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CROSS-DIFFUSION SYSTEMS AND FAST-REACTION LIMITS

ESTHER S. DAUS, LAURENT DESVILLETTES, AND ANSGAR JÜNGEL

Abstract. The rigorous asymptotics from reaction-cross-diffusion systems for three species with known entropy to cross-diffusion systems for two variables is investigated. The equations are studied in a bounded domain with no-flux boundary conditions. The global existence of very weak (integrable) solutions and the rigorous fast-reaction limit are proved. The limiting system inherits the entropy structure with an entropy that is not the sum of the entropies of the components. Uniform estimates are derived from the entropy inequality and the duality method.

1. Introduction

The analysis of cross-diffusion systems with unknowns $u_1, \ldots, u_n$ often relies on the existence of a convex Lyapunov functional, called here an entropy, which provides suitable gradient estimates [6, 9, 10, 16]. Given the partial differential system, the difficulty is to identify such an entropy functional. Often, it is of the form $\int_\Omega \sum_{i=1}^n h_i(u_i)dx$ for convex functions h_i, which only depend on u_i; see the examples in the aforementioned references. In this paper, we identify entropy functionals for certain cross-diffusion systems that are generally not the sum of all $h_i(u_i)$.

Our approach is to consider first reaction-cross-diffusion systems for which a Lyapunov functional is known to exist and which is of the form $\int_\Omega \sum_{i=1}^n h_i(u_i)dx$. Then we perform the limit of vanishing relaxation times that are related to the reaction terms. The limiting system consists of cross-diffusion equations, which possesses an entropy inherited from the original system and where the variables u_i are related by an algebraic relation coming from the reaction terms. This strategy enlarges the class of cross-diffusion systems with an entropy structure by providing examples for which the entropy cannot be easily found in another way.

As an example of this approach, we consider reaction-cross-diffusion equations whose reaction terms correspond to one reversible reaction of the form $A \rightleftharpoons B + C$. More

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specifically, we study the equations

$$\partial_t u_i^\varepsilon - \Delta F_i(u^\varepsilon) = Q_i(u^\varepsilon) \quad \text{in } \Omega,\ t>0,\ i=1,2,3, \tag{1}$$

supplemented with no-flux boundary and initial conditions,

$$\nabla F_i(u^\varepsilon)\cdot\nu = 0 \quad \text{on } \partial\Omega, \quad u_i^\varepsilon(0) = u_i^I \quad \text{in } \Omega,\ i=1,2,3. \tag{2}$$

Here, $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$, $\Omega\subset\mathbb{R}^d$ $(d\ge 1)$ is a bounded domain with smooth boundary, and ν is the exterior unit normal vector to $\partial\Omega$. The unknowns u_i^ε can be interpreted as chemical concentrations, but generally they are just densities in some diffusive system whose application is not specified. The nonlinear functions contain cross-diffusion terms,

$$\begin{aligned} F_1(u^\varepsilon) &= f_1(u_1^\varepsilon) + f_{12}(u_1^\varepsilon, u_2^\varepsilon),\\ F_2(u^\varepsilon) &= f_2(u_2^\varepsilon) + f_{21}(u_1^\varepsilon, u_2^\varepsilon),\\ F_3(u^\varepsilon) &= f_3(u_3^\varepsilon), \end{aligned} \tag{3}$$

and the reaction terms are given by

$$\begin{aligned} Q_1(u^\varepsilon) &= -\varepsilon^{-1}\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big),\\ Q_2(u^\varepsilon) &= \varepsilon^{-1}\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big),\\ Q_3(u^\varepsilon) &= \varepsilon^{-1}\big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big). \end{aligned} \tag{4}$$

The constraints on functions f_i, f_{ij}, and q_i are specied in Assumptions (A1)-(A5) below. The parameter $\varepsilon>0$ models the inverse of a reaction rate or, generally, a relaxation time.

Without diffusion terms, the corresponding system of ordinary differential equations is known to possess the Lyapunov functional

$$h(u) := \sum_{i=1}^3 \int_1^{u_i} \log q_i(s)ds. \tag{5}$$

When $q_i(u_i) = u_i$, we recover the physical entropy for the reaction $A \rightleftharpoons B + C$, i.e. $h(u) = \sum_{i=1}^3 u_i(\log u_i - 1)$. The functional $\int_\Omega h(u)dx$ is still a Lyapunov functional if the diffusion terms are given by $\Delta f_i(u_i)$. In this paper, we allow for the cross-diffusion terms $f_{12}(u_1,u_2)$ and $f_{21}(u_1,u_2)$. Clearly, an additional assumption is then needed to guarantee that (5) is still an entropy for (1). We show that this is the case under a "weak cross-diffusion" condition; see Assumption (A5) below.

The fast-reaction limit $\varepsilon\to 0$ in (1) leads formally to the system

$$\begin{aligned} \partial_t(u_1+u_2) &= \Delta(F_1(u)+F_2(u)),\\ \partial_t(u_1+u_3) &= \Delta(F_1(u)+F_3(u)),\\ q_1(u_1) &= q_2(u_2)q_3(u_3) \quad \text{in } \Omega,\ t>0. \end{aligned}$$

Under certain conditions on q_i, this system can be formulated in terms of the variables $v = u_1+u_2$ and $w = u_1+u_3$, leading to

$$\partial_t v = \Delta G_1(v,w), \quad \partial_t w = \Delta G_2(v,w), \tag{6}$$

where

$$G_1(v,w) = (F_1+F_2)(u_1,u_2,u_3), \quad G_2(v,w) = (F_1+F_3)(u_1,u_2,u_3), \quad \text{and}$$
$$u_1 = q_1^{-1}\big(q_2(u_2(v,w))q_3(u_3(v,w))\big), \quad u_2 = u_2(v,w), \quad u_3 = u_3(v,w).$$

Formally, the limit entropy

$$\int_\Omega h_0(v,w)dx = \int_\Omega \bigg\{ \int_1^{q_1^{-1}(q_2(u_2(v,w))q_3(u_3(v,w)))} \log q_1(s)ds + \int_1^{u_2(v,w)} \log q_2(s)ds + \int_1^{u_3(v,w)} \log q_3(s)ds \bigg\} dx$$

is a Lyapunov functional for (6). A simple example is given in Remark 13.

Fast-reaction limits in reaction-diffusion equations have been studied since about 20 years. These limits are of importance in mass-action kinetics chemistry to reduce a system of many components to a (nonlinear) system with less equations. One of the first papers is [14], where a fast-reaction limit in a system consisting of one parabolic and one ordinary differential equation was performed. Later, the fast-reaction limit in a two-species diffusion system was shown, leading to a nonlinear diffusion equation [1]. Fast irreversible reactions for two species, studied in [2], led to a Stefan-type limit problem with a moving interface, which represents the chemical reaction front. Systems for three species with Lotka-Volterra-type interactions [20] or with reversible reactions [3] were also analyzed. A unified approach for self-similar fast-reaction limits was given in [8].

In [13], the fast-reaction limit in a system containing a parabolic equation on the domain boundary (volume surface diffusion model) was proved. Here, the limit problem is the heat equation with a dynamic boundary condition. A combination of the fast-reaction limit and homogenization techniques has given a two-scale reaction-diffusion system with a moving boundary traveling within the microstructure [18]. Finally, asymptotic limits related to fast reactions were investigated in reaction-diffusion equations from population dynamics [7, 15, 19]. Here, the small parameter describes an averaged time within which two types of species convert to each other. If the conversion is of nonlinear type, the limit problem becomes a cross-diffusion system.

A three-species system with power-like reaction functions q_i was investigated in [3, 4], proving the existence of mild solutions employing a semigroup approach [4] and the fast-reaction limit using entropy and duality techniques [3].

The main difference between our approach and the results of [3, 4], and the main novelty of this paper, is that we allow for cross-diffusion terms in the original reaction-diffusion system, at least "weak cross-diffusion" as specified in Assumption (A5). Interestingly, the Lyapunov functional structure is still kept when adding cross-diffusion to a certain extent. This leads to a much larger set of cross-diffusion systems than known up to now, for which a Lyapunov functional can be produced.

Before we detail our main results, we need some assumptions. First, we introduce the notation

$$\mathbb{R}_+ = [0,\infty), \quad \mathbb{R}_+^* = (0,\infty), \quad Q_T = \Omega\times(0,T).$$

The functions F_i and Q_i are extended continuously to $\mathbb{R}^3$ by setting $F_i(u) = F_i(|u_1|, |u_2|, |u_3|)$ and $Q_i(u) = Q_i(|u_1|, |u_2|, |u_3|)$ for any $u = (u_1, u_2, u_3) \in \mathbb{R}^3$. Finally, we set $|u|^2 = \sum_{i=1}^3 u_i^2$ for $u = (u_1, u_2, u_3) \in \mathbb{R}^3$.

Assumptions. We impose the following conditions.

(A1) Nonlinear diffusion: $f_i \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ satisfies $f_i(s) = g_i(s)s$ for $s \geq 0$, where $g_i \in C^0(\mathbb{R}_+;\mathbb{R}_+) \cap C^1(\mathbb{R}_+^*;\mathbb{R}_+^*)$, and $f_i'(s) \geq \kappa_1 > 0$ for all $s \geq 0$ and for some $\kappa_1 > 0$.

(A2) Cross-diffusion: $f_{ij} \in C^0(\mathbb{R}_+^2;\mathbb{R}_+)$ satisfies $f_{ij}(s_1, s_2) = g_{ij}(s_1, s_2)s_i$, where $g_{ij} \in C^0(\mathbb{R}_+^2;\mathbb{R}_+) \cap C^1((\mathbb{R}_+^*)^2;\mathbb{R}_+^*)$, and $\partial_1 f_{ij}(s_1, s_2) \geq 0$, $\partial_2 f_{ij}(s_1, s_2) \geq 0$ for all s_1, $s_2 \geq 0$, $i, j = 1, 2$, $i \neq j$.

(A3) Reaction terms I: $q_i \in C^1(\mathbb{R}_+)$ satisfies $q_i(0) = 0$, $q_i'(s) > 0$ for all $s > 0$, and $q_i(s_0) \geq 1$ for some $s_0 > 0$.

(A4) Reaction terms II: There exist $C_q > 0$, $\widetilde{C}_q > 0$ such that for all $s = (s_1, s_2, s_3) \in \mathbb{R}_+^3$,

$$\lim_{|s|\to\infty} \frac{q_1(s_1) + q_2(s_2)q_3(s_3)}{\sum_{i=1}^3 F_i(s) \sum_{i=1}^3 s_i + 1} = 0, \quad \frac{q_i(s_i)(1 + q_i(s_i))}{q_i'(s_i)f_i'(s_i)(F_i(s)s_i + 1)} \leq C_q,$$

$$\frac{\sum_{i=1}^3 \int_1^{s_i} \log(1 + q_i(v))\, dv}{\sum_{i=1}^3 F_i(s) \sum_{i=1}^3 s_i + 1} \leq \widetilde{C}_q.$$

(A5) Weak cross-diffusion: There exists $\eta_0 > 0$ and $\delta \in (0, 1)$ such that for all $\eta \in [0, \eta_0]$ and s_1, $s_2 \geq 0$,

$$\begin{aligned}&\left(\frac{q_1'(s_1)\partial_2 f_{12}(s_1, s_2)}{q_1(s_1)(1 + \eta q_1(s_1))} + \frac{q_2'(s_2)\partial_1 f_{21}(s_1, s_2)}{q_2(s_2)(1 + \eta q_2(s_2))}\right)^2 \\ &\quad \leq 2(1 - \delta)^2 \frac{q_1'(s_1)q_2'(s_2)(f_1'(s_1) + \partial_1 f_{12}(s_1, s_2))(f_2'(s_2) + \partial_2 f_{21}(s_1, s_2))}{q_1(s_1)q_2(s_2)(1 + \eta q_1(s_1))(1 + \eta q_2(s_2))}.\end{aligned}$$

(A6) Initial data: $u_i^I \in L^\infty(\Omega)$ and there exists $\kappa_2 > 0$ such that $u_i^I \geq \kappa_2 > 0$ in Ω.

Remark 1 (Discussion of the assumptions)**.** We indicate where the main assumptions are needed in the existence proof.

- Assumption (A1): The lower bound $f_i'(s) \geq \kappa_1 > 0$ and $f_i(0) = 0$ imply that $f_i(s) \geq \kappa_1 s$ for all $s \geq 0$. This means that we require some amount of standard diffusion in the problem. This assumption implies a uniform L^2 bound for the approximate solutions; see Lemma 7.
- Assumption (A2): This is a structure condition on the diffusion matrix. It allows us to show that F is a homeomorphism on $\mathbb{R}_+^3$ (see Lemma 15), which is needed in the approximate scheme.
- Assumption (A3): This condition is satisfied, for instance, for power-type functions q_i with exponent larger than or equal to one. It ensures that the entropy built out of the q_i is well-behaved.
- Assumption (A4): The conditions relate the reaction and diffusion terms. Together with the duality estimate, they yield the uniform integrability of Q_i. Note that because of this assumption, it is not possible to handle reaction terms which grow too fast when

the unknowns become large. The third bound is needed to show that the regularized entropy density is bounded from below; see the arguments before (27).

- Assumption (A5): The weak cross-diffusion condition allows us to prove nonlinear gradient estimates. Expanding the square on the left-hand side of the inequality of the assumption and choosing $\delta > 0$ such that $2(1-\delta)^2 < 1$, we see that this assumption implies that

$$\partial_2 f_{12}(s_1,s_2)\partial_1 f_{21}(s_1,s_2) < \big(f_1'(s_1) + \partial_1 f_{12}(s_1,s_2)\big)\big(f_2'(s_1) + \partial_2 f_{21}(s_1,s_2)\big). \tag{7}$$

  It means that the determinant of the diffusion matrix $F'(u)$ is positive. This information is needed to show that F is a homeomorphism on $\mathbb{R}^3_+$; see Lemma 15. Note that assumption (A5) is typically satisfied when the derivatives of the cross diffusion terms f_{12} and f_{21} are assumed to be small when compared to the derivatives of the standard diffusion terms f_1, f_2; in other words when the cross diffusion is dominated in some sense by the standard diffusion.
- Assumption (A6): The positivity assumption on the initial data is necessary to prove the nonnegativity of u_i. By using an approximation argument, we may relax this condition to $u_i^I \geq 0$ in Ω, but we leave the technical details to the reader.

Note that for instance, the functions

$$\begin{aligned} &f_i(u_i) = \alpha_i u_i + u_i^\delta, \quad q_i(u_i) = u_i^\beta, \quad \beta \geq 1, \quad i = 1,2,3, \\ &f_{12}(u_1,u_2) = \alpha u_1^\gamma u_2, \quad f_{21}(u_1,u_2) = \alpha u_1 u_2^\gamma, \end{aligned}$$

satisfy Assumptions (A1)-(A5) if $\delta > 1$ is sufficiently large and $\alpha > 0$ is sufficiently small; see Lemma 14 for details. □

The first main result is the global-in-time existence of very weak (i.e. integrable) solutions to equations 1, for a given $\varepsilon > 0$.

Theorem 2 (Global existence of solutions)**.** *Let Ω be a bounded open subset of $\mathbb{R}^d$ with a smooth boundary, let assumptions (A1)-(A6) hold, and let $\varepsilon > 0$, $T > 0$. Then there exists a very weak solution $u_i^\varepsilon \in L^2(\Omega_T)$ to (1)-(2) such that*

$$u_i^\varepsilon \geq 0 \text{ in } \Omega_T, \quad F_i(u^\varepsilon),\, Q_i(u^\varepsilon) \in L^1(Q_T),$$

and for all $\phi_i \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{2,\infty}(\Omega))$ with $\nabla\phi_i \cdot \nu = 0$ on $\partial\Omega$,

$$\begin{aligned} -\int_0^\infty \int_\Omega u_i^\varepsilon \partial_t \phi_i dxdt - \int_0^\infty \int_\Omega F_i(u^\varepsilon)\Delta\phi_i^\varepsilon dxdt \\ = \int_0^\infty \int_\Omega Q_i(u^\varepsilon)\phi_i dxdt + \int_\Omega u_i^I(x)\phi_i(x,0)dx, \end{aligned} \tag{8}$$

(and $u_i^\varepsilon(0) = u_i^I$ in $H^m(\Omega)'$, $i = 1,2,3$, where $m > 2 + d/2$). Moreover, this solution satisfies the entropy inequality

$$\int_\Omega h(u^\varepsilon(t))dx + \delta \int_0^t \int_\Omega \sum_{i=1}^3 |\nabla[J_i(u_i^\varepsilon)]|^2 dxd\sigma \leq \int_\Omega h(u^I)dx, \tag{9}$$

where $h(u^\varepsilon)$ is the entropy given by (5)*, δ is defined in Assumption (A5), and*

$$J_i(s) = \int_0^s \min\left\{1, \left(\frac{q_i'(y)f_i'(y)}{q_i(y)(1+q_i(y))}\right)^{1/2}\right\}dy, \quad s \geq 0, \quad i = 1,2,3. \tag{10}$$

Remark 3. (i) A global existence result for cross-diffusion systems of type (1) was also shown in [10]. Compared to that work, we allow for more general reaction terms (in principle we allow for arbitrarily fast growing reactions) and we make explicit the entropy structure of the diffusion system. In [10, Section 5.2], a specific three-species system was studied too, but assuming vanishing reactions and special diffusion coefficients.

(ii) In principle, the existence result (Theorem 2) may be generalized to n-species systems. However, the structure conditions between reaction and diffusion and the weak cross-diffusion assumption would become much more involved. Therefore, we restricted ourselves to the three-species case.

(iii) We have proved the existence of very weak solutions (for the starting systems as well as for the final systems). One may ask whether there are more regular solutions. Stará and John [23] have shown that there are solutions to cross-diffusion systems whose Hölder continuity breaks down after finite time. Thus generally, classical solutions cannot be expected. However, L^p regularity for $p > 2$ may be possible using the duality method [10] or L^∞ bounds can be proved for specific systems of volume-filling type [16]. Note also that some bounds are available on first-order derivatives in x for both the starting and final system, thanks to the entropy estimates (these bounds are, however, not easily written in terms of Sobolev spaces). □

The proof of Theorem 2 is based on a regularization procedure, entropy estimates, and a duality method. More precisely, we replace the time derivative by the implicit Euler discretization with time step size $\tau > 0$ and regularize the reactions Q_i with parameter $\eta > 0$ to make them bounded (say, Q_i^η). The existence of solutions u_i^k, which approximate $u_i(\cdot, k\tau)$, is shown by techniques similarly as in [10]. Using a regularized version of the entropy (5), h^η, we derive the discrete entropy inequality (see Lemma 8)

$$\begin{aligned}\int_\Omega h^\eta(u^k)dx + \tau\sum_{j=1}^k\sum_{i=1}^3\int_\Omega |\nabla[J_i(u_i^j)]|^2dx \\ + \frac{\tau}{\varepsilon}\sum_{j=1}^k\int_\Omega Q^\eta(u^\varepsilon)\cdot(h^\eta)'(u^\varepsilon)dx \leq \int_\Omega h(u^I)dx.\end{aligned} \tag{11}$$

This gives *a priori* estimates independent of the regularization parameters η and τ as well as the relaxation time ε. Further L^2 bounds are obtained from the duality method of [21], here in the discrete version of [10, Lemma 2.12]. Thanks to the discrete Aubin-Lions lemma of [11], we obtain the relative compactness of the sequence of approximate solutions. This allows us to perform the limit $(\eta, \tau) \to 0$ in the approximate problem.

The second main result is the fast-reaction limit.

Theorem 4 (Fast-reaction limit)**.** *Let Ω be a bounded open subset of $\mathbb{R}^d$ with a smooth boundary, $T>0$, and let assumptions (A1)-(A6) hold. We suppose that $q_1(u_1^I)=q_2(u_2^I)q_3(u_3^I)$ in Ω, and that the functions*

$$\mathbb{R}_+^2\to\mathbb{R},\quad (u_2,u_3)\mapsto \frac{1}{u_i}q_1^{-1}(q_2(u_2)q_3(u_3)),\quad i=2,3, \tag{12}$$

are continuous. Furthermore, let $u^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon,u_3^\varepsilon)$ be the very weak solution to (1)-(2) constructed in Theorem 2. Then there exists a subsequence, which is not relabeled, such that, as $\varepsilon\to 0$,

$$u_i^\varepsilon\to u_i\quad \text{strongly in } L^1(Q_T),\ i=1,2,3.$$

The limit $u_i\in L^1(Q_T)$ is a very weak solution to the system

$$\partial_t(u_1+u_2)=\Delta(F_1(u)+F_2(u)),\quad \partial_t(u_1+u_3)=\Delta(F_1(u)+F_3(u)), \tag{13}$$

$$q_1(u_1)=q_2(u_2)q_3(u_3)\quad \text{in } Q_T. \tag{14}$$

Moreover, it satisfies the entropy inequality

$$\int_\Omega h(u(t))dx+\delta\int_0^t\int_\Omega\sum_{i=1}^3|\nabla J_i(u_i)|^2dxd\sigma\le\int_\Omega h(u^I)dx, \tag{15}$$

where h and J_i are defined in (5) and (10), respectively.

The proof is based on the following ideas: From the entropy inequality (see the discrete version (11)), we deduce immediately that

$$q_1(u_1^\varepsilon)-q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\to 0\quad \text{strongly in } L^1(Q_T). \tag{16}$$

Here, we need the condition $q_1(u_1^I)=q_2(u_2^I)q_3(u_3^I)$, which prevents a boundary (more precisely, an initial) layer. We cannot directly apply the Aubin-Lions lemma to u_i^ε, since the bounds for $\partial_t u_i^\varepsilon$ depend on ε. However, $\partial_t(u_1^\varepsilon+u_2^\varepsilon)$ and $\partial_t(u_1^\varepsilon+u_3^\varepsilon)$ are uniformly bounded, showing, together with the gradient estimate from (9), that $u_1^\varepsilon+u_2^\varepsilon\to v$ and $u_1^\varepsilon+u_3^\varepsilon\to w$ strongly in $L^1(Q_T)$. The key idea is to prove that the mapping $(u_2,u_3)\mapsto(u_1+u_2,u_1+u_3)$ can be inverted (see Lemma 12). For this argument, we need the continuity of the functions in (12). We deduce that $u_2^\varepsilon\to u_2$ and $u_3^\varepsilon\to u_3$ a.e. in Q_T and consequently $u_1^\varepsilon=q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\to u_1$ a.e. in Q_T.

The paper is organized as follows. Theorems 2 and 4 are proved in Sections 2 and 3, respectively. In the appendix, we collect some technical and auxiliary results.

2. Proof of the existence result

We start here the

Proof of Theorem 2: We first show the existence of solutions to an approximate problem. Let $T>0$, $N\in\mathbb{N}$, $\tau=T/N$, and $\eta\in(0,1)$. We assume throughout this section that Assumptions (A1)-(A6) hold. Given $u^{k-1}=(u_1^{k-1},u_2^{k-1},u_3^{k-1})\in L^\infty(\Omega;\mathbb{R}_+^3)$, we wish to solve the following implicit Euler scheme with bounded reaction terms:

$$\tau^{-1}(u_i^k-u_i^{k-1})-\Delta F_i(u^k)=Q_i^\eta(u^k)\quad \text{in } \Omega,\ i=1,2,3, \tag{17}$$

together with the no-flux boundary conditions

$$\nabla F_i(u^k)\cdot\nu = 0 \quad \text{on } \partial\Omega,\ i=1,2,3. \tag{18}$$

When $k=1$, we set $u^{k-1}=u^I$. The regularized reaction terms are defined by

$$Q_i^\eta(u^k) = \frac{\sigma_i}{\varepsilon}\left(\frac{q_1(u_1^k)}{1+\eta q_1(u_1^k)} - \frac{q_2(u_2^k)}{1+\eta q_2(u_2^k)}\frac{q_3(u_3^k)}{1+\eta q_3(u_3^k)}\right),$$

where $\sigma_1=-1$ and $\sigma_2=\sigma_3=1$. They satisfy the following properties. First, a straightforward estimation gives

$$|Q_i^\eta(s)| \le \frac{1}{\varepsilon}\frac{1+\eta}{\eta^2} \le \frac{2}{\varepsilon\eta^2} =: K_1(\varepsilon,\eta) \quad \text{for all } s_i\ge 0,\ i=1,2,3. \tag{19}$$

Second, let $Q_i^\eta = Q_{i,+}^\eta - Q_{i,-}^\eta$, where $Q_{i,+}^\eta\ge 0$ and $Q_{i,-}^\eta\ge 0$. Then there exists a constant $K_2(\varepsilon,\eta)>0$ such that

$$Q_{i,-}^\eta(s) \le K_2(\varepsilon,\eta)s_i \quad \text{for all } s_i\ge 0,\ i=1,2,3. \tag{20}$$

Indeed, this estimate is clear for large values of s_i because of the boundedness of $Q_{i,-}^\eta$; for small values of s_i, it follows from $q_i(s_i)=q_i(0)+q_i'(\xi)s_i = q_i'(\xi)s_i$ for some $0<\xi<s_i$.

We also need the following property of F (shown in Lemma 15 in the appendix): $F:\mathbb{R}_+^3\to\mathbb{R}_+^3$ is a C^1-diffeomorphism from $(\mathbb{R}_+^*)^3$ into itself and a homeomorphism from $\mathbb{R}_+^3$ into itself.

2.1. **Existence for scheme** (17)**.** We prove that there exists a strong solution to (17)-(18) (under the assumptions of Theorem 2).

Lemma 5. *Let $0<\tau\le 1/K_1(\varepsilon,\eta)=\varepsilon\eta^2/2$. Then there exists a solution $u^k\in C^0(\overline{\Omega};\mathbb{R}^3)$ to* (17)-(18) *such that $F(u^k)\in W^{2,p}(\Omega;\mathbb{R}^3)$ for all $p<\infty$.*

Proof. The proof is a modification of the proof of Theorem 2.5 in [10]. Since our estimates are partially different, we present a full proof. The idea is to define a fixed-point operator whose compactness follows from the compactness of an elliptic solution operator.

Step 1: Definition of the fixed-point operator. Let $K_{p,\Omega}>0$ be the elliptic regularity constant defined in Lemma 17. Introduce for $u=(u_1,u_2,u_3)\in L^\infty(\Omega;\mathbb{R}^3)$,

$$\overline{M}(u) := \max\left\{\tau K_{p,\Omega}, \frac{1}{\kappa_1}\left(1+\tau\max_{i=1,2,3}\sup_\Omega\frac{Q_{i,-}^\eta(u)}{|u_i|}\right)\right\}.$$

The constant κ_1 is defined in Assumption (A1). Because of (20), $Q_{i,-}^\eta(u)/|u_i|$ is finite and so, $\overline{M}(u)$ is finite too.

We define the fixed-point operator $\Lambda:[0,1]\times L^\infty(\Omega;\mathbb{R}^3)\to L^\infty(\Omega;\mathbb{R}^3)$ by

$$\Lambda(\sigma,u) = \Phi\circ(\sigma\Theta)\circ\Psi(u),$$

where (recall that F_i and Q^η are continuously extended to $\mathbb{R}^3$ by setting $F_i(u)=F_i(|u_1|,|u_2|,|u_3|)$, $Q_i^\eta(u)=Q_i^\eta(|u_1|,|u_2|,|u_3|)$),

$$\Psi: L^\infty(\Omega;\mathbb{R}^3)\to L^\infty(\Omega;\mathbb{R}_+^3)\times(\tau K_{p,\Omega},\infty),$$
$$\Psi(u) = \big(u^{k-1}+\overline{M}(u)F(u)-u+\tau Q^\eta(u),\overline{M}(u)\big),$$

$$\begin{aligned}
&\Theta : L^\infty(\Omega;\mathbb{R}^3_+) \times (\tau K_{p,\Omega},\infty) \to L^\infty(\Omega;\mathbb{R}^3_+),\\
&\qquad \Theta(u,M) = (M\,\mathrm{Id} - \tau\Delta)^{-1}u \text{ with no-flux boundary conditions},\\
&\Phi : L^\infty(\Omega;\mathbb{R}^3_+) \to L^\infty(\Omega;\mathbb{R}^3_+), \quad \Phi(u) = F^{-1}(u).
\end{aligned}$$

A computation shows that any fixed point of $\Lambda(\sigma,\cdot)$ solves

$$\overline{M}(u)F_i(u) - \tau\Delta F_i(u) = \sigma\big(u_i^{k-1} + \overline{M}(u)F_i(u) - u_i + \tau Q_i^\eta(u)\big), \quad i = 1,2,3, \tag{21}$$

which means that for $\sigma = 1$, this fixed point solves (17)-(18).

We have to show that the functions Φ, Θ, and Ψ are well defined. Indeed, by Assumption (A1), the definition of $\overline{M}(u)$, and the property $F_i(u) \ge \kappa_1|u_i|$,

$$\begin{aligned}
&\overline{M}(u)F_i(u) - u_i + \tau Q_i^\eta(u)\\
&\ge |u_i|\left(1 + \tau \max_{i=1,2,3}\sup_\Omega \frac{Q_{i,-}^\eta(u)}{|u_i|}\right) - |u_i| + \tau\big(Q_{i,+}^\eta(u) - Q_{i,-}^\eta(u)\big) \ge \tau Q_{i,+}^\eta(u) \ge 0.
\end{aligned}$$

We deduce that Ψ is well defined. If $\Theta(u,M) = v$ for some $u \in L^\infty(\Omega;\mathbb{R}^3_+)$ then $Mv - \tau\Delta v = u \ge 0$ in Ω and $\nabla v\cdot\nu = 0$ on $\partial\Omega$. Using $v^- = \min\{0,v\}$ as a test function in the weak formulation of this elliptic equation, we see that $v \ge 0$ in Ω. Furthermore, with the test function $(v-\mu)^+ = \max\{0, v-\mu\}$, where $\mu = \|u\|_{L^\infty(\Omega)}/M$, it follows that

$$\tau\int_\Omega |\nabla(v-\mu)^+|^2 dx = \int_{\{v>\mu\}} (u - Mv)(v-\mu)^+ dx \le 0,$$

and hence, $\|v\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}/M$. This shows that Θ is well defined. Finally, Φ is well defined since F is a homeomorphism on $\mathbb{R}^3_+$; see Lemma 15.

We check the properties of Λ needed to apply the Leray-Schauder fixed-point theorem. Clearly, $\Lambda(0,u) = 0$ for all $u \in L^\infty(\Omega;\mathbb{R}^3)$. The continuity of Λ follows from the continuity of the functions Ψ, Θ, and Φ proved in Lemma 2.6 in [10]. By Lemma 17 of the appendix, $\Theta(u,M) \in W^{2,p}(\Omega;\mathbb{R}^3)$ for any $p < \infty$. Since the embedding $W^{2,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact for $p > d/2$, we deduce that $\Theta : L^\infty(\Omega;\mathbb{R}^3_+) \times (\tau K_{p,\Omega},\infty) \to L^\infty(\Omega;\mathbb{R}^3_+)$ is compact too, and the same holds for Λ.

It remains to show a uniform L^∞ estimate for any fixed point u (that is, such that $\Lambda(\sigma,u) = u$). Note that any fixed point is nonnegative and thus, $F_i(u) \ge 0$ in Ω.

Step 2: L^1 estimate for $F_i(u)$. We claim that there exists a constant $C > 0$, depending on $\|u^{k-1}\|_{L^\infty(\Omega)}$ and Ω, such that

$$0 \le \sum_{i=1}^3 \int_\Omega F_i(u)dx \le C. \tag{22}$$

Indeed, the fixed point u solves (21) in Ω and $\nabla F_i(u)\cdot\nu = 0$ on $\partial\Omega$. Summing (21) for $i = 1,2,3$ and denoting $|F(u)|_1 := \sum_{i=1}^3 F_i(u)$, (21) leads to

$$(1-\sigma)\overline{M}(u)|F(u)|_1 + \sigma|u|_1 - \tau\Delta|F(u)|_1 = \sigma|u^{k-1}|_1 + \sigma\tau|Q^\eta(u)|_1. \tag{23}$$

Multiplying this equation by $|F(u)|_1$ and integrating over Ω yields

$$\begin{aligned}(1-\sigma)\int_\Omega \overline{M}(u)|F(u)|_1^2dx + \tau\int_\Omega |\nabla|F(u)|_1|^2dx + \sigma\int_\Omega |u|_1|F(u)|_1dx \\ = \sigma\int_\Omega |u^{k-1}|_1|F(u)|_1dx + \sigma\tau\int_\Omega |Q^\eta(u)|_1|F(u)|_1dx.\end{aligned} \tag{24}$$

If $\sigma = 0$, we have $F_i(u) = 0$ for $i = 1, 2, 3$, and $u = 0$. Therefore we consider $\sigma \neq 0$. Neglecting the first two integrals in (24) and dividing this equation by σ, it follows that

$$\int_\Omega |u|_1|F(u)|_1dx \le \int_\Omega |u^{k-1}|_1|F(u)|_1dx + \tau\int_\Omega |Q^\eta(u)|_1|F(u)|_1dx.$$

Then, by (19) and $\tau K_1(\varepsilon,\eta) \le 1$,

$$\begin{aligned}\int_\Omega |u|_1|F(u)|_1dx &\le 3\|u^{k-1}\|_{L^\infty(\Omega)}\int_\Omega |F(u)|_1dx + 3\tau K_1(\varepsilon,\eta)\int_\Omega |F(u)|_1dx \\ &\le 3\big(\|u^{k-1}\|_{L^\infty(\Omega)}+1\big)\int_\Omega |F(u)|_1dx.\end{aligned}$$

Let $R > 0$. If $|u|_1 \le R$, the continuity of F_i gives $|F(u)|_1 \le \omega(R)$, where ω is a modulus of continuity. Therefore,

$$\begin{aligned}\int_\Omega |u|_1|F(u)|_1dx &\le 3\big(\|u^{k-1}\|_{L^\infty(\Omega)}+1\big)\bigg(\int_{\{|u|_1>R\}}|F(u)|_1dx + \int_{\{|u|_1\le R\}}|F(u)|_1dx\bigg) \\ &\le 3\big(\|u^{k-1}\|_{L^\infty(\Omega)}+1\big)\bigg(\frac{1}{R}\int_\Omega |u|_1|F(u)|_1dx + \omega(R)|\Omega|\bigg),\end{aligned}$$

where $|\Omega|$ denotes the measure of Ω. We choose $R = 6(\|u^{k-1}\|_{L^\infty(\Omega)}+1)$ and obtain

$$\frac{1}{2}\int_\Omega |u|_1|F(u)|_1dx \le 3\big(\|u^{k-1}\|_{L^\infty(\Omega)}+1\big)\omega(R)|\Omega| = \frac{R}{2}\omega(R)|\Omega|.$$

We use this estimate in

$$\begin{aligned}\int_\Omega |F(u)|_1dx &= \int_{\{|u|_1>R\}}|F(u)|_1dx + \int_{\{|u|_1\le R\}}|F(u)|_1dx \\ &\le \frac{1}{R}\int_\Omega |u|_1|F(u)|_1dx + \omega(R)|\Omega| \le 2\omega(R)|\Omega|,\end{aligned}$$

which proves (22).

Step 3: L^∞ estimate for u_i. We use estimate (19) for Q^η in (23):

$$-\tau\Delta|F(u)|_1 \le (1-\sigma)\overline{M}(u)|F(u)|_1 + \sigma|u|_1 - \tau\Delta|F(u)|_1 \le \sigma|u^{k-1}|_1 + 3\sigma\tau K_1(\varepsilon,\eta).$$

As the right-hand side is in $L^\infty(\Omega)$, we can apply Lemma 18 in the appendix to conclude that

$$\||F(u)|_1\|_{L^\infty(\Omega)} \le C\bigg(\frac{3}{\tau}\|u^{k-1}\|_{L^\infty(\Omega)} + 3K_1(\varepsilon,\eta) + \||F(u)|_1\|_{L^1(\Omega)}\bigg).$$

Then, taking into account Assumption (A1) and (22),

$$\|u_i\|_{L^\infty(\Omega)} \le \kappa_1 \||F(u)|_1\|_{L^\infty(\Omega)} \le C(\varepsilon,\eta,\Omega,\|u^{k-1}\|_{L^\infty(\Omega)}),$$

which shows the desired estimate (uniform with respect to the considered fixed points).

Hence, we can apply the Leray-Schauder theorem and infer the existence of a solution to (17)-(18).

It remains to verify the continuity of u_i^k. We know that $F_i(u^k) \in W^{2,p}(\Omega)$ for all $p < \infty$. Thus, choosing $p > d/2$, $F_i(u^k) \in C^0(\overline{\Omega})$. Since F is a homeomorphism, we conclude that $u_i^k \in C^0(\overline{\Omega})$. □

2.2. **A priori estimates for scheme** (17)**.** We show several *a priori* estimates which are (except for Lemma 6 below) uniform in η and τ. Some of these estimates are also uniform with respect to ε and will be used in Section 3. We denote by $C(\delta_1,\ldots,\delta_n)$ a generic positive constant depending on the parameters $\delta_1,\ldots,\delta_n$, whose value may change from occurence to occurence. We begin with

Lemma 6 (Positivity of u_i^k)**.** *Let $\tau < 1$. Then there exists a constant $\delta(\varepsilon,\eta,\tau) > 0$ depending on ε, η, and τ such that*

$$u_i^k \ge \delta(\varepsilon,\eta,\tau) \quad \text{in } \Omega,\ i = 1,2,3. \tag{25}$$

Proof. We proceed by induction. By Assumption (A6), $u_i^I \ge \kappa_2 > 0$ in Ω. Let $u_i^{k-1} \ge \gamma > 0$ for some $\gamma > 0$, $i = 1,2,3$. Using (20), we find that

$$Q_i^\eta(u^k) = Q_{i,+}^\eta(u^k) - Q_{i,-}^\eta(u^k) \ge -K_2(\varepsilon,\eta)u_i^k.$$

Thus, choosing $M \ge (\tau K_2(\varepsilon,\eta)+1)/\kappa_1$,

$$Q_i^\eta(u^k) + \frac{M}{\tau}F_i(u^k) \ge Q_i^\eta(u^k) + \frac{\kappa_1 M}{\tau}u_i^k \ge \frac{u_i^k}{\tau},$$

and consequently, using the scheme and the induction hypothesis,

$$\frac{M}{\tau}F_i(u^k) - \Delta F_i(u^k) = \frac{u_i^{k-1}}{\tau} - \frac{u_i^k}{\tau} + Q_i^\eta(u^k) + \frac{M}{\tau}F_i(u^k) \ge \frac{u_i^{k-1}}{\tau} \ge u_i^{k-1} \ge \gamma.$$

By the minimum principle (see Step 1 of the proof of Lemma 5 for the same argument), $F_i(u^k) \ge \tau\gamma/M$ for $i = 1,2,3$. We know from the proof of Lemma 5 that $u_i^k \in L^\infty(\Omega)$. By Assumption (A1), this gives $F_i(u^k) \le Cu_i^k$, where $C > 0$ depends on the L^∞ bound of u^k. We infer that $u_i^k \ge \tau\gamma/(CM) =: \delta(\varepsilon,\eta,\tau) > 0$. □

Lemma 7 (Uniform L^2 estimate)**.** *There exists a constant $C > 0$ independent of ε, η, and τ such that*

$$\tau\sum_{k=1}^N \int_\Omega \sum_{i=1}^3 F_i(u^k)\sum_{i=1}^3 u_i^k dx \le C, \quad \tau\sum_{k=1}^N \|u_i^k\|_{L^2(\Omega)}^2 \le C, \quad i = 1,2,3.$$

Proof. These bounds are a consequence of the duality estimate stated in Lemma 16. Indeed, we set $v^k = 2u_1^k + u_2^k + u_3^k$ and

$$\mu^k = \frac{2F_1(u^k) + F_2(u^k) + F_3(u^k)}{2u_1^k + u_2^k + u_3^k}.$$

Note that $\mu^k v^k \in H^2(\Omega)$. Then $(v^k - v^{k-1})/\tau = \Delta(\mu^k v^k)$, since the weighted sum of the reaction terms vanishes. Consequently, the following estimates do not depend on ε nor η. Lemma 16 gives

$$\tau \sum_{k=1}^N \int_\Omega \mu^k (v^k)^2 dx \le C\bigg(1 + \tau \sum_{k=1}^N \int_\Omega \mu^k dx\bigg), \tag{26}$$

where $C > 0$ depends only on u^I, Ω, and $T = N\tau$.

It remains to estimate $\int_\Omega \mu^k dx$. Let $L > 0$ and define $S = \{s = (s_1, s_2, s_3) \in \mathbb{R}_+^3 : 2s_1 + s_2 + s_3 \le L\}$ and

$$\begin{aligned}\mu(L) &:= \sup_{s\in S} \frac{2F_1(s) + F_2(s) + F_3(s)}{2s_1 + s_2 + s_3} \\ &= \sup_{s\in S} \frac{2s_1(g_1(s_1) + g_{12}(s_1, s_2)) + s_2(g_2(s_2) + g_{21}(s_1, s_2)) + s_3 g_3(g_3)}{2s_1 + s_2 + s_3}.\end{aligned}$$

Clearly, $\mu(L)$ is finite. It follows that

$$\int_\Omega \mu^k dx = \int_{\{v^k \le L\}} \mu^k dx + \int_{\{v^k > L\}} \mu^k dx \le \mu(L)|\Omega| + \frac{1}{L^2} \int_\Omega \mu^k (v^k)^2 dx.$$

Inserting this estimate into (26) and using $k\tau \le T$, we arrive at

$$\bigg(1 - \frac{C}{L^2}\bigg)\tau \sum_{k=1}^N \int_\Omega \mu^k (v^k)^2 dx \le C\big(1 + T\mu(L)|\Omega|\big).$$

Choosing $L > 0$ sufficiently large, this yields the first estimate in the statement. For the L^2 bound, we observe that $F_i(u^k) \ge \kappa_1 u_i^k$ so that, for all $j = 1, 2, 3$,

$$\tau\kappa_1 \sum_{k=1}^N \int_\Omega (u_j^k)^2 dx \le \tau \sum_{k=1}^N \int_\Omega \sum_{i=1}^3 F_i(u^k) \sum_{i=1}^3 u_i^k dx \le C(L, T, \Omega),$$

which concludes the proof. □

We now introduce the regularized entropy density

$$h^\eta(u^k) = \sum_{i=1}^3 \int_1^{u_i^k} \log\bigg(\frac{q_i(s)}{1 + \eta q_i(s)}\bigg) ds, \quad \eta > 0.$$

We need to show that $\int_\Omega h^\eta dx$ is bounded from below uniformly in η (since otherwise, the following estimates would depend on η). Indeed, we have $h^\eta(u) = h(u) - \sum_{i=1}^3 \int_1^u \log(1 +$

$\eta q_i(s))ds$, and $\int_\Omega h(u)\,dx$ is bounded from below. Now, for $\eta \in [0,1]$, by Assumption (A4) and Lemma 7, it holds that (for $u_i \geq 1$ for all i)

$$\begin{aligned}\sum_{i=1}^3 \int_\Omega \int_1^{u_i} \log(1+\eta q_i(s))dsdx &\leq \sum_{i=1}^3 \int_\Omega \int_1^{u_i} \log(1+q_i(s))\,dsdx \\ &\leq \widetilde{C}_q \int_\Omega \sum_{i=1}^3 F_i(u) \sum_{i=1}^3 u_i\,dx \leq C.\end{aligned}$$

This means that the integral $\int_\Omega h^\eta dx$ is bounded from below uniformly in η, showing our claim. Since $q_i' \geq 0$, the function h^η is convex. The construction of h^η allows for the control of the reaction terms since

$$\begin{aligned}Q^\eta(u^k)\cdot(h^\eta)'(u^k) = &-\frac{1}{\varepsilon}\left(\frac{q_1(u_1^k)}{1+\eta q_1(u_1^k)} - \frac{q_2(u_2^k)q_3(u_3^k)}{(1+\eta q_2(u_2^k))(1+\eta q_3(u_3^k))}\right) \\ &\times\left(\log\frac{q_1(u_1^k)}{1+\eta q_1(u_1^k)} - \log\frac{q_2(u_2^k)q_3(u_3^k)}{(1+\eta q_2(u_2^k))(1+\eta q_3(u_3^k))}\right) \leq 0.\end{aligned} \tag{27}$$

Lemma 8 (Entropy estimate)**.** *Let $0<\eta\leq\min\{1,\eta_0\}$, where η_0 is defined in Assumption (A5). Then (with $\delta>0$ from Assumption (A5))*

$$\begin{aligned}\int_\Omega h^\eta(u^k)dx + \delta\sum_{k=1}^N \int_\Omega \sum_{i=1}^3 \frac{q_i'(u_i^k)f_i'(u_i^k)}{q_i(u_i^k)(1+q_i(u_i^k))}|\nabla u_i^k|^2 dx \\ -\tau\sum_{k=1}^N \int_\Omega Q^\eta(u^k)\cdot(h^\eta)'(u^k)dx \leq \int_\Omega h(u^I)dx.\end{aligned} \tag{28}$$

Proof. We know from Lemma 6 that u_i^k is strictly positive, so $(\partial h^\eta/\partial u_i)(u^k)$ is an admissible test function in the weak formulation of (17):

$$\begin{aligned}\int_\Omega (u^k-u^{k-1})\cdot(h^\eta)'(u^k)dx + \tau\int_\Omega \nabla u^k : (h^\eta)''(u^k)F'(u^k)\nabla u^k dx \\ = \tau\int_\Omega Q^\eta(u^k)\cdot(h^\eta)'(u^k)dx,\end{aligned} \tag{29}$$

where ":" is the Frobenius matrix product. Summing (29) over $k=1,\ldots,j$ and taking into account (27) and

$$\int_\Omega \big(h^\eta(u^k)-h^\eta(u^{k-1})\big)dx \leq \int_\Omega (u^k-u^{k-1})\cdot(h^\eta)'(u^k)dx,$$

which follows from the convexity of h^η, we find that

$$\begin{aligned}\int_\Omega h^\eta(u^j)dx + \tau\sum_{k=1}^j\int_\Omega \nabla u^k : (h^\eta)''(u^k)F'(u^k)\nabla u^k dx \\ - \tau\sum_{k=1}^j\int_\Omega Q^\eta(u^k)\cdot(h^\eta)'(u^k)dx \le \int_\Omega h^\eta(u^I)dx.\end{aligned} \tag{30}$$

A straightforward computation yields

$$\begin{aligned}&\int_\Omega \nabla u^k : (h^\eta)''(u^k)F'(u^k)\nabla u^k dx \\ &\quad = \int_\Omega \big(T_1|\nabla u_1^k|^2 + T_2|\nabla u_2^k|^2 + T_3|\nabla u_3^k|^2 + T_4\nabla u_1^k\cdot\nabla u_2^k\big)dx,\end{aligned}$$

where

$$\begin{aligned}T_1 &= \frac{q_1'(u_1^k)}{q_1(u_1^k)(1+\eta q_1(u_1^k))}\big(f_1'(u_1^k) + \partial_1 f_{12}(u_1^k,u_2^k)\big), \\ T_2 &= \frac{q_2'(u_2^k)}{q_2(u_2^k)(1+\eta q_2(u_2^k))}\big(f_2'(u_2^k) + \partial_2 f_{21}(u_1^k,u_2^k)\big), \\ T_3 &= \frac{q_3'(u_3^k)}{q_3(u_3^k)(1+\eta q_3(u_3^k))}f_3'(u_3^k), \\ T_4 &= \frac{q_1'(u_1^k)}{q_1(u_1^k)(1+\eta q_1(u_1^k))}\partial_2 f_{12}(u_1^k,u_2^k) + \frac{q_2'(u_2^k)}{q_2(u_2^k)(1+\eta q_2(u_2^k))}\partial_1 f_{21}(u_1^k,u_2^k).\end{aligned}$$

Set $\alpha = (1-\delta)T_1/T_4$, where δ comes from Assumption (A5). Then Young's inequality and $T_4^2 \le 2(1-\delta)^2 T_1T_2$ (see Assumption (A5)) show that, for $0<\eta\le\eta_0$,

$$\begin{aligned}T_4\nabla u_1^k\cdot\nabla u_2^k &\ge -T_4\alpha|\nabla u_1^k|^2 - \frac{T_4}{4\alpha}|\nabla u_2^k|^2 = -(1-\delta)T_1|\nabla u_1^k|^2 - \frac{T_4^2}{4(1-\delta)T_1}|\nabla u_2^k|^2 \\ &\ge -(1-\delta)T_1|\nabla u_1^k|^2 - (1-\delta)T_2|\nabla u_2^k|^2.\end{aligned}$$

(Observe that Assumption (A5) could be weakened to $T_4^2 \le 4(1-\delta)^2T_1T_2$, but then we would need to impose (7) as an additional constraint.) We deduce that

$$\int_\Omega \nabla u^k : (h^\eta)''(u^k)F'(u^k)\nabla u^k dx \ge \delta\int_\Omega\big(T_1|\nabla u_1^k|^2 + T_2|\nabla u_2^k|^2\big)dx + \int_\Omega T_3|\nabla u_3^k|^2.$$

Hence, inserting this estimate into (30), observing that $\eta\le 1$ and $h^\eta(u^I)\le h(u^I)$, and including the reaction terms, the result is shown. □

Lemma 9 (Estimate for the discrete time derivative)**.** *Let $m > 2+d/2$. Then there exists a constant $C(\varepsilon)>0$ independent of η and τ, but depending on ε, such that*

$$\tau\sum_{k=1}^N \big\|\tau^{-1}(u_i^k - u_i^{k-1})\big\|_{H^m(\Omega)'} \le C(\varepsilon), \quad i=1,2,3.$$

Proof. Using the duality estimate from Lemma 7, we obtain

$$\begin{aligned}\tau\sum_{k=1}^N\int_\Omega F_i(u^k)dx &\le \tau\sum_{k=1}^N\int_{\{u^k\le 1\}}F_i(u^k)dx+\tau\sum_{k=1}^N\int_{\{u^k>1\}}F_i(u^k)dx\\ &\le \tau\sum_{k=1}^N\int_{\{u^k\le 1\}}F_i(u^k)dx+\tau\sum_{k=1}^N\int_{\{u^k>1\}}F_i(u^k)u_i^k dx\le C.\end{aligned}$$

Furthermore, by Assumption (A4) and Lemma 7 again,

$$\begin{aligned}\tau\sum_{k=1}^N\int_\Omega |Q_i^\eta(u^k)|dx &\le \frac{\tau}{\varepsilon}\sum_{k=1}^N\int_\Omega\big(q_1(u_1^k)+q_2(u_2^k)q_3(u_3^k)\big)dx\\ &\le \frac{C\tau}{\varepsilon}\sum_{k=1}^N\int_\Omega\bigg(\sum_{j=1}^3F_j(u^k)\sum_{j=1}^3u_j^k+1\bigg)\le C(\varepsilon).\end{aligned}$$

Since $m>2+d/2$, we know that $\phi_i\in H^m(\Omega)\hookrightarrow W^{2,\infty}(\Omega)$. Therefore, we can write

$$\tau\sum_{k=1}^N\tau^{-1}(u_i^k-u_i^{k-1})\phi_i dx=\tau\sum_{k=1}^N\int_\Omega F_i(u^k)\Delta\phi_i dx+\tau\sum_{k=1}^N\int_\Omega Q_i^\eta(u^k)\phi_i dx.$$

In view of the two previous estimates, we infer that

$$\tau\sum_{k=1}^N\big\|\tau^{-1}(u_i^k-u_i^{k-1})\big\|_{H^m(\Omega)'}=\sup_{\|\phi_i\|_{H^m(\Omega)}=1}\sum_{k=1}^N\bigg|\int_\Omega(u_i^k-u_i^{k-1})\phi_i dx\bigg|\le C(\varepsilon),$$

showing the desired bound. □

Lemma 10 (Uniform $W^{1,1}$ estimate)**.** *There exists a constant $C>0$ independent of ε, η, and τ such that*

$$\tau\sum_{k=1}^N\|u_i^k\|_{W^{1,1}(\Omega)}\le C.$$

Proof. The Cauchy-Schwarz inequality gives

$$\begin{aligned}\tau\sum_{k=1}^N\|\nabla u_i^k\|_{L^1(\Omega)} &\le \bigg(\tau\sum_{k=1}^N\int_\Omega\frac{q_i'(u_i^k)f_i'(u_i^k)}{q_i(u_i^k)(1+q_i(u_i^k))}|\nabla u_i^k|^2dx\bigg)^{1/2}\\ &\quad\times\bigg(\tau\sum_{k=1}^N\int_\Omega\frac{q_i(u_i^k)(1+q_i(u_i^k))}{q_i'(u_i^k)f_i'(u_i^k)}dx\bigg)^{1/2}.\end{aligned}$$

In view of Lemma 8, the first factor is bounded uniformly in ε, η, and τ. By Assumption (A4) and the duality estimate in Lemma 7, it follows that

$$\tau\sum_{k=1}^N\|\nabla u_i^k\|_{L^1(\Omega)}\le C\bigg(\tau\sum_{k=1}^N\int_\Omega\bigg(\sum_{i=1}^3F_i(u^k)\sum_{i=1}^3u_i^k+1\bigg)dx\bigg)^{1/2}\le C.$$

Taking into account the uniform L^2 bound, we can conclude. □

2.3. **Limit** $(\eta,\tau)\to 0$**.** Let $u_i^{(\tau)}(x,t)=u_i^k(x)$ for $x\in\Omega$, $t\in((k-1)\tau,k\tau]$, $i=1,2,3$, be piecewise constant functions in time, and set $u^{(\tau)}=(u_1^{(\tau)},u_2^{(\tau)},u_3^{(\tau)})$. We introduce the time shift operator $(\sigma_\tau u^{(\tau)})(x,t)=u^{k+1}(x)$ for $x\in\Omega$, $t\in((k-1)\tau,k\tau]$. Let $\phi_i:(0,T)\to H^m(\Omega)$ be a piecewise constant function such that $m>2+d/2$, $\phi_i(t)=0$ for $((N-1)\tau,N\tau]$, and $\nabla\phi_i\cdot\nu=0$ on $\partial\Omega$, $t>0$. Then the weak formulation of (17) reads as

$$\begin{aligned}-\int_0^T\int_\Omega u_i^{(\tau)}(\sigma_\tau\phi_i-\phi_i)dxdt-\int_0^T\int_\Omega F_i(u^{(\tau)})\Delta\phi_i dxdt\\ =\int_0^T\int_\Omega Q_i^\eta(u^{(\tau)})\phi_i dxdt+\int_\Omega u_i^I(x)\phi_i(x,0)dx.\end{aligned}\tag{31}$$

Lemmas 9 and 10 give the (uniform with respect to τ, η) bounds

$$\tau^{-1}\|\sigma_\tau u_i^{(\tau)}-u_i^{(\tau)}\|_{L^1(0,T-\tau;H^m(\Omega)')}+\|u_i^{(\tau)}\|_{L^1(0,T;W^{1,1}(\Omega))}\le C(\varepsilon).$$

Observing that the embedding $W^{1,1}(\Omega)\hookrightarrow L^1(\Omega)$ is compact, we can apply the Aubin-Lions lemma in the version of [11] to infer the existence of a subsequence, which is not relabeled, such that, as $(\eta,\tau)\to 0$,

$$u_i^{(\tau)}\to u_i\quad\text{strongly in } L^1(Q_T),$$

recalling that $Q_T=\Omega\times(0,T)$. According to Lemma 5, the limit has to be performed in such a way that $\tau\le\varepsilon\eta^2/2$ is verified. Possibly for a subsequence, the convergence also holds a.e. in Q_T. The positivity estimate from Lemma 6 implies that $u_i\ge 0$ in Q_T. Moreover,

$$F_i(u^{(\tau)})\to F_i(u),\quad Q^\eta(u^{(\tau)})\to Q(u)\quad\text{a.e. in } Q_T.$$

Assumption (A4) implies that $(Q_i(u^{(\tau)}))$ is equi-integrable. More precisely, we have for all $\delta>0$ the existence of $R_0>0$ such that for all $R\ge R_0$ and $|u|\ge R$,

$$\frac{Q_i(u)}{\sum_{i=1}^3 F_i(u)\sum_{i=1}^3 u_i+1}\le\delta.$$

Then we conclude from Lemma 7 that

$$\begin{aligned}\int_0^T\int_{\{u^{(\tau)}\ge R\}}Q_i^\eta(u^{(\tau)})dxdt&\le\int_0^T\int_{\{u^{(\tau)}\ge R\}}\big(q_1(u_1^{(\tau)})+q_2(u_1^{(\tau)})q_3(u_1^{(\tau)})\big)dxdt\\&\le\delta\int_0^T\int_\Omega\bigg(\sum_{i=1}^3F_i(u^{(\tau)})\sum_{i=1}^3u_i^{(\tau)}+1\bigg)dxdt\le C\delta,\end{aligned}$$

where $C>0$ is independent of ε (and τ, η). This shows the equi-integrability of $(Q_i(u^{(\tau)}))$. The same conclusion holds for $(F_i(u^{(\tau)}))$ since

$$\int_0^T\int_{\{u^{(\tau)}\ge R\}}F_i(u^{(\tau)})dxdt\le\frac{1}{R}\int_0^T\int_\Omega F_i(u^{(\tau)})u_i^{(\tau)}dxdt\le\frac{C}{R}.\tag{32}$$

We deduce from Vitali's convergence theorem that

$$F_i(u^{(\tau)})\to F_i(u),\quad Q_i^\eta(u^{(\tau)})\to Q_i(u)\quad\text{strongly in } L^1(Q_T),\ i=1,2,3.$$

Therefore, we can perform the limit $(\eta,\tau)\to 0$ in (31), showing that $u=(u_1,u_2,u_3)$ solves (8) for all $\phi_i\in H^1(0,T;L^2(\Omega))\cap L^\infty(0,T;H^m(\Omega))$, $i=1,2,3$. By a density argument, we see that the weak formulation also holds for all $\phi_i\in L^\infty(0,T;W^{2,\infty}(\Omega))$ with $\nabla\phi_i\cdot\nu=0$ on $\partial\Omega$.

It remains to verify the entropy inequality (9). We have from (28):

$$\int_\Omega h^\eta(u^{(\tau)})dx+\delta\sum_{i=1}^3\int_0^T\int_\Omega|\nabla[J_i(u_i^{(\tau)})]|^2dxdt\le\int_\Omega h(u^I)dx, \tag{33}$$

where J_i is defined in (10). The a.e. convergence of $u_i^{(\tau)}\to u_i$ implies that $J_i(u_i^{(\tau)})\to J_i(u_i)$ a.e. in Q_T. Moreover, $J_i(s)\le s$, and thanks to the uniform L^2 bound for $u_i^{(\tau)}$, we deduce that $(J_i(u_i^{(\tau)}))$ is bounded in $L^2(0,T;H^1(\Omega))$. Up to a subsequence, we have $\nabla J_i(u_i^{(\tau)})\rightharpoonup\nabla J_i(u_i)$ weakly in $L^2(Q_T)$. As h^η is convex and continuous, it is weakly lower semicontinuous [5, Corollary 3.9] and

$$\int_\Omega h^\eta(u)dx\le\liminf_{\tau\to 0}\int_\Omega h(u^{(\tau)})dx.$$

Since (h^η) converges to h monotonically, we infer from the monotone convergence theorem that

$$\int_\Omega h(u)dx\le\liminf_{(\eta,\tau)\to 0}\int_\Omega h(u^{(\tau)})dx.$$

Therefore, observing that the square of the L^2 norm is also weakly lower semicontinuous, we may pass to the limit $(\eta,\tau)\to 0$ in (33) to conclude (9).

3. Proof of the fast-reaction limit

In the previous section, we have shown some *a priori* estimates for the approximate solution $u_i^{(\tau)}$, which are also independent of ε. Indeed, by Lemmas 7 and 8,

$$\int_0^T\int_\Omega(u_i^{(\tau)})^2dxdt+\int_0^T\int_\Omega F_i(u^{(\tau)})u_i^{(\tau)}dxdt\le C, \tag{34}$$

$$\int_0^T\int_\Omega\frac{q_i'(u_i^{(\tau)})f_i'(u_i^{(\tau)})}{q_i(u_i^{(\tau)})(1+q_i(u_i^{(\tau)}))}|\nabla u_i^{(\tau)}|^2dxdt\le C,\quad i=1,2,3, \tag{35}$$

$$\begin{aligned}&\int_0^T\int_\Omega\left(\frac{q_1(u_1^{(\tau)})}{1+\eta q_1(u_1^{(\tau)})}-\frac{q_2(u_2^{(\tau)})q_3(u_3^{(\tau)})}{(1+\eta q_2(u_2^{(\tau)}))(1+\eta q_3(u_3^{(\tau)}))}\right)\\&\qquad\times\left(\log\frac{q_1(u_1^{(\tau)})}{1+\eta q_1(u_1^{(\tau)})}-\log\frac{q_2(u_2^{(\tau)})q_3(u_3^{(\tau)})}{(1+\eta q_2(u_2^{(\tau)}))(1+\eta q_3(u_3^{(\tau)}))}\right)dxdt\le\varepsilon C.\end{aligned} \tag{36}$$

As mentioned in Section 2.3, estimates (34) and (35) yield the bound

$$\|J_i(u_i^{(\tau)})\|_{L^2(0,T;H^1(\Omega))}\le C, \tag{37}$$

which is uniform in ε, η, and τ. We need more uniform bounds to be able to pass to the limit $\varepsilon\to 0$.

We start here the

Proof of Theorem 4 : We systematically denote by $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ a very weak solution to (1)-(2) constructed in Theorem 2. We first state the

Lemma 11 (ε-uniform estimates)**.** *There exists $C > 0$ independent of ε such that, for $i = 1, 2, 3$,*

$$\|u_i^\varepsilon\|_{L^2(Q_T)} + \|F_i(u^\varepsilon)u_i^\varepsilon\|_{L^1(Q_T)} \le C, \tag{38}$$

$$\|J_i(u_i^\varepsilon)\|_{L^2(0,T;H^1(\Omega))} + \|u_i^\varepsilon\|_{L^1(0,T;W^{1,1}(\Omega))} \le C, \tag{39}$$

$$\big\|q_i(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big\|_{L^1(Q_T)} \le C\sqrt{\varepsilon}. \tag{40}$$

Proof. The first estimate in (38) follows immediately from (34) by performing the limit (η, τ) and using the weakly lower semicontinuity of the L^2 norm. The results in Subsection 2.3 imply that $F_i(u^{(\tau)})u_i^{(\tau)} \to F_i(u^\varepsilon)u_i^\varepsilon$ a.e. in Q_T. Then Fatou's lemma and the second estimate in (34) yield

$$\begin{aligned}\int_0^T\int_\Omega F_i(u^\varepsilon)u_i^\varepsilon dxdt &= \int_0^T\int_\Omega \liminf_{\tau\to 0} F_i(u^{(\tau)})u_i^{(\tau)} dxdt \\ &\le \liminf_{\tau\to 0}\int_0^T\int_\Omega F_i(u^{(\tau)})u_i^{(\tau)} dxdt \le C,\end{aligned}$$

and this implies the second estimate in (38).

We have shown in the proof of Theorem 2 that there exists a subsequence of $(u_i^{(\tau)})$ (not relabeled) such that $u_i^{(\tau)} \to u_i^\varepsilon$ in $L^1(Q_T)$ and a.e. Since J_i is continuous, we have $J_i(u_i^{(\tau)}) \to J_i(u_i^\varepsilon)$ a.e. Estimate (37) then implies that, up to a subsequence, $J_i(u_i^{(\tau)}) \rightharpoonup J_i(u_i^\varepsilon)$ weakly in $L^2(0,T;H^1(\Omega))$. Because of the weakly lower semi-continuity of the norm, we infer from (37) that the first estimate in (39) holds. The $W^{1,1}$ bound for (u_i^ε) follows as in the proof of Lemma 10, using the L^2 bound of $\nabla J_i(u_i^\varepsilon)$.

It follows from the a.e. convergence $q_i(u_i^{(\tau)}) \to q_i(u_i^\varepsilon)$, Fatou's lemma, and estimate (36) that

$$\int_0^T\int_\Omega \big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big)\log\left(\frac{q_1(u_1^\varepsilon)}{q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)}\right)dxdt \le \varepsilon C.$$

Thus, the elementary inequality $4(a^{1/2} - b^{1/2})^2 \le (a-b)\log(a/b)$ gives

$$\begin{aligned}&\big\|q_1(u_1^\varepsilon)^{1/2} - \big(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big)^{1/2}\big\|_{L^2(Q_T)}^2 \\ &\quad\le \frac{1}{4}\int_0^T\int_\Omega \big(q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big)\log\left(\frac{q_1(u_1^\varepsilon)}{q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)}\right)dxdt \le \varepsilon C.\end{aligned}$$

We now use Assumption (A4) and (34) to infer that

$$\begin{aligned}&\big\|q_1(u_1^\varepsilon) - q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big\|_{L^1(Q_T)} \\ &\quad\le \big\|q_1(u_1^\varepsilon)^{1/2} - \big(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big)^{1/2}\big\|_{L^2(Q_T)}\big\|q_1(u_1^\varepsilon)^{1/2} + \big(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\big)^{1/2}\big\|_{L^2(Q_T)}\end{aligned}$$

$$\le C\sqrt{\varepsilon}\|q_1(u_1^\varepsilon)+q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\|_{L^1(Q_T)}^{1/2}$$
$$\le C\sqrt{\varepsilon}\bigg(\int_0^T\int_\Omega\sum_{i=1}^3 F_i(u^\varepsilon)\sum_{i=1}^3 u_i^\varepsilon dxdt\bigg)^{1/2}\le C\sqrt{\varepsilon},$$

which concludes the proof. □

Unfortunately, the estimate on the discrete time derivative in Lemma 9 is not independent of ε, which prevents the direct use of the Aubin-Lions lemma. We overcome this problem by applying this lemma to $u_1^\varepsilon+u_2^\varepsilon$ and $u_1^\varepsilon+u_3^\varepsilon$ and by exploiting estimate (40). Indeed, these sums solve

$$\partial_t(u_1^\varepsilon+u_i^\varepsilon)=\Delta\big(F_1(u^\varepsilon)+F_i(u^\varepsilon)\big),\quad i=2,3.$$

Estimate (38) shows that $(F_i(u^\varepsilon))$ is bounded in $L^1(Q_T)$. Consequently, $\Delta F_i(u^\varepsilon)$ is bounded in $L^1(0,T;W^{2,\infty}(\Omega)')$, i.e.

$$\|\partial_t(u_1^\varepsilon+u_i^\varepsilon)\|_{L^1(0,T;W^{2,\infty}(\Omega)')}\le C,\quad i=2,3.$$

Using this estimate together with the $W^{1,1}$ bound (39) for $u_1^\varepsilon+u_i^\varepsilon$, we can apply the Aubin-Lions lemma of [22] to find a subsequence, which is not relabeled, such that, as $\varepsilon\to 0$,

$$u_1^\varepsilon+u_2^\varepsilon\to v_2,\quad u_1^\varepsilon+u_3^\varepsilon\to v_3\quad\text{strongly in } L^1(Q_T). \tag{41}$$

Moreover, by (40),

$$q_1(u_1^\varepsilon)-q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)\to 0\quad\text{strongly in } L^1(Q_T). \tag{42}$$

We claim that these convergences are sufficient to infer the strong convergence of (u_i^ε) in $L^1(Q_T)$ for $i=1,2,3$. To show this, we need the following auxiliary result:

Lemma 12 (Inversion of $q(u_1)=q_2(u_2)q_3(u_3)$)**.** *The function $g:\mathbb{R}_+^2\to\mathbb{R}_+^2$,*

$$g(u_2,u_3)=\begin{pmatrix}u_2+q_1^{-1}(q_2(u_2)q_3(u_3))\\ u_3+q_1^{-1}(q_2(u_2)q_3(u_3))\end{pmatrix}$$

is a homeomorphism on $\mathbb{R}_+^2$.

Proof. The proof is based on Proposition 6.1 of [10]. In order to use this proposition, we write g as $g(u_2,u_3)=(a_2(u_2,u_3)u_2,a_3(u_2,u_3)u_3)^\top$, with

$$a_i(u_2,u_3)=1+\frac{1}{u_i}q_1^{-1}(q_2(u_2)q_3(u_3)),\quad i=2,3.$$

Note that by Assumption (A3), the inverse of q_1 exists on $\mathbb{R}_+$ and that by hypothesis (12), a_2 and a_3 are continuous on $\mathbb{R}_+^2$. They are bounded from below, $a_i(u_2,u_3)\ge 1$ for all (u_2,u_3). Moreover, $u\mapsto u_ia_i(u)$ is increasing in each variable, $g\in C^1((\mathbb{R}_+^*)^2;(\mathbb{R}_+^*)^2)$, and the determinant of its Jacobian is strictly positive:

$$\det g'=1+(q_1^{-1})'(q_2(u_2)q_3(u_3))\big(q_2'(u_2)q_3(u_3)+q_2(u_2)q_3'(u_3)\big)\ge 1.$$

Then Proposition 6.1 in [10] shows that g is a homeomorphism on $\mathbb{R}_+^2$. □

We come back to the proof of Theorem 4. We proceed with the limit $\varepsilon \to 0$. Limit (42) and the continuity of q_1^{-1} imply that

$$u_1^\varepsilon - q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)) \to 0 \quad \text{strongly in } L^1(Q_T).$$

We deduce from (41) that

$$\begin{aligned} u_2^\varepsilon + q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)) &\to v_2 \quad \text{strongly in } L^1(Q_T), \\ u_3^\varepsilon + q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)) &\to v_3 \quad \text{strongly in } L^1(Q_T). \end{aligned}$$

Clearly, all these convergences also hold a.e. in Q_T (maybe only for a subsequence). Lemma 12 shows that g is invertible and hence,

$$(u_2^\varepsilon, u_3^\varepsilon) = g^{-1}\big(u_2^\varepsilon + q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)), u_3^\varepsilon + q_1^{-1}(q_2(u_2^\varepsilon)q_3(u_3^\varepsilon))\big).$$

We infer from the continuity of g^{-1} that

$$(u_2^\varepsilon, u_3^\varepsilon) \to g^{-1}(v_2, v_3) \quad \text{a.e. in } Q_T.$$

By (38), the convergence also holds in $L^p(Q_T)$ for all $p < 2$.

We set

$$(u_2, u_3) := g^{-1}(v_2, v_3), \quad u_1 := q_1^{-1}(q_2(u_2)q_3(u_3)).$$

The uniform integrability of $q_1(u_1^\varepsilon)$ and $q_2(u_2^\varepsilon)q_3(u_3^\varepsilon)$ from Assumption (A4) and Vitali's theorem now imply that

$$q_1(u_1^\varepsilon) \to q_1(u_1), \quad q_2(u_2^\varepsilon)q_3(u_3^\varepsilon) \to q_2(u_2)q_3(u_3) \quad \text{strongly in } L^1(Q_T).$$

This shows that (14) holds. Furthermore, the uniform integrability of $F_i(u^\varepsilon)$ from (38) and the above convergences give

$$F_i(u^\varepsilon) \to F_i(u) \quad \text{strongly in } L^1(Q_T), \quad i = 1, 2, 3.$$

We can now perform the limit $\varepsilon \to 0$ in the equations

$$\begin{aligned} \int_0^T \int_\Omega (u_1^\varepsilon + u_i^\varepsilon)\partial_t \phi_i dxdt + \int_0^T \int_\Omega (F_i(u^\varepsilon) + F_i(u^\varepsilon))\Delta\phi_i dxdt \\ = -\int_\Omega (u_1^I + u_i^I)(x)\phi_i(x,0)dx, \quad i = 2, 3, \end{aligned}$$

to conclude that $u_1 + u_2$ and $u_2 + u_3$ solve (13).

Estimate (39) and the strong convergence of (u_i^ε) in $L^1(Q_T)$ allow us to pass to the inferior limit $\varepsilon \to 0$ in (9) to conclude that (15) holds. Here, we use the weakly lower semicontinuity of the integrals as in the end of the proof of Theorem 2. This concludes the proof of Theorem 4.

Remark 13 (Example)**.** We consider $q_i(s) = s$. Then $u_1 = u_2 u_3$ and the limiting system becomes

$$\partial_t(u_2u_3 + u_2) = \Delta\big(F_1(u) + F_2(u)\big), \quad \partial_t(u_2u_3 + u_3) = \Delta\big(F_1(u) + F_3(u)\big).$$

Set $v = u_1 + u_2 = u_2 + u_2u_3$ and $w = u_1 + u_3 = u_3 + u_2u_3$. The mapping $(u_2, u_3) \mapsto (v, w)$ can be inverted explicitly:

$$\begin{aligned} u_2(v,w) &= -\frac{1}{2}(w - v + 1) + \frac{1}{2}\sqrt{(w - v + 1)^2 + 4v}, \\ u_3(v,w) &= u_2(w,v) = -\frac{1}{2}(v - w + 1) + \frac{1}{2}\sqrt{(v - w + 1)^2 + 4w}. \end{aligned}$$

It follows that $u_1 = u_2u_3 = w - u_3(v, w)$. Then we can write

$$\partial_t v = \Delta G_2(v,w), \quad \partial_t w = \Delta G_3(v,w),$$

where $G_i(v, w) = (F_1 + F_i)(w - u_3(v, w), u_2(v, w), u_3(v, w))$. This system has an entropy structure. Indeed, let

$$\begin{aligned} h_0(v,w) = (w - u_3(v,w))\big(\log(w - u_3(v,w)) - 1\big) + u_2(v,w)\big(\log u_2(v,w) - 1\big) \\ + u_3(v,w)\big(\log u_3(v,w) - 1\big). \end{aligned}$$

Then

$$\int_\Omega h_0(v(t), w(t))dx + \delta \sum_{i=1}^3 \int_0^t \int_\Omega |\nabla[J_i(u(v,w))]|^2 dxds \le \int_\Omega h_0(u^I)dx,$$

where J_i is given by (10). For example, if $f_i(s) = s$, we can compute J_i explicitly. If $s \ge s_0 := (-1 + \sqrt{5})/2$, we have $1/\sqrt{s(1+s)} \le 1$ and thus,

$$\begin{aligned} J_i(s) &= \int_0^s \min\left\{1, \frac{1}{\sqrt{y(1+y)}}\right\} dy \\ &= s_0 + \left\{\log\left(\frac{1}{2} + s + \sqrt{s + s^2}\right) - \log\left(\frac{1}{2} + s_0 + \sqrt{s_0 + s_0^2}\right)\right\} \mathbf{1}_{\{s > s_0\}}, \end{aligned}$$

where $\mathbf{1}_{\{s>s_0\}}$ is the characteristic function on $\{s > s_0\}$. Note that the entropy h_0 and its associated inequality would be difficult to find without the help of the fast-reaction limit. □

Appendix A. Auxiliary results

First, we give an example of functions which satisfy Assumptions (A1)-(A5).

Lemma 14 (Assumptions (A1)-(A5))**.** *The functions*

$$\begin{aligned} f_i(u_i) &= \alpha_i u_i + u_i^\delta, \quad q_i(u_i) = u_i^\beta, \quad i = 1,2,3, \\ f_{12}(u_1,u_2) &= \alpha u_1^\gamma u_2, \quad f_{21}(u_1,u_2) = \alpha u_1 u_2^\gamma, \end{aligned}$$

satisfy Assumptions (A1)-(A5) if

$$\beta \ge 1, \quad \gamma \ge 1, \quad \delta \ge 1 + 4\max\{\beta, \gamma - 1\}, \quad 1024\alpha^2 \le \min\{\alpha_1, \alpha_2, \delta\}.$$

Proof. Assumptions (A1)-(A3) are satisfied since β, γ, $\delta \geq 1$, Moreover, Assumption (A4) holds if $\delta > 2\beta - 1$. It remains to verify Assumption (A5). Multiply the corresponding inequality by $q_1(u_1)q_2(u_2)(1+\eta q_1(u_1))(1+\eta q_2(u_2))/(q_1'(u_1)q_2'(u_2))$ and abbreviate both sides by

$$\begin{aligned} L &:= \frac{q_1(u_1)q_2(u_2)}{q_1'(u_1)q_2'(u_2)}(1+\eta q_1(u_1))(1+\eta q_2(u_2)) \\ &\quad \times \left(\frac{q_1'(u_1)\partial_2 f_{12}(u_1,u_2)}{q_1(u_1)(1+\eta q_1(u_1))} + \frac{q_2'(u_2)\partial_1 f_{21}(u_1,u_2)}{q_2(u_2)(1+\eta q_2(u_2))}\right)^2, \\ R &:= 2(1-\delta)^2\big(f_1'(u_1)+\partial_1 f_{12}(u_1,u_2)\big)\big(f_2'(u_2)+\partial_2 f_{21}(u_1,u_2)\big). \end{aligned}$$

We have to show that $L \leq R$ for some $\delta \in (0,1)$. We choose $\delta = 1 - 1/\sqrt{2}$. First, we estimate the right-hand side R:

$$R \geq 2(1-\delta)^2(\alpha_1 + \delta u_1^{\delta-1})(\alpha_2 + \delta u_2^{\delta-1}) \geq \min\{\alpha_1,\alpha_2,\delta\}(1+u_1^{\delta-1})(1+u_2^{\delta-1}).$$

For the left-hand side L, we use $\eta \leq 1$ and the elementary inequalities $s \leq 1 + s^\beta$, $(1+s^\varepsilon)(1+s^\eta) \leq 2(1+s^{\max\{\varepsilon,\eta\}})^2$ for $s \geq 0$, $\beta \geq 1$, and ε, $\eta > 0$:

$$\begin{aligned} L &\leq \frac{q_1(u_1)q_2(u_2)}{q_1'(u_1)q_2'(u_2)}(1+q_1(u_1))(1+q_2(u_2))\left(\frac{q_1'(u_1)}{q_1(u_1)}\partial_2 f_{12} + \frac{q_2'(u_2)}{q_2(u_2)}\partial_1 f_{21}\right)^2 \\ &= \alpha^2 u_1 u_2 (1+u_1^\beta)(1+u_2^\beta)(u_1^{\gamma-1}+u_2^{\gamma-1})^2 \\ &\leq 2\alpha^2(1+u_1^\beta)^2(1+u_2^\beta)^2(u_1^{2(\gamma-1)}+u_2^{2(\gamma-1)}) \\ &\leq 8\alpha^2(1+u_1^{2\beta})(1+u_2^{2\beta})\big((1+u_1^{2(\gamma-1)})+(1+u_2^{2(\gamma-1)})\big) \\ &\leq 16\alpha^2(1+u_1^{2\max\{\beta,\gamma-1\}})^2(1+u_2^{2\beta}) + 16\alpha^2(1+u_1^{2\beta})(1+u_2^{2\max\{\beta,\gamma-1\}})^2 \\ &\leq 64\alpha^2(1+u_1^{2\max\{\beta,\gamma-1\}})^2(1+u_2^{2\max\{\beta,\gamma-1\}})^2 \\ &\leq 256\alpha^2(1+u_1^{4\max\{\beta,\gamma-1\}})(1+u_2^{4\max\{\beta,\gamma-1\}}) \\ &\leq 1024\alpha^2(1+u_1^{\delta-1})(1+u_2^{\delta-1}), \end{aligned}$$

if $4\max\{\beta,\gamma-1\} \leq \delta - 1$. Then $L \leq R$ if additionally $1024\alpha^2 \leq \min\{\alpha_1,\alpha_2,\delta\}$. Note that these conditions are far from being optimal. □

We then turn to the

Lemma 15 (F is a homeomorphism)**.** *The function $F = (F_1,F_2,F_3) : \mathbb{R}^3_+ \to \mathbb{R}^3_+$ defined in* (3) *is a C^1-diffeomorphism from $(\mathbb{R}^*_+)^3$ into itself and a homeomorphism from $\mathbb{R}^3_+$ into itself.*

Proof. We follow the strategy of [17, Section 4.2]. The proof of [17] is valid for functions $F_i(s)/s_i$ whose variables separate. Since this is not the case in our situation, we need to modify the proof.

Step 1: F is a C^1-diffeomorphism on $(\mathbb{R}^*_+)^3$. Introduce the function $\Phi = (\Phi_1,\Phi_2,\Phi_3) : \mathbb{R}^3 \to \mathbb{R}^3$, $\Phi_i(s) = \log(F_i(\exp s))$, where log and exp are defined coordinate-wise. By the

Hadamard-Lévy theorem, a C^1 self-mapping is a C^1-diffeomorphism if and only if it is proper and has no critical points. The proof that Φ is proper is exactly as in [17, Section 4.2], since here, the separability of variables is not needed. To show that Φ has no critical points, we compute the determinant

$$\det F'(s) = f_3'(s_3)\Big(\big(f_1'(s_1)+\partial_1 f_{12}(s_1,s_2)\big)\big(f_2'(s_2)+\partial_2 f_{21}(s_1,s_2)\big) - \partial_2 f_{12}(s_1,s_2)\partial_1 f_{21}(s_1,s_2)\Big) > 0,$$

which is positive because of Assumption (A5) (see Remark 1). Since both $\log : (\mathbb{R}_+^*)^3 \to \mathbb{R}^3$ and $\exp : \mathbb{R}^3 \to (\mathbb{R}_+^*)^3$ have no critical points, we conclude that also Φ has no critical points. By the Hadamard-Lévy theorem, $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1-diffeomorphism and so does $F : (\mathbb{R}_+^*)^3 \to (\mathbb{R}_+^*)^3$.

Step 2: F is bijective on $\mathbb{R}_+^3$. It remains to treat the boundary of $\mathbb{R}_+^3$. To this end, we split it as $\partial\mathbb{R}_+^3 = \{0,0,0\} \cup V$, where $V = V_1 \cup \cdots \cup V_6$ and the sets V_i are either a quarter-plane or a half-line. Since F_i can be written as the product of s_i and some nonnegative function, we have $F(V_i) \subset V_i$. We show that F is bijective on each V_i.

As the six cases are similar, we give only a proof for $V_3 = \mathbb{R}_+^* \times \mathbb{R}_+^* \times \{0\}$. The result follows when we have shown that $F \in C^1(V_3; V_3)$ is proper and has no critical points. Let $s = (s_1, s_2, 0) \in V_3$ and define the induced vector $\bar{s} = (s_1, s_2) \in (\mathbb{R}_+^*)^2$ and the induced function $\bar{F} : (\mathbb{R}_+^*)^2 \to (\mathbb{R}_+^*)^2$, $\bar{F}_i(\bar{s}) = F_i(s_1, s_2, 0)$, $i = 1, 2$. Clearly, $\bar{F}$ is proper on $(\mathbb{R}_+^*)^2$ since F is proper on $(\mathbb{R}_+^*)^3$. We prove that $\bar{F}$ is C^1 and has no critical points. The first property is clear and the second one follows from

$$\det \bar{F}'(\bar{s}) = \big(f_1'(\bar{s}_1)+\partial_1 f_{12}(\bar{s})\big)\big(f_2'(\bar{s}_2)+\partial_2 f_{21}(\bar{s})\big) - \partial_2 f_{12}(\bar{s})\partial_1 f_{21}(\bar{s}) > 0.$$

By the Hadamard-Lévy theorem, $\bar{F}$ is a C^1-diffeomorphism on $(\mathbb{R}_+^*)^2$. By construction of $\bar{F}$, this implies that F is bijective on V_3.

Step 3: F^{-1} is continuous on $\mathbb{R}_+^3$. Let $(y_n) \subset (\mathbb{R}_+^*)^3$ be such that $y_n \to y$ as $n \to \infty$. If $y \in (\mathbb{R}_+^*)^3$, then we already know that $F^{-1}(y_n) \to F^{-1}(y)$. Thus, let $y \in V$. Since F is proper, $(F^{-1}(y_n))$ is bounded and there exists a subsequence (not relabeled) such that $F^{-1}(y_n) \to \widetilde{y}$ for some $\widetilde{y}$. As F is one-to-one and continuous, we obtain $F(\widetilde{y}) = y$. Consequently, $F^{-1}(y_n) \to F^{-1}(y)$. We infer that F^{-1} is continuous on $\mathbb{R}_+^3$, which concludes the proof. □

The following result, used in Lemma 7, is proved in [10, Lemma 2.12].

Lemma 16 (Discrete duality estimate)**.** *Let $N \in \mathbb{N}$, $\rho > 0$, and $\tau > 0$ be such that $\rho\tau < 1$ and set $T := N\tau$. Let $\mu^1, \ldots, u^N$ be nonnegative integrable functions and let $u^0, \ldots, u^N$ be nonnegative bounded functions satisfying $\mu^k u^k \in H^2(\Omega)$ and, for $1 \le k \le N$,*

$$\frac{1}{\tau}(u^k - u^{k-1}) - \Delta(\mu^k u^k) \le \rho u^k \quad \text{in } \Omega, \quad \nabla(\mu^k u^k)\cdot\nu = 0 \quad \text{on } \partial\Omega.$$

Then there exists a constant $C > 0$ only depending on $\|u^0\|_{L^2(\Omega)}$, Ω, ρ, and T, such that

$$\tau \sum_{k=1}^N \int_\Omega \mu^k (u^k)^2 dx \le C\bigg(1 + \tau \sum_{k=1}^N \int_\Omega \mu^k dx\bigg).$$

Finally, we recall two useful regularity results for elliptic equations.

Lemma 17 (Theorem 2.3.3.6 in [12])**.** *Let $p \in (1,\infty)$ and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Then there exist positive constants $K_{p,\Omega}$ and $C_{p,\Omega}$ such that for all $M > K_{p,\Omega}$ and all $u \in W^{2,1}(\Omega)$ satisfying*

$$Mu - \Delta u = g \in L^p(\Omega), \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

it holds that

$$\|u\|_{W^{2,p}(\Omega)} \le C_{p,\Omega} \|g\|_{L^p(\Omega)}.$$

Lemma 18 (Lemma 6.6 in [10])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $f \in L^p(\Omega)$ with $p > d/2$, and let $u \in H^2(\Omega)$ satisfy $u \ge 0$ in Ω and*

$$-\Delta u \le f \quad \text{in } \Omega, \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega.$$

Then there exists a positive constant only depending on Ω such that

$$\|u\|_{L^\infty(\Omega)} \le C\big(\|f\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}\big).$$

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