HAL open science

Derived geometry of the first formal neighbourhood of a smooth analytic cycle

Julien Grivaux

▶ To cite this version:

Julien Grivaux. Derived geometry of the first formal neighbourhood of a smooth analytic cycle. Advances in Mathematics, 2020, 361, pp.106924 -. 10.1016/j.aim.2019.106924 . hal-03488359

HAL Id: hal-03488359

https://hal.science/hal-03488359v1

Submitted on 21 Dec 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Version of Record: https://www.sciencedirect.com/science/article/pii/S0001870819305390
Manuscript_e571a1b74835bcbf35de9e221df7b0d1

DERIVED GEOMETRY OF THE FIRST FORMAL NEIGHBOURHOOD OF A SMOOTH ANALYTIC CYCLE

JULIEN GRIVAUX

Abstract. If X is a smooth scheme of characteristic zero or a complex analytic manifold, and S is a locally split infinitesimal thickening of X, we compute explicitly the derived self-intersection of X in S.

Contents

1. Introduction 2
2. The dg-category of complexes 7
2.1. Generalities on mapping cones 7
2.2. Iterated cones for dg functors 9
2.3. Lax monoidal functors 10
3. Construction of dg-endofunctors of $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$ 11
3.1. Canonical functors 11
3.1.1. Main construction 12
3.1.2. Alternative construction 14
3.1.3. Derived invariance 15
3.2. Comparison 16
3.2.1. The octahedron triangle 16
3.2.2. Structure theorem 18
3.2.3. Derived equalizers 19
4. Square zero extensions 23
4.1. General properties 23
4.1.1. Setting 23
4.1.2. The functors $\mathrm{Tor}_B^p(*, A)$ 24
4.1.3. Principal parts 26
4.2. Local obstruction theory 28
4.2.1. Admissible complexes 28
4.2.2. The category $\mathrm{D}^{\mathrm{adm}}(B)$ 30
4.2.3. The local HKR class 32
5. Deformation theory 34
5.1. Infinitesimal thickenings 34
5.2. The global extension theorem 37
5.3. The case of a single sheaf 40
6. Structure of derived self intersections 41
6.1. Preliminar material 41
6.2. Main theorem 43
Appendix A. Multiplicativity of principal parts 48
Appendix B. Derived equalizers via model categories 49
References 50

Date: 15/11/2015.

© 2019 published by Elsevier. This manuscript is made available under the CC BY NC user license
https://creativecommons.org/licenses/by-nc/4.0/

1. Introduction

The Hochschild-Kostant-Rosenberg isomorphism, introduced in [15] for regular algebras and extended later on in a series of papers (e.g. [28], [4], [32], [18][1]) to different geometric settings, can be stated as follows:

Theorem. *If X is either a complex manifold or a smooth scheme over a field of characteristic zero, and if δ is the diagonal injection, then there is a canonical formality isomorphism*

$$\mathbb{L}\delta^*\mathcal{O}_X \simeq \bigoplus_{p=0}^{\dim X} \Omega^p_X[p]$$

in the bounded derived category of coherent sheaves on X.

This result turns out to be extremely useful in algebraic and complex geometry as well as in deformation quantization, we refer the interested reader to the non exhaustive list of papers [10], [25], [27], [12], [5], [8], [11], [22], [19], [3] as well as references therein.

After the pioneering unpublished contribution of Kashiwara [18], there has been a lot of efforts in recent years to understand more general forms of this isomorphism, corresponding to arbitrary closed immersions instead of the diagonal embedding. It started with the work of Arinkin and Căldăraru [1], and was carried on by lot of others including Calaque, Tu, Habliczek, Yu and the author (*see* [6], [2], [33], [14], [13]).

In the present paper, we won't deal with arbitrary closed immersions into an ambient smooth scheme, but rather in the corresponding first order thickening. Let $\mathbf{k}$ be a fixed base field of characteristic zero. We state the results in the algebraic setting, but all of them remain true in the analytic setting as well. One of the principal existing result in this theory is due to Arinkin and Căldăraru:

Theorem A. [1] *If X is a smooth $\mathbf{k}$-scheme and $j\colon X \hookrightarrow S$ is a first-order thickening of X by a locally free sheaf $\mathcal{I}$, then for any locally free sheaf $\mathcal{V}$ on X, the derived pullback $\mathbb{L}j^*(j_*\mathcal{V})$ is formal if and only if $\mathcal{I}$ and $\mathcal{V}$ extend to locally free sheaves on S.*

The main ingredient in the proof is the identification of three cohomology classes attached to a locally free sheaf $\mathcal{V}$ that live in the cohomology group $\mathrm{H}^2(X, \mathcal{H}om(\mathcal{V}, \mathcal{I}\otimes\mathcal{V}))$, whose construction we recall now:

(a) The sheaf of sets on X associating to any open subscheme U of X the set of locally free $\mathcal{O}_S$-extensions of $\mathcal{V}$ on U is an abelian gerbe whose automorphism sheaf is $\mathcal{H}om(\mathcal{V}, \mathcal{I}\otimes\mathcal{V})$, so it defines a class in $\mathrm{H}^2(X, \mathcal{H}om(\mathcal{V}, \mathcal{I}\otimes\mathcal{V}))$.

(b) There is a distinguished truncation triangle

$$\mathcal{I}\otimes\mathcal{V}[1] \longrightarrow \tau^{\geq -1}\,\mathbb{L}j^*(j_*\mathcal{V}) \longrightarrow \mathcal{V} \xrightarrow{+1}$$

yielding a morphism from $\mathcal{V}$ to $\mathcal{I}\otimes\mathcal{V}[2]$ in $\mathrm{D}^{\mathrm{b}}(X)$, which is the same as a class in $\mathrm{H}^2(X, \mathcal{H}om(\mathcal{V}, \mathcal{I}\otimes\mathcal{V}))$.

(c) If η is the extension class of the conormal exact sequence of the embedding j in $\mathrm{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{I})$[2], then the Yoneda product of the Atiyah class of $\mathcal{V}$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{V}, \Omega^1_X\otimes\mathcal{V})$ with η yields a class in $\mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{V}, \mathcal{I}\otimes\mathcal{V})$ which is again $\mathrm{H}^2(X, \mathcal{H}om(\mathcal{V}, \mathcal{I}\otimes\mathcal{V}))$.

[1]For a reproduction of this letter adressed to P. Schapira, *see* the book [19, Chap. 5].

[2]The class η is called the Kodaira-Spencer class in [16], it is zero exactly iff the thickening S is trivial.

The main breakthrough in Arinkin-Căldăraru's approach is the identification between the classes defined in (a) and (b). The relation between (a) and (c) has already been settled earlier on for arbitrary complexes of sheaves by Huybrechts and Thomas [16], refining previous works of Lieblich [23] and Lowen [24]. In the absolute smooth case, their result can be stated as follows:

Theorem B. [16] *Let X be a smooth $\mathbf{k}$-scheme and let $j\colon X \hookrightarrow S$ be a first-order thickening of X by a locally free sheaf. Then the essential image of*

$$\mathbb{L}j^*\colon \mathrm{D}^{\mathrm{perf}}(S) \longrightarrow \mathrm{D}^{\mathrm{perf}}(X)$$

consists of elements $\mathcal{V}_\bullet$ in $\mathrm{D}^-(X)$ such that the composition

$$\mathcal{V}_\bullet \xrightarrow{\mathrm{at}_{\mathcal{V}_\bullet}} \Omega^1_X \otimes \mathcal{V}_\bullet[1] \xrightarrow{\eta\otimes\mathrm{id}} \mathcal{I}\otimes\mathcal{V}_\bullet[2]$$

vanishes.

The present object of this paper is twofold: first we present generalizations of Theorems A and B for arbitrary sheaves on S, which are neither locally free nor push-forwards of sheaves on X. However, we want to emphasize that we don't generalize Theorem B in full generality, because we are only dealing with the case of a smooth ambient scheme in order to avoid considerations about the cotangent complex.

A crucial tool introduced in the paper is a generalization to complexes of sheaves on S of the Yoneda product of the Atiyah and Kodaira-Spencer classes: for any complex of sheaves $\mathcal{K}_\bullet$ in $\mathrm{C}^-(S)$ we define a morphism[3]

$$\Theta_{\mathcal{K}_\bullet}\colon \mathrm{Tor}^0_{\mathcal{O}_S}(\mathcal{K}_\bullet,\mathcal{O}_X) \longrightarrow \mathrm{Tor}^1_{\mathcal{O}_S}(\mathcal{K}_\bullet,\mathcal{O}_X)[2]$$

in $\mathrm{D}^-(X)$, which is the connexion morphism attached to a canonical distinguished triangle

$$\mathrm{Tor}^1_{\mathcal{O}_X}(\mathcal{K}_\bullet,\mathcal{O}_X)[1] \longrightarrow \mathrm{cone}\, j^*\{\Omega^1_S\otimes\mathcal{K}_\bullet \to \mathrm{P}^1_S(\mathcal{K}_\bullet)\} \longrightarrow j^*\mathcal{K}_\bullet \xrightarrow{+1}$$

where P^1_S is the principal parts functor.

In our setting, we replace strict perfect complexes on S by a larger class of complexes, called bounded admissible complexes: these are the bounded complexes $\mathcal{K}_\bullet$ such that the complex $\mathrm{Tor}^1_{\mathcal{O}_S}(\mathcal{K}_\bullet,\mathcal{O}_X)$ is quasi-isomorphic to zero. Up to quasi-isomorphism, bounded admissible complexes and perfect complexes have a very simple common description: a complex $\mathcal{K}_\bullet$ in $\mathrm{D}^-(X)$ is quasi-isomorphic to a bounded admissible complex (resp. is a perfect complex) if and only if $\mathbb{L}j^*\mathcal{K}_\bullet$ is cohomologically bounded (resp. is perfect). However admissible sheaves (even coherent ones) form a much larger class than locally free ones, as we can explain in a simple example: assume that D is a smooth divisor in X with trivial conormal bundle. If $\overline{D}$ is its first formal neighbourhood in X, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_D \longrightarrow \mathcal{O}_{\overline{D}} \longrightarrow \mathcal{O}_D \longrightarrow 0$$

If S is the trivial thickening of X by $\mathcal{O}_X$, we can endow $\mathcal{O}_{\overline{D}}$ with an action of $\mathcal{O}_S$: the action of the ideal $\mathcal{O}_X$ is given by

$$\mathcal{O}_X\otimes\mathcal{O}_{\overline{D}} \longrightarrow \mathcal{O}_X\otimes\mathcal{O}_D \simeq \mathcal{O}_D \longrightarrow \mathcal{O}_{\overline{D}}.$$

With this structure, $\mathcal{O}_{\overline{D}}$ is an admissible sheaf. This construction works in greater generality: for any sheaf $\mathcal{F}$ on X, and any extension class δ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{F})$, any sheaf representing

[3]The fonctors $\mathrm{Tor}^i_{\mathcal{O}_S}(*,\mathcal{O}_X)$ are not the usual hypertor functors, but simply the canonical extension to complexes of the functors $\mathrm{Tor}^i_{\mathcal{O}_S}(*,\mathcal{O}_X)\colon \mathrm{Sh}(S)\to\mathrm{Sh}(X)$.

this extension class can be endowed with a natural action of $\mathcal{O}_S$, and becomes an admissible sheaf .

Theorem 1.1. *Let X be a smooth $\mathbf{k}$-scheme and $j\colon X \hookrightarrow S$ be a first-order thickening of X by a locally free sheaf. For any bounded complex $\mathcal{K}_\bullet$ of $\mathcal{O}_S$-modules, the following properties are equivalent:*

- *The morphism $\Theta_{\mathcal{K}_\bullet}$ vanishes.*
- *The morphism $\mathbb{L}j^*\mathcal{K}_\bullet \to j^*\mathcal{K}_\bullet$ admits a right inverse in $\mathrm{D}^-(X)$.*
- *There exists a bounded admissible complex $\mathcal{L}_\bullet$ and a morphism in $\mathrm{D}^{\mathrm{b}}(S)$ from $\mathcal{L}_\bullet$ to $\mathcal{K}_\bullet$ such that the composition*
$$\mathbb{L}j^*\mathcal{L}_\bullet \longrightarrow \mathbb{L}j^*\mathcal{K}_\bullet \longrightarrow j^*\mathcal{K}_\bullet$$
is an isomorphism in $\mathrm{D}^-(X)$.

Even in the case where $\mathcal{K}_\bullet$ is the push-forward of a perfect complex on X, this gives a new and lighter proof of Theorem B. We also want to emphasize that the equivalent conditions in Theorem 1.1 do not depend only on the isomorphism class of $\mathcal{K}_\bullet$ in $\mathrm{D}^-(S)$, unlike the situation described in Theorem B. However, we can make the link with the two settings as follows: we construct a suitable localization $\mathrm{D}^{\mathrm{adm}}(S)$ of $\mathrm{C}^-(S)$, which is finer than the usual localization that gives rise to the derived category $\mathrm{D}^-(X)$, such that:

– The Tor functors $\mathrm{Tor}^i_{\mathcal{O}_S}(\,*\,,\mathcal{O}_X)\colon \mathrm{C}^-(S) \to \mathrm{C}^-(X)$ factor through triangulated functors from $\mathrm{D}^{\mathrm{adm}}(S)$ to $\mathrm{D}^-(X)$.

– The standard push forward functor $j_*\colon \mathrm{D}^-(X) \to \mathrm{D}^-(S)$ lifts to the admissible derived category $\mathrm{D}^{\mathrm{adm}}(S)$.

– The morphism Θ can be interpreted as a natural transformation in the following diagram

$$\begin{tikzcd}
\mathrm{D}^-(X) \arrow[r, "j_*"] \arrow[rr, bend left, "\mathrm{id}_{\mathrm{D}^-(X)}"] \arrow[rr, bend right, "\mathcal{I}[2]\overset{\mathbb{L}}{\otimes} *"'] & \mathrm{D}^{\mathrm{adm}}(S) \arrow[r, bend left, "\mathrm{Tor}^0_{\mathcal{O}_S}(*,\mathcal{O}_X)"] \arrow[r, bend right, "\mathrm{Tor}^1_{\mathcal{O}_S}(*,\mathcal{O}_X)[2]"'] \arrow[r, phantom, "\Downarrow\Theta"] & \mathrm{D}^-(X)
\end{tikzcd}$$

A geometric example[4] for which the morphism $\Theta_{\mathcal{V}}$ is nonzero for some line bundle $\mathcal{V}$ on X has been constructed by Arinkin and Căldăraru in [1, §4]. We can produce examples that are in some sense much worse, since the morphism $\Theta_{\mathcal{V}}$ doesn't vanish even locally. For instance, assume again that S is the trivial thickening of X by $\mathcal{O}_X$, and fix an exact sequence
$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$
of coherent sheaves on X such that the associated map
$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

[4]By "geometric" we mean that S is the first formal neighbourhood of X in some ambiant smooth scheme.

is not surjective. Arguing in an affine neighbourhood of the point in the support of the cokernel, we can assume that the map

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

is not surjective either. Any sheaf $\mathcal{V}$ corresponding to some extension class in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$, can be endowed with an action of $\mathcal{O}_S$ as follows: the square zero ideal acts by

$$\mathcal{O}_X \otimes \mathcal{V} \simeq \mathcal{V} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{V}$$

and $\mathrm{Tor}^1_{\mathcal{O}_S}(\mathcal{V},\mathcal{O}_X)$ is isomorphic to $\mathcal{N}$. Then we have an exact sequence

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{Ext}^2_{\mathcal{O}_X}(\mathcal{F},\mathcal{N})$$

where the last map is exactly $\mathcal{V} \to \Theta_{\mathcal{V}}$. As soon as $\mathcal{V}$ does not lie in the image of the first map, $\Theta_{\mathcal{V}}$ is nonzero.

We now get back to Theorem A. We give a necessary and sufficient condition for the formality of a derived pullback, as well as an intrinsic interpretation of $\Theta_{\mathcal{V}}$:

Theorem 1.2. *Let X be a smooth $\mathbf{k}$-scheme and let $j\colon X \hookrightarrow S$ be a first-order thickening of X by a locally free sheaf. If $\mathcal{V}$ is a sheaf of $\mathcal{O}_S$-modules, then $\Theta_{\mathcal{V}}$ is the connexion morphism attached to the distinguished truncation triangle*

$$\mathrm{Tor}^1_{\mathcal{O}_S}(\mathcal{V},\mathcal{O}_X)[1] \longrightarrow \tau^{\geq -1}\mathbb{L}j^*\mathcal{V} \longrightarrow j^*\mathcal{V} \xrightarrow{+1}$$

The object $\mathbb{L}j^*\mathcal{V}$ is formal in $\mathrm{D}^-(X)$ if and only $\Theta_{\mathcal{V}}$ and $\{\Theta_{\mathrm{Tor}^p_{\mathcal{O}_S}(\mathcal{V},\mathcal{O}_X)}\}_{p\geq 0}$ vanish. If $\mathcal{V}$ is the push-forward of a torsion free coherent sheaf on X, these conditions are equivalent to the vanishing of $\Theta_{\mathcal{V}}$ and $\Theta_{\mathcal{I}}$.

The morphism Θ is the key to understand more completely the functor $\mathbb{L}j^*$, which is the second and principal purpose of the paper. Let $\aleph$ be the endofunctor of $\mathrm{D}^-(X)$ defined by $\aleph(\mathcal{V}) = \mathbb{L}j^*(j_*\mathcal{V})$. The functor $\aleph$ is a locally (but in general not globally) trivial twist of the formal functor $\mathcal{V} \to \bigoplus_{p\geq 0} \mathcal{I}^{\otimes p} \otimes \mathcal{V}[p]$. We construct bounded approximations of $\aleph$ as follows: let H be the exact endofunctor of $\mathrm{C}^-(X)$ defined by

$$H(\mathcal{V}_\bullet) = \mathrm{cone}\,\{\Omega^1_X \otimes \mathcal{V}_\bullet \longrightarrow \mathrm{P}^1_X(\mathcal{V}_\bullet)\}.$$

Then H is naturally endowed with a morphism to the identity functor. For any positive integer n, we denote by $H^{[n]}$ the equalizer of the n natural maps from H^n to H^{n-1} induced by this morphism. Then we prove the following structure theorem:

Theorem 1.3. *Let X be a smooth $\mathbf{k}$-scheme and let $j\colon X \hookrightarrow S$ be a first-order thickening of X by a locally free sheaf. Then the sequence $(H^{[n]})_{n\geq 0}$ induces a projective system of lax multiplicative endofunctors of $\mathrm{D}^-(X)$, and there is a canonical multiplicative isomorphism*

$$\aleph \simeq \varprojlim_n H^{[n]}$$

which is compatible with the generalized HKR isomorphism constructed by Arinkin and Căldăraru in [1] *when S is globally trivial.*

Although some applications of our results will appear in a forthcoming paper, let us give some motivation to compute the functor $\aleph$. The first motivation comes from the work of Kapranov [17] and Markarian [25]: they construct a structure of derived Lie algebra on the shifted tangent bundle $TX[-1]$ of any complex manifold X, the derived Lie bracket being given by the Atiyah class[5]. In the case of the diagonal embedding, this derived Lie

[5]This is the geometric counterpart of Quillen's theorem [26].

structure has been studied in the framework of Lie groupoids to prove the geometric Duflo isomorphism conjectured by Kontsevich (*see* [9], [7]), and its extension to arbitrary closed embeddings is widely open and of high interest. We believe that the explicit description of $\aleph$ can lead to substantial progress on this question.

The second motivation originates from Kontsevich's homological mirror symmetry conjecture [21]: if X is a closed submanifold of a complex manifold Y, then the global Ext groups $\mathrm{Ext}^i_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_X)$ are the counterpart in the B-model of the Flœr homology groups, and are strongly related to the generalized HKR isomorphism for this closed immersion.

The last and perhaps more important motivation, that overlaps with the two previous ones, is that the object $\aleph(\mathcal{O}_X)$ is the structural sheaf of the derived fiber product $X \times^{\mathrm{h}}_S X$, this operation being performed in the category of derived algebraic schemes[6]. It is of real interest to understand what geometric information can be extracted from this derived scheme.

Let us now present the organization of the paper.

– §2 recalls well-known constructions on the category of complexes of an additive category, and its use is mainly to fix the notation and conventions.

– The entire §3 sets the categorical framework in order to find a reasonable candidate for the functor $\aleph$. In §3.1, we explain how the formal objects $\bigoplus_{p=0}^n G^p[-p]$ attached to a dg endofunctor G of $\mathrm{C}^b(\mathcal{C})$ can be twisted by a closed dg morphism $\Theta\colon \mathrm{id}_{\mathrm{C}^b(\mathcal{C})} \to G$, thus defining dg-endofunctors $(F_n)_{n\geq 0}$ of $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$. This is the content of Theorem 3.3. §3.1.2 is not strictly necessary from a logical point of view and can be skipped at first reading, but it makes perfectly clear why this construction cannot be performed neither in $\mathrm{K}^{\mathrm{b}}(\mathcal{C})$ nor in $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$. Indeed, explicit homotopies at n^{th} step are needed to construct F_{n+1} from F_n. In §3.2, we prove that the functors F_n constructed in the previous part are naturally isomorphic to the equalizers of the n natural maps from Δ_G^n to Δ_G^{n-1} induced by the morphism $\Delta_G \to \mathrm{id}$, where Δ_G is the cone of Θ shifted by minus one (Theorem 3.12).

– §4 deals with algebraic properties of modules over trivial square zero extensions of commutative $\mathbf{k}$-algebras. In §4.1, we prove a few crucial properties for such modules: if B is a trivial square zero extension of a commutative $\mathbf{k}$-algebra A and V is a B-module, then the A-module $\mathrm{Tor}^1_B(V, A)$ admits a very simple description (Corollary 4.5), and the higher Tor modules $\mathrm{Tor}^p_B(V, A), p \geq 2$ can also be explicitly computed (Proposition 4.6). The most important result we prove is the vanishing theorem for principal parts (Theorem 4.7). In §4.2.1, we introduce special classes of complexes of B-modules: admissible and n-admissible complexes. These complexes are a substitute for bounded flat resolutions or strict complexes (Proposition 4.11) and for perfect complexes if $n = +\infty$ (Corollary 4.20 and Proposition 4.22). However, n-admissible resolutions are much more easy to construct canonically than flat resolutions (Corollary 4.14 and Theorem 4.15). In §4.2.2, we define the admissible triangulated category $\mathrm{D}^{\mathrm{adm}}(S)$, which is a substitute for the derived category of perfect S-modules. Then we prove a structure theorem (Proposition 4.21) allowing to reconstruct any complex of B-modules up to an isomorphism in $\mathrm{D}^{\mathrm{adm}}(S)$ from elementary bricks that are objects and morphisms in $\mathrm{D}^-(A)$. In §4.2.3, we define the HKR morphism attached to a complexes of B-modules, and give equivalent algebraic conditions equivalent to its vanishing (Theorem 4.25), which is the local version of Theorem 1.1.

[6]For an overview of derived algebraic geometry, *see* [30]

– §5 generalizes the construction of §4 to the geometric setting. The main result is Theorem 5.6, which is a refined version of Theorem 1.1. Then we deduce Theorem 1.2, which is obtained by combining Theorem 5.7 and Corollary 5.9.

– The last section §6 is entirely devoted to the proof of Theorem 6.5, which is a refined version of Theorem 1.3.

Acknowledgments I would like to thank Richard Thomas for many useful comments, and Bertrand Toën for is invaluable help.

2. The dg-category of complexes

2.1. **Generalities on mapping cones.** Let $\mathcal{C}$ be an additive category. We introduce the following standard notation:

- The categories of complexes of elements of $\mathcal{C}$ which are arbitrary, bounded, bounded from above and bounded from below are denoted by $\mathrm{C}(\mathcal{C})$, $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$, $\mathrm{C}^{-}(\mathcal{C})$, and $\mathrm{C}^{+}(\mathcal{C})$ respectively.
- The corresponding homotopy categories are denoted by $\mathrm{K}(\mathcal{C})$, $\mathrm{K}^{\mathrm{b}}(\mathcal{C})$, $\mathrm{K}^{-}(\mathcal{C})$ and $\mathrm{K}^{+}(\mathcal{C})$.
- If $\mathcal{C}$ is abelian, the corresponding derived categories are denoted by $\mathrm{D}(\mathcal{C})$, $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$, $\mathrm{D}^{-}(\mathcal{C})$ and $\mathrm{D}^{+}(\mathcal{C})$.

The category $\mathrm{C}(\mathcal{C})$ is a k-linear dg-category: for any complexes K and L and for any integer n we have

$$\mathrm{Hom}^n_{\mathrm{dg}}(K,L) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(K^p, L^{p+n}),$$

the differential

$$\delta_n \colon \mathrm{Hom}^n_{\mathrm{dg}}(K,L) \longrightarrow \mathrm{Hom}^{n+1}_{\mathrm{dg}}(K,L)$$

being given by the formula

$$\delta_n(f) = d_L \circ f + (-1)^{n+1} f \circ d_K.$$

All three categories $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$, $\mathrm{C}^{-}(\mathcal{C})$, $\mathrm{C}^{+}(\mathcal{C})$ are dg subcategories of $\mathrm{C}(\mathcal{C})$.

For any objects complexes K and L, we use dashed arrows for morphisms in $\mathrm{Hom}^0(K,L)$, and plain arrows for morphisms in $Z^0(\mathrm{Hom}(K,L))$, that is for closed morphisms of degree zero.

For any arbitrary dg morphism $\varphi \colon K \dashrightarrow L$, we denote by $[\varphi]$ its differential considered as an element in $Z^0(\mathrm{Hom}(K, L[1]))$.

Let $f \colon K \longrightarrow L$ be a morphism of complexes of $\mathcal{C}$. We denote by

$$\begin{cases} \kappa \colon \mathrm{cone}\,(f) \dashrightarrow L \\ \sigma \colon K \dashrightarrow \mathrm{cone}\,(f)[-1] \end{cases}$$

the natural projections. The following lemma is straightforward:

Lemma 2.1.

(i) *The composition* $\mathrm{cone}\,(f) \longrightarrow K[1] \longrightarrow L[1]$ *is* $[\kappa]$.
(ii) *The composition* $K \longrightarrow L \longrightarrow \mathrm{cone}\,(f)$ *is* $[\sigma]$.

Let us now consider another morphism $T \xrightarrow{u} \operatorname{cone}(f)$ and assume that the composite map

$$T \xrightarrow{u} \operatorname{cone}(f) \xrightarrow{\pi} K[1]$$

is homotopic to zero.

Lemma 2.2. *If $\rho\colon T \dashrightarrow K$ satisfies $[\rho] = \pi \circ u$, then the map $\hat{u}$ defined by $\hat{u} = \kappa \circ u - f \circ \rho$ is a morphism of complexes, and the diagram*

$$\begin{tikzcd}
 & L \arrow[d, "j"] \\
T \arrow[ur, "\widehat{u}"] \arrow[r, "u"] \arrow[dr, "{[\rho]}"'] & \operatorname{cone}(f) \arrow[d, "\pi"] \\
 & K[1] \arrow[d, "+1"] \\
 &
\end{tikzcd}$$

commutes in the homotopy category $\mathrm{K}(\mathcal{C})$. *More precisely,* $u - j \circ \hat{u} = [\sigma \circ \rho]$.

Proof. Let us write $u = (\alpha, \beta)$. Then $\alpha = [\rho] = -d_K \circ \rho + \rho \circ d_T$ and $\beta \circ d_T = f \circ \alpha + d_L \circ \beta$. Hence

$$\begin{aligned}
\hat{u} \circ d_T &= \beta \circ d_T - f \circ \rho \circ d_T \\
&= (f \circ \alpha + d_L \circ \beta) - (f \circ d_K \circ \rho + f \circ \alpha) \\
&= d_L \circ (\beta - f \circ \rho) \\
&= d_L \circ \hat{u}.
\end{aligned}$$

Now $u - j \circ \hat{u} = (\alpha, f \circ \rho) = [\sigma \circ \rho]$ since

$$[\sigma \circ \rho] = [(\rho, 0)] = (-d_K \circ \rho, f \circ \rho) + (\rho \circ d_T, 0).$$

□

Lemma 2.3. *For any morphism $f\colon K \longrightarrow L$, we have a canonical isomorphism*

$$\operatorname{cone}\{L \longrightarrow \operatorname{cone}(f)\}[-1] \simeq K \oplus \operatorname{cone} \mathrm{id}_L[-1].$$

Proof. The complex $Z = \operatorname{cone}\{L \longrightarrow \operatorname{cone}(f)\}[-1]$ is $L \oplus K \oplus L[-1]$ with differential given by the matrix

$$d_Z = \begin{pmatrix} d_L & 0 & 0 \\ 0 & d_K & 0 \\ -\mathrm{id} & -f & -d_L \end{pmatrix}.$$

The second projection defines an epimorphism from Z to K, which admits a retraction given by $(f, -\mathrm{id}, 0)$. Hence $Z \simeq K \oplus T$ where $T = L \oplus L[-1]$ endowed with the differential

$$d_T = \begin{pmatrix} d_L & 0 \\ -\mathrm{id} & -d_L \end{pmatrix}.$$

□

2.2. **Iterated cones for dg functors.** If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two additive categories, let us recall some elementary facts:

- The objects of the category $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}_1), \mathrm{C}^{\mathrm{b}}(\mathcal{C}_2))$ are the additive functors from $\mathrm{C}^{\mathrm{b}}(\mathcal{C}_1)$ to $\mathrm{C}^{\mathrm{b}}(\mathcal{C}_2)$ that commute with shift and cones.
- The restriction morphism from $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$ to $\mathcal{C}$ obtained by assigning to any bounded complex its degree zero coefficient yields an equivalence of categories
$$\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}_1), \mathrm{C}^{\mathrm{b}}(\mathcal{C}_2)) \longrightarrow \mathrm{Fct}\,(\mathcal{C}_1, \mathrm{C}^{\mathrm{b}}(\mathcal{C}_2))$$
where on the right hand side we consider all additive functors.
- Let us denote by $\mathrm{Fct}^*_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}_1), \mathrm{C}^{\mathrm{b}}(\mathcal{C}_2))$ the subcategory of *bounded* dg-functors, that is elements of $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}_1), \mathrm{C}^{\mathrm{b}}(\mathcal{C}_2))$ corresponding to $\varinjlim_n \mathrm{Fct}_{\mathrm{add}}(\mathcal{C}_1, \mathrm{C}^{[-n,n]}(\mathcal{C}_2))$ via the above equivalence. Then there are extension functors from $\mathrm{Fct}^*_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}_1), \mathrm{C}^{\mathrm{b}}(\mathcal{C}_2))$ to $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^*(\mathcal{C}_1), \mathrm{C}^*(\mathcal{C}_2))$ for $* \in \{\varnothing, +, -\}$.

For any bounded complex
$$\cdots \xrightarrow{t^{n-2}} T^{n-1} \xrightarrow{t^{n-1}} T^n \xrightarrow{t^n} T^{n+1} \xrightarrow{t^{n+1}} \cdots$$
of objects of $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^b(\mathcal{C}_1), \mathrm{C}^b(\mathcal{C}_2))$, we define a dg functor Ψ_T as follows: for any bounded complex K we put
$$\Psi_T(K) = \bigoplus_{i\in\mathbb{Z}} T^i(K)[-i],$$
the differential being given on each factor $T^i(K)$ by
$$t^i_K + (-1)^i\, T^i(\delta_K).$$
This defines a complex since the composition of two successive differentials on the factor $T^i(K)[-i]$ is given by
$$\begin{pmatrix} (-1)^i T^i(\delta_K) & 0 \\ t^i_K & (-1)^{i+1} T^{i+1}(\delta_K) \\ 0 & t^{i+1}_K \end{pmatrix} \begin{pmatrix} (-1)^i T^i(\delta_K) \\ t^i_K \end{pmatrix}$$
which is
$$\begin{pmatrix} T^i(\delta^2_K) \\ (-1)^i (t^i_K \circ T^i(\delta_K) - T^{i+1}(\delta_K) \circ t^i_K) \\ t^{i+1}_K \circ t^i_K \end{pmatrix}$$
and all three components vanish. The functor
$$\Psi\colon \mathrm{C}^{\mathrm{b}}(\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^b(\mathcal{C}_1), \mathrm{C}^b(\mathcal{C}_2))) \longrightarrow \mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}_1), \mathrm{C}^{\mathrm{b}}(\mathcal{C}_2))$$
is a dg functor, where on the left hand side $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^b(\mathcal{C}_1), \mathrm{C}^b(\mathcal{C}_2))$ is considered as an additive category (and not its dg enhancement); we call Ψ_T the iterated cone of T.

Let us explain how iterated mapping cones can be constructed by taking successive ordinary cones. For a bounded complex T of dg functors, let $p = \max\{i \in \mathbb{Z} \text{ such that } T^{i+1} \neq 0\}$, and let T' be the complex of functors obtained by removing the last functor T^{p+1}. We define a morphism
$$\Lambda\colon \Psi_{T'} \dashrightarrow T^{p+1}[-p]$$
as follows: Λ_K is zero on all factors $T^i(K)[-i]$ for $0 \leq i \leq p-1$ and $-t^p_K$ on $T^p(K)[-p]$.

Lemma 2.4. *The morphism Λ is closed, and $\Psi_T = \operatorname{cone}\Lambda[-1]$.*

Proof. The quantity $\Lambda_K \circ \delta_{\Psi_{T'}(K)}$ obviously vanishes on all the components $T^i(K)[-i]$ if $0 \leq i \leq p-2$. On the component $T^{p-1}(K)[-(p-1)]$, it is $-t_K^p \circ t_K^{p-1}$, which is also zero. Lastly, on the component $T^p(K)[-p]$, it is $(-1)^{p+1}t_K^p \circ T^p(\delta_K)$, which is equal to $(-1)^{p+1}T^{p+1}(\delta_K) \circ t_K^p$. Since $\delta_{T^{p+1}(K)[-p]} = (-1)^p T^{p+1}(\delta_K)$, we get that

$$\delta_{T^{p+1}(K)[-p]} \circ \Lambda_K - \Lambda_K \circ \delta_{\Psi_{T'}(K)} = 0.$$

The last point is straightforward (the shift by minus one explains the minus sign in the morphism Λ). □

2.3. **Lax monoidal functors.** In this section, we recall the notion of lax monoidal functors between tensor categories. These functors form a weaker class than the usual monoidal functors (also called tensor functors), as introduced for instance in [20, §4.2].

Let $(\mathcal{S}_1, \otimes)$ and $(\mathcal{S}_2, \otimes)$ be unital tensor categories, with unit elements $\mathbf{1}_{\mathcal{S}_1}$ and $\mathbf{1}_{\mathcal{S}_2}$, and let H be an additive functor from $\mathcal{S}_1$ to $\mathcal{S}_2$. Besides, assume to be given two morphisms

$$\mathfrak{m}\colon H(\star) \otimes H(\star\star) \longrightarrow H(\star \otimes \star\star)$$
$$\mu\colon \mathbf{1}_{\mathcal{S}_2} \longrightarrow H(\mathbf{1}_{\mathcal{S}_1})$$

where in the first line $\mathfrak{m}$ is a natural transformation between functors from $\mathcal{S}_1 \times \mathcal{S}_1$ to $\mathcal{S}_2$.

Definition 2.5. The triple $(H, \mathfrak{m}, \mu)$ defines a lax monoidal functor if $\mathfrak{m}$ is associative (that is diagram [20, 4.2.2] commutes), and if the compositions

$$H(K) \simeq H(K) \otimes \mathbf{1}_{\mathcal{S}_2} \xrightarrow{\mathrm{id}\otimes\mu} H(K) \otimes H(\mathbf{1}_{\mathcal{S}_1}) \xrightarrow{\mathfrak{m}_{K,\mathbf{1}_{\mathcal{S}_1}}} H(K \otimes \mathbf{1}_{\mathcal{S}_1}) \simeq H(K)$$
$$H(K) \simeq \mathbf{1}_{\mathcal{S}_2} \otimes H(K) \xrightarrow{\mu\otimes\mathrm{id}} H(\mathbf{1}_{\mathcal{S}_1}) \otimes H(K) \xrightarrow{\mathfrak{m}_{\mathbf{1}_{\mathcal{S}_1},K}} H(\mathbf{1}_{\mathcal{S}_1} \otimes K) \simeq H(K)$$

are the identity morphisms.

Remark 2.6.

(i) If H is a lax monoidal endofunctor of $\mathcal{S}$, then $H(\mathbf{1}_{\mathcal{S}})$ is a ring object in $\mathcal{S}$ and for any element K in $\mathcal{S}$, $H(K)$ is a left and right module over this ring object.

(ii) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are additive categories, then any bounded dg functor from $\mathrm{C}^{\mathrm{b}}(\mathcal{C}_1)$ to $\mathrm{C}^{\mathrm{b}}(\mathcal{C}_2)$ having a lax monoidal structure extends naturally to a lax monoidal dg functor from $\mathrm{C}^*(\mathcal{C}_1)$ to $\mathrm{C}^*(\mathcal{C}_2)$ where $* \in \{\varnothing, +, -\}$.

Definition 2.7. Let $(H_1, \mathfrak{m}_1, \mu_1)$ and $(H_2, \mathfrak{m}_2, \mu_2)$ be two lax monoidal functors. A morphism $\varphi\colon H_1 \to H_2$ is multiplicative if the two following diagrams commute

$$\begin{tikzcd}
H_1(K) \otimes H_1(L) \arrow[r, "\mathfrak{m}_1"] \arrow[d, "\varphi_K \otimes \varphi_L"] & H_1(K \otimes L) \arrow[d, "\varphi_{K\otimes L}"] \\
H_2(K) \otimes H_2(L) \arrow[r, "\mathfrak{m}_2"] & H_2(K \otimes L)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & H_1(\mathbf{1}_{\mathcal{S}_1}) \arrow[dd, "\varphi_{\mathbf{1}_{\mathcal{S}_1}}"] \\
\mathbf{1}_{\mathcal{S}_2} \arrow[ur, "\mu_1"] \arrow[dr, "\mu_2"'] & \\
 & H_2(\mathbf{1}_{\mathcal{S}_1})
\end{tikzcd}$$

If $H_1\colon \mathcal{S}_1 \to \mathcal{S}_2$ and $H_2\colon \mathcal{S}_2 \to \mathcal{S}_3$ are two lax monoidal functors, then so is $H_1 \circ H_2$, the multiplication being given by the composition

$$H_2(H_1(K)) \otimes H_2(H_1(L)) \xrightarrow{\mathfrak{m}_2} H_2(H_1(K) \otimes H_1(L)) \xrightarrow{H_2(\mathfrak{m}_1)} H_2(H_1(K \otimes L))$$

and the unit is

$$\mathbf{1}_{\mathcal{S}_3} \xrightarrow{\mu_2} H_2(\mathbf{1}_{\mathcal{S}_2}) \xrightarrow{H_2(\mu_1)} H_2(H_1(\mathbf{1}_{\mathcal{S}_1})).$$

The category of lax monoidal endofunctors of a tensor category $\mathcal{S}$ is itself a tensor category, the tensor structure being the composition.

Let $\mathcal{C}$ be an additive category, let H be a lax monoidal endofunctor of $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$, and assume that H fits into an exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} H \xrightarrow{p} \mathrm{id}_{\mathcal{S}} \longrightarrow 0$$

where p is multiplicative.

Proposition 2.8. *The functor* cone $(N \to H)$ *is naturally a lax monoidal functor, and the natural morphism from* cone $(N \to H)$ *to* $\mathrm{id}_{\mathcal{S}}$ *is multiplicative.*

Proof. For any objects K and L of $\mathcal{C}$, we have a commutative diagram

$$\begin{tikzcd}
N(K) \otimes H(L) \oplus H(K) \otimes N(L) \arrow[r] & H(K) \otimes H(L) \arrow[r] \arrow[d, "\mathfrak{m}_{K,L}"] & K \otimes L \arrow[r] \arrow[d, equal] & 0 \\
0 \to N(K \otimes L) \arrow[r] & H(K \otimes L) \arrow[r] & K \otimes L \arrow[r] & 0
\end{tikzcd}$$

where the first horizontal arrow is $(\iota_K \otimes \mathrm{id}_{H(L)}, -\mathrm{id}_{H(K)} \otimes \iota_L)$ where the two lines are exact. Hence there is a unique morphism

$$N(K) \otimes H(L) \oplus H(K) \otimes N(L) \longrightarrow N(K \otimes L)$$

making the above diagram commutative. We define the multiplicative structure on the functor $Y =$ cone $(N \to H)$ by the composition

$$\begin{tikzcd}
N(K) \otimes N(L) \arrow[r] & N(K) \otimes H(L) \oplus H(K) \otimes N(L) \arrow[r] \arrow[d] & H(K) \otimes H(L) \arrow[d] \\
& N(K) \otimes H(L) \oplus H(K) \otimes N(L) \arrow[r] \arrow[d] & H(K) \otimes H(L) \arrow[d] \\
& N(K \otimes L) \arrow[r] & H(K \otimes L)
\end{tikzcd}$$

We leave to the reader the tedious verification that this morphism is associative. It remains to define the unit morphism of Y. For this, remark that the sequence

$$0 \longrightarrow N(\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}) \longrightarrow H(\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}) \longrightarrow \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} \longrightarrow 0$$

admits a canonical splitting given by μ. This isomorphism yields an isomorphism

$$Y(\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}) \simeq \text{cone } \{N(\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}) \to N(\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})})\} \oplus \mathrm{id}_{\mathcal{C}^{\mathrm{b}}(\mathcal{C})}.$$

and the unit of Y is defined by the inclusion on the second factor. □

3. Construction of dg-endofunctors of $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$

3.1. Canonical functors.

3.1.1. *Main construction.* Let $\mathcal{C}$ be an additive category, and assume to be given a pair (G, Θ) where G is in $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}))$ and

$$\Theta\colon \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} \longrightarrow G$$

is a dg morphism. For any nonnegative integer n we define a dg morphism

$$S_n\colon G^n \longrightarrow G^{n+1}$$

by the formula

$$S_n = \sum_{i=0}^{n} (-1)^{n+i+1} G^{n-i}\left(\Theta_{G^i}\right). \tag{1}$$

Lemma 3.1. *For any nonnegative integer n, $S_{n+1} \circ S_n = 0$.*

Proof. We have

$$S_{n+1} \circ S_n = \sum_{i=0}^{n+1} \sum_{j=0}^{n} (-1)^{i+j+1} G^{n+1-i}(\Theta_{G^i}) \circ G^{n-j}(\Theta_{G^j}). \tag{2}$$

If $i \leq j$, we can write

$$\begin{aligned} G^{n+1-i}(\Theta_{G^i}) \circ G^{n-j}(\Theta_{G^j}) &= G^{n-j}(G^{j-i+1}(\Theta_{G^i})) \circ G^{n-j}(\Theta_{G^i}) \\ &= G^{n-j}(G^{j-i+1}(\Theta_{G^i}) \circ \Theta_{G^i}). \end{aligned}$$

For any morphism $f\colon U \to V$ of complexes of $\mathcal{C}$, we have

$$G(f) \circ \Theta_U = \Theta_V \circ f.$$

We put $f = G^{j-i}(\Theta_{G^i(K)})$, where K is any bounded complex of objects in $\mathcal{C}$. This gives

$$G^{j-i+1}(\Theta_{G^i}) \circ \Theta_{G^i} = \Theta_{G^{j+1}} \circ G^{j-i}(\Theta_{G^i})$$

so that we get

$$G^{n+1-i}(\Theta_{G^i}) \circ G^{n-j}(\Theta_{G^j}) = G^{n-j}(\Theta_{G^{j+1}}) \circ G^{n-i}(\Theta_{G^i}).$$

Hence in the double sum (2), every component indexed by a couple (i, j) for $i \leq j$ cancels with the component indexed by $(j+1, i)$.

□

Definition 3.2.

(i) The functor F_n is the element in $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}))$ obtained as the iterated cone

$$\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} \xrightarrow{S_0} G \xrightarrow{S_1} G^2 \xrightarrow{S_2} \cdots \xrightarrow{S_{n-2}} G^{n-1} \xrightarrow{S_{n-1}} G^n,$$

where the functor $\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$ sits in degree zero.

(ii) The transformation $\Theta_n\colon F_n \dashrightarrow G^{n+1}[-n]$ is defined by the composition

$$(\Theta_n)_K\colon \bigoplus_{i=0}^{n} G^i(K)[-i] \longrightarrow G^n(K)[-n] \xrightarrow{-(S_n)_K} G^{n+1}(K)[-n].$$

(iii) The transformation $\rho_n\colon F_n \dashrightarrow G(F_n)[-1]$ is the map

$$(\rho_n)_K\colon \bigoplus_{i=0}^{n} G^i(K)[-i] \dashrightarrow \bigoplus_{i=1}^{n+1} G^i(K)[-i]$$

that is zero on the first factor K, and the identity morphism on the components $G^i(K)[-i]$ for $1 \leq i \leq n$.

Theorem 3.3. *The elements satisfy the following properties:*

(i) $F_0 = \mathrm{id}$ *and* $\Theta_0 = \Theta$.

(ii) Θ_n *is closed and* $F_n = \mathrm{cone}\,(\Theta_{n-1})[-1]$. *In particular there is an associated natural transformation* Π_n *from* F_n *to* F_{n-1} *and we have a cone exact sequence*

$$0 \longrightarrow G^n[-n] \xrightarrow{\tau_n} F_n \xrightarrow{\Pi_n} F_{n-1} \longrightarrow 0.$$

(iii) $\Theta_{F_n} - G(\tau_n) \circ \Theta_n = [\rho_n]$.

$$\begin{tikzcd} & G^{n+1}[-n] \arrow[d, "G(\tau_n)"] \\ F_n \arrow[ur, "\Theta_n"] \arrow[r, "\Theta_{F_n}"'] & G(F_n) \end{tikzcd}$$

(iv) *If* G *is bounded, then all* F_n *are also bounded.*

Proof. The first point is obvious, and the second point is a consequence of Lemma 2.4. Let us prove the third point.

For any bounded complex K, the differential of $G(F_n(K))[-1]$ on each factor $G^i(K)[-i]$ is equal to

$$-G((S_{i-1})_K) + (-1)^i \delta_{G^i(K)}.$$

Hence if $1 \leq i \leq n-1$, we have on $G^i(K)[-i]$

$$\delta_{G(F_n(K))[-1]} \circ (\rho_n)_K = -G((S_{i-1})_K) + (-1)^i \delta_{G^i(K)}$$

and

$$\begin{aligned} (\rho_n)_K \circ \delta_{F^n(K)} &= (\rho_n)_K \circ ((S_i)_K + (-1)^i \delta_{G^i(K)}) \\ &= (S_i)_K + (-1)^i \delta_{G^i(K)}. \end{aligned}$$

If $i = n$, we must modify the last term: on $G^n(K)[-n]$,

$$(\rho_n)_K \circ \delta_{F^n(K)} = (-1)^n \delta_{G^n(K)}.$$

Besides, by the very definition of the S_i's,

$$G(S_{i-1}) + S_i = -\Theta_{G^i}. \tag{3}$$

Thus $[\rho_n]\colon F_n \to G(F_n)$ is the diagonal morphism

$$[\rho_n]\colon \bigoplus_{i=0}^{n} G^i(K)[-i] \longrightarrow \bigoplus_{i=0}^{n} G^{i+1}(K)[-i]$$

given by the functors Θ_{G^i} for $0 \leq i \leq n-1$, and by $-G((S_{n-1})_K)$ on the last component.

We can now conclude: the morphism Θ_{F_n} is also diagonal, given by the functors Θ_{G^i} for $0 \leq i \leq n$, and the morphism $G(\tau_n) \circ \Theta_n$ is $-S_n$ on the last component and zero elsewhere. Since

$$\Theta_{G^n} - (-S_n) = -G(S_{n-1})$$

we obtain (iii). $\square$

3.1.2. *Alternative construction.* The sequence (F_n, Θ_n) is pretty natural in the homotopy category $\mathrm{K}^{\mathrm{b}}(\mathcal{C})$. If we fix a bounded complex K of objects of $\mathcal{C}$, we have a distinguished triangle

$$G^n(K)[-n] \xrightarrow{(\tau_n)_K} F_n(K) \xrightarrow{(\Pi_n)_K} F_{n-1}(K) \xrightarrow{+1}$$

in $\mathrm{K}^{\mathrm{b}}(\mathcal{C})$ and the diagram below commutes:

$$\begin{tikzcd}
 & G^{n+1}(K)[-n] \arrow[d, "G((\tau_{n-1})_K)"] \\
F_n(K) \arrow[ur, "(\Theta_n)_K"] \arrow[r, "\Theta_{F_n(K)}"'] \arrow[d, "(\Pi_n)_K"'] & G(F_n(K)) \arrow[d, "G((\Pi_n)_K)"] \\
F_{n-1}(K) \arrow[r, "\Theta_{F_{n-1}(K)}"'] & G(F_{n-1}(K)) \arrow[d, "+1"] \\
 &
\end{tikzcd}$$

This diagram suggests an inductive procedure to construct $(\Theta_n)_K$ by lifting the morphism $G((\Pi_n)_K) \circ \Theta_{F_n(K)}$ to $G^{n+1}(K)[-n]$. Unfortunately this lift is not unique. In the remaining part of this section, we explain how to perform an inductive construction at the level of complexes. Since this construction involves the homotopies ρ_n, this explains clearly why the functor F_n cannot directly be constructed on the homotopy category.

Assume that for $0 \leq i \leq n$, (F_i, Θ_i, ρ_i) have already been constructed. We define

$$F_{n+1} = \operatorname{cone} \Theta_n[-1].$$

We have two canonical maps

$$\begin{cases} \kappa_{n+1} \colon F_{n+1} \dashrightarrow G^{n+1}[-n-1] \\ \sigma_{n+1} \colon F_n \dashrightarrow F_{n+1}[-1]. \end{cases}$$

Let us consider the diagram:

$$\begin{tikzcd}
F_{n+1} \arrow[r, "\Theta_{F_{n+1}}"] \arrow[d, "\Pi_{n+1}"'] & G(F_{n+1}) \arrow[d, "G(\Pi_{n+1})"] \\
F_n \arrow[r, "\Theta_{F_n}"] \arrow[dr, "\Theta_n"'] & G(F_n) \\
 & G^{n+1}[-n] \arrow[u, "G(\tau_n)"']
\end{tikzcd}$$

On the one hand, the top square commutes because $\Theta\colon \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} \to G$ is a morphism of functors. On the other hand, thanks to Lemma 2.1 (i),

$$\Theta_n \circ \Pi_{n+1} = [\sigma_{n+1}].$$

Lastly, the bottom triangle of the diagram commutes up to $[\rho_n]$. If we define the morphism $\mu_{n+1}\colon F_{n+1} \dashrightarrow G(F_n)[-1]$ by

$$\mu_{n+1} = G(\tau_n) \circ \kappa_{n+1} + \rho_n \circ \Pi_{n+1}$$

then the composition

$$F_{n+1} \xrightarrow{\Theta_{F_{n+1}}} G(F_{n+1}) \xrightarrow{G(\Pi_{n+1})} G(F_n)$$

is exactly $[\mu_{n+1}]$. We define $\Theta_{n+1}\colon F_{n+1} \dashrightarrow G^{n+2}[-n-1]$ by

$$\Theta_{n+1} = G(\kappa_{n+1}) \circ \Theta_{F_{n+1}(K)} - G(\Theta_n) \circ \mu_{n+1}$$

and $\rho_{n+1}\colon F_{n+1} \dashrightarrow G(F_{n+1})[-1]$ by

$$\rho_{n+1} = G(\sigma_{n+1}) \circ \mu_{n+1}.$$

Thanks to Lemma 2.2, Θ_{n+1} is a closed morphism, and if we consider the diagram

$$\begin{tikzcd}
 & G^{n+2}[-n-1] \arrow[d, "G(\tau_{n+1})"'] & \\
F_{n+1} \arrow[ur, "\Theta_{n+1}"] \arrow[r, "\Theta_{F_{n+1}}"] \arrow[dr, "{[\mu_{n+1}]}"'] & G(F_{n+1}) \simeq \mathrm{cone}\,(G(\Theta_n)[-1]) \arrow[d, "G(\Pi_{n+1})"'] \arrow[u, bend right, "{[G(\kappa_{n+1})]}"'] & \\
 & G(F_n) \arrow[u, bend right, "{[G(\sigma_{n+1})]}"'] &
\end{tikzcd}$$

then $\Theta_{F_{n+1}} - G(\tau_{n+1}) \circ \Theta_{n+1} = [\rho_{n+1}]$.

3.1.3. *Derived invariance.* In this section, we study some specific properties of the functors F_n when G is exact.

Lemma 3.4. *Assume that G is an exact functor. Then all functors F_n are also exact.*

Proof. If G is exact, then so are all the functors G^n. Let us now remark that the cone of a morphism of exact endofunctors of $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$ is also exact. Hence the result follows by induction, since

$$F_{n+1}\,[1] \simeq \mathrm{cone}\ (\Theta_n\colon F_n \longrightarrow G^{n+1}[-n]).$$

□

Definition 3.5. Let G_1 and G_2 two endofunctors of $\mathcal{C}^{\mathrm{b}}(\mathcal{C})$, and let

$$\Gamma\colon G_1 \to G_2$$

be a closed dg morphism. We say that Γ is a quasi-isomorphism if for any bounded complex K of elements of $\mathcal{C}$, the morphism

$$\Gamma(K)\colon G_1(K) \to G_2(K)$$

is a quasi-isomorphism.

Remark 3.6. If $\Gamma\colon G_1 \to G_2$ is a quasi-isomorphism between exact functors, it induces a true isomorphism between the associated endofunctors of $\mathrm{D}^{\mathrm{b}}(\mathcal{C})$.

Proposition 3.7. *Let G_1 and G_2 two exact endofunctors of the category $\mathcal{C}^{\mathrm{b}}(\mathcal{C})$ endowed with morphisms $\Theta_i\colon \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} \to G_i$ for $i = 1, 2$, and let $\Gamma\colon G_1 \to G_2$ be a quasi-isomorphism such that $\Psi \circ \Theta_1 = \Theta_2$. Then for any positive integer n, Γ induces quasi-isomorphisms between F_n^1 and F_n^2.*

Proof. Since G_1 and G_2 are exact, for any positive integer n, the morphism Γ induces a quasi-isomorphism between G_1^n and G_2^n. For any positive integer n, we have a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1^n[-n] & \longrightarrow & F_n^1 & \longrightarrow & F_1^{n-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G_2^n[-n] & \longrightarrow & F_n^2 & \longrightarrow & F_{n-1}^2 & \longrightarrow & 0 \end{array}$$

and the result follows by induction. □

3.2. **Comparison.**

3.2.1. *The octahedron triangle.* For any object K in $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$, we have a diagram in $\mathrm{K}^{\mathrm{b}}(\mathcal{C})$:

$$\begin{tikzcd}
& \operatorname{cone}\Theta_{F_n(K)} \arrow[dr, thick] \arrow[dl, dashed] & \\
F_{n+1}(K)[1] \arrow[ur, thick] \arrow[d, dashed] \arrow[rr, dashed] & & G(F_{n-1}(K)) \arrow[ul, dashed] \arrow[d, dashed] \\
F_n(K) \arrow[rr, "\Theta_{F_n(K)}"] \arrow[dr] & & G(F_n(K)) \arrow[ul] \arrow[dl, dashed] \\
& G^{n+1}(K)[-n] \arrow[ur] \arrow[uul] &
\end{tikzcd}$$

where curved arrows are shifted by one. The octahedron axiom yields a triangle

$$F_{n+1}(K) \longrightarrow \operatorname{cone}\Theta_{F_n(K)}[-1] \longrightarrow G(F_{n-1}(K))[-1] \xrightarrow{+1}$$

This triangle can be lifted at the level of complexes, as shown in the following result:

Theorem 3.8. *There is a canonical exact sequence*

$$0 \longrightarrow F_{n+1} \xrightarrow{\mathfrak{p}_{n+1}} \operatorname{cone}\Theta_{F_n}[-1] \xrightarrow{\nu_n} G(F_{n-1})[-1] \longrightarrow 0$$

such that:

(i) *The map $\mathfrak{p}_{n+1}$ lifts $\Pi_{n+1}\colon F_{n+1} \longrightarrow F_n$.*

(ii) *The map $\Delta_G(\Pi_n) - \mathfrak{p}_n \circ \iota_{F_n}\colon \Delta_G(F_n) \to \Delta_G(F_{n-1})$ factors through ν_n.*

Proof. The morphism of functors $F_n \to G(F_n)$ can be represented by the diagram

$$\begin{tikzcd}
\mathrm{id} \arrow[r, "S_0"] \arrow[d, "\Theta"] & G \arrow[r, "S_1"] \arrow[d, "\Theta_G"] & \cdots \arrow[r] & G^{n-1} \arrow[r, "S_{n-1}"] \arrow[d, "\Theta_{G^{n-1}}"] & G^n \arrow[d, "\Theta_{G_n}"] \\
G \arrow[r, "G(S_0)"'] & G^2 \arrow[r, "G(S_1)"'] & \cdots \arrow[r] & G^n \arrow[r, "G(S_n)"'] & G^{n+1}
\end{tikzcd}$$

Hence cone $\Theta(F_n)[-1]$ is the iterated cone associated with the complex dg-functors

$$\mathrm{id} \xrightarrow{L^0} G \oplus G \xrightarrow{L^1} G^2 \oplus G^2 \xrightarrow{L^2} \cdots \xrightarrow{L^{n-1}} G^n \oplus G^n \xrightarrow{\mathrm{pr}_2 \circ L^n} G^{n+1}$$

where

$$\begin{cases} L^0 = \begin{pmatrix} \Theta \\ -\Theta \end{pmatrix} \\ L^i = \begin{pmatrix} S_i & 0 \\ -\Theta_{G^i} & -G(S_{i-1}) \end{pmatrix} \quad \text{for } 1 \leq i \leq n-1. \end{cases}$$

Now remark that by (3), we have an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & G^i \arrow[r] \arrow[d, "S_i"] & G^i \oplus G^i \arrow[r] \arrow[d, "L^i"] & G^i \arrow[r] \arrow[d, "G(S_{i-1})"] & 0 \\
0 \arrow[r] & G^{i+1} \arrow[r] & G^{i+1} \oplus G^{i+1} \arrow[r] & G^{i+1} \arrow[r] & 0
\end{tikzcd}$$

where the map $G^i \to G^i \oplus G^i$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the one from $G^i \oplus G^i$ to G^i is $((-1)^i, (-1)^{i+1})$. This gives the required exact sequence. Let us now prove the two remaining statements in the Theorem. Point (i) is obvious. For point (ii), the morphism

$$\Delta_G(\Pi_n) - \mathfrak{p}_{n-1} \circ \iota_{F_n} \colon \Delta_G(F_n) \longrightarrow \Delta_G(F_{n-1})$$

can we written as

$$\begin{tikzcd}
\mathrm{id} \arrow[r, "L^0"] \arrow[d, "\mu_0"] & G \oplus G \arrow[r, "L^1"] \arrow[d, "\mu_1"] & G^2 \oplus G^2 \arrow[r, "L^2"] \arrow[d, "\mu_2"] & \cdots \arrow[r, "L^{n-1}"] & G^n \oplus G^n \arrow[r, "L^n"] \arrow[d, "\mu_n"] & G^{n+1} \\
\mathrm{id} \arrow[r, "L^0"] & G \oplus G \arrow[r, "L^1"] & G^2 \oplus G^2 \arrow[r, "L^2"] & \cdots \arrow[r, "\mathrm{pr}_2 \circ L^{n-1}"] & G^n
\end{tikzcd}$$

where

$$\mu_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

if $0 \leq i \leq n-1$ and $\mu_n = \mathrm{pr}_2$. Hence $\Delta_G(\Pi_{n-1}) - \mathfrak{p}_n \circ \iota_{F_n}$ is obtained as the composition

$$\Delta_G(F_n) \xrightarrow{\nu_n} G(F_{n-1})[-1] \hookrightarrow \Delta_G(F_{n-1}).$$

This finishes the proof. □

Corollary 3.9. *The map* $\mathfrak{p}_{n+1} \colon F_{n+1} \to \Delta_G(F_n)$ *is the equalizer of the two morphisms*

$$\begin{cases} \Delta_G(F_n) \xrightarrow{\Delta_G(\Pi_n)} \Delta_G(F_{n-1}) \\ \Delta_G(F_n) \xrightarrow{\iota_{F_n}} F_n \xrightarrow{\mathfrak{p}_{n-1}} \Delta_G(F_{n-1}). \end{cases}$$

3.2.2. *Structure theorem.*

Definition 3.10. Let us fix a pair (G, Θ) as before.

(i) The functor Δ_G is the object in $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}))$ defined by
$$\Delta_G = \mathrm{cone}\,\Theta[-1],$$
it is equipped with a natural left inverse $\iota\colon \Delta_G \to \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$ (so it is a faithful functor).

(ii) The morphism $\mathfrak{p}_n\colon F_n \hookrightarrow \Delta_G(F_{n-1})$ is the lift of Π_n given by Proposition 3.8.

(iii) The monomorphism $\mathrm{j}_n\colon F_n \hookrightarrow \Delta_G^n$ is
$$\mathrm{j}_n\colon F_n \hookrightarrow \Delta_G \circ F_{n-1} \hookrightarrow \cdots \hookrightarrow \Delta_G^{n-1} \circ F_1 \hookrightarrow \Delta_G^n$$

(iv) The maps $(\pi_{n,i})_{1\leq i\leq n}$ are the n natural projections from Δ_G^n to Δ_G^{n-1} induced by the map $\Delta_G \to \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$.

(v) The functor $\Delta_G^{[n]}$ is the equalizer of the n maps $\pi_{n,i}$.

Lemma 3.11. *For any integer n, the map j_n factors through the functor $\Delta_G^{[n]}$, and*
$$(\mathrm{j}_n)_{n\geq 0}\colon (F_n)_{n\geq 0} \longrightarrow (\Delta_G^{[n]})_{n\geq 0}$$
is a morphism of projective systems.

Proof. We proceed by induction. The morphism j_{n+1} can be written as
$$F_{n+1} \xhookrightarrow{\mathfrak{p}_{n+1}} \Delta_G(F_n) \xhookrightarrow{\Delta_G(\mathrm{j}_n)} \Delta_G^{n+1}. \tag{4}$$
Let us consider the following commutative diagram.
$$\begin{tikzcd} F_{n+1} \arrow[r, hook, "\mathfrak{p}_{n+1}"] & \Delta_G(F_n) \arrow[r, hook, "\Delta_G(\mathrm{j}_n)"] \arrow[d, "\iota_{F_n}"] & \Delta_G^{n+1} \arrow[d, "\iota_{\Delta_G^n}"] \\ & F_n \arrow[r, hook, "\mathrm{j}_n"] & \Delta_G^n \end{tikzcd} \tag{5}$$
Since $\iota_{\Delta_G^n} = \pi_{n+1,1}$, the morphism
$$F_{n+1} \xhookrightarrow{\mathrm{j}_{n+1}} \Delta_G^{n+1} \xrightarrow{\pi_{n+1,1}} \Delta_G^n$$
is equal to
$$F_{n+1} \xrightarrow{\Pi_{n+1}} F_n \xhookrightarrow{\mathrm{j}_n} \Delta_G^n.$$
We have $\iota_{\Delta_G^n} = \Delta_G(\pi_{n,1})$, so that the compositions of the up horizontal and right down arrows of the square in diagram (5) is $\Delta_G(\pi_{n,1} \circ \mathrm{j}_n)$. By induction, for any integer i with $1 \leq i \leq n$,
$$\Delta_G(\pi_{n,1} \circ \mathrm{j}_n) = \Delta_G(\pi_{n,i} \circ \mathrm{j}_n) = \pi_{n+1,i+1} \circ \Delta_G(\mathrm{j}_n),$$
so that the morphisms
$$F_{n+1} \xhookrightarrow{\mathrm{j}_{n+1}} \Delta_G^{n+1} \xrightarrow{\pi_{n+1,i+1}} \Delta_G^n$$
are all equal to
$$F_{n+1} \xrightarrow{\Pi_{n+1}} F_n \xhookrightarrow{\mathrm{j}_n} \Delta_G^n.$$
This finishes the proof. □

Theorem 3.12. *The sequence of morphisms*

$$(\mathfrak{j}_n)_{n\geq 0}\colon (F_n)_{n\geq 0} \longrightarrow (\Delta_G^{[n]})_{n\geq 0}$$

defines an isomorphism of projective systems of dg-endofunctors of $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$.

Proof. We argue by induction. As Δ_G is faithful, the equalizer of the n maps

$$\pi_{n+1,i}\colon \Delta_G^{n+1} \longrightarrow \Delta_G^{n} \qquad\qquad 2\leq i\leq n+1$$

is

$$\Delta_G(\mathfrak{j}_n)\colon \Delta_G(F_n) \hookrightarrow \Delta_G^{n+1}$$

If $(\mathfrak{D},\chi)$ is the equalizer of the $(n+1)$ maps $\pi_{n+1,i}$ for $1\leq i\leq n$, then χ factors through $\Delta_G(F_n)$ as shown below

$$\begin{tikzcd}
\mathfrak{D} \arrow[r, hook, "\chi"] \arrow[d, hook, "\widetilde{\chi}"] & \Delta_G^{n+1} \\
\Delta_G(F_n) \arrow[ur, hook] &
\end{tikzcd}$$

and $(\mathfrak{D},\widetilde{\chi})$ is the equalizer of the two maps

$$\begin{cases} \Delta_G(F_n) \hookrightarrow \Delta_G^{n+1} \xrightarrow{\pi_{n+1,i}} \Delta_G^{n} \\ \Delta_G(F_n) \hookrightarrow \Delta_G^{n+1} \xrightarrow{\pi_{n+1,1}} \Delta_G^{n} \end{cases}$$

where in the first morphism, i is any integer such that $2\leq i\leq n+1$ (the corresponding morphism doesn't depend on i). These morphisms can be written as

$$\begin{cases} \Delta_G(F_n) \xrightarrow{\Delta_G(\Pi_n)} \Delta_G(F_{n-1}) \xhookrightarrow{\Delta_G(\mathfrak{j}_{n-1})} \Delta_G^{n} \\ \Delta_G(F_n) \xrightarrow{\iota_{F_n}} F_n \xhookrightarrow{\mathfrak{p}_n} \Delta_G(F_{n-1}) \xhookrightarrow{\Delta_G(\mathfrak{j}_{n-1})} \Delta_G^{n} \end{cases}$$

Since the equalizer remains unchanged after post-composition with a monomorphism, it is isomorphic to $(F_{n+1},\mathfrak{p}_{n+1})$ thanks to Lemma 3.9. This completes the induction step. □

3.2.3. *Derived equalizers.* Let H be an element of $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}))$ endowed with a closed dg morphism $\Psi\colon H \to \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$. For any positive integer n, we denote by $(\pi_{n,i})_{1\leq i\leq n}$ the n natural projections from H^n to H^{n-1} induced by Ψ.

Definition 3.13. If (H,Ψ) is given, let n be a positive integer..

(i) The nth (standard) equalizer of (H,Ψ), denoted by $H^{[n]}$, is the equalizer of the n maps $(\pi_{n,i})_{1\leq i\leq n}$.

(ii) If $\widetilde{H}$ denotes the functor cone Ψ and $\Theta\colon \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} \to \widetilde{H}$ is the associated morphism, the nth derived equalizer of (H,Ψ), denoted by $H^{[[n]]}$, is the functor $\Delta_{\widetilde{H}}^{[n]}$ (*see* Definition 3.10).

Remark 3.14. The word "derived" in the name "derived equalizer" refers to the formalism of Quillen derived functors between model categories. Although not strictly necessary, this approach is explained in Appendix B and sheds light on many considerations about derived equalizers.

Let U denote the element of the center of $\mathrm{Fct}_{\mathrm{dg}}(\mathcal{C})$ defined by

$$U = \mathrm{cone}\left(\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} \longrightarrow \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}\right). \tag{6}$$

and let $\iota\colon U[-1] \to \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$ be the corresponding natural morphism. Thanks to Lemma 2.3, there is an isomorphism

$$\Delta_{\widetilde{H}} \simeq H \oplus U[-1] \tag{7}$$

such that the diagram

$$\begin{tikzcd} \Delta_{\widetilde{H}} \arrow[rr, "\sim"] \arrow[dr] & & H \oplus U[-1] \arrow[dl, "\Psi \oplus \iota"] \\ & \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} & \end{tikzcd}$$

commutes. Hence there is a natural morphism of projective systems

$$(H^{[n]})_{n\geq 0} \longrightarrow (H^{[[n]]})_{n\geq 0}.$$

Let us give a few properties:

Proposition 3.15.

- *If H is an exact dg endofunctor of $\mathcal{C}^{\mathrm{b}}(\mathcal{C})$ endowed with a morphism from H to $\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$, then all functors $H^{[[n]]}$ are also exact.*
- *Assume that $\mathcal{S}$ is a tensor category, and let H be a lax monoidal dg endofunctor of $\mathcal{C}^{\mathrm{b}}(\mathcal{S})$ endowed with a multiplicative morphism from H to $\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{S})}$. Then $(H^{[n]})_{n\geq 0}$ and $(H^{[[n]]})_{n\geq 0}$ form a projective system of lax monoidal functors, and the natural morphisms from $H^{[n]}$ to $H^{[[n]]}$ are multiplicative.*
- *Let H_1 and H_2 be two exact dg endofunctors of the category $\mathcal{C}^{\mathrm{b}}(\mathcal{C})$ endowed with morphisms $\Psi_i\colon H_i \to \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$ for $i = 1, 2$, and let $\Gamma\colon H_1 \to H_2$ be a quasi-isomorphism such that $\Psi_2 \circ \Gamma = \Psi_1$. Then for any positive integer n, Γ induces quasi-isomorphisms between $H_1^{[[n]]}$ and $H_2^{[[n]]}$.*

Proof. The first and third point follow directly by induction using the exact sequence provided in Theorem 3.3 (ii). For the second point, the multiplicaty of $H^{[n]}$ is straightforward. It also implies the multiplicativity of $H^{[[n]]}$, as we shall see now. First we remark that for any bounded complexes K and L of elements of $\mathcal{S}$, there is a natural morphism

$$U(K) \otimes U(L) \longrightarrow U(K \otimes L)[1]$$

given by the morphism

$$\begin{tikzcd} K \otimes L \arrow[r, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] \arrow[d, equal] & K \otimes L \oplus K \otimes L \arrow[r, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] & K \otimes L \\ K \otimes L \arrow[r] & K \otimes L & \end{tikzcd}$$

Now we can endow $H \oplus U[-1]$ with a lax monoidal structure as follows: we define the multiplicative morphism as a matrix of the type

$$\begin{array}{r c} & \begin{array}{cccc} H(K)\otimes H(L) & H(K)\otimes U(L)[-1] & U(K)\otimes H(L)[-1] & U(K)\otimes U(L)[-2] \end{array} \\ \begin{array}{r} H(K\otimes L) \\ U(K\otimes L)[-1] \end{array} & \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & * \end{pmatrix} \end{array}$$

whose components are:

- the morphism $H(K) \otimes H(L) \to H(K \otimes L)$ provided by the lax monoidal structure of H,
- the morphism $H(K) \otimes U(L) \to K \otimes U(L) \xrightarrow{\sim} U(K \otimes L)$,
- the morphism $U(K) \otimes H(L) \to U(K) \otimes L \xrightarrow{\sim} U(K \otimes L)$,
- the morphism $U(K) \otimes U(L) \to U(K \otimes L)[1]$ formerly introduced.

The unit of $\Delta_{\widetilde{H}}$ is defined by the composition

$$\mathbf{1}_{\mathcal{S}} \longrightarrow H(\mathbf{1}_{\mathcal{S}}) \longrightarrow \Delta_{\widetilde{H}}(\mathbf{1}_{\mathcal{S}}).$$

Hence $H^{[[n]]}$ are also lax monoidal functors, and the compatibility of the multiplicative structures follows from the fact that the natural morphism from H to $\Delta_{\widetilde{H}}$ is multiplicative. $\square$

Let us assume to be given three elements A, B and N of $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}))$ fitting in an exact sequence

$$0 \longrightarrow N \longrightarrow A \longrightarrow B \xrightarrow{p} \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})} \longrightarrow 0.$$

and assume that A is right exact. Let

$$H = \text{cone } (A \longrightarrow B).$$

Then H is endowed with a natural morphism $\Psi\colon H \to \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$. Let us now consider the morphism

$$B(p) - p_B\colon B^2 \longrightarrow B.$$

Its image lies in A/N. The composition

$$A \circ B \longrightarrow B^2 \xrightarrow{B(p)-p_B} B$$

is the morphism

$$A \circ B \xrightarrow{A(p)} A \longrightarrow B.$$

Since A is left exact, this proves that the image of $B(p) - p_B$ is exactly A/N. Hence the morphism

$$H(\Psi) - \Psi_H\colon H^2 \longrightarrow H.$$

factors by a surjective morphism

$$\nu\colon H^2 \longrightarrow Z$$

where

$$Z = \text{cone } (A \longrightarrow A/N).$$

Lemma 3.16. *For any positive integer n, the morphisms*

$$\begin{cases} H^{[n]} \to H^{[n-1]} \\ H \circ H^{[n]} \longrightarrow H^2 \circ H^{[n-1]} \xrightarrow{\nu_{H^{[n-1]}}} Z \circ H^{[n-1]} \end{cases}$$

are onto. In particular there is a natural exact sequence

$$0 \longrightarrow H^{[n+1]} \longrightarrow H(H^{[n]}) \longrightarrow Z(H^{[n-1]}) \longrightarrow 0.$$

Proof. We argue by induction. There is a natural morphism $Z \to H$ given by the morphism

$$\begin{array}{ccc} A & \longrightarrow & A/N \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

Besides, the composition

$$Z \circ H \longrightarrow H^2 \xrightarrow{\nu} Z$$

is induced by the morphism $H \longrightarrow \mathrm{id}_{\mathrm{C^b}(\mathcal{C})}$. Thus the composition

$$Z \circ H^{[n-1]} \longrightarrow H \circ H^{[n-1]} \longrightarrow H^2 \circ H^{[n-2]} \xrightarrow{\nu_{H^{[n-2]}}} Z \circ H^{[n-2]}$$

is induced by the natural morphism $H^{[n-1]} \to H^{[n-2]}$ which is onto by induction. Let us now consider the diagram

$$\begin{tikzcd} & & Z(H^{[n-1]}) \arrow[d] \arrow[dr, "\alpha"] & \\ 0 \arrow[r] & H^{[n]} \arrow[r] \arrow[dr, "\gamma"'] & H(H^{[n-1]}) \arrow[r, "\beta"'] \arrow[d] & Z(H^{[n-2]}) \\ & & H^{[n-1]} \arrow[d] & \\ & & 0 & \end{tikzcd}$$

As α is onto, so are β and γ. This completes the induction step. □

Theorem 3.17. *Assume to be given an exact sequence*

$$A \longrightarrow B \xrightarrow{p} \mathrm{id}_{\mathrm{C^b}(\mathcal{C})} \longrightarrow 0$$

in $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C^b}(\mathcal{C}))$*, and let* $H = \mathrm{cone}\ (A \to B)$.

(i) *If A is right exact and if* $\ker\ (A \to B) = \{0\}$*, for any positive integer n, there is an exact sequence*[7]

$$0 \longrightarrow H^{[n+1]} \longrightarrow H(H^{[n]}) \longrightarrow U \circ A\,(H^{[n-1]}) \longrightarrow 0.$$

(ii) *If A and B are exact, for any nonnegative integer n, the functors $H^{[n]}$ and $H^{[[n]]}$ are exact, and the morphism*

$$H^{[n]} \longrightarrow H^{[[n]]}$$

is an isomorphism of endofunctors of $\mathrm{D^b}(\mathcal{C})$.

[7]The fonctor U is defined by (6).

Proof. The first point results directly from Lemma 3.16 and the fact that $Z = U \circ A$. For the second point, the functors $H^{[[n]]}$ are exact thanks to Proposition 3.15. Besides, since Z is exact, the exact sequence

$$0 \longrightarrow H^{[n+1]} \longrightarrow H(H^{[n]}) \longrightarrow Z(H^{[n-1]}) \longrightarrow 0$$

provided by Lemma 3.16 and the nine lemma show by induction that the functors $H^{[n]}$ are also exact. The functor $\widetilde{H}$ is the cone of the complex of functors $A \to B \to \mathrm{id}_{\mathrm{C^b}(\mathcal{C})}$. Hence there is a morphism $Z \to \widetilde{H}[-1]$ given by

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A/N \arrow[d] & \\ A \arrow[r] & B \arrow[r] & \mathrm{id}_{\mathrm{C^b}(\mathcal{C})} \end{tikzcd}$$

which is an isomorphisms of endofunctors of $\mathrm{D^b}(\mathcal{C})$. Let us consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & H^{[n+1]} \arrow[r] \arrow[d] & H(H^{[n]}) \arrow[r] \arrow[d] & Z(H^{[n-1]}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^{[[n+1]]} \arrow[r] & \Delta_{\widetilde{H}}(H^{[[n]]}) \arrow[r] & \widetilde{H}(H^{[[n-1]]})[-1] \arrow[r] & 0 \end{tikzcd}$$

where the first line is exact thanks to Lemma 3.16, and the second line is also exact thanks to Theorem 3.8. It implies directly the required result by induction. □

Lastly, we give a more simple situation where standard and derived equalizers are quasi-isomorphic:

Proposition 3.18. *Let L be an element of $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C^b}(\mathcal{C}))$, and define a couple (H, Ψ) as follows: $H = L \oplus \mathrm{id}_{\mathrm{C^b}}(\mathcal{C})$ and Ψ is the second projection. Then for any nonnegative integer n, the map from $H^{[n]}$ to $H^{[[n]]}$ is a quasi-isomorphism, and $H^{[n]}$ is isomorphic to the functor $\bigoplus_{p=0}^{n} L^p$.*

Proof. Left to the reader. □

4. Square zero extensions

4.1. General properties.

4.1.1. *Setting.* Let A be an algebra over a field k of characteristic zero. If no ring is specified, tensor product will always be taken over A.

Let I be a free A-module of finite rank, and let B be a k-extension of A by I, i.e. we have an exact sequence

$$0 \longrightarrow I \longrightarrow B \xrightarrow{\pi} A \longrightarrow 0 \tag{8}$$

of k-algebras, where $I^2 = 0$. We will always assume that B is trivial (as a $\mathbf{k}$-extension), which means that (8) splits. Hence B is isomorphic to the trivial extension $I \oplus A$ as a $\mathbf{k}$-vector space, the ring structure being given by

$$(i, a).(i', a') = (ia' + ai', aa').$$

Splittings of the sequence (8) are in one to one correspondence with injective $\mathbf{k}$-algebra morphisms $\sigma\colon A \hookrightarrow B$. These morphisms form an affine space over the module $\mathrm{Der}\,(A, I)$ of I-values derivations of A.

Modules over B admit a simple description, that we give now. Let M and N be two A-modules.

(i) Any extension V in $\mathrm{Ext}^1_B(M,N)$ yields a multiplication map

$$\mu_V : I \otimes M \longrightarrow N$$

given by $\mu_V(i \otimes m) = iv$ where v is any lift of m in V.

(ii) If Z and μ are in $\mathrm{Ext}^1_A(M,N)$ and $\mathrm{Hom}_A(I \otimes M, N)$ respectively, there is an associated extension Z_μ in $\mathrm{Ext}^1_B(M,N)$ defined via the choice of a splitting of (8) as follows: as an A-module $Z_\mu = Z$, and the action of I is given by the composition

$$I \otimes Z \twoheadrightarrow I \otimes M \xrightarrow{\mu} N \hookrightarrow Z.$$

Although Z_μ depends of σ, its isomorphism class doesn't.

Lemma 4.1. *The map*

$$\mathrm{Ext}^1_B(M,N) \xrightarrow{\sim} \mathrm{Ext}^1_A(M,N) \oplus \mathrm{Hom}_A(I \otimes M, N)$$

where the second component is $V \to \mu_V$, is a group isomorphism. Its inverse is given by $(Z,\mu) \to Z_\mu$.

The proof is straightforward, we leave it to the reader. It follows from this description that a B-module is simply given by two A-modules M and N, an extension class in $\mathrm{Ext}^1_A(M,N)$, and a *surjective* A-linear morphism from $I \otimes M$ to N.

Lastly, let us present two useful base change operations for B-modules. Let V be a B-module, and put $M = V \otimes_B A$ and $N = IV$. Assume to be given two A-modules U and V and two *surjective* A-linear morphisms $u : U \to M$ and $v \colon N \to V$. Define the B-modules V' and V'' by the cartesian diagrams

$$\begin{array}{ccc} V' & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & M \end{array} \quad \text{and} \quad \begin{array}{ccc} N & \longrightarrow & IV \\ \downarrow & & \downarrow \\ V & \longrightarrow & V'' \end{array}$$

Lemma 4.2. *There are isomorphisms*

$$V' \otimes_B A \simeq U,\ IV' \simeq N,\ V'' \simeq M,\ IV'' \simeq V$$

such that the compositions

$$\begin{cases} U \simeq V' \otimes_B A \longrightarrow M \\ N \longrightarrow IV'' \simeq V \end{cases}$$

are equal to u and v respectively. Besides, the multiplication morphisms of V' and V'' are given by the composition

$$\begin{cases} \mu_{V'} : I \otimes (V' \otimes_B A) \longrightarrow I \otimes M \xrightarrow{\mu_V} N \simeq IV' \\ \mu_{V''} : I \otimes (V' \otimes_B A) \simeq I \otimes M \xrightarrow{\mu_V} N \longrightarrow IV'' \end{cases}$$

Proof. Left to the reader. □

4.1.2. *The functors* $\mathrm{Tor}^p_B(\,*\,, A)$.

Lemma 4.3. *Let M be an A-module. Then $\mathrm{Tor}^1_B(M,A)$ is canonically isomorphic to $I \otimes M$.*

Proof. We can assume that $M = A$, and the lemma follows directly from (8). □

Proposition 4.4. *Let V be in $\mathrm{Ext}^1_B(M,N)$. Then the connection morphism*

$$I \otimes M \simeq \mathrm{Tor}^1_B(M,A) \to \mathrm{Tor}^0_B(N,A) \simeq N$$

is exactly μ_V, and there is an exact sequence

$$0 \longrightarrow \mathrm{Tor}^1_B(V,A) \longrightarrow I \otimes M \xrightarrow{\mu_V} N$$

Proof. Let S be the kernel of the natural morphism from σ^*V ot V induced by the identity of V via the isomorphism

$$\mathrm{Hom}_A(V,V) \simeq \mathrm{Hom}_B(\sigma^*V,V).$$

We consider the diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \otimes N \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & I \otimes M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \sigma^*N \arrow[r] \arrow[d] & \sigma^*V \arrow[r] \arrow[d] & \sigma^*M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

For any A-module P and any positive integer i the module $\mathrm{Tor}^i_B(\sigma^*P,A)$ vanishes. Hence the Tor exact sequence

$$\mathrm{Tor}^1_B(V,A) \longrightarrow \mathrm{Tor}^1_B(M,A) \longrightarrow \mathrm{Tor}^0_B(N,A)$$

can be identified with the exact sequence

$$\ker\,(S \otimes_B A \longrightarrow V) \longrightarrow I \otimes M \longrightarrow N$$

obtained by the snake lemma. By diagram chase, we get the first point of the proposition. Now

$$S = \{(i \otimes v, v') \in \sigma^*V \text{ such that } iv + v' = 0\}$$

so that

$$IS = \{(i \otimes v, 0) \in \sigma^*V \text{ such that } v \in IV\}$$

and we get

$$S \otimes_B A = \{(i \otimes m, v') \in I \otimes M \oplus V \text{ such that } \mu_V(i \otimes m) + v' = 0\}.$$

Hence $\ker\,(S \otimes_B A \longrightarrow V)$ is the kernel of μ_V, and embeds in $I \otimes M$. □

Corollary 4.5. *Let V be a B-module, and let $M = V \otimes_B A$. Then the map*

$$\mathrm{Tor}^1_B(V,A) \longrightarrow \mathrm{Tor}^1_B(M,A) \simeq I \otimes M$$

is injective, and its image is $\ker \mu_V$.

Proof. The B-module V defines a canonical class in $\mathrm{Ext}^1_B(M,IV)$. Hence the result follows from Proposition 4.4. □

Proposition 4.6. *Let V be a B-module, and let $M = V \otimes_B A$. Then for every positive integer p, there is a functorial isomorphism*

$$\mathrm{Tor}_B^p(V, A) \simeq I^{\otimes(p-1)} \otimes \mathrm{Tor}_B^1(V, A).$$

Proof. We prove the result by induction on p. Let us consider the module S introduced in the proof of Proposition 4.4. We have an exact sequence

$$0 \longrightarrow S \longrightarrow \sigma^* V \longrightarrow V \longrightarrow 0 \tag{9}$$

so that $\mathrm{Tor}_B^{p+1}(V, A)$ is isomorphic to $\mathrm{Tor}_B^p(S, A)$. The module $S \otimes_B A$ is isomorphic to $I \otimes M$ and the multiplication map

$$\mu_S \colon I \otimes (S \otimes_B A) \to IS$$

is given via this isomorphism by

$$i' \otimes (i \otimes m) \longrightarrow (-i' \otimes \mu_V(i \otimes m), 0).$$

Hence

$$\mathrm{Tor}_B^1(S, A) \simeq \ker \mu_S \simeq I \otimes \ker \mu_V \simeq I \otimes \mathrm{Tor}_B^1(V, A)$$

and

$$\begin{aligned} \mathrm{Tor}_B^{p+1}(V, A) \simeq \mathrm{Tor}_B^p(S, A) &\simeq I^{\otimes(p-1)} \otimes \mathrm{Tor}_B^1(S, A) \\ &\simeq I^{\otimes p} \otimes \mathrm{Tor}_B^1(V, A). \end{aligned}$$

□

4.1.3. *Principal parts.* The B-module Ω_B^1 fits into a natural (split) exact sequence

$$0 \longrightarrow I \longrightarrow \Omega_B^1 \otimes_B A \xrightarrow{p} \Omega_A^1 \longrightarrow 0$$

which is the conormal sequence associated with the map $B \to A$. We put $E = \Omega_B^1 \otimes_B A$, E is canonically isomorphic to $I \oplus \Omega_A^1$ after a choice of a splitting σ of (8).

Recall that for any module M over a commutative $\mathbf{k}$-algebra R, the module of principal parts $\mathrm{P}_R^1(M)$ is the R-module defined (as a $\mathbf{k}$-vector space) by

$$\mathrm{P}_R^1(M) = \Omega_R^1 \otimes_R M \oplus M$$

where R acts by

$$r(\omega \otimes m, m') = (r\omega \otimes m + dr \otimes m', rm).$$

The main result we prove is:

Theorem 4.7. *Let V be a B-module. Then the map*

$$\mathrm{Tor}_B^1(\mathrm{P}_B^1(V), A) \longrightarrow \mathrm{Tor}_B^1(V, A)$$

vanishes. More precisely, the connexion morphism

$$\ker \mu_V \simeq \mathrm{Tor}_B^1(V, A) \longrightarrow \mathrm{Tor}_B^0(\Omega_B^1 \otimes_B V, A) \simeq E \otimes M$$

is obtained by the chain of inclusions

$$\ker \mu_V \hookrightarrow I \otimes M \hookrightarrow E \otimes M.$$

Proof. It enough to prove the second statement of the theorem. Let $M = V \otimes_B A$. We have a commutative diagram

$$\begin{tikzcd} \mathrm{Tor}^1_B(V,A) \arrow[r] \arrow[d] & \mathrm{Tor}^0_B(\Omega^1_B \otimes_B V, A) \arrow[d, "\sim"] \\ \mathrm{Tor}^1_B(M,A) \arrow[r] & \mathrm{Tor}^1_B(\Omega^1_B \otimes_B M, A) \end{tikzcd}$$

Hence it suffices to prove the result for $V = M$. Via the trivialisation given by σ, the B-module $\mathrm{P}^1_B(M)$ is isomorphic (as a $\mathbf{k}$-vector space) to

$$I \otimes M \oplus \Omega^1_A \otimes M \oplus M,$$

and B acts by the formula

$$(i,a).(i' \otimes m, \omega \otimes m', m'') = (ai' \otimes m + i \otimes m'', a\omega \otimes m' + da \otimes m'', am'').$$

Hence there are two natural exact sequences

$$\begin{cases} 0 \longrightarrow \Omega^1_A \otimes M \longrightarrow \mathrm{P}^1_B(M) \longrightarrow \sigma^* M \longrightarrow 0 \\ 0 \longrightarrow I \otimes M \longrightarrow \mathrm{P}^1_B(M) \longrightarrow \mathrm{P}^1_A(M) \longrightarrow 0 \end{cases}$$

This gives a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega^1_A \otimes M \arrow[r] & \mathrm{P}^1_A(M) \arrow[r] & M \arrow[r] & 0 \\ 0 \arrow[r] & E \otimes M \arrow[r] \arrow[u] \arrow[d] & \mathrm{P}^1_B(M) \arrow[r] \arrow[u] \arrow[d] & M \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\ 0 \arrow[r] & I \otimes M \arrow[r] & \sigma^* M \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

□

As a corollary of this result, there is a natural exact sequence

$$0 \longrightarrow \mathrm{Tor}^1_B(V,A) \longrightarrow E \otimes M \longrightarrow \mathrm{P}^1_B(V) \otimes_B A \longrightarrow M \longrightarrow 0. \tag{10}$$

Definition 4.8. The *residual Atiyah morphism* of a B-module V is the morphism

$$\chi_V \colon V \otimes_B A \longrightarrow IV[1]$$

in $\mathrm{D}^{\mathrm{b}}(B)$ attached to the exact sequence of B-modules

$$0 \longrightarrow IV \longrightarrow V \longrightarrow V \otimes_B A \longrightarrow 0.$$

Let us now fix an A-module M. The principal parts exact sequence

$$0 \longrightarrow E \otimes M \longrightarrow \mathrm{P}^1_B(M) \longrightarrow M \longrightarrow 0$$

defines a morphism

$$\mathrm{at}_B(M) \colon M \longrightarrow E \otimes M[1]$$

in $\mathrm{D}^{\mathrm{b}}(B)$, which is the Atiyah class of M over B.

Proposition 4.9. *For any A-module M and any splitting σ of* (14)*, the morphism*

$$\mathrm{at}_B(M) \colon M \longrightarrow E \otimes M[1] \simeq I \otimes M[1] \oplus \Omega^1_A \otimes M[1]$$

is the couple $\{\chi_{\sigma^* M}, \mathrm{at}_A(M)\}$.

Proof. This follows directly from the diagram

$$\begin{tikzcd}
0 \arrow[r] & I \otimes M \arrow[r] & \sigma^* M \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & E \otimes M \arrow[r] \arrow[u] \arrow[d] & \mathrm{P}^1_B(M) \arrow[r] \arrow[u] \arrow[d] & M \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_X \arrow[r] & \mathrm{P}^1_A(M) \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

appearing in the proof of Theorem 4.7. □

4.2. **Local obstruction theory.**

4.2.1. *Admissible complexes.* In this section, we use the homological grading convention for complexes in order to avoid negative indices. All complexes will be concentrated in non-negative homological degrees. For any complex $K_\bullet$ we denote by $\overline{K}_\bullet$ the complex $K_\bullet \otimes_B A$.

Definition 4.10. Let n be in $\mathbb{N} \cup \{\infty\}$.

- A complex $K_\bullet$ of B-modules is *n-admissible* if for any i such that $0 \leq i \leq n-1$, the A-module $\mathrm{H}_i(\mathrm{Tor}^1_B(K_\bullet, A))$ vanishes.
- For $n = +\infty$, we simply say that $K_\bullet$ is admissible (instead of "∞-admissible").
- We say that that a B-module K is admissible if it admissible as a complex concentrated in degree 0, that is if $\mathrm{Tor}^1_B(K, A)$ vanishes.

Let us denote by $\mathbf{Tor}^p_B(*, A)$ the hypertor functors defined by the usual formula

$$\mathbf{Tor}^p_B(K_\bullet, A) = \mathrm{H}_p(K_\bullet \overset{\mathbb{L}}{\otimes}_B A).$$

Proposition 4.11. *Let $K_\bullet$ be a complex of B-modules and n be in $\mathbb{N} \cup \{+\infty\}$. Then the complex $K_\bullet$ is n-admissible if and only if the natural map*

$$\mathbf{Tor}^i_B(K_\bullet, A) \longrightarrow \mathrm{H}_i(K_\bullet \otimes_B A)$$

is an isomorphism for $0 \leq i \leq n$ and surjective for $i = n+1$.

Proof. By [31, Application 5.7.8], there is a filtration

$$\mathrm{F}_0\mathbf{Tor}^i_B(K_\bullet, A) \subseteq \mathrm{F}_1\mathbf{Tor}^i_B(K_\bullet, A) \subseteq \cdots \subseteq \mathrm{F}_i\mathbf{Tor}^i_B(K_\bullet, A)$$

and a spectral sequence of homological type such that

$$\begin{cases} \mathrm{E}^2_{p,q} = \mathrm{H}_p(\mathrm{Tor}^q_B(K_\bullet, A)) \\ \mathrm{E}^\infty_{p,q} = \mathrm{Gr}_p\mathbf{Tor}^{p+q}_B(K_\bullet, A) \end{cases}$$

The map from $\mathbf{Tor}^i_B(K_\bullet, A)$ to $\mathrm{H}_i(K_\bullet \otimes_B A)$ is the composition

$$\mathbf{Tor}^i_B(K_\bullet, A) \twoheadrightarrow \mathrm{Gr}_i\mathbf{Tor}^i_B(K_\bullet, A) \simeq \mathrm{E}^\infty_{i,0} \hookrightarrow \mathrm{E}^2_{i,0}.$$

If the complex $K_\bullet$ is n-admissible, then $\mathrm{E}^2_{p,1}$ vanish for $0 \leq p \leq n-1$. Thanks to Proposition 4.6, $\mathrm{E}^2_{p,q}$ vanishes as well for $0 \leq p \leq n-1$ and $q \geq 1$. Hence $\mathrm{F}_j\mathbf{Tor}^i_B(K_\bullet, A)$ vanishes for $0 \leq j < i \leq n$. For any $r \geq 2$ and $0 \leq p \leq n+1$, the maps $\mathrm{d}^r_{p,0} \colon \mathrm{E}^r_{p,0} \to \mathrm{E}^r_{p-r,r-1}$ vanishes, and no differential d^r abuts to $\mathrm{E}^r_{p,0}$, so that $\mathrm{E}^2_{p,0} \simeq \mathrm{E}^\infty_{p,0}$. This proves that the map

$$\mathbf{Tor}^i_B(K_\bullet, A) \to \mathrm{H}_i(K_\bullet \otimes_B A)$$

is an isomorphism for $0 \leq i \leq n$, and surjective for $i = n+1$. Conversely, assume that all the maps

$$\mathbf{Tor}_B^i(K_\bullet, A) \to \mathrm{H}_i(K_\bullet \otimes_B A)$$

are isomorphisms if $0 \leq i \leq n$, and are surjective for $i = n+1$. Then $\mathrm{F}_j\mathbf{Tor}_B^i(K_\bullet, A)$ vanishes for $0 \leq j < i \leq n$, and $\mathrm{d}^2_{i,0}$ vanishes for $0 \leq i \leq n+1$.

The first point implies that $\mathrm{E}^\infty_{p,q}$ vanishes as soon as $q \geq 1$ and $p+q \leq n$. Let us now prove by induction on k that $\mathrm{E}^r_{k,1}$ vanishes if $r \geq 2$ and $k \leq n-1$. The module $\mathrm{E}^{r+1}_{k,1}$ is the middle cohomology of the complex

$$\mathrm{E}^r_{k+r,2-r} \xrightarrow{\mathrm{d}^r_{k+r,2-r}} \mathrm{E}^r_{k,1} \xrightarrow{\mathrm{d}^r_{k,1}} \mathrm{E}^r_{k-r,r}$$

Thanks to Proposition (4.6), the last term $\mathrm{E}^2_{k-r,r}$ is isomorphic to $I^{\otimes r} \otimes \mathrm{E}^2_{k-r,1}$, so it vanishes by induction. Besides, $\mathrm{E}^r_{k+r,2-r}$ is always zero if $r \geq 3$, and if $r = 2$ the differential $\mathrm{d}^2_{k+2,0}$ vanishes. Hence $\mathrm{E}^{r+1}_{k,1} \simeq \mathrm{E}^r_{k,1}$, and since $\mathrm{E}^\infty_{k,1}$ vanishes, all terms $\mathrm{E}^r_{k,1}$ vanish as well. □

Given any bounded complex of B-modules and any nonnegative integer n, there is a canonical procedure that allows to produce n-admissible complexes isomorphic to the initial one in $\mathrm{D}^{\mathrm{b}}(B)$.

Definition 4.12. The functor μ is the element of $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(B))$ defined by the formula

$$\mu(K_\bullet) = \text{cone } (\Omega^1_B \otimes_B K_\bullet \longrightarrow \mathrm{P}^1_B(K_\bullet)).$$

The natural morphism from μ to $\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(S)}$ is a quasi-isomorphism.

Proposition 4.13. *Let $K_\bullet$ be a bounded complex of B-modules. Then there are natural isomorphisms*[8]

$$\begin{cases} \mathrm{Tor}^1_B(\mu(K_\bullet), A) \simeq I \otimes \mathrm{Tor}^1_B(K_\bullet, A)[1] \\ \mathrm{Tor}^1_B(\widetilde{\mu}(K_\bullet), A) \simeq I \otimes \mathrm{Tor}^1_B(K_\bullet, A)[2] \oplus \mathrm{Tor}^1_B(K_\bullet, A) \end{cases}$$

in $\mathrm{D}^{\mathrm{b}}(B)$.

Proof. We start by noticing that

$$\mathrm{Tor}^1_B(\mu(K_\bullet), A) = \text{cone } \{\mathrm{Tor}^1_B(\Omega^1_B \otimes_B K_\bullet, A) \to \mathrm{Tor}^1_B(\mathrm{P}^1_B(K_\bullet), A)\}.$$

By Theorem 4.7, there is an exact sequence

$$0 \to \mathrm{Tor}^2_B(K_\bullet, A) \to \mathrm{Tor}^1_B(\Omega^1_B \otimes_{O_B} K_\bullet, A) \to \mathrm{Tor}^1_B(\mathrm{P}^1_B(K_\bullet), A) \to 0$$

and thanks to Proposition 4.6,

$$\mathrm{Tor}^2_B(K_\bullet, A) \simeq I \otimes \mathrm{Tor}^1_B(K_\bullet, A).$$

This gives the first isomorphism. The second one is proven using the same method. □

Corollary 4.14. *If $K_\bullet$ is n-admissible, then $\mu(K_\bullet)$ is $(n+1)$-admissible.*

Proof. If $i \leq n+1$, we have

$$\mathrm{H}_i(\mathrm{Tor}^1_B(\mu(K_\bullet), A)) \simeq \mathrm{H}_{i-1}(I \otimes \mathrm{Tor}^1_B(K_\bullet, A)) = \{0\}.$$

□

[8] For the definition of $\widetilde{\mu}$, *see* Definition 3.13.

Theorem 4.15. *Let n be a nonnegative integer and $K_\bullet$ be an n-admissible complex. Then for any positive integer p, the natural morphism from $\mu^{[p]}(K_\bullet)$ to $K_\bullet$ is a quasi-isomorphism, and $\mu^{[p]}(K_\bullet)$ is $n+p$-admissible.*

Proof. Thanks to Theorem 3.17 (i), there is an exact sequence

$$0 \longrightarrow \mu^{[p+1]}(K_\bullet) \longrightarrow \mu(\mu^{[p]}(K_\bullet)) \longrightarrow U(\Omega^1_B \otimes_B \mu^{[p-1]}(K_\bullet)) \longrightarrow 0.$$

Hence, thanks to Proposition 4.13,

$$\mathrm{Tor}^1_B(\mu^{[p+1]}(K_\bullet), A) \simeq I \otimes \mathrm{Tor}^1_B(\mu^{[p]}(K_\bullet), A)[1].$$

This gives the result. □

4.2.2. *The category* $\mathrm{D}^{\mathrm{adm}}(B)$. Given a bounded complex $K_\bullet$ of B-modules, it is interesting to know if $K_\bullet$ can be reconstructed from the two complexes $\overline{K}_\bullet$ and $\mathrm{Tor}^1_B(K_\bullet, A)$. At the level of complexes, the answer is given by Lemma 4.1: $K_\bullet$ is entirely determined by $K_\bullet$, the submodule $\mathrm{Tor}^1_B(K_\bullet, A)$ of $I \otimes \overline{K}_\bullet$, and the extension class of the exact sequence

$$0 \longrightarrow I \otimes \overline{K}_\bullet / \mathrm{Tor}^1_B(K_\bullet, A) \longrightarrow K_\bullet \longrightarrow \overline{K}_\bullet \longrightarrow 0 \tag{11}$$

in $\mathrm{D}^{\mathrm{b}}(A)$. We can address the same problem in the derived setting: assume to be given a quadruplet $(M_\bullet, N_\bullet, \mu, \delta)$ where:

- $M_\bullet, N_\bullet$ are in $\mathrm{D}^{\mathrm{b}}(A)$,
- μ is in $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(A)}(I \otimes M_\bullet, N_\bullet)$,
- δ is in $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(A)}(M_\bullet, N_\bullet[1])$.

We look for elements $K_\bullet$ in $\mathrm{C}^-(B)$ such that $\overline{K}_\bullet$ and $IK_\bullet$ are isomorphic in $\mathrm{D}^{\mathrm{b}}(A)$ to $M_\bullet$ and $N_\bullet$ respectively, and via this isomorphisms μ is the multiplication class $\mu_{K_\bullet}$, and δ is the extension class of (11). Besides, we want to define a refined notion of quasi-isomorphism in $\mathrm{C}^-(B)$ in order that such a complex $K_\bullet$ be unique up to quasi-isomorphism.

Definition 4.16. Let $\mathfrak{N}$ be the null system in $\mathrm{K}^-(B)$ defined as follows:

$$\mathfrak{N} = \{K_\bullet \text{ in } \mathrm{K}^-(B) \text{ such that } \overline{K}_\bullet \text{ and } \mathrm{Tor}^1_B(K_\bullet, A) \text{ are exact}\}.$$

The admissible derived category $\mathrm{D}^{\mathrm{adm}}(B)$ is the triangulated category defined as the localization of $\mathrm{K}^-(B)$ with respect to the null system $\mathfrak{N}$.

Elements of $\mathfrak{N}$ are exact complexes, but the converse is not true. In fact, elements of $\mathfrak{N}$ are those for which the E^2 page of the hypertor spectral sequence vanishes. Hence a morphism $\varphi\colon K_\bullet \to L_\bullet$ is an isomorphism in $\mathrm{D}^{\mathrm{adm}}(B)$ if and only if $\overline{\varphi}$ and $\mathrm{Tor}^1_B(\varphi, A)$ are quasi-isomorphisms.

Remark 4.17. The null system can also be described as

$$\mathfrak{N} = \{K_\bullet \text{ in } \mathrm{K}^-(B) \text{ such that } K_\bullet \text{ and } \overline{K}_\bullet \text{ are exact}\}.$$

Therefore a morphism of complexes $\varphi\colon K_\bullet \to L_\bullet$ is an isomorphism in the category $\mathrm{D}^{\mathrm{adm}}(B)$ if and only if φ and $\overline{\varphi}$ are quasi-isomorphisms.

Let us give a few properties related to the categories $\mathrm{D}^{\mathrm{adm}}(B)$.

Proposition 4.18. *Let $\varphi\colon K_\bullet \to L_\bullet$ be a quasi-isomorphism between two elements of $\mathrm{C}^-(B)$. Assume that the complexes $K_\bullet$ and $L_\bullet$ are n-admissible of length at most n for some n in $\mathbb{N} \cup \{+\infty\}$. Then φ is an isomorphism in $\mathrm{D}^{\mathrm{adm}}(B)$.*

Proof. Follows from Proposition 4.11. □

Proposition 4.19. *Let $K_\bullet$ be an admissible complex, and assume that there exists an integer n such that $\mathrm{H}_p(\overline{K}_\bullet)$ vanishes for $p > n$. Then $\tau^{\geq -n}K_\bullet$ is admissible and the morphism $K_\bullet \longrightarrow \tau^{\geq -n}K_\bullet$ is an isomorphism in $\mathrm{D}^{\mathrm{adm}}(B)$.*

Proof. Let $N = \ker\{K_n \to K_{n-1}\}$. Then $\tau^{\geq -n}K_\bullet$ can be represented by the complex

$$N \longrightarrow K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_0$$

We consider again the hypertor spectral sequence associated to this complex. Since $K_\bullet$ is admissible, $\mathrm{E}^2_{p,q}$ vanishes for all integers p and q such that $0 \leq p \leq n-1$ and $q \geq 1$. Hence

$$\mathrm{E}^2_{n,1} \simeq \mathrm{E}^\infty_{n,1} \simeq \mathrm{Gr}_n \mathbf{Tor}^{n+1}_B(\tau^{\geq -n}K_\bullet, A).$$

As $K_\bullet$ is admissible, we have isomorphisms

$$\mathbf{Tor}^{n+1}_B(\tau^{\geq -n}K_\bullet, A) \simeq \mathbf{Tor}^{n+1}_B(K_\bullet, A) \simeq \mathrm{H}_{n+1}(\overline{K}_\bullet) = \{0\}.$$

Hence $\mathrm{E}^2_{n,1}$ vanishes, so that $\tau^{\geq -n}K_\bullet$ is admissible. Then the result follows from Proposition 4.18. □

Corollary 4.20. *A complex $K_\bullet$ is isomorphic in $\mathrm{D}^-(B)$ to a bounded admissible complex if and only if the derived pullback $K \overset{\mathbb{L},\ell}{\otimes}_B A$ is cohomologically bounded.*[9]

Proposition 4.21. *Given $(M_\bullet, N_\bullet, \mu, \delta)$, there exists $K_\bullet$ in $\mathrm{C}^-(B)$ corresponding to these data whose isomorphism class in $\mathrm{D}^{\mathrm{adm}}(B)$ is unique. Besides, the map*

$$(M_\bullet, N_\bullet, \mu, \delta) \longrightarrow K_\bullet$$

is functorial, and $K_\bullet$ is admissible if and only if μ is an isomorphism in $\mathrm{D}^-(B)$.

Proof. Let $P_\bullet$ and $Q_\bullet$ be projective resolution of $K_\bullet$ and $N_\bullet$ respectively. We can represent μ and δ by true morphisms $\mu\colon I \otimes P_\bullet \to Q_\bullet$ and $\delta\colon P_\bullet \to Q_\bullet[1]$. By adding if necessary a null homotopic complex to $P_\bullet$, we can even assume that μ becomes is surjective. Let $K_\bullet$ denote the cone of $\delta\colon P_\bullet \to Q_\bullet[1]$ shifted by -1. Repeating the construction of Lemma (4.1), $K_\bullet$ is naturally a complex of B-modules satisfying all required properties.

Let us now discuss uniqueness. Assume that a complex $L_\bullet$ corresponds to $(M_\bullet, N_\bullet, \mu, \delta)$. We have two morphisms $P_\bullet \to \overline{L}_\bullet$ and $Q_\bullet \to IL_\bullet$. By adding to $K_\bullet$ a null-homotopic complex if necessary, we can assume that these two morphisms are surjective. Then Lemma 4.2 implies that $L_\bullet$ is isomorphic to $K_\bullet$ in $\mathrm{D}^{\mathrm{adm}}(B)$. □

To avoid dealing with unbounded projective complexes, it is possible to use perfect complexes instead of admissible ones. Perfect complexes of B-modules admit a very simple characterization, which we give now. We leave the adaptations from the admissible to the perfect setting to the reader.

Proposition 4.22. *Let $K_\bullet$ be a bounded complex of B-modules. Then $K_\bullet$ is perfect if and only if the derived pullback $K_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A$ is perfect.*

[9] The superscript "ℓ" means that the tensor product is derived with respect to the left variable and not as a bifunctor, see [14, §3] for more details on this issue.

Proof. If $K_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A$ is perfect, then Corollary 4.20 shows that $K_\bullet$ it quasi-isomorphic to a bounded admissible complex, so that we can assume that $K_\bullet$ is indeed bounded and admissible. Let us consider the exact sequence

$$0 \longrightarrow I \otimes \overline{K}_\bullet \longrightarrow K_\bullet \longrightarrow \overline{K}_\bullet \longrightarrow 0$$

as an exact sequence of A-modules, the A-module structure on $K_\bullet$ being given after the choice of a retraction σ of the Atiyah sequence (8). It gives a morphism

$$\alpha \colon \overline{K}_\bullet \longrightarrow I \otimes \overline{K}_\bullet[1]$$

Thanks to Proposition 4.21, the complex $K_\bullet$ can be reconstructed from the quadruplet $(\overline{K}_\bullet, I \otimes \overline{K}_\bullet, \mathrm{id}, \alpha)$. Since $\overline{K}_\bullet$ is a perfect complex, it admits a bounded projective resolution. Hence the proof of Proposition 4.21 shows that the complex $K_\bullet$ is isomorphic to a bounded complex of projective B-modules in $\mathrm{D}^{\mathrm{adm}}(B)$. □

4.2.3. *The local HKR class.* Let $K_\bullet$ be a bounded complex of B-modules. We have an exact sequence of A-modules (the A-module-structure on $K_\bullet$ being given by σ):

$$0 \longrightarrow \mathrm{Tor}^1_B(K_\bullet, A) \longrightarrow I \otimes \overline{K}_\bullet \xrightarrow{\mu_{K_\bullet}} K_\bullet \longrightarrow \overline{K}_\bullet \longrightarrow 0. \tag{12}$$

Definition 4.23. For any complex $K_\bullet$ of B-modules, the local HKR class of $K_\bullet$ is the morphism

$$\theta_{K_\bullet} \colon \overline{K}_\bullet \longrightarrow \mathrm{Tor}^1_B(K_\bullet, A)[2]$$

in $\mathrm{D}^{\mathrm{b}}(A)$ associated with (12).

Remark 4.24. The morphism $\theta_{K_\bullet}$ is well defined on the admissible derived category $\mathrm{D}^{\mathrm{adm}}(S)$, but not on $\mathrm{D}^-(S)$. In fact we can see θ as a natural transformation

$$\begin{tikzcd} \mathrm{D}^{\mathrm{adm}}(S) \arrow[r, bend left, "\mathrm{Tor}^0_S(\,*\,,\mathcal{O}_X)"] \arrow[r, bend right, "\mathrm{Tor}^1_S(\,*\,,\mathcal{O}_X)\,[2]"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathrm{D}^-(X) \end{tikzcd}$$

Theorem 4.25. *Let $K_\bullet$ be a bounded complex of B-modules. Then the following properties are equivalent:*

(i) *The local HKR class $\theta_{K_\bullet}$ vanishes.*

(ii) *The map $K_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \longrightarrow K_\bullet \otimes_B A$ admits a right inverse in $\mathrm{D}^-(A)$.*

(iii) *There exists a bounded admissible complex $V_\bullet$ of B-modules and a morphism in $\mathrm{D}^{\mathrm{b}}(B)$ from $V_\bullet$ to $K_\bullet$ such that the induced map*

$$V_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \longrightarrow K_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \longrightarrow K \otimes_B A$$

in $\mathrm{D}^-(A)$ is an isomorphism.

(iv) *There exists a bounded admissible complex $V_\bullet$ of B-modules and a morphism in $\mathrm{D}^{\mathrm{adm}}(B)$ from $V_\bullet$ to $K_\bullet$ such that the induced map $\overline{V}_\bullet \to \overline{K}_\bullet$ in $\mathrm{D}^{\mathrm{b}}(A)$ is an isomorphism.*

(v) *There exists a bounded admissible complex $V_\bullet$ of B-modules and a sub-complex $T_\bullet$ of $I \otimes \overline{V}_\bullet$ such that $K_\bullet$ is isomorphic to $V_\bullet/T_\bullet$ in $\mathrm{D}^{\mathrm{adm}}(B)$.*

Under any of these conditions,

$$\begin{aligned} K_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A &\simeq K_\bullet \otimes_B A \oplus \mathrm{Tor}^1_B(K_\bullet, A) \overset{\mathbb{L},\ell}{\otimes}_B A\,[1] \\ &\simeq K_\bullet \otimes_B A \oplus \bigoplus_{p\geq 0} I^{\otimes p} \otimes \mathrm{Tor}^1_B(K_\bullet, A)[p+1]. \end{aligned}$$

Proof. (i) $\Rightarrow$ (iv) Let us consider the two exact sequences

$$\begin{cases} 0 \longrightarrow IK_\bullet \longrightarrow K_\bullet \longrightarrow \overline{K}_\bullet \longrightarrow 0 \\ 0 \longrightarrow \mathrm{Tor}^1_B(K_\bullet, A) \longrightarrow I \otimes K_\bullet \xrightarrow{\mu_{K_\bullet}} IK_\bullet \longrightarrow 0 \end{cases}$$

They yield two morphisms

$$\alpha\colon \overline{K}_\bullet \to IK_\bullet[1] \quad \text{and} \quad \beta\colon IK_\bullet \to \mathrm{Tor}^1_B(K_\bullet, A)[1]$$

and $\beta \circ \alpha$ is exactly the local HKR class $\theta_{K_\bullet}$. Hence if we consider the exact sequence

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{D^b}(A)}(\overline{K}_\bullet, I \otimes \overline{K}_\bullet[1]) \arrow[r] & \mathrm{Hom}_{\mathrm{D^b}(A)}(\overline{K}_\bullet, IK_\bullet[1]) \arrow[d, "\beta \circ (\star)"] \\ & \mathrm{Hom}_{\mathrm{D^b}(A)}(\overline{K}_\bullet, \mathrm{Tor}^1_B(K_\bullet, A)[2]) \end{tikzcd}$$

the map α can be lifted to a morphism

$$\widetilde{\alpha}\colon \overline{K}_\bullet \to I \otimes \overline{K}_\bullet[1]. \tag{13}$$

Using the notation of Proposition 4.21, there is a morphism of quadruplets

$$(\overline{K}_\bullet, I \otimes \overline{K}_\bullet, \mathrm{id}, \widetilde{\alpha}) \longrightarrow (\overline{K}_\bullet, IK_\bullet, \mu_{K_\bullet}, \alpha).$$

Thanks to Proposition 4.21, we get an admissible complex $V_\bullet$ and a morphism from $V_\bullet$ to $K_\bullet$ in $\mathrm{D^{adm}}(B)$ such that the induced map from $\overline{V}_\bullet$ to $\overline{K}_\bullet$ is an isomorphism in $\mathrm{D^b}(A)$. Thanks to Proposition 4.19, we can replace $V_\bullet$ by a truncation of sufficiently high horder, so that it becomes admissible and bounded.

(iv) $\Rightarrow$ (v) We can assume that the morphism $V_\bullet \to K_\bullet$ is a true morphism of complexes. Let us now consider the diagram

$$\begin{tikzcd} & I \otimes \overline{V}_\bullet \arrow[r] \arrow[d] & V_\bullet \arrow[r] \arrow[d] & \overline{V}_\bullet \arrow[r] \arrow[d, "\sim"] & 0 \\ 0 \arrow[r] & I \otimes \overline{K}_\bullet / \mathrm{Tor}^1_B(K_\bullet, A) \arrow[r] & K_\bullet \arrow[r] & \overline{K}_\bullet \arrow[r] & 0 \end{tikzcd}$$

and let $T_\bullet$ denotes the kernel of the left vertical arrow. Then the natural map $V_\bullet/T_\bullet \to K_\bullet$ is a quasi-isomorphism, and the induced map

$$\overline{V_\bullet/T_\bullet} \longrightarrow K_\bullet$$

is an isomorphism in $\mathrm{D^b}(A)$. Hence $K_\bullet$ is isomorphic to the complex $V_\bullet/T_\bullet$ in the admissible derived category $\mathrm{D^{adm}}(B)$ (*see* Remark 4.17).

(v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (i) There is a natural B-linear morphism from $\sigma^* K_\bullet$ (where $K_\bullet$ is considered as a A-module) to $K_\bullet$. Let $S_\bullet$ denote its kernel. For every resolution $L_\bullet$ of $K_\bullet$, the corresponding map $L_\bullet \otimes_B A \to K_\bullet \otimes_B A$ admits a right inverse given by the composition

$$K_\bullet \otimes_B A \longrightarrow K_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \simeq L_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \longrightarrow L \otimes_B A.$$

Applying this with $L_\bullet$ being equal to $S_\bullet \to \sigma^* K_\bullet$, we get that the complex $S_\bullet \otimes_B A \to K_\bullet$ is isomorphic to $\overline{K}_\bullet \oplus \mathrm{Tor}^1_B(K_\bullet, A)[1]$ in $\mathrm{D}^-(A)$. Now the natural morphism

$$\begin{array}{ccc} I \otimes K_\bullet & \longrightarrow & K_\bullet \\ \downarrow & & \| \\ S_\bullet \otimes_B A & \longrightarrow & K_\bullet \end{array}$$

is a quasi-isomorphism (*see* the proof of Proposition 4.4). Hence $\theta_{K_\bullet}$ vanishes.

Let us now prove the last statement of the theorem. We can replace $K_\bullet$ by a complex of the form $P_\bullet / T_\bullet$ where $P_\bullet$ is a bounded complex of projective B-modules, and $T_\bullet$ is a subcomplex of the complex $I \otimes \overline{P}_\bullet$. Then $\mathrm{Tor}^1_B(K_\bullet, A)$ is isomorphic to $T_\bullet$. Now we have a distinguished triangle

$$T_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \longrightarrow P_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \longrightarrow K_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \xrightarrow{+1}$$

The second map is a splitting of the map $K_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A \longrightarrow K_\bullet \otimes_B A$, so that $Q_{K_\bullet}$ is isomorphic to $T_\bullet \overset{\mathbb{L},\ell}{\otimes}_B A\,[1]$. This gives the result. $\square$

5. Deformation theory

5.1. **Infinitesimal thickenings.** If $(X, \mathcal{O}_X)$ is any ring space, we introduce the following standard notation:

- $\mathrm{C}^{\mathrm{b}}(X)$ (resp. $\mathrm{C}^-(X)$) is the category of bounded (resp. bounded from above) complexes of sheaves of $\mathcal{O}_X$-modules.
- $\mathrm{K}^{\mathrm{b}}(X)$ and $\mathrm{K}^-(X)$ are the homotopy categories of $\mathrm{C}^{\mathrm{b}}(X)$ and $\mathrm{C}^-(X)$ respectively.
- $\mathrm{D}^{\mathrm{b}}(X)$ and $\mathrm{D}^-(X)$ are the derived categories of $\mathrm{C}^{\mathrm{b}}(X)$ and $\mathrm{C}^-(X)$ respectively.

Let $\mathbf{k}$ be a field of characteristic zero and let $(X, \mathcal{O}_X)$ be a $\mathbf{k}$-ringed space that is either a smooth $\mathbf{k}$-scheme or a smooth complex manifold (in this case $\mathbf{k} = \mathbb{C}$). Let $\mathcal{I}$ be a locally free sheaf of finite rank on X.

Definition 5.1. An infinitesimal thickening of X by $\mathcal{I}$ is a sheaf of $\mathbf{k}_X$-algebras $\mathcal{O}_S$ on X fitting into an exact sequence of sheaves of $\mathbf{k}_X$-algebras

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{14}$$

where $\mathcal{I}$ satisfies $\mathcal{I}^2 = 0$, that is locally split in the category of sheaves of $\mathbf{k}_X$-algebras.

The local splitting condition means that $\mathcal{O}_S$ is locally isomorphic to the trivial k_X-extension of $\mathcal{O}_X$ by $\mathcal{I}$, which is the sheaf $\mathcal{I} \oplus \mathcal{O}_X$ endowed with the ring structure

$$(i, f).(i', f') = (if' + i'f, ff').$$

In geometrical terms, if we consider $S = (X, \mathcal{O}_S)$ as a ringed space, then there is a natural closed immersion $j\colon X \to S$ that admits locally a right inverse. If we work in the algebraic

category, the map $X \to S$ is locally of the form $\operatorname{Spec} A \to \operatorname{Spec} B$ where B is the trivial $\mathbf{k}$-extension of A by the free A-module $\Gamma(\operatorname{Spec} A, \mathcal{I})$.

Let us introduce again some notation: for any sheaf of $\mathcal{O}_S$-modules $\mathcal{F}$ on X, we put

- $\overline{\mathcal{F}} = j^*\mathcal{F}$,
- $\operatorname{Tor}^i_S(\mathcal{F}, \mathcal{O}_X) = \operatorname{Tor}^i_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_X)$.

If $(X, \mathcal{I})$ are given, the isomorphism classes of infinitesimal thickenings of X by $\mathcal{I}$ are classified by the cohomology group $\mathrm{H}^1(X, \mathcal{D}er(\mathcal{O}_X, \mathcal{I}))$, and

$$\mathrm{H}^1(X, \mathcal{D}er(\mathcal{O}_X, \mathcal{I})) \simeq \mathrm{H}^1(X, \mathcal{H}om(\Omega^1_X, \mathcal{I})) \simeq \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{I}).$$

We can see this latter space as the space of morphisms in $\mathrm{D}^{\mathrm{b}}(X)$ from Ω^1_X to $\mathcal{I}[1]$. Hence every such ringed space S is given (up to isomorphism) by a morphism

$$\eta \colon \Omega^1_X \longrightarrow \mathcal{I}[1] \tag{15}$$

in the derived category $\mathrm{D}^{\mathrm{b}}(X)$ of coherent sheaves on X[10]. In the sequel, we will therefore consider an infinitesimal thickening of X as a triplet $(X, \mathcal{I}, \eta)$ where η is a morphism in $\mathrm{D}^{\mathrm{b}}(X)$ from Ω^1_X to $\mathcal{I}[1]$. Let $\mathcal{E} = j^*\Omega^1_S$. We can write down the conormal exact sequence of j, which is

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{E} \longrightarrow \Omega^1_X \longrightarrow 0, \tag{16}$$

and its extension class is precisely η. This shows how to extract η intrinsically from the pair (X, S).

A particular case of this construction is the following one: fix a closed embedding $i \colon X \to Y$ of complex manifolds, and define S as the first formal neighbourhood of X in Y. Then η is the extension class of the conormal exact sequence

$$0 \longrightarrow \mathrm{N}^*_{X/Y} \longrightarrow \Omega^1_{Y|X} \longrightarrow \Omega^1_X \longrightarrow 0.$$

Many notions that have been introduced in §4 for the local case admit a straightforward adaptation in the geometric setting, and some need to be refined. Let us be more specific:

- All the results in §4.1 remain unchanged, the most important ones being Proposition 4.4 and Theorem 4.7.
- The theory of admissible complexes developed in §4.2.1 remains unchanged. Concerning §4.2.2, the derived category $\mathrm{D}^{\mathrm{adm}}(S)$ is well-defined. However, Proposition 4.21 only holds when the thickening S is trivial (that is when j admits a global retraction). Lastly, the characterization of perfect complexes (Proposition 4.22) is still valid since it is a local property on X.
- The material of §4.2.3 can not be directly adapted unless S is globally trivial. We will explain in the remaining part of the section how to define an analog of the local HKR class is the geometric setting, in order that Theorem 4.25 be valid.

For any complex $\mathcal{K}_\bullet$ of $\mathcal{O}_S$-modules, we have an exact sequence

$$0 \longrightarrow \operatorname{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \longrightarrow \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet \longrightarrow j^*\mathrm{P}^1_S(\mathcal{K}_\bullet) \longrightarrow \overline{\mathcal{K}}_\bullet \longrightarrow 0 \tag{17}$$

that corresponds to the sheaf version of (10).

[10] The extension class corresponding to η is called the Kodaira-Spencer class in [16].

Definition 5.2. For any complex $\mathcal{K}_\bullet$ in $\mathrm{C}^-(S)$, the geometric HKR class of $\mathcal{K}_\bullet$ is the morphism

$$\Theta_{\mathcal{K}_\bullet} \colon \overline{\mathcal{K}}_\bullet \longrightarrow \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[2]$$

given by (17).

Let us now explain why this definition generalizes the local HKR class introduced in Definition 4.23.

Proposition 5.3. *If S is globally trivial (that is if the embedding j admits a retraction $\sigma \colon S \to X$) the global HKR class $\Theta_{\mathcal{K}_\bullet}$ is the extension class associated with the exact complex of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \longrightarrow \mathcal{I} \otimes \overline{\mathcal{K}}_\bullet \longrightarrow \sigma_* \mathcal{K}_\bullet \longrightarrow \overline{\mathcal{K}}_\bullet \longrightarrow 0$$

corresponding to the multiplication map.

Proof. After the choice of a splitting σ, we have a commutative diagram of complexes of sheaves of $\mathcal{O}_X$-modules

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \Omega^1_X \otimes \overline{\mathcal{K}}_\bullet \arrow[r] \arrow[d] & \Omega^1_X \otimes \overline{\mathcal{K}}_\bullet \arrow[d] & & \\
0 \arrow[r] & \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \arrow[r] \arrow[d, equal] & \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet \arrow[r] \arrow[d] & j^*\mathrm{P}^1_S(\mathcal{K}_\bullet) \arrow[r] \arrow[d] & \overline{\mathcal{K}}_\bullet \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \arrow[r] & \mathcal{I} \otimes \overline{\mathcal{K}}_\bullet \arrow[r] \arrow[d] & \sigma_*\mathcal{K}_\bullet \arrow[r] \arrow[d] & \overline{\mathcal{K}}_\bullet \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd}$$

where all lines and columns are exact. □

Proposition 5.4. *For anycomplex $\mathcal{K}_\bullet$ in $\mathrm{C}^-(X)$, the morphism $\Theta_{\mathcal{K}_\bullet}$ is the composition*

$$\mathcal{K}_\bullet \xrightarrow{\mathrm{at}_{\mathcal{K}_\bullet}} \Omega^1_X \otimes \mathcal{K}_\bullet[1] \xrightarrow{\mathrm{id} \otimes \eta} \mathcal{I} \otimes \mathcal{K}_\bullet[2]$$

where $\mathrm{at}_{\mathcal{K}_\bullet}$ *denotes the Atiyah class of* $\mathcal{K}_\bullet$.

Proof. The morphisms $\mathrm{at}_{\mathcal{K}_\bullet}$ and $\mathrm{id} \otimes \eta$ correspond to the extension classes of the short exact sequences

$$\begin{cases} 0 \longrightarrow \Omega^1_X \otimes \mathcal{K}_\bullet \longrightarrow \mathrm{P}^1_X(\mathcal{K}_\bullet) \longrightarrow \mathcal{K}_\bullet \longrightarrow 0 \\ 0 \longrightarrow \mathcal{I} \otimes \mathcal{K}_\bullet \longrightarrow \mathcal{E} \otimes \mathcal{K}_\bullet \longrightarrow \Omega^1_X \otimes \mathcal{K}_\bullet \longrightarrow 0. \end{cases}$$

Their Yoneda product is the exact sequence

$$0 \longrightarrow \mathcal{I} \otimes \mathcal{K}_\bullet \longrightarrow \mathcal{E} \otimes \mathcal{K}_\bullet \longrightarrow \mathrm{P}^1_X(\mathcal{K}_\bullet) \longrightarrow \mathcal{K}_\bullet \longrightarrow 0$$

and the corresponding morphism from $\mathcal{K}_\bullet$ to $\mathcal{I} \otimes \mathcal{K}_\bullet[2]$ in the derived category is $\Theta_{\mathcal{K}_\bullet}$. □

Corollary 5.5. *For any element $\mathcal{K}_\bullet$ and $\mathcal{L}_\bullet$ in $\mathrm{C}^-(X)$, the morphism*

$$\Theta_{\mathcal{K}_\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{L}_\bullet} \colon \mathcal{K}_\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{L}_\bullet \longrightarrow \mathcal{I} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{K}_\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{L}_\bullet [2]$$

is equal to $\Theta_{\mathcal{K}_\bullet} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathrm{id}_{\mathcal{L}_\bullet} + \mathrm{id}_{\mathcal{K}_\bullet} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \Theta_{\mathcal{L}_\bullet}$.

Proof. This follows from the analogous formula for the Atiyah morphism, which is well-known (*see* [25, Lemma 2]). □

5.2. **The global extension theorem.** In this section, we state and prove the geometric version of Theorem 4.25.

Theorem 5.6. *For any bounded complex $\mathcal{K}_\bullet$ of $\mathcal{O}_S$-modules, the following properties are equivalent:*

(i) *The HKR class $\Theta_{\mathcal{K}_\bullet}$ vanishes.*

(ii) *The morphism $\mathbb{L}j^*\mathcal{K}_\bullet \to j^*\mathcal{K}_\bullet$ admits a right inverse in $\mathrm{D}^-(X)$.*

(iii) *There exists a bounded admissible complex $\mathcal{L}_\bullet$ and a morphism in $\mathrm{D}^{\mathrm{b}}(S)$ from $\mathcal{L}_\bullet$ to $\mathcal{K}_\bullet$ such that the composition*
$$\mathbb{L}j^*\mathcal{L}_\bullet \longrightarrow \mathbb{L}j^*\mathcal{K}_\bullet \longrightarrow j^*\mathcal{K}_\bullet$$
is an isomorphism in $\mathrm{D}^-(X)$.

(iv) *There exists a bounded admissible complex $\mathcal{L}_\bullet$ and a morphism in $\mathrm{D}^{\mathrm{adm}}(S)$ from $\mathcal{L}_\bullet$ to $\mathcal{K}_\bullet$ such that the induced morphism from $\overline{\mathcal{L}}_\bullet$ to $\overline{\mathcal{K}}_\bullet$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}(X)$.*

(v) *There exists a bounded admissible complex $\mathcal{L}_\bullet$ and a sub-complex $\mathcal{T}_\bullet$ of $\mathcal{I} \otimes \overline{\mathcal{L}}_\bullet$ such that $\mathcal{K}_\bullet$ is isomorphic to $\mathcal{L}_\bullet/\mathcal{T}_\bullet$ in $\mathrm{D}^{\mathrm{adm}}(S)$.*

If any of these properties hold, there is an isomorphism
$$\mathbb{L}j^*\mathcal{K}_\bullet \simeq j^*\mathcal{K}_\bullet \oplus \mathbb{L}j^* \operatorname{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1].$$

Proof. The implications (iv) ⇒ (iii) ⇒ (ii) are straightforward.

(ii) ⇒ (i) Let $\mathcal{Q}_{\mathcal{K}_\bullet}$ be the cone of the morphism $\mathbb{L}j^*\mathcal{K}_\bullet \to j^*\mathcal{K}_\bullet$ shifted by -1, so that there is an exact triangle
$$\mathbb{L}j^*\mathcal{K}_\bullet \longrightarrow j^*\mathcal{K}_\bullet \longrightarrow \mathcal{Q}_{\mathcal{K}_\bullet}[1] \xrightarrow{+1} \tag{18}$$
Let $\widetilde{\mathcal{K}}_\bullet$ be the cone of the natural morphism from $\Omega^1_S \otimes \mathcal{K}_\bullet$ to $\mathrm{P}^1_S(\mathcal{K}_\bullet)$. We fix a projective resolution $\mathcal{P}_\bullet \to \mathcal{K}_\bullet$ of $\mathcal{K}_\bullet$. Since $\widetilde{\mathcal{K}}_\bullet$ and $\mathcal{K}_\bullet$ are isomorphic in the derived category, there exists a morphism from $\mathcal{P}_\bullet$ to $\widetilde{\mathcal{K}}_\bullet$ such that the diagram
$$\begin{tikzcd} \mathcal{P}_\bullet \arrow[r] \arrow[d] & \mathcal{K}_\bullet \arrow[d, equal] \\ \widetilde{\mathcal{K}}_\bullet \arrow[r] & \mathcal{K}_\bullet \end{tikzcd}$$
commutes. Let $\mathcal{M}_\bullet$ (resp. $\mathcal{N}_\bullet$) be the cone of the the morphism $\mathcal{P}_\bullet \to \mathcal{K}_\bullet$ (resp. $\widetilde{\mathcal{K}}_\bullet \to \mathcal{K}_\bullet$). Then we have a morphism of distinguished triangles
$$\begin{tikzcd} j^*\mathcal{P}_\bullet \arrow[r] \arrow[d] & j^*\mathcal{K}_\bullet \arrow[r] \arrow[d] & j^*\mathcal{M}_\bullet \arrow[r, "+1"] \arrow[d] & {} \\ j^*\widetilde{\mathcal{K}}_\bullet \arrow[r] & j^*\mathcal{K}_\bullet \arrow[r] & j^*\mathcal{N}_\bullet \arrow[r, "+1"] & {} \end{tikzcd}$$
in the homotopy category $\mathrm{K}^{\mathrm{b}}(X)$. Remark that the first triangle is isomorphic to (18) in $\mathrm{D}^{\mathrm{b}}(X)$. Now $j^*\mathcal{N}_\bullet$ is the iterated cone of the morphisms of complexes
$$\Omega^1_X \otimes j^*\mathcal{K}_\bullet \longrightarrow j^*\mathrm{P}^1_S(\mathcal{K}_\bullet) \longrightarrow j^*\mathcal{K}_\bullet$$

so it is isomorphic to $\mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[2]$ in $\mathrm{D}^{\mathrm{b}}(X)$, and via this isomorphism the morphism $j^*\mathcal{K}_\bullet \to j^*\mathcal{N}_\bullet$ is nothing but $\Theta_{\mathcal{K}_\bullet}$. Hence we get a commutative diagram of morphisms

$$\begin{tikzcd} j^*\mathcal{K}_\bullet \arrow[rr] \arrow[dr, "\Theta_{\mathcal{K}_\bullet}"'] & & \mathcal{Q}_{\mathcal{K}_\bullet}[1] \arrow[dl] \\ & \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[2] & \end{tikzcd}$$

in the derived category $\mathrm{D}^{\mathrm{b}}(X)$. If the natural morphism from $\mathbb{L}j^*\mathcal{K}_\bullet$ to $j^*\mathcal{K}_\bullet$ admits a right inverse, the connexion morphism from $j^*\mathcal{K}_\bullet$ to $\mathcal{Q}_{\mathcal{K}_\bullet}[1]$ associated to (18) vanishes, and so does $\Theta_{\mathcal{K}_\bullet}$.

(i) $\Rightarrow$ (iv). This is the most difficult step. Let us give first an overview of the strategy of the proof. First we construct two different morphisms from $\overline{\mathcal{K}}_\bullet$ to $\mathcal{E} \otimes \overline{\mathcal{K}}_\bullet[1]$. Then we show that their difference pre-composed by the map $\mathcal{K}_\bullet \to \overline{\mathcal{K}}_\bullet$, denoted by Ψ, factors through $\mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1]$ via the chain of inclusions

$$\mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \hookrightarrow \mathcal{I} \otimes \overline{\mathcal{K}}_\bullet \hookrightarrow \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet.$$

Then we consider the distinguished triangle

$$\mathcal{K}_\bullet \xrightarrow{\Psi} \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1] \longrightarrow \mathcal{L}_\bullet[1] \xrightarrow{+1}$$

and we prove that the composition

$$\mathbb{L}j^*\mathcal{L}_\bullet \longrightarrow \mathbb{L}j^*\mathcal{K}_\bullet \longrightarrow j^*\mathcal{K}_\bullet$$

is an isomorphism in $\mathrm{D}^-(X)$.

First morphism. We consider the two exact sequences

$$\begin{cases} 0 \longrightarrow \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet / \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \longrightarrow j^*\mathrm{P}^1_S(\mathcal{K}_\bullet) \longrightarrow \overline{\mathcal{K}}_\bullet \longrightarrow 0 \\ 0 \longrightarrow \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \longrightarrow \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet \longrightarrow \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet / \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \longrightarrow 0 \end{cases}$$

They yield two morphisms

$$\begin{cases} \gamma \colon \overline{\mathcal{K}}_\bullet \to \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet / \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1] \\ \delta \colon \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet / \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X) \to \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1] \end{cases}$$

and $\delta \circ \gamma = \Theta_{\mathcal{K}_\bullet}$. The exact sequence

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\overline{\mathcal{K}}_\bullet, \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet[1]) \arrow[r] & \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\overline{\mathcal{K}}_\bullet, \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet / \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1]) \arrow[d] \\ & \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\overline{\mathcal{K}}_\bullet, \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)) \end{tikzcd}$$

shows that the map γ can be lifted to a morphism

$$\widetilde{\gamma} \colon \overline{\mathcal{K}}_\bullet \longrightarrow \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet[1].$$

Second morphism. The exact sequence of principal parts

$$0 \longrightarrow \Omega^1_S \otimes \mathcal{K}_\bullet \longrightarrow \mathrm{P}^1_S(\mathcal{K}_\bullet) \longrightarrow \mathcal{K}_\bullet \longrightarrow 0$$

gives a morphism

$$\epsilon \colon \mathcal{K}_\bullet \longrightarrow \Omega^1_S \otimes \mathcal{K}_\bullet[1] \longrightarrow \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet[1].$$

Comparison. Let us consider the following diagram in $\mathrm{D}^{\mathrm{b}}(B)$:

$$\begin{tikzcd}
\mathcal{K}_\bullet \arrow[dr, "\epsilon"] \arrow[drr, dashed] \arrow[dd, "\pi"'] & & \\
& \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet[1] \arrow[r] & \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet / \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1] \\
\overline{\mathcal{K}}_\bullet \arrow[ur, "\widetilde{\gamma}"] \arrow[urr, dashed] & &
\end{tikzcd}$$

Although the interior triangle almost never commutes, we claim that the big (dotted) triangle commutes. This follows from the commutativity of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_S \otimes \mathcal{K}_\bullet \arrow[r] \arrow[d] & \mathrm{P}^1_S(\mathcal{K}_\bullet) \arrow[r] \arrow[d] & \mathcal{K}_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet / \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1] \arrow[r] & j^* \mathrm{P}^1_S(\mathcal{K}_\bullet) \arrow[r] & \overline{\mathcal{K}}_\bullet \arrow[r] & 0
\end{tikzcd}$$

Hence $\epsilon - \widetilde{\gamma} \circ \pi$ can be lifted to a map

$$\Psi \colon \mathcal{K}_\bullet \longrightarrow \mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1].$$

We denote by $\mathcal{L}_\bullet$ the cone of Ψ shifted by -1, it is equipped with a natural morphism with values in $\mathcal{K}_\bullet$.

Local description. Let us assume that we are in the globally split case, so that $j \colon X \to S$ admits a global retraction σ. Thanks to Theorem 4.25, $\mathcal{K}_\bullet$ is isomorphic in $\mathrm{D}^{\mathrm{adm}}(X)$ to a complex of the form $\mathcal{M}_\bullet / \mathcal{T}_\bullet$ where $\mathcal{M}_\bullet$ is admissible and $\mathcal{T}_\bullet$ is a sub-complex of $\mathcal{I} \otimes \overline{\mathcal{M}}_\bullet$, which is isomorphic to $\mathrm{Tor}^1_{\mathcal{O}_S}(\mathcal{K}_\bullet, \mathcal{O}_X)$ in $\mathrm{D}^-(X)$.

We first describe the morphism ϵ. Thanks to the functoriality of the Atiyah class, we can write ϵ as the composition

$$\mathcal{K}_\bullet \longrightarrow \overline{\mathcal{K}}_\bullet \xrightarrow{\mathrm{at}_S(\overline{\mathcal{K}}_\bullet)} \Omega^1_S \otimes \overline{\mathcal{K}}_\bullet[1] \simeq \mathcal{E} \otimes \overline{\mathcal{K}}_\bullet[1]$$

and this map can be explicitly described using Proposition 4.9. It follows from Proposition 4.21 that the residual Atiyah morphism and the extension class of the exact sequence

$$0 \longrightarrow \mathcal{I} \otimes \overline{\mathcal{M}}_\bullet \longrightarrow \mathcal{M}_\bullet \longrightarrow \overline{\mathcal{M}}_\bullet \longrightarrow 0 \tag{19}$$

differ from the element of $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\overline{\mathcal{M}}_\bullet, \mathcal{I} \otimes \overline{\mathcal{M}}_\bullet[1])$ which is the extension class of (19) considered as an exact sequence of $\mathcal{O}_X$-modules (via the retraction σ). Now we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{T}_\bullet \arrow[r] \arrow[d] & \mathcal{M}_\bullet \arrow[r] \arrow[d] & \mathcal{M}_\bullet / \mathcal{T}_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I} \otimes \overline{\mathcal{M}}_\bullet \arrow[r] & \mathcal{M}_\bullet \arrow[r] & \overline{\mathcal{M}}_\bullet \arrow[r] & 0
\end{tikzcd}$$

so that we can see ϵ as the difference of the extensions classes of the exact sequence

$$0 \longrightarrow \mathcal{T}_\bullet \longrightarrow \mathcal{M}_\bullet \longrightarrow \mathcal{M}_\bullet / \mathcal{T}_\bullet \longrightarrow 0 \tag{20}$$

considered first as an exact sequence of $\mathcal{O}_S$-modules, and then as an exact sequence of $\mathcal{O}_X$-modules. On the other hand $\widetilde{\gamma}$ is the sum of the extension class of the sequence (19), considered as a sequence of $\mathcal{O}_X$-modules, and of an arbitrary morphism in $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\overline{\mathcal{M}}_\bullet, \mathcal{T}_\bullet[1])$. Hence, it follows that Ψ can be written as the sum of two terms:

- The extension class of (20).
- A morphism that can be written as the composition
$$\mathcal{M}_\bullet/\mathcal{T}_\bullet \longrightarrow \overline{\mathcal{M}}_\bullet \xrightarrow{\xi} \mathcal{T}_\bullet[1]$$
where ξ is an arbitrary element in $\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X)}(\overline{\mathcal{M}}_\bullet, \mathcal{T}_\bullet[1])$.

We can give a local description of the morphism
$$\mathbb{L}j^*\Psi\colon \mathbb{L}j^*\mathcal{K}_\bullet \longrightarrow \mathbb{L}j^*\,\mathrm{Tor}^1_S(\mathcal{K}_\bullet, \mathcal{O}_X)[1].$$
It is the morphism
$$\overline{\mathcal{M}}_\bullet \oplus \bigoplus_{p\geq 0} \mathcal{I}^{\otimes p} \otimes \mathcal{T}_\bullet[p+1] \longrightarrow \bigoplus_{p\geq 0} \mathcal{I}^{\otimes p} \otimes \mathcal{T}_\bullet[p+1]$$
given in matrix form by
$$\begin{array}{r c}
 & \begin{array}{ccccc} \overline{\mathcal{M}}_\bullet & \mathcal{T}_\bullet[1] & \mathcal{I}\otimes\mathcal{T}_\bullet[2] & \mathcal{I}^{\otimes 2}\otimes\mathcal{T}_\bullet[3] & \cdots \end{array} \\
\begin{array}{r} \mathcal{T}_\bullet[1] \\ \mathcal{I}\otimes\mathcal{T}_\bullet[2] \\ \mathcal{I}^{\otimes 2}\otimes\mathcal{T}_\bullet[3] \\ \vdots \end{array} &
\begin{pmatrix} \xi & \mathrm{id} & & & \\ & * & \mathrm{id} & & \\ & & * & \mathrm{id} & \\ & & & \ddots & \ddots \end{pmatrix}
\end{array}$$

This map admits a left inverse, so that there is an isomorphism
$$\mathbb{L}j^*(\mathcal{M}_\bullet/\mathcal{T}_\bullet) \simeq \overline{\mathcal{M}}_\bullet \oplus \mathbb{L}j^*\,\mathcal{T}_\bullet[1]$$
for which $\mathbb{L}j^*\Psi$ is the second projection.

End of the proof. If follows from the previous local description that the composition
$$\mathbb{L}j^*\mathcal{A}_\bullet \longrightarrow \mathbb{L}j^*\mathcal{K}_\bullet \longrightarrow j^*\mathcal{K}_\bullet$$
is an isomorphism in $\mathrm{D}^-(X)$. Thanks to Corollary 4.20, we can assume that $\mathcal{A}$ is bounded and admissible.

(iv) $\Rightarrow$ (v) Same proof as in the local case. □

5.3. **The case of a single sheaf.** In this section, we deal with the case where $\mathcal{K}_\bullet$ is concentrated in a single degree. The main result we prove is the following:

Theorem 5.7. *Let $\mathcal{K}$ be a sheaf of $\mathcal{O}_S$-modules. Then the cone of $\Theta_{\mathcal{K}}[-1]$ is isomorphic in $\mathrm{D}^{\mathrm{b}}(A)$ to $\tau^{\geq -1}\,\mathbb{L}j^*\,\mathcal{K}$.*

Proof. The complex $\Omega^1_S \otimes \mathcal{K} \to \mathrm{P}^1_S(\mathcal{K})$ is a resolution of $\mathcal{K}$. Besides, this complex is 1-admissible (*see* Definition 4.10): indeed, thanks to Theorem (4.7), the map
$$\mathrm{Tor}^1_S(\Omega^1_S \otimes \mathcal{K}, \mathcal{O}_X) \longrightarrow \mathrm{Tor}^1_S(\mathrm{P}^1_S(\mathcal{K}), \mathcal{O}_X)$$
is surjective. Therefore, $\tau^{\geq -1}\,\mathbb{L}j^*\mathcal{K}$ is isomorphic to
$$\mathcal{E} \otimes j^*\mathcal{K} \longrightarrow j^*\mathrm{P}^1_S(\mathcal{K})$$
in $\mathrm{D}^{\mathrm{b}}(X)$. This gives the result. □

In this situation, we can complete the picture of Theorem 4.7 by the two following results:

Theorem 5.8. *For any sheaf $\mathcal{V}$ of $\mathcal{O}_S$-modules, the following properties are equivalent:*

(i) *The HKR class $\Theta_{\mathcal{V}}$ vanishes.*

(ii) *The morphism $\mathbb{L}j^*\mathcal{V} \to j^*\mathcal{V}$ admits a right inverse in $\mathrm{D}^-(X)$.*

(iii) *The object $\tau^{\geq -1}\,\mathbb{L}j^*\mathcal{V}$ is formal in $\mathrm{D}^-(X)$.*

(iv) *The sheaf $\mathcal{V}$ extends to an admissible sheaf on S.*

Under any of these conditions, $\mathbb{L}j^*\mathcal{V} \simeq j^*\mathcal{V} \oplus \mathbb{L}j^*\operatorname{Tor}^1_S(\mathcal{V}, \mathcal{O}_X)[1]$.

Proof.

(ii) $\Leftrightarrow$ (iii) Obvious.

(i) $\Leftrightarrow$ (ii) and (iv) $\Rightarrow$ (ii) Follows from Theorem 5.6.

(i) $\Rightarrow$ (iv) According to Theorem 5.6, there exists an admissible complex $\mathcal{L}_\bullet$ concentrated in negative degrees and a morphism from $\mathcal{L}_\bullet$ to $\mathcal{V}$ in $\mathrm{D}^{\mathrm{adm}}(S)$ such that the composition

$$j^*\mathcal{L}_\bullet \longrightarrow j^*\mathcal{V}$$

is an isomorphism. According to Proposition 4.19, we can replace $\mathcal{L}_\bullet$ by its last truncation $\mathcal{H}_0(\mathcal{L}_\bullet)$, which is still admissible. □

Corollary 5.9. *Let $\mathcal{V}$ be a sheaf of $\mathcal{O}_S$-modules. Then $\mathbb{L}j^*\mathcal{V}$ is formal in $\mathrm{D}^-(X)$ if and only $\Theta_{\mathcal{V}}$ and $\{\Theta_{\operatorname{Tor}^p_{\mathcal{O}_S}(\mathcal{V},\mathcal{O}_X)}\}_{p\geq 0}$ vanish.*

6. Structure of derived self intersections

6.1. **Preliminar material.** We fix a pair (X, S) where S is a locally trivial thickening of X.

Lemma 6.1. *The direct image functor $j_*\colon \mathrm{C}^-(X) \to \mathrm{C}^-(S)$ factorizes through a functor*

$$j_*\colon \mathrm{D}^-(X) \to \mathrm{D}^{\mathrm{adm}}(S)$$

that lifts the usual push forward functor j_* at the level of derived categories.

Proof. We must prove that for quasi-isomorphism $\varphi\colon \mathcal{V}_\bullet \to \mathcal{W}_\bullet$ between elements in $\mathrm{C}^-(X)$, $j_*\varphi$ is an isomorphism in the admissible derived category $\mathrm{D}^{\mathrm{adm}}(S)$. This is straightforward, since $\overline{j_*\varphi}$ equals φ, and $\operatorname{Tor}^1_S(j_*\varphi, \mathcal{O}_X)$ equals $\mathrm{id}_{\mathcal{I}} \otimes \varphi$. □

As a corollary, we see that $\Theta_{\mathcal{V}_\bullet}$ is well defined for any object $\mathcal{V}_\bullet$ of $\mathrm{D}^-(X)$. This fact can also be deduced easily from Proposition 5.4, which gives an explicit description of $\Theta_{\mathcal{V}_\bullet}$.

We define an endofunctor $\aleph$ of $\mathrm{D}^-(X)$ as follows: for any complex $\mathcal{V}_\bullet$ in $\mathrm{D}^-(X)$, we put

$$\aleph(\mathcal{V}_\bullet) = \mathbb{L}j^*\left(j_*\mathcal{V}_\bullet\right).$$

Let us give a few properties of $\aleph$:

- The functor $\aleph$ is continuous (i.e. commutes with arbitrary limits and colimits).
- Given a nonzero complex $\mathcal{V}_\bullet$ in $\mathrm{D}^{\mathrm{b}}(X)$, $\aleph(\mathcal{V}_\bullet)$ is *never* bounded.

- The functor $\aleph$ carries a natural lax monoidal structure: if $\mathcal{V}_\bullet$ and $\mathcal{W}_\bullet$ are sheaves on $\mathcal{O}_X$-modules, the product map is given by the composition

$$\begin{tikzcd} j^*j_*\mathcal{V}_\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} j^*j_*\mathcal{W}_\bullet \arrow[d, "\sim"] \\ j^*(j_*\mathcal{V}_\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} j_*\mathcal{W}_\bullet) \arrow[d] \\ j^*j_*(\mathcal{V}_\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} j^*j_*\mathcal{W}_\bullet) \arrow[r] & j^*j_*(\mathcal{V}_\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{W}_\bullet) \end{tikzcd}$$

  and the unit is $\mathcal{O}_X \simeq j^*\mathcal{O}_S \simeq \mathbb{L}j^*\mathcal{O}_S \longrightarrow \mathbb{L}j^*(j_*\mathcal{O}_X)$.
- The ring object $\aleph(\mathcal{O}_X)$ in $\mathrm{D}^-(X)$ is the structural sheaf of the derived intersection of X in S.

In the single sheaf case, we can provide a simple formality criterion under some additional hypotheses:

Theorem 6.2. *Let $\mathcal{V}$ be a coherent sheaf of $\mathcal{O}_X$-modules which is not a torsion sheaf. Then the following properties are equivalent:*

(i) $\aleph(\mathcal{V})$ *is formal in* $\mathrm{D}^-(X)$.

(ii) $\tau^{\geq -2}\,\aleph(\mathcal{V})$ *is formal in* $\mathrm{D}^{\mathrm{b}}(X)$.

(iii) $\Theta_{\mathcal{V}}$ *and* $\Theta_{\mathcal{I}}$ *vanish.*

Proof.

(i) $\Rightarrow$ (ii) Obvious.

(ii) $\Rightarrow$ (iii) If $\tau^{\geq 2}$ is formal, then Theorem 5.8 implies that $\Theta_{\mathcal{V}}$ vanishes and that

$$\aleph(\mathcal{V}) \simeq \mathcal{V} \oplus \aleph(\mathcal{I} \otimes \mathcal{V})[1].$$

Hence $\tau^{\geq -1}\,\aleph(\mathcal{I} \otimes \mathcal{V})$ is formal, so that $\Theta_{\mathcal{I}\otimes\mathcal{V}}$ vanishes. Using Corollary 5.5, the derived trace of $\Theta_{\mathcal{I}\otimes\mathcal{V}}$ with respect to the factor $\mathcal{V}$ is $r \times \Theta_{\mathcal{I}}$ where r is the generic rank of $\mathcal{V}$. As r is nonzero, $\Theta_{\mathcal{I}}$ vanishes.

(iii) $\Rightarrow$ (i) If $\Theta_{\mathcal{I}}$ and $\Theta_{\mathcal{V}}$ vanish, then all the classes $\Theta_{\mathcal{I}^{\otimes p}\otimes\mathcal{V}}$ vanish, so that $\aleph(\mathcal{V})$ is formal thanks to Corollary 5.9. □

Assume that the thickening S is globally trivial, that is the morphism j admits a global retraction $\sigma\colon S \to X$. Then every complex of sheaves $\mathcal{V}_\bullet$ in $\mathrm{C}^-(X)$ admits an admissible resolution $K_{\mathcal{V}_\bullet}$, which is $\sigma^*\mathcal{V}_\bullet \otimes K_{\mathcal{O}_X}$ where $K_{\mathcal{O}_X}$ is the complex

$$\cdots \longrightarrow \sigma^*\mathcal{I}^{\otimes 3} \longrightarrow \sigma^*\mathcal{I}^{\otimes 2} \longrightarrow \sigma^*\mathcal{I} \longrightarrow \mathcal{O}_S.$$

This gives a distinguished HKR isomorphism

$$\Gamma_\sigma\colon \aleph(\mathcal{V}_\bullet) \xrightarrow{\sim} \bigoplus_{p\geq 0} \mathcal{I}^{\otimes p} \otimes \mathcal{V}_\bullet\,[p]$$

in $\mathrm{D}^-(X)$. Besides, via this isomorphism, the lax monoidal structure on $\aleph$ is simply given by the shuffle product (*see* [1, Proposition 1.10]). We now come back to the general case, and set the following definition:

Definition 6.3. The Arinkin-Căldăraru functor, denoted by H, is the element of $\mathrm{Fct}^*_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(X))$ defined by

$$H(\mathcal{V}_\bullet) = \mathrm{cone}\,(\Omega^1_X \otimes \mathcal{V}_\bullet \longrightarrow \mathrm{P}^1_X(\mathcal{V}_\bullet)).$$

Thanks to Propositions A.2 and 2.8, the functor H is exact, and is naturally a lax monoidal functor of $\mathrm{C}^b(X)$. Hence:

- According to Theorem 3.17 (ii), the functors $H^{[n]}$ and $H^{[[n]]}$ are exact and bounded. They are also naturally lax monoidal functors thanks to Proposition 3.15.
- By Proposition 3.15, the natural morphism from $H^{[n]}$ to $H^{[[n]]}$ is multiplicative. Theorem 3.17 (ii) implies that this morphism is a quasi-isomorphism.
- All these structures extend on $\mathrm{C}^-(X)$, and can be defined on $\mathrm{D}^-(X)$ using flat resolutions.

Let us discuss the case where S is a globally trivial thickening of X. Let T be the element of $\mathrm{Fct}^*_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(X))$ defined by

$$T(\mathcal{V}_\bullet) = \mathcal{I} \otimes \mathcal{V}_\bullet\,[1] \oplus \mathcal{V}_\bullet.$$

Thanks to Proposition 3.18,

- $T^{[n]}$ is naturally isomorphic to $\mathcal{V}_\bullet \to \bigoplus_{p\geq 0} \mathcal{I}^{\otimes p} \otimes \mathcal{V}_\bullet\,[p]$.
- The natural map from $T^{[n]}$ to $T^{[[n]]}$ is a quasi-isomorphism.

Proposition 6.4. *If there exists a global retraction $\sigma\colon S \to X$, then there is a natural exact sequence* [11]

$$0 \longrightarrow U(\Omega^1_X \otimes \mathcal{V}_\bullet) \longrightarrow H(\mathcal{V}_\bullet) \longrightarrow T(\mathcal{V}_\bullet) \longrightarrow 0$$

of dg-endofunctors of $\mathrm{C}^{\mathrm{b}}(X)$, and for all nonnegative integer n the map from $H^{[n]}$ to $T^{[n]}$ is a quasi-isomorphism.

Proof. The first part follows directly from the exact sequence

$$0 \longrightarrow \Omega^1 \otimes \mathcal{V}_\bullet \longrightarrow \mathrm{P}^1_S(\mathcal{V}_\bullet) \longrightarrow \sigma^*\mathcal{V}_\bullet \longrightarrow 0$$

obtained in the local case in the proof of Theorem 4.7. For the second part we have a commutative diagram

$$\begin{array}{ccc} H^{[n]} & \longrightarrow & T^{[n]} \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ H^{[[n]]} & \xrightarrow{\sim} & T^{[[n]]} \end{array}$$

where the bottom horizontal map is a quasi-isomorphism because of Proposition 3.15. Hence the top horizontal map is a quasi-isomorphism. □

6.2. **Main theorem.** In this section, we compute explicitly the functor $\aleph$. Let us recall that the functor μ has been defined in Definitions 4.12 (in the local case), the definition remains the same in the geometric case, and it is a bounded dg endofunctor of $\mathrm{C}^{\mathrm{b}}(X)$. Besides, there is a natural morphism

$$j^* \circ \mu \circ j_* \longrightarrow j^* \circ (j_* \circ j^*) \circ \mu \circ j_* \xrightarrow{\sim} H$$

[11] The functor U is defined by (6).

of dg-endofunctors of $\mathrm{C}^{\mathrm{b}}(X)$. For any nonnegative integer n, we define a natural transformation $\chi_n \colon \aleph \to H^{[[n]]}$ as follows: for any complex $\mathcal{V}_\bullet$ in $\mathrm{C}^-(X)$ the morphism $\chi_n(\mathcal{V}_\bullet)$ is the composition.

$$\aleph(\mathcal{V}_\bullet) \longrightarrow j^*\mu^{[n]}(j_*\mathcal{V}_\bullet) \longrightarrow H^{[n]}(\mathcal{V}_\bullet).$$

Theorem 6.5. *Assume to be given a pair (X, S) where X is either a smooth scheme over a field of characteristic zero or a complex manifold, and S is a locally trivial infinitesimal thickening of S. Then the following properties are valid:*

(i) *The morphisms $\chi_n \colon \aleph \to H^{[[n]]}$ are multiplicative.*

(ii) *For any complex $\mathcal{V}_\bullet$ concentrated in negative degrees, le local homology morphism $\mathcal{H}_p(\chi_n(\mathcal{V}_\bullet))$ is an isomorphism for $0 \leq p \leq n$.*

(iii) *The sequence of morphisms $(\chi_n)_{n\geq 0}$ define a multiplicative isomorphism*

$$\aleph \simeq \varprojlim_n H^{[n]}.$$

(iv) *If S is globally trivial and σ is an associated retraction of j, the composition*

$$\aleph \simeq \varprojlim_n H^{[n]} \longrightarrow \varprojlim_n T^{[n]} \simeq \bigoplus_{p\geq 0} \mathcal{I}^{\otimes p} \otimes (*)\,[p]$$

is the generalized HKR isomorphism Γ_σ.

Proof.

(i) The functor $\mu^{[n]}$ is a lax multiplicative endofunctor of $\mathcal{C}^-(S)$, and the morphism from $\mu^{[n]}$ to $\mathrm{id}_{C^-(S)}$ is also multiplicative. If $P_{\mathcal{V}_\bullet}$ and $P_{\mathcal{W}_\bullet}$ are two flat resolutions over X of complexes $\mathcal{V}_\bullet$ and $\mathcal{W}_\bullet$ in $\mathrm{C}^-(X)$, we have a commutative diagram

$$\begin{tikzcd}
\mu^{[n]}(j_*P_{\mathcal{V}_\bullet}) \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} \mu^{[n]}(j_*P_{\mathcal{W}_\bullet}) \arrow[r] \arrow[d] & \mu^{[n]}(j_*P_{\mathcal{V}_\bullet}) \otimes_{\mathcal{O}_Y} \mu^{[n]}(j_*P_{\mathcal{W}_\bullet}) \arrow[r] \arrow[d] & \mu^{[n]}(j_*P_{\mathcal{V}_\bullet} \otimes_{\mathcal{O}_Y} j_*P_{\mathcal{W}_\bullet}) \arrow[d] \\
j_*P_{\mathcal{V}_\bullet} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} j_*P_{\mathcal{W}_\bullet} \arrow[r] & j_*P_{\mathcal{V}_\bullet} \otimes_{\mathcal{O}_Y} j_*P_{\mathcal{W}_\bullet} \arrow[r, equal] & j_*P_{\mathcal{V}_\bullet} \otimes_{\mathcal{O}_Y} j_*P_{\mathcal{W}_\bullet}
\end{tikzcd}$$

The morphism from $j^* \circ \mu^{[n]} \circ j_*$ to $H^{[n]}$ being multiplicative, we get a commutative diagram

$$\begin{tikzcd}
\aleph(\mathcal{V}_\bullet) \overset{\mathbb{L}}{\otimes} \aleph(\mathcal{W}) \arrow[rrr] & & & \aleph(\mathcal{V}_\bullet \overset{\mathbb{L}}{\otimes} \mathcal{W}_\bullet) \\
\mathbb{L}j^*\mu^{[n]}(j_*P_{\mathcal{V}_\bullet}) \overset{\mathbb{L}}{\otimes} \mathbb{L}j^*\mu^{[n]}(j_*P_{\mathcal{W}_\bullet}) \arrow[r] \arrow[u, "\sim"] \arrow[d] & \mathbb{L}j^*\mu^{[n]}(j_*P_{\mathcal{V}_\bullet}) \otimes \mathbb{L}j^*\mu^{[n]}(j_*P_{\mathcal{W}_\bullet}) \arrow[r] \arrow[d] & \mathbb{L}j^*\mu^{[n]}(j_*P_{\mathcal{V}_\bullet} \otimes_{\mathcal{O}_Y} j_*P_{\mathcal{W}_\bullet}) \arrow[u, "\sim"'] \arrow[d] & \\
j^*\mu^{[n]}(j_*P_{\mathcal{V}_\bullet}) \overset{\mathbb{L}}{\otimes} \mu^{[n]}(j_*P_{\mathcal{W}_\bullet}) \arrow[r] \arrow[d] & j^*\mu^{[n]}(j_*P_{\mathcal{V}_\bullet}) \otimes j^*\mu^{[n]}(j_*P_{\mathcal{W}_\bullet}) \arrow[r] \arrow[d] & j^*\mu^{[n]}(j_*P_{\mathcal{V}_\bullet} \otimes_{\mathcal{O}_Y} j_*P_{\mathcal{W}_\bullet}) \arrow[d] & \\
H^{[n]}(P_{\mathcal{V}_\bullet}) \overset{\mathbb{L}}{\otimes} H^{[n]}(P_{\mathcal{W}_\bullet}) \arrow[r] & H^{[n]}(P_{\mathcal{V}_\bullet}) \otimes H^{[n]}(P_{\mathcal{W}_\bullet}) \arrow[r] & H^{[n]}(P_{\mathcal{V}_\bullet} \otimes P_{\mathcal{W}_\bullet}) &
\end{tikzcd}$$

which proves that χ_n is multiplicative.

(ii) As $\aleph$ is cocomplete, we can assume without loss of generality that $\mathcal{V}_\bullet$ is bounded. Recall that the functor. Let us denote by $\widetilde{\Theta}$ the natural morphism from $\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(S)}$ to $\widetilde{\mu}$[12]. There is a

[12]For the definition of $\widetilde{\mu}$, *see* Definition 3.13.

natural morphism

$$\widetilde{\mu} \circ j_* \longrightarrow j_* \circ j^* \circ \widetilde{\mu} \circ j_* \xrightarrow{\sim} j_* \circ \widetilde{H}$$

of dg functors from $\mathrm{C}^{\mathrm{b}}(X)$ to $\mathrm{C}^{\mathrm{b}}(S)$. For any nonnegative integer n, this gives a morphism

$$\alpha_n \colon \widetilde{\mu}^n \circ j_* \simeq \widetilde{\mu} \circ \widetilde{\mu}^{n-1} \circ j_* \longrightarrow \widetilde{\mu} \circ j_* \circ \widetilde{H}^{n-1}.$$

Now we have a commutative diagram

$$\begin{tikzcd}
\widetilde{\mu}^n \circ j_* \arrow[r, "S_n^{\widetilde{\mu}} \circ j_*"] \arrow[d, "\alpha_n"'] & \widetilde{\mu}^{n+1} \circ j_* \arrow[d, "\alpha_{n+1}"] \\
\widetilde{\mu} \circ j_* \circ \widetilde{H}^{n-1} \arrow[r, "W_n"'] & \widetilde{\mu} \circ j_* \circ \widetilde{H}^n
\end{tikzcd}$$

where the morphism $S_n^{\widetilde{\mu}}$ is the alternated sum defined by (1)[13], and the bottom morphism W_n is the sum of $\widetilde{\mu} \circ j_* (S_{n-1}^{\widetilde{H}})$ and of the morphism obtained as the composition

$$\widetilde{\mu} \circ j_* \circ \widetilde{H}^{n-1} \longrightarrow j_* \circ \widetilde{H}^n \xrightarrow{\widetilde{\Theta}_{j_* \circ \widetilde{H}^n}} \widetilde{\mu} \circ j_* \circ \widetilde{H}^n.$$

Let $\mathfrak{F}_n$ denote the dg functor from $\mathrm{C}^{\mathrm{b}}(X)$ to $\mathrm{C}^{\mathrm{b}}(S)$ defined as the iterated cone of the complex

$$j_* \xrightarrow{W_0} \widetilde{\mu} \circ j_* \xrightarrow{W_1} \widetilde{\mu} \circ j_* \circ \widetilde{H} \longrightarrow \cdots \xrightarrow{W_{n-1}} \widetilde{\mu} \circ j_* \circ \widetilde{H}^{n-1} \ .$$

Thanks to the previous discussion, we have a chain of natural transformations

$$\widetilde{\mu}^{[n]} \circ j_* \longrightarrow \widetilde{\mu}^{[[n]]} \circ j_* \longrightarrow \mathfrak{F}_n.$$

For any complex $\mathcal{V}_\bullet$ in $\mathrm{C}^{\mathrm{b}}(X)$ concentrated in nonpositive degrees, the corresponding morphisms

$$\widetilde{\mu}^{[n]}(j_* \mathcal{V}_\bullet) \longrightarrow \widetilde{\mu}^{[[n]]}(j_* \mathcal{V}_\bullet) \longrightarrow \mathfrak{F}_n(\mathcal{V}_\bullet) \tag{21}$$

are all quasi-isomorphisms: this follows from Proposition 4.15 and from the fact that $\widetilde{\mu}$ is quasi-isomorphic to zero. We now claim that $\mathfrak{F}_n(\mathcal{V}_\bullet)$ is an n-admissible complex, which is a purely local problem. Hence we can assume that S is globally trivial. If $\widetilde{T}$ is the shift functor defined by[14]

$$\widetilde{T}(\mathcal{V}_\bullet) = \mathcal{I} \otimes \mathcal{V}_\bullet[2]$$

then there is a natural quasi-isomorphism from $\widetilde{H}$ to $\widetilde{T}$, and the diagram

$$\begin{tikzcd}
& \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(X)} \arrow[dl, "\Theta"'] \arrow[dr, "0"] & \\
\widetilde{H} \arrow[rr] & & \widetilde{T}
\end{tikzcd}$$

commutes. Hence, if $\mathfrak{F}_n^{\mathrm{loc}}$ is the iterated cone of the complex

$$j_* \xrightarrow{W_0^{\mathrm{loc}}} \widetilde{\mu} \circ j_* \xrightarrow{W_1^{\mathrm{loc}}} \widetilde{\mu} \circ j_* \circ \widetilde{T} \longrightarrow \cdots \xrightarrow{W_{n-1}^{\mathrm{loc}}} \widetilde{\mu} \circ j_* \circ \widetilde{T}^{n-1}$$

[13]Since we are going to use the construction of §3.1 for the couples $(\widetilde{\mu}, \widetilde{\Theta})$ and $(\widetilde{H}, \Theta)$, we put a superscript to distinguish them.

[14]This definition doesn't match with Definition 3.13, however it differs from it by an element in the center of $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(X))$.

where W_p^{loc} is given by the composition

$$W_p^{\mathrm{loc}}\colon \widetilde{\mu}\circ j_*\circ \widetilde{T}^{p-1}\longrightarrow j_*\circ\widetilde{H}\circ\widetilde{T}^{p-1}\longrightarrow j_*\circ\widetilde{T}^p\xrightarrow{\widetilde{\Theta}_{j_*\circ\widetilde{T}^p}}\widetilde{\mu}\circ j_*\circ\widetilde{T}^p,$$

there is a natural quasi-isomorphism from $\mathfrak{F}_n(\mathcal{V}_\bullet)$ to $\mathfrak{F}_n^{\mathrm{loc}}(\mathcal{V}_\bullet)$. Besides, thanks to Lemma 6.1, all arrows from $\widetilde{\mu}\circ j_*\circ\widetilde{H}^p$ to $\widetilde{\mu}\circ j_*\circ\widetilde{T}^p$ induce isomorphisms in the admissible derived category $\mathrm{D}^{\mathrm{adm}}(S)$, so the morphism from $\mathfrak{F}_n(\mathcal{V}_\bullet)$ to $\mathfrak{F}_n^{\mathrm{loc}}(\mathcal{V}_\bullet)$ is an isomorphism in $\mathrm{D}^{\mathrm{adm}}(S)$. Hence, the claim is equivalent to the fact that $\mathfrak{F}_n^{\mathrm{loc}}(\mathcal{V}_\bullet)$ is n-admissible.

Thanks to Proposition 4.13, for any complex $\mathcal{K}_\bullet$ of $\mathcal{O}_X$-modules, we have two commutative diagrams

$$\begin{tikzcd}
\mathrm{Tor}_S^1(\widetilde{\mu}\circ j_*(\mathcal{K}_\bullet),\mathcal{O}_X) \arrow[r] \arrow[d, "\sim"] & \mathrm{Tor}_S^1(j_*\circ\widetilde{H}(\mathcal{K}_\bullet),\mathcal{O}_X) \arrow[r] & \mathrm{Tor}_S^1(j_*\circ\widetilde{T}(\mathcal{K}_\bullet),\mathcal{O}_X) \arrow[d, "\sim"] \\
\mathcal{I}^{\otimes 2}\otimes\mathcal{K}_\bullet[2]\oplus\mathcal{I}\otimes\mathcal{K}_\bullet \arrow[rr, "\begin{pmatrix}1 & 0\end{pmatrix}"] & & \mathcal{I}^{\otimes 2}\otimes\mathcal{K}_\bullet[2]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Tor}_S^1(j_*\mathcal{K}_\bullet,\mathcal{O}_X) \arrow[r, "\widetilde{\Theta}_{j_*\mathcal{K}_\bullet}"] \arrow[d, "\sim"] & \mathrm{Tor}_S^1(\widetilde{\mu}\circ j_*(\mathcal{K}_\bullet),\mathcal{O}_X) \arrow[d, "\sim"] \\
\mathcal{I}\otimes\mathcal{K}_\bullet \arrow[r, "\begin{pmatrix}0\\1\end{pmatrix}"] & \mathcal{I}^{\otimes 2}\otimes\mathcal{K}_\bullet[2]\oplus\mathcal{I}\otimes\mathcal{K}_\bullet
\end{tikzcd}$$

This gives the diagram

$$\begin{tikzcd}
\mathrm{Tor}_S^1(\widetilde{\mu}\circ j_*\circ\widetilde{T}^{p-1}(\mathcal{V}_\bullet),\mathcal{O}_X) \arrow[r, "\mathrm{Tor}_S^1(W_p^{\mathrm{loc}},\mathcal{O}_X)"] \arrow[d, "\sim"] & \mathrm{Tor}_S^1(\widetilde{\mu}\circ j_*\circ\widetilde{T}^p(\mathcal{V}_\bullet),\mathcal{O}_X) \arrow[d, "\sim"] \\
\mathcal{I}^{\otimes p+1}\otimes\mathcal{V}_\bullet[2p]\oplus\mathcal{I}^{\otimes p}\otimes\mathcal{V}_\bullet[2p-2] \arrow[r, "\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}"] & \mathcal{I}^{\otimes p+2}\otimes\mathcal{V}_\bullet[2p+2]\oplus\mathcal{I}^{\otimes p+1}\otimes\mathcal{V}_\bullet[2p]
\end{tikzcd}$$

and we get

$$\mathrm{Tor}_S^1(\mathfrak{F}_n^{\mathrm{loc}}(\mathcal{V}_\bullet),\mathcal{O}_X)\simeq\mathcal{I}^{\otimes n+1}\otimes\mathcal{V}_\bullet[n]$$

so that $\mathfrak{F}_n^{\mathrm{loc}}(\mathcal{V}_\bullet)$ is n-admissible. This proves the claim. The chain of morphisms (21) induces a commutative diagram

$$\begin{tikzcd}
j^*\mu^{[n]}(j_*\mathcal{V}_\bullet) \arrow[r] \arrow[d] \arrow[rr, bend left, "\sim"] & j^*\mu^{[[n]]}(j_*\mathcal{V}_\bullet) \arrow[r] \arrow[d] & j^*\mathfrak{F}_n(\mathcal{V}_\bullet) \arrow[d, equal] \\
H^{[n]}(\mathcal{V}_\bullet) \arrow[r, "\sim"] & H^{[[n]]}(\mathcal{V}_\bullet) \arrow[r, equal] & H^{[[n]]}(\mathcal{V}_\bullet)
\end{tikzcd}$$

where the top horizontal row is an isomorphism as $\mu^{[n]}(\mathcal{V}_\bullet)$ and $\mathfrak{F}_n(\mathcal{V}_\bullet)$ are both n-admissible, and the bottom horizontal row is an isomorphism thanks to Theorem 3.17.

(iii) follows directly from (ii).

(iv) Let S be the dg functor from $\mathrm{C}^{\mathrm{b}}(X)$ to $\mathrm{C}^b(S)$ defined by

$$S(V_\bullet)=\mathrm{cone}\,(\mathcal{I}\otimes\mathcal{V}_\bullet\longrightarrow\sigma^*\mathcal{V}_\bullet).$$

There is a natural morphism $\mu \circ j_*$ to S. If $\widetilde{S} = \operatorname{cone}(S \to j_*)$, there is a natural morphism $\widetilde{S} \to \widetilde{T}$ such that the diagram

$$\begin{tikzcd}
\widetilde{\mu} \circ j_* \arrow[rr] \arrow[dr] & & \widetilde{S} \arrow[dl] \\
& \widetilde{T} &
\end{tikzcd}$$

commutes. Hence we get another commutative diagram

$$\begin{tikzcd}
\widetilde{\mu} \circ j_* \arrow[r] \arrow[d] & j_* \circ \widetilde{H} \arrow[r] \arrow[d] & j_* \circ \widetilde{T} \arrow[r] \arrow[d] & \widetilde{\mu} \circ \widetilde{T} \arrow[d] \\
\widetilde{S} \arrow[r] \arrow[rrr, bend right, "\gamma"'] & j_* \circ \widetilde{T} \arrow[r, equal] & j_* \circ \widetilde{T} \arrow[r] & \widetilde{S} \circ \widetilde{T}
\end{tikzcd}$$

where for any $\mathcal{V}_\bullet$, $\gamma_{\mathcal{V}_\bullet}$ is the composition

$$\begin{tikzcd}
& & \mathcal{I} \otimes \mathcal{V}_\bullet \arrow[r] \arrow[d] & \sigma^* \mathcal{V}_\bullet \arrow[r] & \mathcal{V}_\bullet \\
\mathcal{I}^{\otimes 2} \otimes \mathcal{V}_\bullet \arrow[r] & \sigma^*(\mathcal{I} \otimes \mathcal{V}_\bullet) \arrow[r] & \mathcal{I} \otimes \mathcal{V}_\bullet & &
\end{tikzcd}$$

If $\mathfrak{K}_n$ is the functor from $\mathrm{C}^{\mathrm{b}}(X)$ to $\mathrm{C}^{\mathrm{b}}(S)$ defined as the iterated cone of the functors

$$j_* \longrightarrow \widetilde{S} \xrightarrow{\gamma} \widetilde{S} \circ \widetilde{T} \xrightarrow{\gamma_{\widetilde{T}}} \cdots \xrightarrow{\gamma_{\widetilde{T}^{n-2}}} \widetilde{S} \circ \widetilde{T}^{n-1},$$

then there is a natural morphism from $\mathfrak{F}_n^{\mathrm{loc}}$ to $\mathfrak{K}_n$, and if $\Gamma_\sigma^{(n)}$ is the composition

$$\aleph(\mathcal{V}_\bullet) \xrightarrow{\Gamma_\sigma} \bigoplus_{p \geq 0} \mathcal{I}^{\otimes p} \otimes \mathcal{V}_\bullet\,[p] \longrightarrow \bigoplus_{p=0}^{n} \mathcal{I}^{\otimes p} \otimes \mathcal{V}_\bullet\,[p]$$

we have a commutative diagram

$$\begin{tikzcd}
H^{[n]} \arrow[rrr, bend left] & & & \\
& j^* \mathfrak{F}_n^{\mathrm{loc}} \arrow[r] \arrow[dd] & H^{[[n]]} \arrow[dd] & \\
\aleph \arrow[uu, "\chi_n"] \arrow[ur] \arrow[dr] \arrow[dd, "\Gamma_\sigma^{(n)}"'] & & & \\
& j^* \mathfrak{K}_n \arrow[r] & T^{[[n]]} & \\
T^{[n]} \arrow[rrr, bend right] & & &
\end{tikzcd}$$

This finishes the proof. $\square$

Appendix A. Multiplicativity of principal parts

Let X be a smooth scheme over a field of characteristic zero (or on a complex manifold). Our aim is to prove that the principal parts functor P^1_X is naturally a lax monoidal functor. Although we didn't find explicitly the material of this section in the literature, the method we use can be found in a slightly different form in [25].

Let D be the diagonal in X^2, let W be the subscheme of X^2 defined by

$$\mathcal{O}_W = \mathcal{O}_{X^2}/\mathcal{I}_D^2,$$

let p_i the two projections from X^2 to X, and let q_i (resp q_{ij}) be the three projections from X^3 to X (resp. from X^3 to X^2). Then for any sheaves $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{O}_X$-modules,

$$\begin{aligned}
\mathrm{P}^1_X(\mathcal{F}) \otimes \mathrm{P}^1_X(\mathcal{G}) &= p_{1*}(\mathcal{O}_W \otimes p_2^*\mathcal{F}) \otimes \mathrm{P}^1_X(\mathcal{G}) \\
&= p_{1*}(\mathcal{O}_W \otimes p_2^*\mathcal{F} \otimes p_1^*\,\mathrm{P}^1_X(\mathcal{G})) \\
&= p_{1*}(\mathcal{O}_W \otimes p_2^*\mathcal{F} \otimes p_1^*\,p_{1*}(\mathcal{O}_W \otimes p_2^*\mathcal{G})) \\
&= p_{1*}(\mathcal{O}_W \otimes p_2^*\mathcal{F} \otimes q_{12*}\,q_{13}^*(\mathcal{O}_W \otimes p_2^*\mathcal{G})) \\
&= q_{1*}(q_{12}^*\,\mathcal{O}_W \otimes q_{13}^*\,\mathcal{O}_W \otimes q_2^*\mathcal{F} \otimes q_3^*\mathcal{G}).
\end{aligned}$$

Let $\delta\colon X \times X \to X \times X^2 = X^3$ be the partial diagonal injection on the two last factors of X^3 and T be the image of δ. Then we get a morphism

$$\begin{aligned}
\mathrm{P}^1_X(\mathcal{F}) \otimes \mathrm{P}^1_X(\mathcal{G}) &\to q_{1*}(q_{12}^*\,\mathcal{O}_W \otimes q_{13}^*\,\mathcal{O}_W \otimes \mathcal{O}_T \otimes q_2^*\mathcal{F} \otimes q_3^*\mathcal{G}) \\
&= q_{1*}(q_{12}^*\,\mathcal{O}_W \otimes q_{13}^*\,\mathcal{O}_W \otimes \delta_*\mathcal{O}_{X^2} \otimes q_2^*\mathcal{F} \otimes q_3^*\mathcal{G}) \\
&= q_{1*}\delta_*(\delta^*(q_{12}^*\,\mathcal{O}_W \otimes q_{13}^*\,\mathcal{O}_W) \otimes \delta^*(q_2^*\mathcal{F} \otimes q_3^*\mathcal{G})) \\
&= p_{1*}(\mathcal{O}_W \otimes p_2^*\mathcal{F} \otimes p_2^*\mathcal{G}) \\
&= \mathrm{P}^1_X(\mathcal{F} \otimes \mathcal{G})
\end{aligned}$$

which is a morphism of bifunctors

$$\mathfrak{m}\colon \mathrm{P}^1_X(\star) \otimes \mathrm{P}^1_X(\star\star) \longrightarrow \mathrm{P}^1_X(\star \otimes \star\star).$$

Lemma A.1. *The morphism* $\mathfrak{m}$ *is associative.*

Proof. For any positive integer n, let us denote by Δ_{ij} the partial diagonal in X^n corresponding to the equality of the ith and j^{th} components, and let $\overline{\Delta}_{ij}$ be its first formal neighbourhood in X^n. For $n = 3$, there is a natural morphism

$$\mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\overline{\Delta}_{13}} \longrightarrow \mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\overline{\Delta}_{13}} \otimes \mathcal{O}_{\Delta_{23}}$$

between subsheaves of X^3. This morphism, interpreted as a morphism of correspondences from X^2 to X, is exactly $\mathfrak{m}$. Then the associativity of $\mathfrak{m}$ follows from the commutativity of the diagram of subsheaves of X^4

$$\begin{tikzcd}
\mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\overline{\Delta}_{13}} \otimes \mathcal{O}_{\overline{\Delta}_{14}} \arrow[r] \arrow[d] & \mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\overline{\Delta}_{13}} \otimes \mathcal{O}_{\overline{\Delta}_{14}} \otimes \mathcal{O}_{\Delta_{23}} \arrow[d] \\
\mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\overline{\Delta}_{13}} \otimes \mathcal{O}_{\overline{\Delta}_{14}} \otimes \mathcal{O}_{\Delta_{24}} \arrow[r] & \mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\overline{\Delta}_{13}} \otimes \mathcal{O}_{\overline{\Delta}_{14}} \otimes \mathcal{O}_{\Delta_{234}}
\end{tikzcd}$$

viewed as correspondences between X^3 and X. □

The sheaf $\mathrm{P}^1_X(\mathcal{O}_X)$ is canonically isomorphic to $\Omega^1_X \oplus \mathcal{O}_X$. Hence the second inclusion defines a natural morphism

$$\mu\colon \mathcal{O}_X \longrightarrow \mathrm{P}^1_X(\mathcal{O}_X).$$

Proposition A.2. *The couple* $(\mathfrak{m}, \mu)$ *endows the principal parts functor* P^1_X *witht he structure of a lax monoidal functor.*

Proof. We must check that the properties of Definition 2.5 are satisfied. For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules, let us describe the composition

$$\mathrm{P}^1_X(\mathcal{F}) \xrightarrow{\mathrm{id}\otimes\mu} \mathrm{P}^1_X(\mathcal{F}) \otimes \mathrm{P}^1_X(\mathcal{O}_\mathcal{X}) \xrightarrow{\mathfrak{m}} \mathrm{P}^1_X(\mathcal{F}).$$

The unit morphism $\mathcal{O}_X \to \mathrm{P}^1_X(\mathcal{O}_X)$ is given by the morphism

$$\mathcal{O}_X \longrightarrow p_{1*}\,\mathcal{O}_{X^2} \longrightarrow p_{1*}\,\mathcal{O}_{\overline{\Delta}_{12}}.$$

Let us consider the diagram

$$\begin{tikzcd}
q_{1*}(\mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\overline{\Delta}_{13}} \otimes q_2^*\mathcal{F}) \arrow[r] & q_{1*}(\mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\overline{\Delta}_{13}} \otimes \mathcal{O}_{\Delta_{23}} \otimes q_2^*\mathcal{F}) \arrow[d, equal] \\
q_{1*}(\mathcal{O}_{\overline{\Delta}_{12}} \otimes q_2^*\mathcal{F}) \arrow[u] \arrow[r] & q_{1*}(\mathcal{O}_{\overline{\Delta}_{12}} \otimes \mathcal{O}_{\Delta_{23}} \otimes q_2^*\mathcal{F}) \arrow[d, equal] \\
p_{1*}(\mathcal{O}_{\overline{\Delta}_{12}} \otimes p_2^*\mathcal{F}) \arrow[u] \arrow[uu, bend left] \arrow[r, equal] & p_{1*}(\mathcal{O}_{\overline{\Delta}_{12}} \otimes p_2^*\mathcal{F})
\end{tikzcd}$$

The top horizontal arrow is the map $\mathrm{P}^1_X(\mathcal{F}) \otimes \mathrm{P}^1_X(\mathcal{O}_X) \to \mathrm{P}^1_X(F)$, and the top round arrow is the map $\mathrm{P}^1_X(\mathcal{F}) \to \mathrm{P}^1_X(\mathcal{F}) \otimes \mathrm{P}^1_X(\mathcal{O}_X)$. This proves the first property of Definition 2.5. The second one is proven in the same way. □

Appendix B. Derived equalizers via model categories

In this section, we explain briefly how to use model categories to prove that the derived equalizers introduced in §3.2.3 can interpreted as specific derived Quillen functors.

Let $\mathcal{M}$ be a model category. For any object a in $\mathcal{M}$, we denote by $\mathcal{M}/a$ the model category of objects lying over a. For any morphism $\varphi\colon a \to b$ in $\mathcal{M}$, the push forward functor

$$\varphi_*\colon \mathcal{M}/a \longrightarrow \mathcal{M}/b$$

is a left Quillen functor, it admits a right adjoint: we call it the pullback functor and denote it by

$$\varphi^*\colon \mathcal{M}/b \longrightarrow \mathcal{M}/a.$$

If n in a positive integer and $b = \Pi_{i\in\{1,n\}}\, a = a^n$, we have

$$\mathrm{Hom}_\mathcal{M}(a, a^n) \simeq \mathrm{Hom}_\mathcal{M}(a, a)^n.$$

Hence there is a natural map $\mathrm{i}_n\colon a \to a^n$ corresponding via the above isomorphisms to $(\mathrm{id}_\mathcal{M}, \ldots, \mathrm{id}_\mathcal{M})$. and the pullback functor

$$\mathrm{i}_n^*\colon \mathcal{M}/a^n \longrightarrow \mathcal{M}/a$$

is a right Quillen functor. The functor i_n^* admits a very simple description: an object in $\mathcal{M}/a^n$ consists of an object m on $\mathcal{M}$ together with n maps in $\mathrm{Hom}_\mathcal{C}(m, a)$. Then its image by i_n is the equalizer of these n maps.

We can derive these functors, obtaining a pair of adjoint functors

$$\mathrm{Ho}(\mathcal{M}/a) \underset{\mathrm{Ri}_n^*}{\overset{\mathrm{Li}_n*}{\rightleftarrows}} \mathrm{Ho}(\mathcal{M}/a^n)$$

Explicitly, the functor Ri_n* is obtained as follows: for any object $c \to a^n$ in $\mathcal{M}_{a^n}$, we take an object c' such that the composition

$$c' \longrightarrow c \longrightarrow a^n$$

is a fibration in $\mathcal{M}$. Then $\mathrm{Ri}_n^*(c) = \mathrm{i}_n^*(c')$.

We apply this construction to a very specific situation corresponding to the setting of derived equalizers: let $\mathcal{C}$ be a $\mathbf{k}$-linear category and let $\mathcal{M}$ be the category of dg modules on $\mathrm{C}^{\mathrm{b}}(\mathcal{C}) \otimes \mathrm{C}^b(\mathcal{C})^{\mathrm{op}}$. Then $\mathcal{M}$ can be described as follows: its objects are dg functors from $\mathrm{C}^{\mathrm{b}}(\mathcal{C}) \otimes \mathrm{C}^b(\mathcal{C})^{\mathrm{op}}$ to the category $\mathrm{C}(\mathbf{k})$ of complexes of $\mathbf{k}$-vector spaces, and its morphisms are natural transformations between dg functors.

As any category of dg-modules, $\mathcal{M}$ has a natural model category structure defined by Toën and Vaquié (*see* [29, Def. 3.1]), where weak equivalences and fibrations admit the following description: if $\Psi\colon U \to V$ is a natural transformation betwen two objects of $\mathcal{M}$ considered as dg fonctors, then Ψ is a weak equivalence (resp. a fibration) if and only if for any object K of $\mathrm{C}^{\mathrm{b}}(\mathcal{C}) \otimes \mathrm{C}^b(\mathcal{C})^{\mathrm{op}}$, $\Psi(K)$ is a quasi-isomorphism (resp. $\Psi(K)$ is surjective). There is a fully faithful embedding

$$\iota\colon \mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C})) \longrightarrow \mathcal{M}$$

given by

$$\iota(F)(K \otimes L) = \mathrm{Hom}_{\mathrm{C}(\mathbf{k})}(L, F(K)).$$

Let $\varphi\colon F \to G$ be a natural transformation between to dg endofunctors of $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$. Then $\iota(\varphi)$ is a weak equivalence (resp. a fibration) in the model category $\mathcal{M}$ if and only if for any object K of $\mathrm{C}^{\mathrm{b}}(\mathcal{C})$, the morphism φ_K is a quasi-isomorphism[15] (resp. φ_K is surjective).

Assume to be given a couple (H, Ψ) where H is in $\mathrm{Fct}_{\mathrm{dg}}(\mathrm{C}^{\mathrm{b}}(\mathcal{C}))$ and $\Psi\colon H \to \mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$ is a natural transformation. For any nonnegative integer n, H^n is endowed with n natural maps to $\mathrm{id}_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$, so that we can consider $\iota(H^n)$ as an element in the category $\mathcal{M}/\mathrm{id}^n_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$.

Proposition B.1. *Assume to be given a triplet $(\mathcal{C}, H, \Psi)$. Then for any nonnegative integer n, the following assertions are valid:*

- $\iota(H^{[n]}) = \mathrm{i}_n^*\{\iota(H^n)\}$
- $\Delta^n_{\widetilde{H}}$ *is a fibrant element in* $\mathcal{M}/\mathrm{id}^n_{\mathrm{C}^{\mathrm{b}}(\mathcal{C})}$ *and the natural map* $\Delta^n_{\widetilde{H}} \to H^n$ *is a weak equivalence.*
- $\iota(H^{[[n]]})$ *is isomorphic to* $\mathrm{Ri}_n^*\{\iota(H^n)\}$.

Proof. The first assertion is straightforward, and the third assertion is a direct consequence of the second. The second assertion follows from equation (7). □

References

[1] Dima Arinkin and Andrei Căldăraru. When is the self-intersection of a subvariety a fibration? *Adv. Math.*, 231(2):815–842, 2012.

[2] Dima Arinkin, Andrei Căldăraru, and Marton Hablicsek. On the formality of derived intersections. *Preprint*, 2014.

[3] P. Bressler, R. Nest, and B. Tsygan. Riemann-Roch theorems via deformation quantization. I, II. *Adv. Math.*, 167(1):1–25, 26–73, 2002.

[4] Ragnar-Olaf Buchweitz and Hubert Flenner. The global decomposition theorem for Hochschild (co-)homology of singular spaces via the Atiyah-Chern character. *Adv. Math.*, 217(1):243–281, 2008.

[15] This means that φ is a quasi-isomorphism as defined in Definition 3.5.

[5] D. Calaque and M. Van den Bergh. Hochschild cohomology and Atiyah classes. *Adv. Math.*, 224(5):1839–1889, 2010.
[6] Damien Calaque, Andrei Căldăraru, and Junwu Tu. On the Lie algebroid of a derived self-intersection. *Adv. Math.*, 262:751–783, 2014.
[7] Damien Calaque and Carlo A. Rossi. *Lectures on Duflo isomorphisms in Lie algebra and complex geometry*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2011.
[8] Damien Calaque, Carlo A. Rossi, and Michel Van den Bergh. Căldăraru's conjecture and Tsygan's formality. *Ann. of Math. (2)*, 176(2):865–923, 2012.
[9] Damien Calaque, Carlo A. Rossi, and Michel Van den Bergh. Căldăraru's conjecture and Tsygan's formality. *Ann. of Math. (2)*, 176(2):865–923, 2012.
[10] Andrei Căldăraru. The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism. *Adv. Math.*, 194(1):34–66, 2005.
[11] Vasiliy Dolgushev, Dmitry Tamarkin, and Boris Tsygan. The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal. *J. Noncommut. Geom.*, 1(1):1–25, 2007.
[12] Julien Grivaux. On a conjecture of Kashiwara relating Chern and Euler classes of o-modules. *J. Differential Geom.*, 90(2):267–275, 2012.
[13] Julien Grivaux. Formality of derived intersections. *Documenta Math.*, 19:1003–1016, 2014.
[14] Julien Grivaux. The Hochschild-Kostant-Rosenberg isomorphism for quantized analytic cycles. *Int. Math. Res. Not. IMRN*, (4):865–913, 2014.
[15] G. Hochschild, Bertram Kostant, and Alex Rosenberg. Differential forms on regular affine algebras. *Trans. Amer. Math. Soc.*, 102:383–408, 1962.
[16] Daniel Huybrechts and Richard P. Thomas. Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes. *Math. Ann.*, 346(3):545–569, 2010.
[17] M. Kapranov. Rozansky-Witten invariants via Atiyah classes. *Compositio Math.*, 115(1):71–113, 1999.
[18] M. Kashiwara. Unpublished letter to Pierre Schapira, 1991.
[19] M. Kashiwara and P. Schapira. *Deformation quantization modules*. Astérisque. Société mathématique de France, 2012.
[20] Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.
[21] Maxim Kontsevich. Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139. Birkhäuser, Basel, 1995.
[22] Maxim Kontsevich. Deformation quantization of Poisson manifolds. *Lett. Math. Phys.*, 66(3):157–216, 2003.
[23] Max Lieblich. Moduli of complexes on a proper morphism. *J. Algebraic Geom.*, 15(1):175–206, 2006.
[24] Wendy Lowen. Obstruction theory for objects in abelian and derived categories. *Comm. Algebra*, 33(9):3195–3223, 2005.
[25] Nikita Markarian. The Atiyah class, Hochschild cohomology and the Riemann-Roch theorem. *J. Lond. Math. Soc. (2)*, 79(1):129–143, 2009.
[26] Daniel Quillen. On the (co-) homology of commutative rings. In *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, pages 65–87. Amer. Math. Soc., Providence, R.I., 1970.
[27] Ajay C. Ramadoss. The relative Riemann-Roch theorem from Hochschild homology. *New York J. Math.*, 14:643–717, 2008.
[28] Richard G. Swan. Hochschild cohomology of quasiprojective schemes. *J. Pure Appl. Algebra*, 110(1):57–80, 1996.
[29] Bertrand Toën. The homotopy theory of dg-categories and derived Morita theory. *Invent. Math.*, 167(3):615–667, 2007.
[30] Bertrand Toën. Derived algebraic geometry. *EMS Surv. Math. Sci.*, 1(2):153–240, 2014.
[31] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.
[32] Amnon Yekutieli. The continuous Hochschild cochain complex of a scheme. *Canad. J. Math.*, 54(6):1319–1337, 2002.
[33] Shilin Yu. *The Dolbeault dga of a formal neighborhood*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–The Pennsylvania State University.

CNRS, I2M (Marseille) & IHÉS
E-mail address: jgrivaux@math.cnrs.fr