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# Cloud-rain predator-prey interactions: analyzing some properties of the Koren-Feingold model and introduction of a new species-competition bulk system with a Hopf bifurcation

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## Abstract

This paper deals with some properties of predator-prey cloud-rain models. Focus is put on scaling and on some mathematical features such as stability and limit cycles. Precisely, the Koren-Feingold delay differential equation model is first investigated and it is shown that it has no limit cycles. Then, by considering another point of view (*i.e.* species competition dynamics) for parametrizing cloud-rain processes, a system of ordinary differential equations to model these processes is formulated. Some examples are given to illustrate that this model reproduces in a realistic way the essential macroscopic behavior of a cloud-rain system. The model has a Hopf bifurcation at which certain properties of cloud-rain interactions in the model are represented. This is an important point to prepare for further examination of cloud synchronization in a cloud field by Kuramoto model, for instance.

*Keywords:* Cloud-rain interactions, Species competition dynamics, Predator-prey model, Cloud physics

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1 **1. Introduction**

2 Clouds are complex nonlinear dynamic systems with many degrees of free-  
 3 dom and interactions across a vast range of spatio-temporal scales. Despite  
 4 this complexity, certain non-trivial aspects of their macrobehavior are pre-  
 5 dictable without a consideration of the full complexity of the dynamic system  
 6 (Feingold et al., 2010; Koren and Feingold, 2011; Feingold and Koren, 2013;  
 7 Koren et al., 2017; Mülmenstädt and Feingold, 2018). This implies that mo-  
 8 dels that capture the essential physics have the potential to contribute to our  
 9 understanding of clouds and their interactions. Such models are thus always  
 10 welcome, especially in conjunction with more detailed models. This is of im-  
 11 portance in the context of the climate system (*e.g.* Held, 2005) since, as is  
 12 well known, clouds play a major role in the climate system (IPCC, 2013).

An example of macrobehavior that can be tackled by nonlinear dynamics  
 is the interaction between cloud droplets and rain. Recently, Koren and Fein-  
 gold (2011) (hereafter KF11) analyzed cloud-rain coupling by means of the  
 following non-linear system of delay differential equations (DDEs):

$$\begin{cases} \frac{dH}{dt} = \frac{H_0 - H}{\tau_1} - \frac{\alpha H^2(t - T)}{c_1 N_d(t - T)} \\ \frac{dN_d}{dt} = \frac{N_0 - N_d}{\tau_2} - \alpha c_2 H^3(t - T) \end{cases} \quad (1)$$

13 with  $c_1 = 2 \times 10^{-6} \text{ mm m}^{-2}$ ,  $c_2 = 0.3 \text{ m}^{-1}$ , and  $\alpha = 2 \text{ mm m}^{-6} \text{ d}^{-1}$  (see also  
 14 Table 1 in appendix). Here,  $H$  is cloud depth (in m) and  $N_d$  is cloud droplet  
 15 concentration (in  $\text{cm}^{-3}$ ). These are the two macroscopic degrees of freedom.  
 16 In addition,  $t$  is time and  $T$  is an arbitrary constant delay (both in minutes).  
 17 The constants  $\tau_1$  and  $\tau_2$  are timescales of the order of ten minutes.  $H_0$  is a  
 18 height that represents the full environmental potential for cloud development  
 19 (*i.e.* the maximum cloud depth possible) and  $N_0$  is the background concen-  
 20 tration of aerosol (*i.e.* the maximum concentration of cloud droplets that  
 21 can be reached). According to KF11, this system exhibits predator-prey like  
 22 behavior. In particular, two examples of cloud behavior are provided which  
 23 represent (*i*) an oscillator limit cycle and (*ii*) a damped oscillator.

24 In the first part of this paper, we explore some physical and mathema-  
 25 tical properties of Eq.1 that we consider important from the perspective of  
 26 nonlinear dynamics that have thus far not been addressed. More precisely,  
 27 our analysis, which is presented in Section 2, concentrates on scale analysis,

28 form of the equations, stability analysis, and the question of the existence of  
29 limit cycles by means of the Busenberg theorem applied to DDEs.

30 In the second part of this paper, we introduce a bulk model which follows  
31 the initial idea of modeling the macrobehavior of a cloud-rain system and has  
32 properties in common with species competition dynamics. In contrast to the  
33 KF11 model, it is a system of ODEs rather than DDEs. However, the model  
34 presented here reproduces the behavior of the KF11 model, and is derived  
35 from a physically-based parametrization. This model differs from the KF11  
36 model not only in its structure, but in the variety of its dynamics which we  
37 examine through the lense of nonlinear dynamics, *e.g.* linear stability analysis  
38 and bifurcation theory. This exploration yields regions of parameter space in  
39 which clouds grow at the expense of rain and *vice versa*.

40 It must be emphasized that this model does not consider the full com-  
41 plexity of cloud-precipitation interactions; rather, we examine certain predic-  
42 table elements of the macrobehavior of the full complex dynamical system.  
43 In particular, the model we propose incorporates population dynamics in a  
44 natural way, which includes exchanges with the surrounding environment.  
45 Our model is presented in Section 3 and examples are given in order to show  
46 that it is physically realistic and able to reproduce the macroscopic behavior  
47 of a cloud-rain system. Section 4 provides a discussion and conclusions.

## 48 **2. Analysis of some physical and mathematical properties of the** 49 **KF11 model**

### 50 *2.1. Scale or order-of-magnitude analysis*

51 The system (1) can be solved numerically by means of the pydelay package  
52 of Python (<http://pydelay.sourceforge.net/>) or MATLAB's dde23 al-  
53 gorithm. The PyDDE solver can also be useful. These methods are based on  
54 the Bogacki-Shampine method which is a 3(2) Runge-Kutta scheme adapted  
55 to DDEs. However, before diving into numerical computations, we analyze  
56 the magnitudes of the different terms in the system of equations.

In order to have a consistent set of equations in terms of units, we replace  
 $\alpha$  and  $c_2$  in Eq. (1) by  $\alpha' = 10^{-3} \alpha / (60 \times 24)$  and  $c'_2 = 10^{-6} c_2$  (see appendix).  
Typically, for cloud height,  $H_0 \sim 500$  m and so, by definition of  $H_0$ ,  $0 < H < H_0$ .  
For cloud droplet concentration,  $N_0 \sim 400 \text{ cm}^{-3}$  and so, by definition  
of  $N_0$ ,  $N_d < N_0$ . A reasonable minimum for  $N_d$  ( $N_{d,min}$ ) is about  $50 \text{ cm}^{-3}$ .  
Hence, considering the values of the constants involved in Eq. (1), *i.e.*  $\tau_1 =$

$\tau_2 \sim 60$  min, we find:

$$\frac{H_0 - H}{\tau_1} < \frac{H_0}{\tau_1} \approx 8 \text{ m min}^{-1} \quad \frac{\alpha' H^2}{c_1 N_d} < \frac{\alpha' H_0^2}{c_1 N_{d,min}} \approx 3500 \text{ m min}^{-1}$$

and

$$\frac{N_0 - N_d}{\tau_2} < \frac{N_0}{\tau_2} \approx 7 \text{ cm}^{-3} \text{ min}^{-1} \quad \alpha' c_2' H^3 < \alpha' c_2' H_0^3 \approx 0.5 \text{ cm}^{-3} \text{ min}^{-1}$$

57 It follows that, since  $H$  and  $N_d$  are physically expected to maintain or-  
 58 ders or magnitude as mentioned above during the evolution of the cloud  
 59 rain system, that  $\alpha' H^2/(c_1 N_d)$  and  $(N_0 - N_d)/\tau_2$  are the dominant terms  
 60 in Eq. (1). This is particularly clear for the equation governing the evolu-  
 61 tion of  $H$ . Concerning the equation for  $dN_d/dt$ , having a ratio close to 1  
 62 implies that  $c_2$  be  $10^4$  or  $10^5$  higher than the value given in Table 1. Conse-  
 63 quently, the evolution of  $N_d$  is either an increasing or a decreasing expo-  
 64 nential function according to the sign (positive or negative, respectively)  
 65 of  $N_0 - N_d$ . The examples chosen by KF11 have positive signs. Precisely,  
 66  $N_d(t) - N_0 = [N_d(0) - N_0] \exp(-t/\tau_2)$ , where  $N_d(0) = N_d(t = 0)$ . It ensues  
 67 that  $N_d$  evolves exponentially towards  $N_0$ . With respect to  $dH/dt$ , the do-  
 68 minant term is always negative and, unless  $N_d$  becomes very small, it will  
 69 remain the dominant term in the evolution of  $H$ . Hence,  $H$  is a decreasing  
 70 function which must stop at  $H = 0$  by definition. This scale analysis, based  
 71 only on the physical realistic orders of magnitude of  $H$  and  $N_d$  is independent  
 72 of any constant delay in the equation. Some numerical computations with the  
 73 numerical scheme mentioned above have confirmed the evolution described  
 74 above (Fig. 1). Nonetheless, we have obtained some oscillatory behavior in  
 75 rare cases by varying arbitrarily some constants in Eq. (1) as the constant  
 76  $c_1$  around  $10^{-3} \text{ mm m}^{-1}$  and the constant  $c_2$  around  $10^4 \text{ m}^{-1}$  or  $10^5 \text{ m}^{-1}$ . We  
 77 have observed that these oscillations are, in addition, extremely sensitive to  
 78 the value of  $c_2$  chosen since they disappear if  $c_2$  is changed even slightly (we  
 79 recover the evolution described above).

## 80 2.2. Form of the system of equations

It is worth noting that the system (1) of two equations is not formally similar to a predator-prey system, *stricto sensu*. Indeed, in its basic form, the Lotka-Volterra system is the following ( $x$  and  $y$  are the two degrees of

freedom):

$$\begin{cases} \frac{dx}{dt} = ax - bxy \\ \frac{dy}{dt} = -cy + dxy \end{cases} \quad (2)$$

with  $\{a, b, c, d\}$  a set of four positive constants. The signature  $(+, - | -, +)$ , or sometimes  $(+, - | +, -)$ , is also present in more elaborate versions of Lotka-Volterra models (*e.g.* Lipowski and Lipowska, 2000) since it is a characteristic of species-competition models, with or without predation. However, the system 1 has the form:  $dH/dt = -AH - BH^2/N_d$  and  $dN_d/dt = -CN_d - DH^3$  with  $\{A, B, C, D\}$  a set of four positive constants. So, system (1) does not have the general form of a Lotka-Volterra system since the signature is fully negative,  $(-, - | -, -)$ . It follows that  $\dot{H}$  and  $\dot{N}_d$  are negative if  $(H, N_d)$  always satisfy the two inequalities:

$$\frac{H}{\tau_1} + \frac{\alpha'H^2}{c_1N_d} > \frac{H_0}{\tau_1} \sim 10 \text{ m min}^{-1}$$

and

$$\frac{N_d}{\tau_2} + \alpha'c_2'H^3 > \frac{N_0}{\tau_2} \sim 1 - 10 \text{ cm}^{-3} \text{ min}^{-1}$$

81 So, with regard to the typical orders of magnitude of the two degrees of  
 82 freedom ( $H \sim 400 \text{ m}$ ,  $N_d \sim 100 \text{ cm}^{-3}$ ), care is needed in the choice of the  
 83 constants (*i.e.* in the model parametrization) in order to avoid always de-  
 84 creasing functions  $H(t)$  and  $N_d(t)$ .

### 85 2.3. Stability analysis

To determine further some of the properties of the system (1), we briefly discuss the stability of the model (with no delay) near its equilibrium points. These are independent of any delays, so that a system of DDEs has the same equilibrium points as the corresponding system (zero delay) of ODEs. However, investigating the stability for DDEs is a more complex task (*e.g.* Engelborghs et al., 2000, and references therein) which is beyond the scope of this paper (some elements can be found in Koren et al. (2017, Section IV)). Nonetheless, it is still instructive to analyze stability in the simplest case of no delay. A quick calculation yields that the KF11 system has an equilibrium point given by

$$N_d^e = N_0 - \alpha'c_2'\tau_2(H^e)^3$$

and

$$H^e = \frac{c_1 N_d^e}{2\alpha' \tau_1} \left[ -1 + \left( 1 + \frac{4\alpha' \tau_1 H_0}{c_1 N_d^e} \right)^{1/2} \right] \quad (3)$$

Physically, because  $N_d^e$  (and  $H^e$ ) must be positive, the value of  $\alpha' c_2'$  is constrained:  $\alpha' c_2' < N_0 / [\tau_2 (H^e)^3]$ . The behavior of Eq. (1) near this equilibrium point can be obtained by evaluating the Jacobian matrix  $J$  at  $(H^e, N_d^e)$  and finding its eigenvalues. These are given by the well-known formula:

$$\lambda_{\pm} = \frac{1}{2} \left[ T_J \pm (T_J^2 - 4\Delta_J)^{1/2} \right]$$

where  $T_J$  and  $\Delta_J$  are respectively the trace and the determinant of  $[J]$ . A straightforward calculation yields:

$$T_J = - \left( \frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{2\alpha' H^e}{c_1 N_d^e} \right)$$

and

$$\Delta_J = \frac{1}{\tau_2} \left( \frac{1}{\tau_1} + \frac{2\alpha' H^e}{c_1 N_d^e} \right) + 3\alpha'^2 c_2' \frac{(H^e)^4}{c_1 (N_d^e)^2}$$

86 Since  $T_J < 0$  and  $\Delta_J > 0$ , the equilibrium point  $(H^e, N_d^e)$  is a stable node  
 87 or a stable focus (spiral) according to the sign of  $(T_J^2 - 4\Delta_J)$ . We note that  
 88 if the value of  $\alpha' c_2'$  does not satisfy the above constraint, then  $T_J$  can be  
 89 negative, which means that Eq. (1) describes an unstable dynamical system.  
 90 In addition, since the sign of  $T_J$  never changes, there is no local bifurcation.  
 91 A similar analysis that corroborates the above result has been performed  
 92 recently by Jiang and Wang (2014).

#### 93 2.4. Existence of limit cycles

94 One of the important points of this paper is the possibility of limit cycle  
 95 solutions of the KF11 delay differential equations (DDEs). This needs ri-  
 96 gorous examination. First, we note that the KF11 equations with constant  
 97 delay  $T$  do not have any limit cycle solutions with period  $T$ . To see this, we  
 98 consider either the Bendixson-Dulac theorem or the special case known as  
 99 Bendixson's criterion (*e.g.* Minorsky, 1962; Glansdorff and Prigogine, 1971).  
 100 To illustrate the use and consequences of this theorem to the problem at  
 101 hand, we first rewrite the KF11 system:  $dH/dt = f[H(t - T), N_d(t - T)]$   
 102 and  $dN_d/dt = g[H(t - T), N_d(t - T)]$ , where the functions  $f$  and  $g$  are

103 given by the rhs (right hand sides) in Eq. (1). The Bendixson-Dulac theo-  
 104 rem may be applied to this system by considering the sign of the expression  
 105  $\partial_H(\phi f) + \partial_{N_d}(\phi g)$ , where  $\phi(H, N_d) = 1$  (this case is Bendixson's criterion).

106 Now, suppose that this set of equations has a non-constant  $T$ -periodic  
 107 solution  $[H(t), N_d(t)]$ . Then, it follows that  $dH/dt = f[H(t), N_d(t)]$  and  
 108  $dN_d/dt = g[H(t), N_d(t)]$ . A simple calculation shows that  $\partial_H(f) + \partial_{N_d}(g)$   
 109 is always negative. We can therefore conclude by the Bendixson-Dulac theo-  
 110 rem that no solutions of period  $T$  exist in the simply connected region defined  
 111 by positive values of  $H$  and  $N_d$ .

112 Since the KF11 equations are DDEs, a customary application of the  
 113 Bendixson-Dulac theorem to prove the non-existence of limit cycles is inade-  
 114 quate. However, this theorem has been generalized (Busenberg and van den  
 115 Driessche, 1993, Section 4) to higher dimensional ODEs, including DDEs like  
 116 those proposed by KF11, for instance. This generalized "Bendixson-Dulac"  
 117 theorem extends the concept of limit cycle to that of simple loop solutions,  
 118 *i.e.* any continuous solution of the system of equations whose orbit contains  
 119 a closed curve. Here, we apply this generalized theorem to the KF11 DDEs  
 120 in order to demonstrate the absence of loop solutions.

To illustrate the use and consequences of this generalized theorem to the  
 problem at hand, especially theorem 4.1 in Busenberg and van den Driessche  
 (1993), we first rewrite the KF11 system:

$$\begin{cases} \frac{dH}{dt} &= F_1(H, N_d) h_t(H_t, N_{d,t}) + l_1(H, N_d) \\ \frac{dN_d}{dt} &= F_2(H, N_d) k_t(H_t, N_{d,t}) + l_2(H, N_d) \end{cases}$$

where  $F_1(H, N_d) = -\alpha/c_1$ ,  $F_2(H, N_d) = -\alpha c_2$ ,  $h_t(H_t, N_{d,t}) = H_t^2/N_{d,t}$  and  
 $k_t(H_t, N_{d,t}) = H_t^3$  are time-delayed functions (the subscript "t" stands for  
 "time-delayed"),  $l_1(H, N_d) = (H_0 - H)/\tau_1$ , and  $l_2(H_d, N_d) = (N_0 - N_d)/\tau_2$ .  
 Theorem 4.1 in Busenberg and van den Driessche (1993) gives conditions  
 under which the KF11 DDEs have no simple loop solution. To verify the  
 first condition, we define the vector function  $g(H, N_d) = (g_1, g_2, 0)$ . Here,  
 $g_1 = \alpha c_2 A^3 + (N_0 - N_d)/\tau_2$ ,  $g_2 = -\alpha A^2/(c_1 B) - (H_0 - H)/\tau_1$ ,  $A$  and  $B$  being  
 constants. For notational simplicity, let  $(u, v)$  represent the coordinates of the



curve in Theorem 4.1 of Busenberg and van den Driessche (1993). Then:

$$\begin{aligned}
g(H, N_d) &\cdot \left( \frac{-\alpha u^2}{c_1 v} + l_1, -\alpha c_2 u^3 + l_2, 0 \right) \\
&= \frac{\alpha^2 c_2 A^2 u^2}{c_1} \left( \frac{u}{B} - \frac{A}{v} \right) + \alpha c_2 l_1 (u^3 + A^3) \\
&\quad - \frac{\alpha l_2}{c_1} \left( \frac{A^2}{B} + \frac{u^2}{v} \right) \\
&\leq 0
\end{aligned}$$

121 This follows from the following observations. The third term always contri-  
122 butes negatively, the second positively, to the sum. The first contributes  
123 negatively whenever  $uv < AB$ . Because  $c_1$  is small, the first and third terms  
124 dominate the second so that, whenever  $uv < AB$ , the entire sum is negative.  
125 Since  $A$  and  $B$  can be taken as large as wished, the region is finally  $\mathbb{R}_+^2$ . To  
126 verify the second condition, we calculate  $\partial_H g_2 - \partial_{N_d} g_1 = 1/\tau_1 + 1/\tau_2 > 0$  for  
127 any solution  $(H, N_d)$  in the region specified above. So, an application of the  
128 Busenberg-van den Driessche theorem implies that there is no simple loop  
129 solution traversed in the clockwise sense (the analogue for DDEs of the limit  
130 cycle of ODEs) for the positive values of  $H$  and  $N_d$ . A similar argument with  
131 vector function  $-g$  works in the opposite sense as well. In other words, there  
132 is no solution of the KF11 system which enclose a smooth oriented simple  
133 closed curve.

### 134 3. A new species-competition bulk model

#### 135 3.1. Description

136 In order to retain the simplicity of the KF11 model with ODEs instead of  
137 DDEs, we propose an approach which incorporates population competition,  
138 interaction, and carrying capacities in a natural way. Here, the populations  
139 are given by cloud water content  $L_c$  ( $\text{g cm}^{-3}$ ), rain water content  $L_r$  ( $\text{g cm}^{-3}$ ),  
140 and cloud droplet number concentration  $N_d$  ( $\text{cm}^{-3}$ ). We emphasize that these  
141 variables are considered here as macroscopic averages and that our model  
142 captures some of the macroscale properties of cloud-precipitation interactions  
143 by means of three relatively simple equations in three variables, rather than a  
144 fine-scale description with complete microphysical details. The model consists  
145 of three coupled differential equations, one each for  $L_c$ ,  $L_r$ , and  $N_d$ . The  
146 system includes three main processes that contribute to the cloud budgets of

147  $L_c$ ,  $L_r$ , and  $N_d$ , namely: (1) sources/sinks which represent exchanges with the  
 148 surroundings, and two internal processes, (2) autoconversion of cloud water  
 149 to rain water, and (3) accretion of cloud water by rain water.

150 The various terms in the equations were suggested by the Seifert and  
 151 Beheng (2001) parametrization for simulating autoconversion and accretion.  
 152 Other terms can be added to the equations, such as self-collection for ins-  
 153 tance, but to retain some measure of simplicity we use only the terms al-  
 154 ready mentioned and which we describe further. This parametrization is a  
 155 straightforward consequence of the stochastic collection equation (STE) with  
 156 polynomial kernel. For this reason, it seems to us that this approach is more  
 157 rigorous than many other methods which consist in using heuristic (or em-  
 158 pirical) parametrizations. Details on the STE can be found in Seifert and  
 159 Beheng (2001) and Pruppacher and Klett (1997) and references therein. It is  
 160 worth mentioning that the goal of Seifert and Beheng (2001) has been to fill  
 161 a gap between heuristic parametrizations and detailed microphysical (com-  
 162 putationally expensive) schemes. The terms representing autoconversion and  
 163 accretion are given by the following expressions (for a comprehensive over-  
 164 view see the references):

– Autoconversion: the contribution to  $dL_c/dt$  and  $dL_r/dt$  is given by

$$\left. \frac{dL_c}{dt} \right|_{auto} = -K L_c^4 N_d^{-2} = - \left. \frac{dL_r}{dt} \right|_{auto} \quad (4)$$

– Accretion: the contribution to  $dL_c/dt$  and  $dL_r/dt$  is given by

$$\left. \frac{dL_c}{dt} \right|_{accret} = -k_r L_c L_r = - \left. \frac{dL_r}{dt} \right|_{accret} . \quad (5)$$

165 We note that these expressions are adapted from the Seifert and Beheng  
 166 (2001) parametrization by considering the variables as bulk, cloud-scale vari-  
 167 ables. The various parameters in the equations above are defined as follows:  
 168  $K = [k_c/(20x^*)] (\nu + 2)(\nu + 4)(\nu + 1)^{-2}$ , where  $x^*$  is a cloud drop mass se-  
 169 parating the cloud droplets from raindrops,  $k_c$  ( $k_r$ ) is a constant from the  
 170 cloud (rain) water kernel, and  $\nu$  is the shape parameter of the gamma dis-  
 171 tribution. See Seifert and Beheng (2001) for complete details. By combining  
 172 these equations, neglecting spatial dependence and adding source/sink terms  
 173 for cloud and rain water contents, we arrive at the following system of bulk

174 coupled first-order differential equations for  $L_c$ ,  $L_r$  and  $N_d$ :

$$\begin{cases} \frac{dL_c}{dt} &= A_c L_c - B_c(N_d)L_c^4 - k_r L_c L_r \\ \frac{dL_r}{dt} &= -A_r L_r + B_c(N_d)L_c^4 + k_r L_c L_r \\ \frac{dN_d}{dt} &= f(N_d, L_c, L_r) \end{cases} \quad (6)$$

Here,  $B_c(N_d) = KN_d^{-2}$ , and  $A_c$  and  $A_r$  are two positive constants that can be chosen so that the source/sink terms represent the background meteorological conditions. In particular,  $A_r$  represents the rain out process. These two constants give the timescales,  $\tau_c = 1/A_c$  and  $\tau_r = 1/|A_r|$ , of cloud water content and rain water content evolutions respectively. In addition,  $f(N_d, L_c, L_r)$  is a function which can be defined according to the question being investigated. Here, we choose to define  $f$  as follows:

$$\frac{dN_d}{dt} = A_c(N_0 - N_d) - \frac{4}{3}k_c L_c^2 - k_r L_r N_d. \quad (7)$$

175 On the rhs, the first term represents a supply of cloud droplets from the  
 176 surroundings,  $N_0$  being the background aerosol concentration that feeds the  
 177 system (nucleation). Considered alone, this term causes an exponential in-  
 178 crease (decrease) of  $N_d$  with  $N_0$  as horizontal asymptote if initially  $N_d$  is lo-  
 179 wer (higher) than  $N_0$ . This represents cloud droplet concentration and when  
 180 considered alone it tends toward a constant background aerosol population  
 181 concentration. Moreover, this term (or similar ones) should be dominant in  
 182 the very early stages of the cloud system evolution. Indeed, at the initial time  
 183 (and close to  $t = 0$ ),  $N_d$  is zero or very small (clear air condition) and must  
 184 increase enough in order that the cloud can further evolve and produce rain.  
 185 The two other terms in Eq. (7) come from Seifert and Beheng (2001, Eqs.  
 186 A-5, A-6, and A-9) and represents respectively cloud droplet self-collection  
 187 and accretion.

188 It has to be recalled that the state variables considered ( $L_c$ ,  $L_r$ , and  $N_d$ )  
 189 in this bulk model are macroscopic averages over the whole cloud. They  
 190 can also be seen as local variables over a volume element of the cloud for  
 191 which they are more or less uniform. A further step would be to introduce in  
 192 the equation an explicit inside-cloud location (*e.g.* altitude  $z$ ) dependence.  
 193 Nonetheless, as showed below, the present model describes some macroscale  
 194 features of could-precipitation coupling.

195 3.2. Comment on the form of the system of equations

196 We note that the first two equations in (6) constitute a predator-prey  
197 system modified to include autoconversion. To see this, observe that the  
198 constants  $\{a, b, c, d\}$  in the Lotka-Volterra equations (Eq. 2) have the follo-  
199 wing formal correspondence with the model presented here:  $\{a = A_c, b =$   
200  $-d = k_r, c = A_r\}$ . Hence, the source/sink and accretion terms form a stan-  
201 dard predator-prey system when considered alone. The quartic terms not  
202 only represent autoconversion of cloud droplets to rain given in the Seifert  
203 and Beheng (2001) parametrization, but also ensure that the cloud droplet  
204 population does not exhibit unbounded growth in the absence of competition.  
205 Thus, it plays a similar role to the carrying capacity term used in population  
206 dynamics to modify some of the unrealistic features of the original Lotka-  
207 Volterra system.

208 The entire system of equations is not rigorously equivalent to a three-  
209 species competition model. As already discussed, the first two equations  
210 constitute a modified Lotka-Volterra system. The autoconversion terms which  
211 modify the traditional predator-prey system have coefficients driven by the  
212 third equation, *viz.* that for  $N_d$ . The interpretation of the varying coefficients  
213 is that the carrying capacities of the water contents vary according to the  
214 state of  $N_d$ , which is more realistic for clouds.

215 It is worth mentioning that  $L_c$  and  $L_r$  are, for physical reasons, necessa-  
216 rily bounded. So, in virtue of the Poincaré-Bendixson theorem, the first two  
217 equations of system (6) have a solution that either converges towards a limit  
218 or presents an asymptotic behaviour that can take the form of a limit cycle.  
219 [Boundedness has not been proved in a mathematical sense but, rather, we](#)  
220 [rely on numerical evidence \(see some examples below, in Subsection 3.4\) and](#)  
221 [on the fact that the present system of equations has much in common with](#)  
222 [other predator-prey models which are known to have bounded solutions. Ob-](#)  
223 [viously, if  \$k\_r\$  and  \$B\_c\$  are zero, the evolution of  \$L\_c\$  is exponentially divergent.](#)  
224 [The terms involving non-zero values of  \$k\_r\$  and  \$B\_c\$  prevent unbounded evolu-](#)  
225 [tion within certain parameter ranges, as is the case in predator-prey models.](#)  
226 [It is far beyond the scope of this paper to address rigorously \(\*i.e.\* mathema-](#)  
227 [tically speaking\) boundedness of solutions, but that could be formulated and](#)  
228 [explored deeply in a future specific work.](#)

229 3.3. Stability analysis

230 3.3.1. Equilibrium points

The system

$$\begin{cases} \frac{dL_c}{dt} = A_c L_c - B_c(N_d) L_c^4 - k_r L_c L_r \\ \frac{dL_r}{dt} = -A_r L_r + B_c(N_d) L_c^4 + k_r L_c L_r \end{cases} \quad (8)$$

has two equilibrium points  $X_e = \{L_c^{(e)}, L_r^{(e)}\}$ . In this section we treat  $N_d$  as a parameter. The first one is trivial:  $X_e^{(1)} = 0$ . There is no cloud nor *a fortiori* rain. Using Eq. (7),  $N_d^{(e)} = N_0$ , which means that the atmospheric volume considered has a constant CCN (cloud condensation nuclei) loading (that does not allow cloud formation). The second equilibrium point  $X_e^{(2)}$  is such that:

$$A_c L_c^{(e)} = |A_r| L_r^{(e)} \quad \text{and} \quad B_c(N_d) \frac{|A_r|^3}{A_c^4} (L_r^{(e)})^3 + \frac{k_r}{A_c} L_r^{(e)} - 1 = 0 \quad (9)$$

Since the two first terms of this third order polynomial are positive, there is only one real positive solution  $L_r^{(e)}$  – the other two are complex conjugates. The expression of  $N_d^{(e)}$  follows from those of  $L_c^{(e)}$  and  $L_r^{(e)}$  using Eq. (7) Plugging  $L_c^{(e)}$  and  $L_r^{(e)}$  into Eq. (7) yields:

$$N_d^{(e)} = \left( A_c + \frac{k_r A_c}{|A_r|} L_c^{(e)} \right)^{-1} \left[ A_c N_0 - \frac{4}{3} k_c (L_c^{(e)})^2 \right] \quad (10)$$

The eigenvalues of the jacobian matrix  $J$  of the system (6) are:

$$\lambda_{\pm} = \frac{1}{2} \left[ T_J \pm (T_J^2 - 4\Delta_J)^{1/2} \right]$$

where the trace  $T_J$  and the determinant  $\Delta_J$  of  $[J]$  can be written as follows:

$$T_J = \text{Tr}(J) = A_c - |A_r| + k_r (L_c^{(e)} - L_r^{(e)}) - 4B_c(N_d) (L_c^{(e)})^3 \quad (11)$$

and

$$\Delta_J = \det(J) = -A_c |A_r| + k_r (A_c L_c^{(e)} + |A_r| L_r^{(e)}) + 4|A_r| B_c(N_d) (L_c^{(e)})^3 \quad (12)$$

231 They must be evaluated for each of the two equilibrium points.

232 3.3.2. *Stability analysis around  $X_e^{(1)}$*

233 In this case,  $T_J = A_c - |A_r|$  and  $\Delta_J = -A_c|A_r| < 0$ . The two eigenvalues  
 234 have opposite sign,  $\lambda_+ \lambda_- < 0$ , with  $\lambda_+ = A_c > 0$  and  $\lambda_- = -|A_r| < 0$ . The  
 235 state  $X_e^{(1)}$  is thus a saddle point.

236 3.3.3. *Stability analysis around  $X_e^{(2)}$*

In this case

$$T_J = (3A_c + |A_r|) \left( \frac{k_r L_r^{(e)}}{A_c} - 1 \right) \quad \text{and} \quad \Delta_J = -2A_c |A_r| \left( \frac{k_r L_r^{(e)}}{A_c} - \frac{3}{2} \right)$$

It follows that the equilibrium point  $X_e^{(2)} = (L_c^{(e)}, L_r^{(e)})$  is stable if:

$$\frac{k_r L_r^{(e)}}{A_c} < 1 \quad \text{i.e.} \quad L_r^{(e)} < \frac{A_c}{k_r} \quad \text{and} \quad L_c^{(e)} < \frac{|A_r|}{k_r} \quad (13)$$

237 Otherwise, for  $k_r L_r^{(e)}/A_c > 1$ ,  $X_e^{(2)}$  is an unstable equilibrium point.

238 The identity  $k_r L_r^{(e)}/A_c = 1$ , which separates a stable regime from an  
 239 unstable one, has the straightforward consequence that  $B_c [N_d^{(e)}] (L_c^{(e)})^4 =$   
 240 0, *i.e.*  $B_c = 0$  since  $L_c^{(e)} \neq 0$ . Physically, this critical value means that  
 241 the net rate of the autoconversion process is zero: formation of a raindrop  
 242 from cloud droplets only is compensated over a given time interval by the  
 243 formation of cloud droplets from a raindrop only. Thus, the system maintains  
 244 constant rain and cloud water contents whose values — see Eq. (13) —  
 245 depend only on the exchange rates with the surroundings ( $A_c$  and  $A_r$ ) and  
 246 on the cloud-to-rain reaction constant ( $k_r$ ). Below the critical value ( $B_c < 0$ ),  
 247 the autoconversion process results in net production of cloud droplets from  
 248 raindrops. It constitutes a nonlinear amplification term for cloud droplets and  
 249 a nonlinear damping term for raindrops, *i.e.* disturbances from equilibrium  
 250 grow. In other words, the cloud is growing such that cloud droplets become  
 251 more numerous while the number of raindrops vanishes. This behavior is  
 252 reversed above the critical value ( $B_c > 0$ ): the non linear term compensates  
 253 any deviation from the equilibrium point (stability). In the case  $B_c < 0$ ,  
 254 without any cloud-rain interaction ( $k_r = 0$ ), cloud (rain) water content is an  
 255 increasing (decreasing) function.

256 The other characteristic value is  $k_r L_r^{(e)}/A_c = 1.5$ . Below (above) this  
 257 value,  $\Delta_J$  is positive (negative). So  $X_e^{(2)}$  is a saddle point if  $L_r^{(e)} = 1.5A_c/k_r$ .

The nature (node, focus or spiral) of  $X_e^{(2)}$  is provided by the sign of  $T_J^2 - 4\Delta_J$ :

$$T_J^2 - 4\Delta_J = \left[ (3A_c + |A_r|)^2 \left( \frac{k_r L_r^{(e)}}{A_c} - 1 \right)^2 \right] + 4|A_r|A_c \left( 2\frac{k_r L_r^{(e)}}{A_c} - 3 \right)$$

258 If  $k_r L_r^{(e)}/A_c = 1$ ,  $T_J^2 - 4\Delta_J = -4|A_r|A_c < 0$  which means that  $X_e^{(2)}$  is a focus.

For  $k_r L_r^{(e)}/A_c \neq 1$ , the sign of  $T_J^2 - 4\Delta_J$  depends on the number  $\mathcal{N}_\tau = |A_r|/A_c$  through the second order polynomial equation:

$$(3A_c + |A_r|)^2 \left( \frac{k_r L_r^{(e)}}{A_c} \right)^2 - 2 \left( \frac{k_r L_r^{(e)}}{A_c} \right) [(3A_c + |A_r|)^2 - 4A_c|A_r|] + (|A_r| - 3A_c)^2$$

The two solutions are easy to find:

$$Z_\pm = \frac{(3A_c + |A_r|)^2 - 4|A_r|A_c}{(3A_c + |A_r|)^2} \pm \frac{2 [(9A_c^2 + |A_r|A_c) (|A_r|^2 + A_c|A_r|)]^{1/2}}{(3A_c + |A_r|)^2}$$

259 Three cases can be distinguished at first sight:

- 260 -  $|A_r| = A_c$ : so  $Z_\pm = (3 \pm \sqrt{5})/4 > 0$ . Between these two values,  $T_J^2 -$
- 261  $4\Delta_J < 0$ , so that  $X_e^{(2)}$  is a focus. Outside of this interval, we have a
- 262 node
- 263 -  $|A_r| \gg A_c$ : so  $Z_\pm = 1 \pm 2(A_c/|A_r|)^{1/2} > 0$ . Same as previously.
- 264 -  $|A_r| \ll A_c$ : so  $Z_\pm = 1 \pm (2/3)(|A_r|/A_c)^{1/2} > 0$ . Same as previously.

### 265 3.3.4. Andronov-Hopf bifurcation

266 It follows from what precedes that the eigenvalues of the jacobian matrix  
 267 cross the imaginary axis for  $B_c = 0$ . At this point, they are pure imaginary  
 268 numbers,  $\lambda_\pm = i(A_c|A_r|)^{1/2}$ . There, the system has a limit cycle. Just after  
 269 ( $B_c < 0$ ), the spiral is unstable whilst it is stable just before ( $B_c > 0$ ).  
 270 This behavior is precisely a *Hopf bifurcation*: an unstable focus gives birth  
 271 to a stable focus (and *vice versa*) through a limit cycle. The existence of  
 272 such a bifurcation is an important point for further investigations of cloud  
 273 organization in a coupled-cloud field at the mesoscale (Kuramoto, 2003).

### 274 3.4. Examples

275 We now illustrate the behavior of this set of three equations with some  
 276 examples. The values chosen for  $k_r$  and  $k_c$  are those given in Seifert and Be-  
 277 heng (2001) which follow from approximations to the collection kernel in Long

278 (1974), *i.e.*  $k_c \approx 5.66 \times 10^{11} \text{ cm}^3 \text{ g}^{-2} \text{ min}^{-1}$  and  $k_r \approx 3.47 \times 10^5 \text{ cm}^3 \text{ g}^{-1} \text{ min}^{-1}$ .  
 279 The value of  $x^* = 2.6 \times 10^{-7} \text{ g}$ , following Seifert and Beheng (2001). We  
 280 take  $\nu = 2$ , which is a typical value for warm clouds (*e.g.*, Pruppacher  
 281 and Klett, 1997, chap. 2). With these values, the parameter  $K \approx 2.905 \times$   
 282  $10^{17} \text{ cm}^3 \text{ g}^{-3} \text{ min}^{-1}$ .

283 In addition, we take  $A_c = 0.02 \text{ min}^{-1}$  and  $A_r = 0.1 \text{ min}^{-1}$ . So, cloud  
 284 water content and rain water content are expected to evolve on the timescales  
 285  $\tau_c = 50 \text{ min}$  and  $\tau_r = 10 \text{ min}$ . The latter is lower than the former since raining  
 286 out is a rapid process compared to the evolution of cloud water content. These  
 287 values are chosen for illustrative purposes.

### 288 3.4.1. First example: periodic behavior

289 We choose as initial conditions  $N_d(0) = 10 \text{ cm}^{-3}$ ,  $L_c(0) = 10^{-9} \text{ g cm}^{-3}$ ,  
 290  $L_r(0) = 0$ , and  $N_0 = 50 \text{ cm}^{-3}$ . The system exhibits oscillatory behavior after  
 291 approximately 300 min, with a period of about 200 min, which is physically  
 292 realistic (Fig. 2a). Both  $N_d$  and  $L_c$  increase initially (the former faster than  
 293 the latter) until rain appears. Soon thereafter, rain water content peaks and  
 294 droplet number and mass concentration decrease drastically in a short in-  
 295 terval of time. Minimum values of  $L_c$  are reached and rain water content  
 296 decreases to zero. Then  $N_d$  and  $L_c$  start to increase again, and so on. This  
 297 periodicity appears clearly as a limit cycle on the plot  $(L_c, L_r)$  displayed in  
 298 Fig. 2b. It is interesting to mention that peaks in rain water content are de-  
 299 layed from  $L_c$  peaks by about 20 – 30 min, which is a very realistic value for  
 300 rain to form from a cloud droplet by collection (Fig. 2c). We also note that  
 301 similar oscillations and limit cycles are obtained for other initial conditions  
 302 (not shown). [Note that the model presented here represents well a timescale](#)  
 303 [of precipitation production that has to be explicitly and arbitrarily accounted](#)  
 304 [for \*via\* delays in the more complicated DDE framework.](#)

### 305 3.4.2. Second example: damped behavior

306 If  $N_0$  is lower than  $N_d(0)$ , the feeding of the system is not sufficient  
 307 to maintain the cloud-rain system in a harmonic (or quasi) oscillation and a  
 308 damping is observed (Fig. 3):  $N_d$ ,  $L_c$ , and  $L_r$  still oscillate but their respective  
 309 amplitudes decrease progressively until a steady state is reached. Initially  $N_d$   
 310 decreases exponentially as expected, and then decreases more rapidly when  
 311  $L_c$  and  $L_r$  peak for the first time.

312 Damping can also be obtained by introducing in the model a decrea-  
 313 sing evolution for the CCN supply, for instance  $N_0 = 50 \exp(-0.002 t)$ . This



314 means that the initial CCN burden is divided by around 2.72 in 500 min,  
315 *i.e.* 8 hours. Keeping the same initial conditions as previously, we get the  
316 behavior displayed in Figure 4, which is typical of a damped oscillator:  $N_d$   
317 increases until a certain time  $t$  (approximately 120 min) at which the expo-  
318 nential function becomes significant. Then  $N_d$  decreases dramatically. The  
319 same behavior is seen for  $L_c$  and  $L_r$  which increase until  $t = 500$  min and  
320 then exhibit decaying amplitude oscillations that eventually ( $t > 1\,500$  min)  
321 die out. Again, we note that the peaks of  $L_r$  are delayed with respect to those  
322 of  $L_c$  by about 20 – 30 min.

323 The same damping behavior is observed (not shown) for different initial  
324 conditions, in particular if  $L_r \neq 0$ , provided the values are realistic.

#### 325 4. Discussion and conclusion

326 Certain aspects of a system’s bulk behavior arise out of many small-  
327 scale interactions. These aspects may be investigated with simple models  
328 that faithfully represent the bulk behavior, but do not consider directly the  
329 full complexity of the system. In conjunction with more detailed models and  
330 simulations, models which represent bulk behavior can also enhance our un-  
331 derstanding of complex dynamical systems. The model presented here, which  
332 has aspects of species competition in biology, is a tool with which to examine  
333 the complexity of cloud-rain interactions in a way that highlights certain as-  
334 pects of their behavior in a simple but realistic manner. This model is not  
335 meant to supplant more detailed simulations of cloud-precipitation interac-  
336 tions but to represent certain predictable elements of the macrobehavior of  
337 the full complex dynamical system.

338 In the first part of the paper, we examined some important mathematical  
339 properties of the KF11 DDEs model which is a similar model devoted to  
340 analyze the bulk behavior of a cloud-rain system. The focus has been put on  
341 scaling and limit cycles or loop solutions. In particular, it has been shown,  
342 through the Busenberg theorem that this model has no proper limit cycle so-  
343 lutions. The model we have introduced in this paper (second part) is another  
344 point of view which retains the idea of modeling cloud-precipitation interac-  
345 tion through population dynamics and exhibits predator-prey behavior with  
346 rain as predator and cloud droplets as prey. The stability analysis has shown  
347 that our model has limit cycles and a Hopf bifurcation.

348 In particular, the model formulated here is a system of ODEs rather  
349 than DDEs. It is derived from a physically-based parametrization. Several

350 examples illustrate that our model reproduces realistically the behaviour of  
 351 an unique cloud-rain system. In particular, rain water content peaks are  
 352 delayed by about 20 – 30 min with respect to cloud water content peaks.  
 353 Some of the novel properties of the model include regions of parameter space  
 354 in which clouds grow at the expense of rain and *vice versa* as determined  
 355 from bifurcation analysis.

356 Further properties of our model such as ability to represent various phy-  
 357 sical situations, and sensitivity tests to parametrization schemes similar to  
 358 those performed by Wacker (1995) for mixed and ice clouds, will be explored  
 359 in more detail in future work. Due to its properties, this cloud-rain model  
 360 is intended to be further developed for examination of cloud organization at  
 361 larger scales in a coupled-cloud field.

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## 367 Appendix 1: Unit analysis

368 The system (1) is derived from four empirical equations which we have  
 369 summarized in Table 1 together with their numbering in KF11.

Eq. number in KF11	Equation	Constants
[2]	$LWP = 0.5 c_1 H^2$	$c_1 = 2 \times 10^{-6} \text{ mm m}^{-2}$
[5]	$dLWP/dt = -R$	None
[4]	$R = \alpha H^3 N_d^{-1}$	$\alpha = 2 \text{ mm m}^{-6} \text{ d}^{-1}$
[9]	$dN_d/dt = -c_2 N_d R$	$c_2 = 0.3 \text{ m}^{-1}$

TAB. 1 – *Equations and constants in KF11 discussed here. In KF11, LWP and R are liquid water path and rainrate, respectively. The corresponding units are explicitly  $\text{g m}^{-2}$  and  $\text{mm d}^{-1}$ .*

Eq. [2] connects liquid water path ( $LWP$ , in  $\text{g m}^{-2}$ ) with cloud depth ( $H$ , in  $\text{m}$ ). The dimension of  $LWP$  is  $[M][L]^{-2}$  (where  $[M]$  and  $[L]$  are mass and length dimensions), which implies that the dimension of  $c_1$  is  $[M][L]^{-4}$

and not  $[L]^{-1}$  (cf. Table 1). Indeed, the well-established relationship is given by  $LWP (\text{g m}^{-2}) = 0.5 c_w H^2$ , where  $c_w = 2 \times 10^{-6} \text{ kg m}^{-4}$  (*e.g.* Geoffroy et al., 2008). Actually,  $c_w = \rho_w c_1$ , where  $\rho_w \approx 10^3 \text{ kg m}^{-3}$  is liquid water density and  $c_1 = 2 \times 10^{-9} \text{ m}^{-1}$  — in Tab. 1,  $c_1 = 2 \times 10^{-6} \text{ mm m}^{-2}$ , *i.e.*  $c_1 (\text{m}^{-1}) = 10^{-3} c_1 (\text{mm m}^{-2})$ . The relationship between  $LWP$  and  $H$  we get is thus:

$$\begin{aligned} LWP (\text{g m}^{-2}) &= \frac{1}{2} c_w (\text{kg m}^{-4}) H^2 (\text{m}^2) \\ &= \frac{1}{2} \rho_w (\text{kg m}^{-3}) c_1 (\text{m}^{-1}) H^2 (\text{m}^2) \end{aligned} \quad (14)$$

If we look now at Eq [5] and the units chosen, the decreasing rate of  $LWP$  due to rain cannot equal exactly the rainrate  $R$  (in  $\text{mm d}^{-1}$ ). In fact,  $dLWP/dt$  is proportional to the precipitation flux density (or rain current  $J_r = -\rho_w R$ ). The coefficient of proportionality is determined by the choice of units. If, as in KF11, we express time in minutes (it is the most natural timescale for cloud macroscopic physics), and keep  $LWP$  in  $\text{g m}^{-2}$  as above, we obtain:

$$\begin{aligned} \frac{dLWP}{dt} (\text{g m}^{-2} \text{ min}^{-1}) &= -\rho_w (\text{g m}^{-3}) R (\text{m min}^{-1}) \\ &= -10^3 \rho_w (\text{kg m}^{-3}) R (\text{m min}^{-1}) \\ &= -\frac{10^3}{60 \times 24} \rho_w (\text{kg m}^{-3}) R (\text{m d}^{-1}) \\ &= -\frac{1}{60 \times 24} \rho_w (\text{kg m}^{-3}) R (\text{mm d}^{-1}) \end{aligned} \quad (15)$$

since  $R (\text{m min}^{-1}) = 10^{-3} R (\text{mm d}^{-1}) / (60 \times 24)$ . Combining Eqs. (14) and (15) yields:

$$\begin{aligned} \frac{dH}{dt} (\text{m min}^{-1}) &= \frac{dLWP/dt (\text{g m}^{-2} \text{ min}^{-1})}{c_w (\text{kg m}^{-4}) H (\text{m})} \\ &= -\frac{\rho_w (\text{kg m}^{-3}) R (\text{mm d}^{-1})}{(60 \times 24) c_w (\text{kg m}^{-4}) H (\text{m})} \\ &= -\frac{1}{60 \times 24} \frac{R (\text{mm d}^{-1})}{c_1 (\text{m}^{-1}) H (\text{m})} \\ &= -\frac{10^3}{60 \times 24} \frac{R (\text{mm d}^{-1})}{c_1 (\text{mm m}^{-2}) H (\text{m})} \end{aligned} \quad (16)$$

370 It is possible, although not common, to divide  $LWP$  by  $\rho_w$  and then to de-  
371 fine a volumetric liquid water path:  $LWP_v (\text{m}) = 10^{-3} LWP (\text{g m}^{-2}) / \rho_w (\text{kg m}^{-3})$ .

372 In this case, Eqs. [2] and [5] of Table 1 are correct provided that, for Eq. [5],  $R$   
 373 be expressed in  $\text{m min}^{-1}$ , or equivalently, considering  $10^{-3} R (\text{mm d}^{-1}) / (60 \times$   
 374  $24)$ . However, we have not found any explicit mention of  $LWP$  in  $\text{m}$  or  $\text{mm}$   
 375 in KF11 and further papers; there  $LWP$  is explicitly expressed in  $\text{g m}^{-2}$   
 376 several times.

The KF11 model also employs two parametrizations which relate  $R$ ,  $N_d$ , and  $dN_d/dt$  (see Eqs. [4] and [9] in Table 1). The units of the latter two quantities are naturally  $\text{cm}^{-3}$  and  $\text{cm}^{-3} \text{min}^{-1}$ , respectively. According to the units used in KF11, the rhs of Eq. [4] is in  $\text{mm m}^{-3} \text{cm}^3 \text{d}^{-1}$ , while the lhs (left hand side) is in  $\text{mm d}^{-1}$ . Although Eq. [4] is dimensionally correct ( $[L][T]^{-1}$ , where  $[T]$  is the time dimension), the units of the rhs and lhs are not the same. Indeed, the relation  $R = \alpha H^3 / N_d$  should have  $N_d$  in  $\text{m}^{-3}$  when  $H$  is in  $\text{m}$ . If we choose to express  $N_d$  in  $\text{cm}^{-3}$ , then we must multiply  $H^3 / N_d$  by  $10^{-6}$ . With the unit of  $\alpha$ ,  $R$  is thus in  $\text{mm d}^{-1}$ :

$$R (\text{mm d}^{-1}) = 10^{-6} \alpha (\text{mm m}^{-6} \text{d}^{-1}) H^3 (\text{m}^3) N_d^{-1} (\text{cm}^3) \quad (17)$$

Thus, for Eq. (16), we obtain:

$$\frac{dH}{dt} (\text{m min}^{-1}) = -\frac{10^{-3}}{60 \times 24} \frac{\alpha (\text{mm m}^{-6} \text{d}^{-1}) H^2 (\text{m}^2)}{c_1 (\text{mm m}^{-2}) N_d (\text{cm}^{-3})} \quad (18)$$

In addition, Wood (2006) (ref. 20 in KF11) gives a value of  $c_2$  which does not correspond to that used in KF11. Aside from the conflation of rain current and rainrate in Wood (2006) — this article seems to use  $P$  as rainrate and divides it by liquid water density ( $\rho_w$ ), whereas  $P$  should be the rain current and  $P/\rho_w$  the rainrate —, the coefficient of proportionality between  $dN_d/dt$  and  $R$  is  $3E_0/4 = 3 \times 10^3 \text{m}^{-1}$  since  $E_0 = 4 \times 10^3 \text{m}^{-1}$ . This value is different from the  $0.3 \text{m}^{-1}$  used in KF11 (assuming that  $P$ , in Wood (2006), is rainrate, and not rain current, which is very few probable due to a dimensional analysis of the Eq. 14 of Wood (2006),  $c_2$  would be  $3 \text{m}^{-1}$ , *i.e.* the previous value of  $c_2$  divided by  $\rho_w$ , which is again different from the value of Table 1). Because, in Eq. [9],  $dN_d/dt$  has to be in  $\text{cm}^{-3} \text{min}^{-1}$ , we have:

$$\frac{dN_d (\text{cm}^{-3} \text{min}^{-1})}{dt} = c_2 N_d (\text{cm}^{-3}) R (\text{m min}^{-1}) \quad (19)$$

377 with  $c_2 = 3 \times 10^3 \text{m}^{-1}$ .

Then, because  $R (\text{m min}^{-1}) = 10^{-3} R (\text{mm d}^{-1}) / (60 \times 24)$ , we obtain, by substituting the rainrate of Eq. (17) into Eq. (19):

$$\frac{dN_d (\text{cm}^{-3} \text{min}^{-1})}{dt} = \frac{10^{-9}}{60 \times 24} c_2 (\text{m}^{-1}) \alpha (\text{mm m}^{-6} \text{d}^{-1}) H^3 (\text{m}^3) \quad (20)$$

Finally, with this set of units, *i.e.* with  $H$  and  $N_d$  in m and  $\text{cm}^{-3}$  respectively, and time in minutes, the consistent (in terms of units) system of equations we get is:

$$\begin{cases} \frac{dH}{dt} = \frac{H_0 - H}{\tau_1} - \frac{10^{-3}}{60 \times 24} \frac{\alpha H^2(t - T)}{c_1 N_d(t - T)} \\ \frac{dN_d}{dt} = \frac{N_0 - N_d}{\tau_2} - \frac{10^{-9}}{60 \times 24} \alpha c_2 H^3(t - T) \end{cases} \quad (21)$$

378 with  $c_1 = 2 \times 10^{-6} \text{ mm m}^{-2}$ ,  $c_2 = 3 \times 10^3 \text{ m}^{-1}$ ,  $\alpha = 2 \text{ mm m}^{-6} \text{ d}^{-1}$ . If we  
 379 consider that  $LWP$  is a volumetric liquid water path, as it might be assumed  
 380 as suggested above, and  $R$  in  $\text{mm d}^{-1}$ , the factor  $10^{-3}$  is replaced by  $10^3$ .

## 381 Appendix 2: Other empirical parametrizations

382 Other empirical parametrizations can also be used to relate  $H$ ,  $N_d$ , and  
 383  $R$ . Here, we would like to suggest three. In particular, we can introduce  
 384 the following three alternative parametrizations for the delay term  $dN_d/dt$   
 385 instead of the one in Wood (2006). (i) The empirical relationship proposed in  
 386 Mechem et al. (2006, Eq. 9):  $dN_d/dt$  ( $\text{cm}^{-3} \text{ d}^{-1}$ ) =  $-69.4 (N_d R)^{0.668}$ , with  $N_d$   
 387 in  $\text{cm}^{-3}$  and  $R$  in  $\text{cm d}^{-1}$ . (ii) The following empirical relationships given  
 388 in Geoffroy et al. (2008):  $J_r$  ( $10^{-6} \text{ kg m}^{-2} \text{ s}^{-1}$ ) =  $a (LWP^\alpha / N_d) - b$ . Here,  
 389  $LWP$  is in  $\text{g m}^{-2}$ ,  $a$ ,  $b$ , and  $\alpha$  are positive constants that take the values  
 390 summarized in Table 2.

$a$	$\alpha$	$b$
$0.3 \times 10^6$	2	$10^{-6}$
$24.37 \times 10^9$	1	0
$21.5 \times 10^3$	1.5	$2.3 \times 10^{-6}$

TAB. 2 – Values of the constants in Geoffroy et al. (2008)

391 When using this parametrization, it is necessary to first use Eq. (14) to  
 392 change  $LWP$  into  $H$ .

(iii) Finally, a parametrization of the delay term  $dN_d/dt$  by taking

$$\frac{N_d}{LWP} \frac{dLWP}{dt} = \frac{2N_d}{H_r} \frac{dH_r}{dt} \quad (22)$$

393 This follows from Eq. (14). The equation for  $dH_r/dt$  can then be substituted  
 394 into this equation to complete the parametrization. This equation with delay

395 is finally substituted for the delay term in the second equation in the KF11  
396 system.

397 When using these different parametrizations, a set of coupled first-order  
398 DDEs similar to Eq. (1) is obtained. The units must be made consistent  
399 when the various empirical relationships are employed. Namely,  $H$  is in m,  
400  $N_d$  is in  $\text{cm}^{-3}$ , and time is expressed in minutes. For the numerical tests we  
401 have performed, these parametrizations have given results similar to those  
402 in Figure 1. However, we have not made a complete test of sensitivity of  
403 the model to changes in parametrization since it is not the purpose of this  
404 paper. Our goal here is to suggest other possibilities to parametrize cloud-  
405 rain interaction processes that could be investigated more deeply in future  
406 research.

- 407 Busenberg, S., van den Driessche, P., 1993. A method for proving the non-  
408 existence of limit cycles. *Jour. Math. Anal. Appl.* 172, 463–479.
- 409 Engelborghs, K., Luzyanina, T., Roose, D., 2000. Numerical bifurcation ana-  
410 lysis of delay differential equations. *Jour. Comp. Appl. Math.* 125, 265–275.
- 411 Feingold, G., Koren, I., 2013. A model of coupled oscillators applied to the  
412 aerosol-cloud-precipitation system. *Nonlin. Processes Geophys.* 20, 1011–  
413 1021.
- 414 Feingold, G., Koren, I., Wang, H., Xue, H., A., B., 2010. Precipitation-  
415 generated oscillations in open cellular cloud fields. *Nature Letters* 466,  
416 849–852.
- 417 Geoffroy, O., Brenguier, J.-L., Sandu, I., 2008. Relationship between drizzle  
418 rate, liquid water path and droplet concentration at the scale of a strato-  
419 cumulus cloud system. *Atmos. Chem. Phys.* 8, 4641–4654.
- 420 Glansdorff, P., Prigogine, I., 1971. *Structure, stabilité et fluctuations*. Masson  
421 et Cie.
- 422 Held, I., 2005. The gap between simulation and understanding in climate  
423 modeling. *Bull. Am. Meteor. Soc.*, 1609–1614.
- 424 IPCC, ., 2013. *Climate Change 2013: The Physical Science Basis*. Contribu-  
425 tion of Working Group I to the Fifth Assessment Report of the Intergo-  
426 vernmental Panel on Climate Change.
- 427 Jiang, Q., Wang, S., 2014. Aerosol Replenishment and Cloud Morphology: A  
428 VOCALS Example. *Journal of the Atmospheric Sciences* 71, 300–311.
- 429 Koren, I., Feingold, G., 2011. Aerosol-cloud-precipitation system as a  
430 predator-prey problem. *PNAS* 108, 12227–12232.
- 431 Koren, I., Tziperman, E., Feingold, G., 2017. Exploring the nonlinear cloud  
432 and rain equation. *Chaos* 27, 013107–1–013107–9.
- 433 Kuramoto, Y., 2003. *Chemical Oscillations, Waves, and Turbulence*. Dover  
434 Publications.
- 435 Lipowski, A., Lipowska, D., 2000. Nonequilibrium phase transition in a lattice  
436 prey-predator system. *Physica A* 276, 456–464.
- 437 Long, A. B., 1974. Solutions to the Droplet Collection Equation for Polyno-  
438 mial Kernels. *J. Atm. Sci.* 31, 1040–1052.
- 439 Mechem, D. B., Robinson, P. C., Kogan, Y. L., 2006. Processing of cloud  
440 condensation nuclei by collision-coalescence in a mesoscale model. *Journal*  
441 *of Geophysical Research: Atmospheres* 111, d18204.
- 442 Minorsky, N., 1962. *Nonlinear Oscillations*. Van Nostrand.

- 443 Mülmenstädt, J., Feingold, G., 2018. The Radiative Forcing of Aerosol–Cloud  
444 Interactions in Liquid Clouds: Wrestling and Embracing Uncertainty. *Current Climate Change Reports* 4, 23–40.  
445
- 446 Pruppacher, H., Klett, J., 1997. *Microphysics of Clouds and Precipitation*.  
447 Atmospheric and Oceanographic Sciences Library. Springer Netherlands.
- 448 Seifert, A., Beheng, K. D., 2001. A double-moment parameterization for simulating autoconversion, accretion and selfcollection. *Atmos. Res.* 59-60,  
449 265–281.  
450
- 451 Wacker, U., 1995. Competition of precipitation particles in a model with  
452 parametrized cloud microphysics. *J. Atmos. Sci.* 52, 2577–2589.
- 453 Wood, R., 2006. Rate of loss of cloud droplets by coalescence in warm clouds.  
454 *J. Geophys. Res.* 111, 1–6.



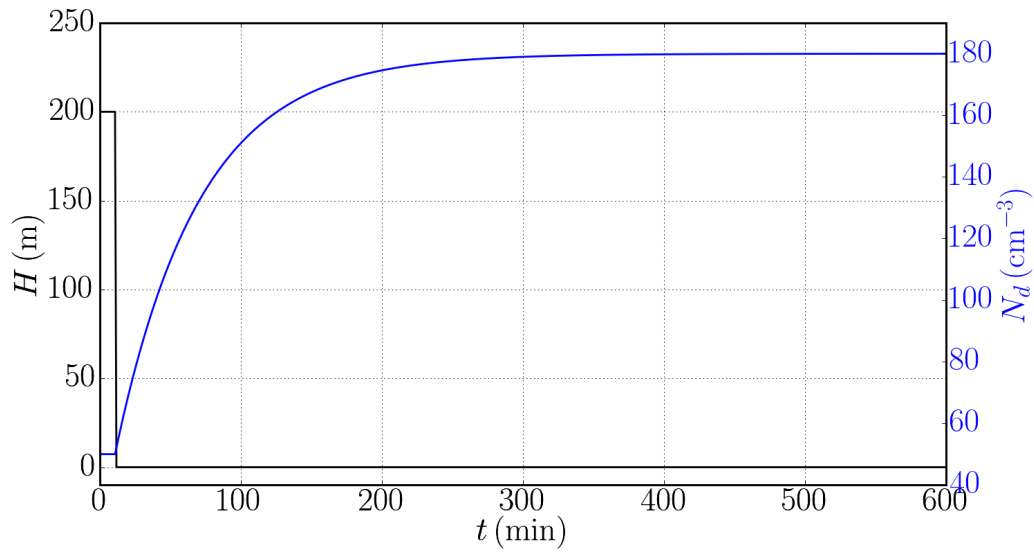
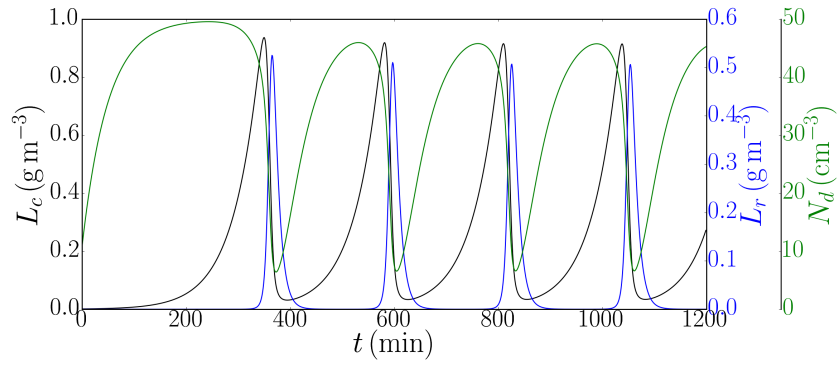
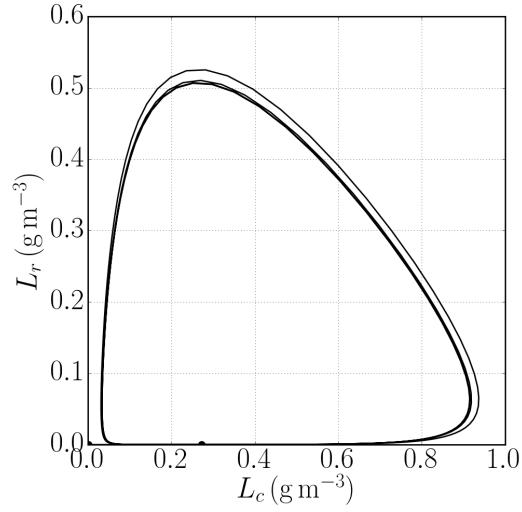


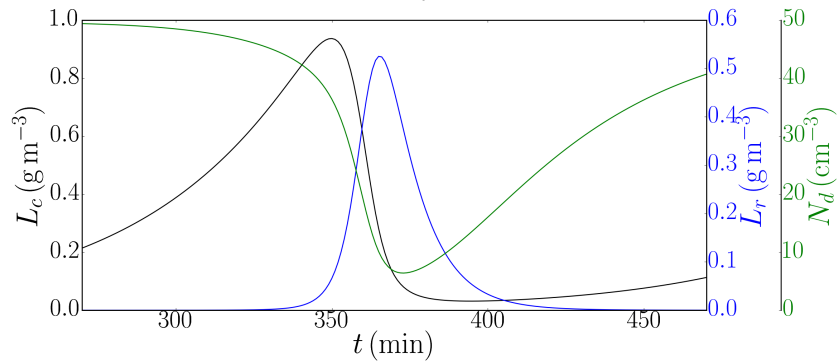
FIG. 1 – Evolution of  $H$  and  $N_d$  according to system (1) with  $H_0 = 530$  m,  $N_0 = 180$  cm<sup>-3</sup>,  $\tau_1 = \tau_2 = 60$  min, and  $T = 12$  min. Initial conditions are  $H(t = 0) = 200$  m and  $N_d(t = 0) = 50$  cm<sup>-3</sup>. We have used the `pydelay` package of Python.



a

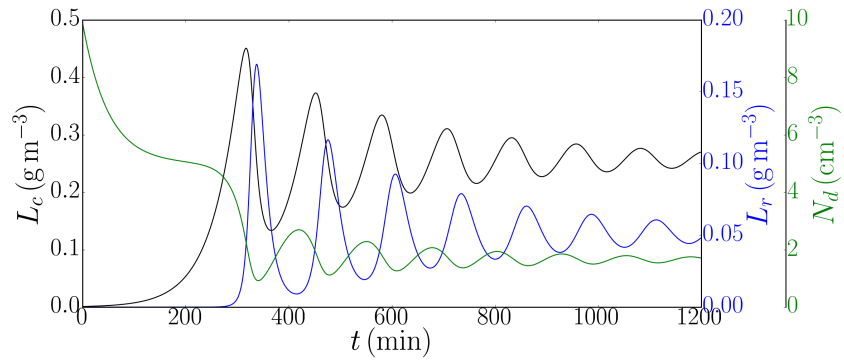


b

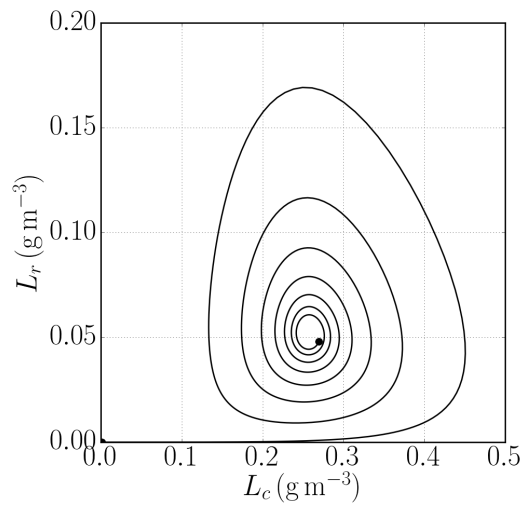


c

FIG. 2 – Evolution of  $N_d$ ,  $L_c$  and  $L_r$  (a) and limit-cycle in the configuration space  $(L_c, L_r)$  (b) for the initial conditions and values of parameters given in the text. On the limit cycle, initial and final time of integration are at the origin and  $(0.25, 0)$  respectively. (c) Zoom of (a).

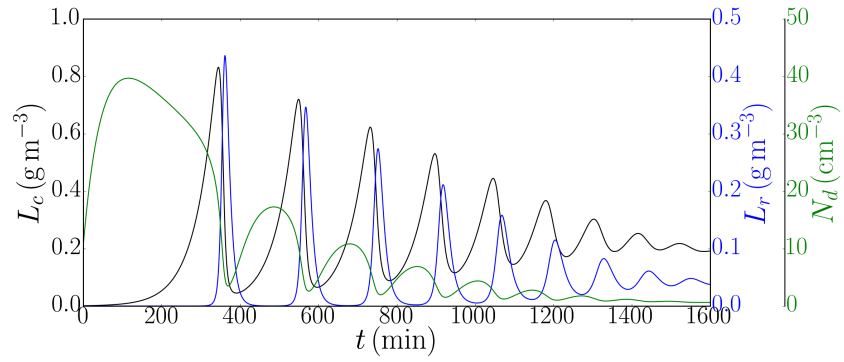


*a*

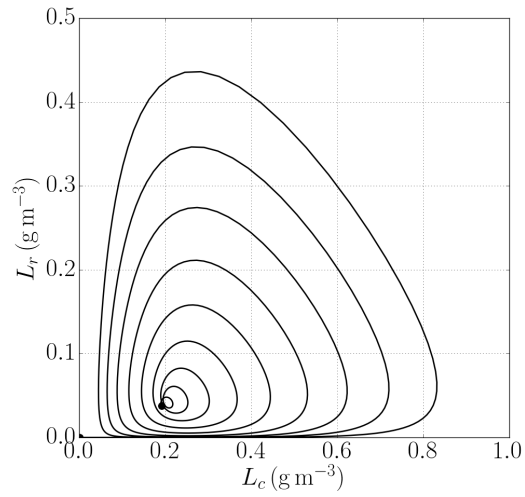


*b*

FIG. 3 – Same as Fig. 2 but for  $N_0 = 5 \text{ cm}^{-3}$ . On the limit cycle, initial time of integration is at the origin and final time of integration is represented by the black small ball at about  $(0.28, 0.05)$ .



*a*



*b*

FIG. 4 – Another kind of damped cloud system. On the limit cycle, initial time of integration is at the origin and final time of integration is represented by the black small ball at about  $(0.2, 0.05)$ .