^d)$, it holds*

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} K_\alpha(X-y)\rho_1(dy) - \int_{\mathbb{R}^d} K_\alpha(Y-y)\rho_2(dy)\right||X-Y|^{q-1}\right] \leq C_{\alpha,q,p}\left(2 + \|\rho_1\|_{L^p} + \|\rho_2\|_{L^p}\right)\mathbb{E}[|X-Y|^q].$$

Proof. First, we notice that

$$\nabla K_\alpha(x) = \left(I_d - \alpha\frac{x\otimes x}{|x|^2}\right)|x|^{-\alpha},$$

hence, denoting $\bar{X}, \bar{Y}$ some independent copies of X, Y, we find

$$|K_\alpha(X-\overline{X}) - K_\alpha(Y-\overline{Y})| \le C_\alpha \left(|X-Y| + |\overline{X}-\overline{Y}|\right)\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right).$$

This yields to

$$\begin{aligned}\mathbb{E}\left[|K_\alpha(X-\overline{X}) - K_\alpha(Y-\overline{Y})||X-Y|^{q-1}\right] \le C\mathbb{E}&\left[\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right)|X-Y|^q\right] \\ &+ C\mathbb{E}\left[\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right)|\overline{X}-\overline{Y}||X-Y|^{q-1}\right] \\ &=: C(I_1+I_2).\end{aligned}$$

$\diamond$ Estimate of I_1: First we easily get for this term, by taking firstly the expectation on $(\overline{X}, \overline{Y})$

$$\begin{aligned}I_1 =& \mathbb{E}_{X,Y}\left[|X-Y|^q \mathbb{E}_{\overline{X},\overline{Y}}\left[\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right)\right]\right] \\ &\le C_{\alpha,p}\left(\|\rho_1\|_{L^p(\mathbb{R}^d)} + \|\rho_2\|_{L^p(\mathbb{R}^d)} + \|\rho_1\|_{L^1(\mathbb{R}^d)} + \|\rho_2\|_{L^1(\mathbb{R}^d)}\right)\mathbb{E}\left[|X-Y|^q\right].\end{aligned}$$

$\diamond$ Estimate of I_2: We use Holder's inequality to find

$$\begin{aligned}I_2 &\le C\mathbb{E}\left[|\overline{X}-\overline{Y}|\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right)^{1/q}\left(\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right)^{1/q}|X-Y|\right)^{q-1}\right] \\ &\le C\mathbb{E}\left[|\overline{X}-\overline{Y}|^q\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right)\right]^{1/q}\mathbb{E}\left[\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right)|X-Y|^q\right]^{(q-1)/q} \\ &= C\mathbb{E}\left[|\overline{X}-\overline{Y}|^q\left(\frac{1}{|X-\overline{X}|^\alpha} + \frac{1}{|Y-\overline{Y}|^\alpha}\right)\right].\end{aligned}$$

Now taking first the expectation w.r.t. (X, Y) and then w.r.t. $(\overline{X}, \overline{Y})$ yields similarly as above

$$I_2 \le C_{\alpha,q}\left(\|\rho_1\|_{L^1\cap L^p(\mathbb{R}^d)} + \|\rho_2\|_{L^1\cap L^p(\mathbb{R}^d)}\right)\mathbb{E}\left[|X-Y|^q\right].$$

Hence putting all those estimates together leads to the desired result, since

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} K_\alpha(X-y)\rho_1(dy) - \int_{\mathbb{R}^d} K_\alpha(Y-y)\rho_2(dy)\right||X-Y|^{q-1}\right] = \mathbb{E}\left[|K_\alpha(X-\overline{X}) - K_\alpha(Y-\overline{Y})||X-Y|^{q-1}\right]$$

□

Then we can obtain the desired stability estimate stated in the

Proposition 4.1. *Let be $T > 0, q \in (1, a)$, and $(\mathcal{X}^1_t)_{t\in[0,T]}$ and $(\mathcal{X}^2_t)_{t\in[0,T]}$ two solutions to equation* (1.7) *build with the same Lévy process, and assume their respective laws $\rho^1_\cdot, \rho^2_\cdot \in L^1(0,T;L^p(\mathbb{R}^d)) \cap \mathcal{P}\left(\mathcal{D}(0,T;(\mathbb{R}^d))\right)$ for some $p > \frac{d}{d-\alpha}$. Then it holds*

$$\mathbb{E}\left[|\mathcal{X}^1_t - \mathcal{X}^2_t|^q\right] \le \mathbb{E}\left[|\mathcal{X}^1_0 - \mathcal{X}^2_0|^q\right]e^{2t + \int_0^t \left(\|\rho^1_s\|_{L^p} + \|\rho^1_s\|_{L^p}\right)ds}.$$

Proof. Since the two solutions are build on the same Lévy process, we get

$$|\mathcal{X}^1_t - \mathcal{X}^2_t|^q = |\mathcal{X}^1_0 - \mathcal{X}^2_0|^q + q\int_0^t |\mathcal{X}^1_s - \mathcal{X}^2_s|^{q-1}\left|\int_{\mathbb{R}^2} K_\alpha(\mathcal{X}^1_s - y)\rho^1_s(dy) - \int_{\mathbb{R}^2} K_\alpha(\mathcal{X}^2_s - y)\rho^2_s(dy)\right| ds.$$

Taking the expectation, using Lemma 4.1 and Gronwall's inequality yields the desired result.

□

Corollary 4.1. *When $a > \alpha$, the set S^a_α defined in* (3.22) *is a singleton.*

Proof. Recall that

$$\mathcal{S}^a_\alpha = \{\rho \in \mathcal{P}\left(\mathcal{D}\left(0,T;\mathbb{R}^d\right)\right) \,|\, \rho_\cdot = \mathcal{L}(X_\cdot),\ X_\cdot \text{ solution to (1.7), and } \int_0^T I_a(\rho_t)dt < \infty\}.$$

is not empty due to Theorem 3.1. But due to Lemma 3.4, if $\rho \in \mathcal{S}^a_\alpha$ then $\rho \in L^1\left(0,T;L^q(\mathbb{R}^d)\right)$, for any $q \in \left(1, \frac{d}{d-a}\right)$. But since $\frac{d}{d-a} > \frac{d}{d-\alpha}$, one can choose $q \in \left(\frac{d}{d-\alpha}, \frac{d}{d-a}\right)$ and due to Proposition 4.1 there is at most one solution to equation (1.7) with initial condition of law $\rho_0 \in \mathcal{P}_\kappa(\mathbb{R}^d)$. □

Theorem 3.1 and Corollary 4.1 conclude the proof of Theorem 2.1.

Appendix A. On the critical sensitivity in the Fair Competition case

In this section, we develop the formal computations of (2.13), in order to give a sharper idea of the critical sensitivity $\chi(a)$ in Fair Competition case of Theorem 2.1. Using the definition of the fractional Laplacian by Fourier multipliers (1.3), we can obtain the identity

$$-(-\Delta)^{a/2}(|x|^\varepsilon) = -2^a \frac{\Gamma\left(\frac{d+\varepsilon}{2}\right)\Gamma\left(\frac{a-\varepsilon}{2}\right)}{\Gamma\left(-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d+a-\varepsilon}{2}\right)}|x|^{\varepsilon-a}, \tag{A.1}$$

Applying Ito's rule to $Z^{i,j}_s$ with $\phi(x) = |x|^\varepsilon$ for some $\varepsilon \in (0,1)$ is not possible, since ϕ defined so is not C^2 (not even C^a), but let us perform the computations for the sake of the discussion. Coming back to (2.13) and taking the expectation formally yields

$$\begin{aligned}\mathbb{E}\left[\left|Z^{i,j}_t\right|^\varepsilon\right] = \mathbb{E}\left[\left|Z^{i,j}_0\right|^\varepsilon\right] - \int_0^t \frac{\chi}{N}\varepsilon\mathbb{E}\left[|Z^{i,j}_s|^{\varepsilon-2}Z^{i,j}_s \cdot \left(\sum_{k\neq i,j}\left(\frac{Z^{i,k}_s}{|Z^{i,k}_s|^a} - \frac{Z^{j,k}_s}{|Z^{j,k}_s|^a}\right)\right)\right] ds \\ - \int_0^t \frac{2\chi\varepsilon}{N}\mathbb{E}\left[|Z^{i,j}_s|^{\varepsilon-a}\right] ds + 2\int_0^t \mathbb{E}\left[-(-\Delta)^{a/2}\phi(Z^{i,j}_s)\right] ds.\end{aligned} \tag{A.2}$$

Recalling (A.1) and the argument used in the proof of Proposition 3.4 yields

$$\mathbb{E}\left[\left|Z^{i,j}_t\right|^\varepsilon\right] \geq \mathbb{E}\left[\left|Z^{i,j}_0\right|^\varepsilon\right] + 2\varepsilon\left(-2^a\frac{\Gamma\left(\frac{d+\varepsilon}{2}\right)\Gamma\left(\frac{a-\varepsilon}{2}\right)}{\Gamma\left(-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d+a-\varepsilon}{2}\right)\varepsilon} - \chi\right)\int_0^t \mathbb{E}\left[|Z^{i,j}_s|^{\varepsilon-a}\right] ds$$

So that, optimizing w.r.t. $\varepsilon \in (0,1)$, the condition becomes

$$\chi < \chi(a) = \sup_{\varepsilon\in(0,1)}\left\{-2^a\frac{\Gamma\left(\frac{d+\varepsilon}{2}\right)\Gamma\left(\frac{a-\varepsilon}{2}\right)}{\Gamma\left(-\frac{\varepsilon}{2}\right)\Gamma\left(\frac{d+a-\varepsilon}{2}\right)\varepsilon}\right\}.$$

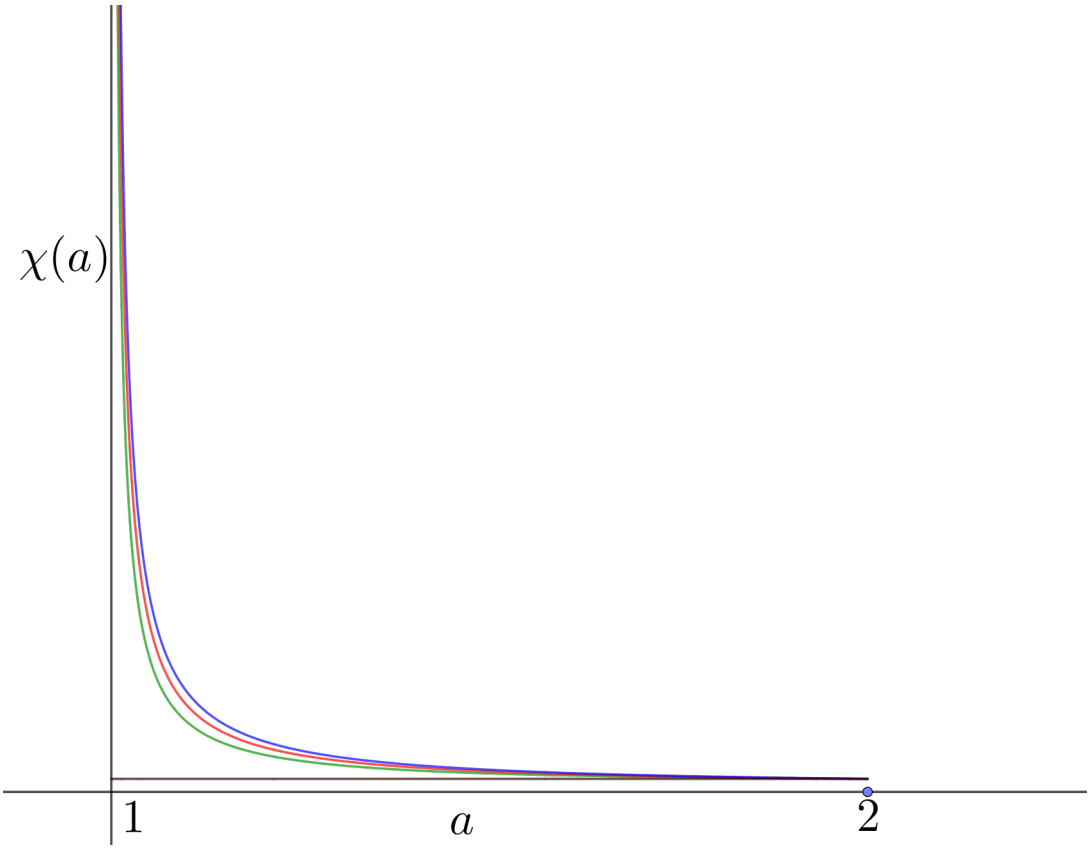

Figure A.2. Critical sensitivity threshold $\chi(a)$ for $a \in (1,2)$, in green for $d = 2$, in red for $d = 3$, in blue for $d = 4$, together with the asymptotic horizontal line $y = 1$.

Since this appendix is rather formal, we limit ourselves to numerically check that this supremum is obtained as $\varepsilon \to 1^-$ which yields

$$\chi(a) = -2^a \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{a-1}{2}\right)}{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{d+a-1}{2}\right)} = 2^{a-1} \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{a-1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+a-1}{2}\right)}.$$

In particular we have $\lim_{a\to 2^-} \chi(a) = 1$, and the limit threshold is consistent with the one obtained for classical diffusion in [16, Theorem 6].

In order to rigorously obtain this threshold, one should be able to prove that

$$\lim_{\eta\to 0^+} \int_0^t \mathbb{E}\left[-(-\Delta)^{a/2}\phi_\eta\left(Z_s^{i,j}\right)\right] ds = \int_0^t \mathbb{E}\left[-(-\Delta)^{a/2}\phi(Z_s^{i,j})\right] ds, \tag{A.3}$$

in the equality after (3.20). This seems more complicated thab in the case $a = 2$, where one enjoys the explicit formula

$$\Delta\phi_\eta(x) = \varepsilon\left(|x|^2 + \eta^2\right)^{\frac{\varepsilon-4}{4}}\left((d+\varepsilon-2)|x|^2 + d\eta^2\right).$$

So that it is clear that for any $x \neq 0$

$$\lim_{\eta\to 0^+} \Delta\phi_\eta(x) = \Delta\phi(x)$$

To the best of the author's knowledge, there is no explicit formula for $-(-\Delta)^{a/2}\phi_\eta$ when $a < 2$, and the limit A.3 has to be obtained by indirect arguments. So we have to bound $-(-\Delta)^{a/2}\phi_\eta$ by below, thus we have to use Lemma 3.2, which implies the technical limitations $a > a^*$ and the implicit definition of $\chi(a)$ in Theorem 2.1 in the Fair Competition case.

Appendix B. Remarks on the case $\alpha \leq 1$

In this section, we focus on the case $\alpha \leq 1$, which implies that the aggregation kernel K_α is not so singular, given that it is $(1-\alpha)$-Holder. It would be interesting to obtain some quantitative result, similarly as [21] in the case $a = 2$. Stating a convergence/consistency result, in the case $\alpha \leq 1$ is rather cheap, given that it consists mostly in a moment control, in view of Lemma 2.1. The bound (2.12) is straightforward in the case $\alpha = 1$. Otherwise it is a matter of propagation of moments of order $(1-\alpha)p$ for some $p > 1$. But due to (3.16) this is possible only if $(1-\alpha)p < a$ i.e. $1 < a + \alpha$. Therefore we can expect a convergence/consistency result in the all area

$$\{(a, \alpha) \in (0, 2) \times (0, 1],\ 1 < a + \alpha\}.$$

Then we divide this area in three parts (see Figure 1).

- $(a, \alpha) \in (1, 2) \times (0, 1)$
  Making the assumption $F_0^N = \rho_0^{\otimes N}$ with $\rho_0 \in L \ln L \cap \mathcal{P}_\kappa(\mathbb{R}^d)$ with $\kappa \in (1, a)$ similarly as in the Diffusion Dominated case in Theorem 2.1, Lemma 3.3 apply and we let $p = \frac{\kappa}{1-\alpha} > 1$. And we obtain the claimed convergence/consistency result.
  Then since Lemmas 3.6 and 4.1 also apply, we can deduce to the uniqueness of the limit point and obtain a complete propagation of chaos result.

- $(a, \alpha) \in (0, 1) \times (0, 1), 1 < a + \alpha$
  In this case it is possible to obtain the bound

  $$\int_0^T \sup_{N \geq 1} \int_{\mathbb{R}^{dN}} \langle x_1 \rangle^\kappa F_t^N dt < \infty,$$

  for some $\kappa \in (1 - \alpha, a)$, but the moment estimate is more complicated than in the proof of Lemma 3.3, as for $\kappa < 1$, m_κ is not convex, and one can not enjoy point (*iii*) of Lemma 3.1. We do not treat this problem here. Nevertheless, should this bound be obtained, it would immediately imply the tightness of the sequence of the empirical measure, and the convergence of a subsequence to some element of the set

  $$\mathcal{S}_\alpha^a = \{\rho \in \mathcal{P}\left(\mathcal{D}\left(0, T; \mathbb{R}^d\right)\right) \mid \rho_\cdot = \mathcal{L}(X_\cdot),\ X_\cdot \text{ solution to (1.7), and } \int_0^T \int_{\mathbb{R}^2} |x|^\kappa \rho_t,\ dt < \infty\}.$$

  - $\alpha < a$
    The moment information is not sufficient to conclude that this set is reduced to a singleton, given the lack of regularity of K_α. It could be interesting to try to extend the result of Corollary 3.1 in the range $a \in (0, 1)$, in order to gain some regularity on the limit point, by passing some fractional Fisher information of low order to the limit.
  - $\alpha \geq a$
    Here we can not go beyond the convergence/consistency result.

  Otherwise, mollifying the kernel K_α near the origin so that it is Lipschitz, uniqueness can be obtained by standard coupling arguments. Hence the above strategy yields the existence of a solution to the nonlinear equation (1.6), with K_α replaced by its mollification. In the classical case (see [31]), this existence of solution to nonlinear drift-diffusion equations is usually proved by a fix point argument in $C([0, T], \mathcal{P}_k(\mathbb{R}^d))$ for some $k > 1$. Here we could not use this strategy given that we can not expect on the solution moment of higher order than $a \leq 1$.

Appendix C. Proof of Proposition 3.2

We begin this section by defining for $\varepsilon > 0$, ψ_ε on $\mathbb{R}^d$ as

$$\psi_\varepsilon(x) = \varepsilon^{-2} e^{-\sqrt{1+\left(\frac{|x|}{\varepsilon}\right)^2}}.$$

which is borrowed from [20]. Observe that for any $x, y \in \mathbb{R}^d \times \mathbb{R}^d$ it holds

$$\begin{aligned}\Phi(\psi_\varepsilon(x), \psi_\varepsilon(y)) &= -(\psi_\varepsilon(x) - \psi_\varepsilon(y))\left(\sqrt{1 + \left(\frac{|x|}{\varepsilon}\right)^2} - \sqrt{1 + \left(\frac{|y|}{\varepsilon}\right)^2}\right) \\ &\leq \varepsilon^{-1}|x - y|\,(\psi_\varepsilon(x) + \psi_\varepsilon(y))\end{aligned} \tag{C.1}$$

Lemma Appendix C.1. *Let be $\pi \in \mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ and for $N \geq 1$ define*

$$\pi^{N,\varepsilon} = \int_{\mathcal{P}(\mathbb{R}^d)} (\rho * \psi_\varepsilon)^{\otimes N} \pi(d\rho).$$

Then for any $x_2, \cdots, x_N := X^{N-1} \in \mathbb{R}^{d(N-1)}$, define $p^\varepsilon(\cdot|X^{N-1})$ the conditional law knowing X^{N-1} under $\pi^{N,\varepsilon}$. Then it holds

(*i*)
$$\|\nabla \ln p^\varepsilon(\cdot|X^{N-1})\|_{L^\infty} \leq \varepsilon^{-1}.$$

(*ii*) *there exist constants $C_{\varepsilon,R}, c_{\varepsilon,R} > 0$ such that for any $x \in \mathbb{R}^d, \mathbf{e} \in \mathbb{S}^{d-1}$ and $u \in [0, R]$*

$$c_{\varepsilon,R} p^\varepsilon(x|X^{N-1}) \leq p^\varepsilon(x + u\mathbf{e}|X^{N-1}) \leq C_{\varepsilon,R} p^\varepsilon(x|X^{N-1}).$$

(*iii*)
$$\Phi(\pi^{N,\varepsilon}(X^N_x), \pi^{N,\varepsilon}(X^N_y)) \leq \pi^{N-1,\varepsilon}(X^{N-1})\Phi(\psi_\varepsilon(x), \psi_\varepsilon(y))$$

Proof. Proof of (*i*):
First note that

$$p^\varepsilon(x|X^{N-1}) = \frac{\pi^{N,\varepsilon}(x, x_2, \cdots, x_N)}{\pi^{N-1,\varepsilon}(x_2, \cdot, x_N)},$$

Indeed due to Fubini's Theorem one can check that

$$\begin{aligned}\int_{\mathbb{R}^d} \pi^{N,\varepsilon}(x, x_2, \cdots, x_N)dx &= \int_{\mathbb{R}^d} \int_{\mathcal{P}(\mathbb{R}^d)} (\rho * \psi_\varepsilon)(x) \prod_{k=2}^N (\rho * \psi_\varepsilon)(x_k)\pi(d\rho)dx \\ &= \int_{\mathcal{P}(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} (\rho * \psi_\varepsilon)(x)dx\right) \prod_{k=2}^N (\rho * \psi_\varepsilon)(x_k)\pi(d\rho) = \pi^{N-1,\varepsilon}(x_2, \cdots, x_N),\end{aligned}$$

and the sequence $(\pi^{N,\varepsilon})$ is compatible. Hence

$$\nabla \ln p^\varepsilon(\cdot|X^{N-1}) = \nabla_1 \ln \pi^{N,\varepsilon}(\cdot, x_2, \cdots, x_N),$$

and we use [20, Lemma 5.9] to conclude the result.

Proof of (*ii*):
Observe that for any $x \in \mathbb{R}^d, \mathbf{e} \in \mathbb{S}^{d-1}$ and $u \in [0, R]$ it holds

$$\begin{aligned}\left|\sqrt{1 + \frac{|x + u\mathbf{e}|^2}{\varepsilon^2}} - \sqrt{1 + \frac{|x|^2}{\varepsilon^2}}\right| &= \varepsilon^{-2} \frac{\left||x + u\mathbf{e}|^2 - |x|^2\right|}{\sqrt{1 + \frac{|x+u\mathbf{e}|^2}{\varepsilon^2}} + \sqrt{1 + \frac{|x|^2}{\varepsilon^2}}} \\ &\leq \varepsilon^{-1} u \frac{\varepsilon^{-1}|x + u\mathbf{e}| + \varepsilon^{-1}|x|}{\sqrt{1 + \frac{|x+u\mathbf{e}|^2}{\varepsilon^2}} + \sqrt{1 + \frac{|x|^2}{\varepsilon^2}}} \leq \varepsilon^{-1} R,\end{aligned}$$

hence

$$\sqrt{1+\frac{|x|^2}{\varepsilon^2}}+\varepsilon^{-1}R \geq \sqrt{1+\frac{|x+u\mathbf{e}|^2}{\varepsilon^2}} \geq \sqrt{1+\frac{|x|^2}{\varepsilon^2}}-\varepsilon^{-1}R, \text{ i.e.}$$
$$e^{\varepsilon^{-1}R}\psi_\varepsilon(x) \geq \psi_\varepsilon(x+\mathbf{e}u) \geq e^{-\varepsilon^{-1}R}\psi_\varepsilon(x),$$

so that for any $\rho \in \mathcal{P}(\mathbb{R}^d)$

$$\rho * \psi_\varepsilon(x+u\mathbf{e}) = \int \psi_\varepsilon(x+u\mathbf{e}-y)\rho(dy) \leq C_{\varepsilon,R}\int \psi_\varepsilon(x-y)\rho(dy) = C_{\varepsilon,R}\rho*\psi_\varepsilon(x),$$
$$\rho * \psi_\varepsilon(x+u\mathbf{e}) \geq c_{\varepsilon,R}\int \psi_\varepsilon(x-y)\rho(dy) = c_{\varepsilon,R}\rho*\psi_\varepsilon(x),$$

and then

$$\begin{aligned}
p^\varepsilon(x+u\mathbf{e}|X^{N-1}) &= \frac{\pi^{N,\varepsilon}(x+u\mathbf{e},x_2,\cdots,x_N)}{\pi^{N-1,\varepsilon}(x_2,\cdot,x_N)} \\
&= \frac{\int_{\mathcal{P}(\mathbb{R}^d)}(\rho*\psi_\varepsilon)(x+u\mathbf{e})(\rho*\psi_\varepsilon)^{\otimes(N-1)}(X^{N-1})\pi(d\rho)}{\int_{\mathcal{P}(\mathbb{R}^d)}(\rho*\psi_\varepsilon)^{\otimes(N-1)}(X^{N-1})\pi(d\rho)} \\
&\leq C_{\varepsilon,R}\frac{\int_{\mathcal{P}(\mathbb{R}^d)}(\rho*\psi_\varepsilon)(x)(\rho*\psi_\varepsilon)^{\otimes(N-1)}(X^{N-1})\pi(d\rho)}{\int_{\mathcal{P}(\mathbb{R}^d)}(\rho*\psi_\varepsilon)^{\otimes(N-1)}(X^{N-1})\pi(d\rho)} = C_{\varepsilon,R}p^\varepsilon(x|X^{N-1}),
\end{aligned}$$

and similarly for the lower bound and the point is proved.

Proof of (*iii*):
By convexity of Φ and Jensen's inequality we successively obtain

$$\begin{aligned}
\Phi(\pi^{N,\varepsilon}(X_x^N),\pi^{N,\varepsilon}(X_y^N)) &= \Phi\left(\int_{\mathcal{P}(\mathbb{R}^d)}\rho*\psi_\varepsilon(x)\prod_{i=2}^N\rho*\psi_\varepsilon(x_i)\pi(d\rho),\int_{\mathcal{P}(\mathbb{R}^d)}\rho*\psi_\varepsilon(y)\prod_{i=2}^N\rho*\psi_\varepsilon(x_i)\pi(d\rho)\right) \\
&\leq \int_{\mathcal{P}(\mathbb{R}^d)}\Phi\left(\rho*\psi_\varepsilon(x)\prod_{i=2}^N\rho*\psi_\varepsilon(x_i),\rho*\psi_\varepsilon(y)\prod_{i=2}^N\rho*\psi_\varepsilon(x_i)\right)\pi(d\rho) \\
&\leq \int_{\mathcal{P}(\mathbb{R}^d)}\prod_{i=2}^N\rho*\psi_\varepsilon(x_i)\Phi\left(\rho*\psi_\varepsilon(x),\rho*\psi_\varepsilon(y)\right)\pi(d\rho) \\
&\leq \int_{\mathcal{P}(\mathbb{R}^d)}\prod_{i=2}^N\rho*\psi_\varepsilon(x_i)\Phi\left(\psi_\varepsilon(x),\psi_\varepsilon(y)\right)\pi(d\rho) = \pi^{N-1,\varepsilon}(X^{N-1})\Phi\left(\psi_\varepsilon(x),\psi_\varepsilon(y)\right)
\end{aligned}$$

□

Before completing the proof we will furthermore use the following consideration. Let $F^N, G^N \in \mathcal{P}(\mathbb{R}^{dN})$ and according to the notations previously introduced for $x,y,x_2,\cdots,x_N \in \mathbb{R}^{2(N+1)}$ denote $X_N^x = (x,x_2,\cdots,x_N) \in \mathbb{R}^{dN}$. Then straightforward computations yields

$$\begin{aligned}
\mathcal{D}^N &:= \theta\Phi(F^N(X_N^x),F^N(X_N^y)) + (1-\theta)\Phi(G^N(X_N^x),G^N(X_N^y)) \\
&- \Phi(\theta F^N(X_N^x)+(1-\theta)G^N(X_N^x),\theta F^N(X_N^y)+(1-\theta)G^N(X_N^y)) \\
&= -\theta\left(F^N(X_N^x)-F^N(X_N^y)\right)\left(\ln\left(\frac{\theta F^N(X_N^x)+(1-\theta)G^N(X_N^x)}{F^N(X_N^x)}\right)-\ln\left(\frac{\theta F^N(X_N^y)+(1-\theta)G^N(X_N^y)}{F^N(X_N^y)}\right)\right) \\
&- (1-\theta)\left(G^N(X_N^x)-G^N(X_N^y)\right)\left(\ln\left(\frac{\theta F^N(X_N^x)+(1-\theta)G^N(X_N^x)}{G^N(X_N^x)}\right)-\ln\left(\frac{\theta F^N(X_N^y)+(1-\theta)G^N(X_N^y)}{G^N(X_N^y)}\right)\right).
\end{aligned}$$

Note that due to the convexity of Φ, $\mathcal{D}^N$ is nonnegative. Denote $f(\cdot|X^{N-1})$ (resp. $g(\cdot|X^{N-1})$) the conditional law w.r.t. to the first component knowing the last $N-1$ under F^N (resp. G^N) i.e.

$$F^N(x, x_2, \cdots, x_N) = f(x|x_2, \cdots, x_N)F^{N-1}(x_2, \cdots, x_N)$$
$$G^N(x, x_2, \cdots, x_N) = g(x|x_2, \cdots, x_N)G^{N-1}(x_2, \cdots, x_N),$$

and define

$$h(t) = \ln\left(\theta + (1-\theta)\frac{g(z_t|.)}{f(z_t|.)}\frac{G^{N-1}}{F^{N-1}}\right),\ z_t = tx + (1-t)y$$

Since $f\nabla\frac{g}{f} = g\nabla\ln\frac{g}{f}$

$$\begin{aligned} h'(t) &= \frac{G^{N-1}f(z_t|.)\nabla\frac{g(|.)}{f(|.)}(z_t)\cdot(x-y)}{\theta f(z_t|.)F^{N-1} + (1-\theta)g(z_t|.)G^{N-1}} \\ &= \frac{G^{N-1}g(z_t|.)\nabla\ln\frac{g(|.)}{f(|.)}(z_t)\cdot(x-y)}{\theta f(z_t|.)F^{N-1} + (1-\theta)g(z_t|.)G^{N-1}}, \end{aligned}$$

we can rewrite

$$\begin{aligned} \mathcal{D}^N = &-\int_0^1 \frac{(1-\theta)\theta G^{N-1}F^{N-1}\left(f(x|) - f(y|)\right)g(z_t|.)\nabla\ln\frac{g(|.)}{f(|.)}(z_t)\cdot(x-y)}{\theta f(z_t|.)F^{N-1} + (1-\theta)g(z_t|.)G^{N-1}}dt \\ &-\int_0^1 \frac{(1-\theta)\theta G^{N-1}F^{N-1}\left(g(x|) - g(y|)\right)f(z_t|.)\nabla\ln\frac{f(|.)}{g(|.)}(z_t)\cdot(x-y)}{\theta f(z_t|.)F^{N-1} + (1-\theta)g(z_t|.)G^{N-1}}dt. \end{aligned} \tag{C.2}$$

We are now in position to prove Lemma 3.2. Following the idea of [15, *Proof of Lemma 4.2*] and [20, *Proof of Lemma 5.10*], we only treat the case $M = 2$ and $\omega_1 = \mathcal{B}_r := \{\rho \in \mathcal{P}_1(\mathbb{R}^2)\,|\,W_1(\rho, f_1) < r\}$ for some $r > 0$ and $f_1 \in \mathcal{P}_1(\mathbb{R}^2)$. For $\pi \in \mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d))$, define

$$\theta := \pi(\omega_1)\,(>0 \quad \text{w.l.o.g.}), \quad F := \theta^{-1}\mathbf{1}_{\omega_1}\pi,\ G := (1-\theta)^{-1}\mathbf{1}_{\omega_1^c}\pi.$$

Our aim is to prove that

$$\tilde{\mathcal{I}}_a(\pi) = \theta\tilde{\mathcal{I}}_a(F) + (1-\theta)\tilde{\mathcal{I}}_a(G),$$

or equivalently, by convexity, that for any fixed $\eta > 0$

$$\theta\tilde{\mathcal{I}}_a(F) + (1-\theta)\tilde{\mathcal{I}}_a(G) - \tilde{\mathcal{I}}_a(\pi) < \eta.$$

Let then be $\eta > 0$ fixed for the rest of the proof and for $N \geq 1, \varepsilon > 0$ define

$$F^{N,\varepsilon} := \int_{\mathcal{P}_1(\mathbb{R}^2)} (\rho * \psi_\varepsilon)^{\otimes N} F(d\rho),\ G^{N,\varepsilon} := \int_{\mathcal{P}_1(\mathbb{R}^2)} (\rho * \psi_\varepsilon)^{\otimes N} G(d\rho)$$

Also note that

$$\pi^{N,\varepsilon} := \int_{\mathcal{P}_1(\mathbb{R}^2)} (\rho * \psi_\varepsilon)^{\otimes N}\pi(d\rho) = \theta F^{N,\varepsilon} + (1-\theta)G^{N,\varepsilon},$$

since F and G have disjunct supports. It is also clear (see for instance the computations done in the proof of Lemma Appendix C.1), that the sequences $(\pi^{N,\varepsilon})_{N\geq1}$, $(F^{N,\varepsilon})_{N\geq1}$ and $(G^{N,\varepsilon})_{N\geq1}$ are compatible, and denote $\pi^\varepsilon, F^\varepsilon$ and G^ε in $\mathcal{P}_\kappa(\mathcal{P}(\mathbb{R}^d)))$ the probability measures which are associated to these sequences by the Hewitt and Savage Theorem.

$$\begin{aligned} \mathcal{K}^N &:= \theta\mathcal{I}_a^N(F^N) + (1-\theta)\mathcal{I}_a^N(G^N) - \mathcal{I}_a^N(\pi^N) \\ &= \int_{\mathbb{R}^{d(N+1)}} |x-y|^{-(d+a)}\left(\theta\Phi(F^N(X_x^N), F^N(X_y^N)) + (1-\theta)\Phi(G^N(X_x^N), G^N(X_y^N)) - \Phi(\pi^N(X_x^N), \pi^N(X_y^N))\right). \end{aligned} \tag{C.3}$$

Let be $R > 0$ such that $2\int_{\mathbb{B}_R^c} |x|^{-(d-1+a)}dx < \eta$. For couples $(x, y) \in \mathbb{R}^4$ such that $|x - y| \leq R$ we use the upper bound provided in (C.2) and on the complementary set, the one of (C.1). Which gives

$$\begin{aligned}
\mathcal{K}^N \leq & \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\geq R} |x-y|^{-(d+a)} \Big(\theta\Phi(F^N(X_x^N), F^N(X_y^N)) + (1-\theta)\Phi(G^N(X_x^N), G^N(X_y^N))\Big) dxdydX^{N-1} \\
& + \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\leq R} |x-y|^{-(d+a)} \Big(\theta\Phi(F^N(X_x^N), F^N(X_y^N)) + (1-\theta)\Phi(G^N(X_x^N), G^N(X_y^N)) - \Phi(\pi^N(X_x^N), \pi^N(X_y^N))\Big) \\
& \leq \int_{\mathbb{R}^{d(N-1)}} \Big(\theta F^{N-1}(X^{N-1}) + (1-\theta)G^{N-1}(X^{N-1})\Big) dX^{N-1} \int_{|x-y|\geq R} |x-y|^{-(d+a)}\Phi(\psi_\varepsilon(x), \psi_\varepsilon(y))dxdy \\
& - \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\leq R} \int_0^1 \frac{(1-\theta)\theta G^{N-1}F^{N-1}\left(f(|x|) - f(|y|)\right) g(z_t|.)\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\cdot(x-y)}{|x-y|^{d+a}\left(\theta f(z_t|.)F^{N-1} + (1-\theta)g(z_t|.)G^{N-1}\right)} dtdxdydX^{N-1} \\
& - \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\leq R} \int_0^1 \frac{(1-\theta)\theta G^{N-1}F^{N-1}\left(g(|x|) - g(|y|)\right) f(z_t|.)\nabla \ln \frac{f(|.)}{g(|.)}(z_t)\cdot(x-y)}{|x-y|^{d+a}\left(\theta f(z_t|.)F^{N-1} + (1-\theta)g(z_t|.)G^{N-1}\right)} dtdxdydX^{N-1}.
\end{aligned} \tag{C.4}$$

Since, by (C.1) it holds

$$\begin{aligned}
\int_{|x-y|\geq R} |x-y|^{-(d+a)}\Phi(\psi_\varepsilon(x), \psi_\varepsilon(y))dxdy &\leq \int_{|x-y|\geq R} |x-y|^{-(d+a)}(\psi_\varepsilon(x) + \psi_\varepsilon(y))dxdy \\
&\leq 2\int_{\mathbb{R}^d} \psi_\varepsilon(x)\left(\int_{y,|x-y|>R}, |x-y|^{-(d-1+a)}dy\right)dx < \eta,
\end{aligned}$$

therefore

$$\begin{aligned}
\mathcal{K}^N \leq \eta & - \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\leq R} \int_0^1 \frac{(1-\theta)\theta G^{N-1}F^{N-1}\left(f(|x|) - f(|y|)\right) g(z_t|.)\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\cdot(x-y)}{|x-y|^{d+a}\left(\theta f(z_t|.)F^{N-1} + (1-\theta)g(z_t|.)G^{N-1}\right)} dtdxdydX^{N-1} \\
& - \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\leq R} \int_0^1 \frac{(1-\theta)\theta G^{N-1}F^{N-1}\left(g(|x|) - g(|y|)\right) f(z_t|.)\nabla \ln \frac{f(|.)}{g(|.)}(z_t)\cdot(x-y)}{|x-y|^{d+a}\left(\theta f(z_t|.)F^{N-1} + (1-\theta)g(z_t|.)G^{N-1}\right)} dtdxdydX^{N-1}.
\end{aligned} \tag{C.5}$$

Let now be $s \in (0, r)$ and define

$$F' = \mathbf{1}_{\mathcal{B}_s} F, \quad \text{and} \quad F'' = F - F'.$$

Then

$$\begin{aligned}
\mathcal{K}^N \leq \eta & - \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\leq R} \int_0^1 \frac{(1-\theta)\theta G^{N-1}F'^{N-1}\left(f(|x|) - f(|y|)\right) g(z_t|.)\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\cdot(x-y)}{|x-y|^{d+a}\left(\theta f(z_t|.)F'^{N-1} + (1-\theta)g(z_t|.)G^{N-1}\right)} dtdxdydX^{N-1} \\
& - \int_{\mathbb{R}^{d(N-1)}} \theta F''^{N-1} \int_{|x-y|\leq R} |f(|x|) - f(|y|)| \int_0^1 \frac{g(z_t)\left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{f(z_t)|x-y|^{d-1+a}} dtdxdydX^{N-1} \\
& - \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\leq R} \int_0^1 \frac{(1-\theta)\theta G^{N-1}F'^{N-1}\left(g(|x|) - g(|y|)\right) f(z_t|.)\nabla \ln \frac{f(|.)}{g(|.)}(z_t)\cdot(x-y)}{|x-y|^{d+a}\left(\theta f(z_t|.)F'^{N-1} + (1-\theta)g(z_t|.)G^{N-1}\right)} dtdxdydX^{N-1} \\
& - \int_{\mathbb{R}^{d(N-1)}} \theta F''^{N-1} \int_{|x-y|\leq R} |g(|x|) - g(|y|)| \int_0^1 \frac{\left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{|x-y|^{d-1+a}} dtdxdydX^{N-1} \\
& := \mathcal{K}_1^N + \mathcal{K}_2^N + \mathcal{K}_3^N + \mathcal{K}_4^N.
\end{aligned}$$

Set now $u = \frac{r+s}{2}$ and $\delta = \frac{r-s}{2}$, and denote

$$\tilde{\mathbb{B}}_u^{N-1} = \left\{(x_2, \cdots, x_N) \in \mathbb{R}^{d(N-1)} \,|\, \frac{1}{N-1}\sum_{i=2}^N \delta_{x_i} \in \mathcal{B}_u\right\},$$

then

$$\mathcal{K}_1^N \leq \int_{\mathbb{R}^{d(N-1)}} \int_{|x-y|\leq R} (\mathbf{1}_{\mathbb{B}^{N-1}} + \mathbf{1}_{\mathbb{B}^{N-1,c}}) \int_0^1 \frac{(1-\theta)\theta G^{N-1} F'^{N-1} |f(x)-f(y)| \, g(z_t|.) \left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{|x-y|^{d-1+a} (\theta f(z_t|.) F'^{N-1} + (1-\theta) g(z_t|.) G^{N-1})} dt dx dy dX^{N-1}$$

$$\leq \int_{\mathbb{R}^{d(N-1)}} (1-\theta)\mathbf{1}_{\mathbb{B}^{N-1}} G^{N-1} \int_{|x-y|\leq R} |f(x)-f(y)| \int_0^1 \frac{g(z_t|.) \left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{|x-y|^{d-1+a} f(z_t|.)} dt dx dy dX^{N-1}$$

$$+ \int_{\mathbb{R}^{d(N-1)}} \theta \mathbf{1}_{\mathbb{B}^{N-1,c}} F'^{N-1} \int_{|x-y|\leq R} |f(x)-f(y)| \int_0^1 \frac{\left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{|x-y|^{d-1+a}} dt dx dy dX^{N-1}.$$

Using Lemma Appendix C.1 we find easily that

$$|f(x)-f(y)| \int_0^1 \frac{\left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{|x-y|^{d-1+a}} dt \leq 2\varepsilon^{-1} |f(x)-f(y)| \, |x-y|^{-(d-1+a)},$$

Therefore

$$\int_{|x-y|\leq R} |f(x)-f(y)| \int_0^1 \frac{\left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{|x-y|^{d-1+a}} dt dx dy \leq \int_{|x-y|\leq R} |\ln f(x) - \ln f(y)| \frac{|f(x)-f(y)|}{|\ln f(x) - \ln f(y)|} |x-y|^{-(d-1+a)} dx dy$$

$$\leq \int_{|x-y|\leq R} \|\nabla \ln f(|)\|_{L^\infty} (f(x)+f(y)) \, |x-y|^{-d+(2-a)} dx dy$$

$$\leq 2\varepsilon^{-1} \int_{\mathbb{R}^d} f(x) \left(\int_{y\in\mathbb{R}^d, |x-y|\leq R} |x-y|^{-d+2-a} dy \right) dx \leq C_{\varepsilon,R,a}.$$

On the other hand

$$|f(x)-f(y)| \int_0^1 \frac{g(z_t|.) \left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{|x-y|^{d-1+a} f(z_t|.)} dt \leq 2\varepsilon^{-1} |f(x)-f(y)| \int_0^1 \frac{g(z_t|.)}{|x-y|^{d-1+a} f(z_t|.)} dt$$

$$\leq 2\varepsilon^{-1} \left| \int_0^1 \nabla f(z_s|) \cdot (x-y) ds \right| \int_0^1 \frac{g(z_t|.)}{|x-y|^{d-1+a} f(z_t|.)} dt$$

$$\leq 2\varepsilon^{-1} |x-y|^{-d+2-a} \int_0^1 \int_0^1 \frac{|\nabla f(z_s|)| \, g(z_t|.)}{f(z_t|.)} dt ds$$

$$\leq 2\varepsilon^{-1} |x-y|^{-d+2-a} \int_0^1 \int_0^1 |\nabla \ln f(z_s|)| \, g(z_t|.) \frac{f(z_s|\cdot)}{f(z_t|.)} dt ds$$

$$\leq 2 \frac{C_{\varepsilon,R}}{c_{\varepsilon,R}} \varepsilon^{-2} |x-y|^{-d+2-a} \int_0^1 g(z_t|.) dt,$$

where we used points $(i) - (ii)$ of Lemma Appendix C.1 to pass to the last line. But

$$\int_0^1 g(z_t|.) dt = \int_0^1 g\left(y + t|x-y| \frac{(x-y)}{|x-y|} |. \right) dt,$$

for $|x-y| \leq R$ by we have by point (ii) of Lemma Appendix C.1

$$\int_0^1 g(z_t|.) dt \leq C_{\varepsilon,R} g(y|.).$$

So that

$$\int_{|x-y|\leq R} |f(x)-f(y)| \int_0^1 \frac{g(z_t|.) \left|\nabla \ln \frac{g(|.)}{f(|.)}(z_t)\right|}{|x-y|^{d-1+a} f(z_t|.)} dt \leq 2\varepsilon^{-2} C_{\varepsilon,R} \int_{|x-y|\leq R} g(y)|x-y|^{-d+2-a} dx dy \leq C_{\varepsilon,a,R}.$$

Finally we obtain

$$\mathcal{K}_1^N \leq C_{\varepsilon,a,R}\left((1-\theta)\int_{\mathbb{R}^{d(N-1)}} \mathbf{1}_{\mathbb{B}^{N-1}} G^{N-1} dX^{N-1} + \theta \int_{\mathbb{R}^{d(N-1)}} \mathbf{1}_{\mathbb{B}^{N-1,c}} F'^{N-1} dX^{N-1}\right).$$

The other terms are treated similarly and we conclude this step with

$$\begin{aligned}\mathcal{K}^N \leq \eta + C_{\varepsilon,a,R}&\left((1-\theta)\int_{\mathbb{R}^{d(N-1)}} \mathbf{1}_{\mathbb{B}^{N-1}} G^{N-1} dX^{N-1} + \theta \int_{\mathbb{R}^{d(N-1)}} \mathbf{1}_{\mathbb{B}^{N-1,c}} F'^{N-1} dX^{N-1}\right)\\ &+ C_{\varepsilon,a,R}\theta \int_{\mathbb{R}^{d(N-1)}} F''^{N-1} dX^{N-1}.\end{aligned}$$

⋄ *Step five*

The end of the proof is then exactly taken from [20, Lemma 5.10]. Nevertheless we reproduce it here for the sake of completeness. First we treat the third trerm in the above r.h.s. by observing that $F^{''} = \mathbf{1}_{\mathcal{B}_r\setminus\mathcal{B}_s} F$. Therefore

$$\int_{\mathbb{R}^{d(N-1)}} F''^{N-1}(X^{N-1}) dX^{N-1} = \int_{\mathcal{P}(\mathbb{R}^d)} \mathbf{1}_{\mathcal{B}_r\setminus\mathcal{B}_s}(\rho_\varepsilon) F(d\rho).$$

Due to Lebesgue's dominated convergence Theorem, the r.h.s. in the above identity converges to 0. Therefore one can chose some $s < r$ such that

$$C_\varepsilon c_a \theta \int_{\mathbb{R}^{d(N-1)}} F''^{N-1}(X^{N-1}) dX^{N-1} < \eta,$$

uniformly in N. Then for $X^{N-1} \notin \tilde{\mathbb{B}}_u^{N-1}$ and $\rho \in \mathcal{B}_s$ we find that

$$\begin{aligned}W_1\left(\frac{1}{N-1}\sum_{i=2}^N \delta_{x_i}, \rho * \psi_\varepsilon\right) &\geq W_1\left(\frac{1}{N-1}\sum_{i=2}^N \delta_{x_i}, f_1\right) - W_1(f_1,\rho) - W_1(\rho, \rho*\psi_\varepsilon)\\ &\geq u - s - c\varepsilon \geq \frac{\delta}{2},\end{aligned}$$

for any $\varepsilon > 0$ small enough. Therefore using a Chebychev-like argument it holds

$$\begin{aligned}\int_{\tilde{\mathbb{B}}_u^{N-1,c}} F'^{N-1}(X^{N-1}) dX^{N-1} &= \int_{\mathcal{P}(\mathbb{R}^d)}\left(\int_{\mathbb{R}^{d(N-1)}} \mathbf{1}_{\tilde{\mathbb{B}}_u^{N-1,c}} \rho_\varepsilon^{\otimes(N-1)}\right) F'(d\rho)\\ &\leq \frac{2}{\delta}\int_{\mathcal{P}(\mathbb{R}^d)}\left(\int_{\mathbb{R}^{d(N-1)}} W_1\left(\frac{1}{N-1}\sum_{i=2}^N \delta_{x_i}, \rho_\varepsilon\right)\rho_\varepsilon^{\otimes(N-1)}(dX^{N-1})\right) F'(d\rho)\end{aligned}$$

We claim that there is a constant C depending only on κ (see [14, Theroem 1] in case $d = 2, p = 1, q = \kappa < 2$) such that it holds

$$\int_{\mathbb{R}^{d(N-1)}} W_1\left(\frac{1}{N-1}\sum_{i=2}^N \delta_{x_i}, \rho_\varepsilon\right)\rho_\varepsilon^{\otimes(N-1)}(dX^{N-1}) \leq C\left(M_\kappa(\rho_\varepsilon)\right)^{\frac{1}{\kappa}}(N-1)^{-\left(1-\frac{1}{\kappa}\right)}. \tag{C.6}$$

Note that [20, Remark 2.12] provides the same result with the exponent $1-\frac{1}{\kappa}$ replaced with $\gamma \in \left(0, \frac{1}{3+\frac{2}{\kappa}}\right)$, but the rate of convergence does not play any role in the proof. Summing up (C.6) w.r.t. F', yields

$$\begin{aligned}\int_{\tilde{\mathbb{B}}_u^{N-1,c}} F'^{N-1}(X^{N-1}) dX^{N-1} &\leq \frac{C}{\delta(N-1)^{\left(1-\frac{1}{\kappa}\right)}}\int_{\mathcal{P}(\mathbb{R}^d)} \left(M_\kappa(\rho_\varepsilon)\right)^{\frac{1}{\kappa}} F'(d\rho)\\ &\leq \frac{C}{\delta(N-1)^{\left(1-\frac{1}{\kappa}\right)}}\left(\int_{\mathcal{P}(\mathbb{R}^d)} M_\kappa(\rho)\pi(d\rho) + M_\kappa(\psi_\varepsilon)\right)^{\frac{1}{\kappa}},\end{aligned}$$

since

$$\begin{aligned}M_\kappa(\rho_\varepsilon) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle x\rangle^\kappa \rho(x-y)\psi_\varepsilon(y)dxdy &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\langle x+y\rangle^\kappa \rho(x)\psi_\varepsilon(y)dxdy\\ &\leq 2^\kappa\left(\int_{\mathbb{R}^d}\langle x\rangle^\kappa\rho(x)dx + \int_{\mathbb{R}^d}\langle y\rangle^\kappa\psi_\varepsilon(y)dy\right)\end{aligned}$$

Treating in the exact same fashion the integral w.r.t. G^{N-1} concludes this step with

$$\forall \eta > 0,\ \exists N_\eta,\ \text{s.t.}\ \forall N \geq N_\eta,\ \mathcal{K}^N \leq \eta.$$

$\diamond$ Final step
Gathering all the estimates obtained in the previous steps yields for any $\varepsilon > 0$

$$\lim_{N\to+\infty} \left| \mathcal{I}_a^N(\pi^{N,\varepsilon}) - \theta \mathcal{I}_a^N(F^{N,\varepsilon}) - (1-\theta)\mathcal{I}_a^N(G^{N,\varepsilon}) \right| = 0.$$

Hence we deduce

$$\begin{aligned}
\tilde{\mathcal{I}}_a(\pi^\varepsilon) &= \sup_{N\geq 1} \mathcal{I}_a^N(\pi^{N,\varepsilon}) = \lim_{N\to+\infty} \mathcal{I}_a^N(\pi^{N,\varepsilon}) \\
&= \theta \lim_{N\to+\infty} \mathcal{I}_a^N(F^{N,\varepsilon}) + (1-\theta) \lim_{N\to+\infty} \mathcal{I}_a^N(G^{N,\varepsilon}) \\
&= \theta \sup_{N\geq 1} \mathcal{I}_a^N(F^{N,\varepsilon}) + (1-\theta) \sup_{N\geq 1} \mathcal{I}_a^N(G^{N,\varepsilon}) \\
&= \theta \tilde{\mathcal{I}}_a(F^\varepsilon) + (1-\theta)\tilde{\mathcal{I}}_a(G^\varepsilon).
\end{aligned}$$

But using the convexity of the functional Φ and Jensen's inequality yields

$$\tilde{\mathcal{I}}_a(\pi^\varepsilon) = \sup_{N\geq 1} \mathcal{I}_a^N(\pi^{N,\varepsilon}) \leq \sup_{N\geq 1} \mathcal{I}_a^N(\pi^N) = \tilde{\mathcal{I}}_a(\pi).$$

Morever it is clear from the fact that the functionals $(\mathcal{I}_a^N)_{N\geq 1}$ are l.s.c. w.r.t. the weak convergence in $\mathcal{P}(\mathbb{R}^{dN})$, that $\tilde{\mathcal{I}}_a$ is l.s.c. w.r.t. the weak convergence in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$. But since $\pi^\varepsilon \overset{*}{\rightharpoonup} \pi$ in $\mathcal{P}(\mathcal{P}(\mathbb{R}^d))$ we get that

$$\lim_{\varepsilon\to 0} \tilde{\mathcal{I}}_a(\pi^\varepsilon) = \tilde{\mathcal{I}}_a(\pi).$$

Therefore

$$\tilde{\mathcal{I}}_a(\pi) = \theta \tilde{\mathcal{I}}_a(F) + (1-\theta)\tilde{\mathcal{I}}_a(G),$$

which concludes the proof.

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References

[1] D. Applebaum. Lévy processes and stochastic calculus, Cambridge studies in advanced mathematics 93, (2004).
[2] P. Biler, T. Cieslak, G. Karch, J. Zienkiewicz. Local criteria for blowup in two-dimensional chemotaxis models, Discrete and Continuous Dynamical Systems - Series A, Vol.37, p1841 - 1856, (2017).
[3] P. Billingsley. Convergence of probability measures, Wiley series in probability and statistics, (1999).
[4] A. Blanchet, J. Dolbeaut, B. Perthame. Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions Electronic Journal of Differential Equations, Texas State University, Department of Mathematics, 44, 32 pp, (2006).
[5] V. Calvez, N. Bournaveas. The one-dimensional Keller-Segel model with fractional diffusion of cells Nonlinearity, Vol. 23, (2010).
[6] V. Calvez, J. Carrillo, F. Hoffmann. Equilibria of homogeneous functionals in the fair-competition regime Nonlinear Analysis, (2016).
[7] E. Carlen, Superadditivity of Fisher's information and logarithmic Sobolev inequalities, J. Funct. Anal., 101 (1991), pp. 194211.
[8] J. Carrillo, Y.-P Choi, M. Hauray. The derivation of swarming models: Mean-field limit and Wasserstein distances CISM International Centre for Mechanical Sciences, Vol. 533, Springer-Verlag Wien, (2014), 1-46.
[9] E. Cinlar. Probability and stochastics, Graduate Texts in Mathematics 261, Springer (2011).
[10] E. Di Nezza, G. Palatucci, E. Valdinoci. Hitchhiker's guide to fractional Sobolev spaces, Bulletin des Sciences Mathmatique, Vol. 136 (2012) no.5.

[11] G. Egana, S. Mischler. Uniqueness and long time asymptotic for the Keller-Segel equation: the parabolic-elliptic case, Arch. Ration. Mech. Anal. 220 (2016).
[12] M. Erbar. Gradient flows of the entropy for jump processes, Annales de lInstitut Henri Poincaré (2014).
[13] C. Escudero. The fractional KellerSegel model, Nonlinearity, Vol. 19, (2006).
[14] N. Fournier, A. Guillin. On the rate of convergence in Wasserstein distance of the empirical measure. Probability Theory and Related Fields, 162, (2015), 707–738.
[15] N. Fournier, M. Hauray, S. Mischler. Propagation of chaos for the 2D viscous vortex model. J. Eur. Math. Soc., Vol. 16, No 7, 1423-1466, 2014.
[16] N. Fournier, B. Jourdain. Stochastic particle approximation of the Keller-Segel equation and two-dimensional generalization of Bessel processes. Accepted at Ann. Appl. Probab.
[17] A.-C. Garcia, and P. Pickl. Microscopic derivation of the Keller-Segel equation in the sub-critical regime, Preprint https://arxiv.org/abs/1703.04376.
[18] D. Godinho, C. Quininao. Propagation of chaos for a subcritical Keller-Segel Model, Annales de lInstitut Henri Poincaré (2013).
[19] R. Granero-Belichon. On the fractional Fisher information with applications to a hyperbolic-parabolic system of chemotaxis, Journal of Differential Equations, vol. 262, n. 4, pp. 32503283, (2017).
[20] M. Hauray, S. Mischler. On Kac's chaos and related problems, J. Funct. Anal., Volume 266, P. 60556157,(2014).
[21] T. Holding, Propagation of chaos for Hlder continuous interaction kernels via Glivenko-Cantelli. Preprint, arXiv:1608.02877, (2016).
[22] H. Huang, J.-G. Liu. Well posedness for the Keller-Segel equation with fractional laplacian and the theory of propagation of chaos. Kinetic and Related Models (2016).
[23] W. Jager and S. Luckhaus, On Explosions of Solutions to a System of Partial Differential Equations Modelling Chemotaxis, transactions of the american mathematical society Volume 329, Number 2, February 1992
[24] M. K.-H. Kiessling, The Hartree limit of Borns ensemble for the ground state of a bosonic atom or ion, J. Math. Phys., 53 (2012), p. 095223.
[25] M. Kwasnicki. Ten equivalent definitions of the fractional laplace operator. Frac. Calc. Appl. Anal. 20(1) (2017): 751.
[26] L. Laflèche Fractional Fokker-Planck Equation with General Confinement Force. preprint, arxiv:1803.02672 (2018).
.
[27] E. H. Lieb and M. Loss, Analysis, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2nd ed., 2001.
[28] H. Osada. Propagation of chaos for the two-dimensional Navier-Stokes equation. Probabilistic methods in mathematical physics, 303-334, (1987).
[29] D. Robinson and D. Ruelle, Mean entropy of states in classical statistical mechanics, Commun. Math. Phys., 5 (1967), pp. 288300.
[30] N. Rougerie, De Finetti theorems, mean-field limits and Bose-Einstein condensation. arXiv:1506.05263, 2014. LMU lecture notes.
[31] A.-S. Sznitman, Topics in propagation of chaos , In *École d'Été de Probabilités de Saint-Flour XIX–-1989*, volume 1464, chapter Lecture Notes in Math., pages 165–251. Springer, Berlin, 1991.
[32] G. Toscani, The fractional Fisher information and the central limit theorem for stable laws, Ricerche Mat., 65 (1) 71-91 (2016).
[33] C. Villani, A review of mathematical topics in collisional kinetic theory, Handbook of Mathematical Fluid Dynamics (Vol. 1), 2002)
[34] C. Villani, Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equation without cut-off, Revista Matematica Iberoamericana Vol. 15, N. 2, 1999