



HAL
open science

A blowup solution of a complex semi-linear heat equation with an irrational power

Giao Ky Duong

► **To cite this version:**

Giao Ky Duong. A blowup solution of a complex semi-linear heat equation with an irrational power. *Journal of Differential Equations*, 2019, 267, pp.4975 - 5048. 10.1016/j.jde.2019.05.024 . hal-03488267

HAL Id: hal-03488267

<https://hal.science/hal-03488267v1>

Submitted on 20 Dec 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

A BLOWUP SOLUTION OF A COMPLEX SEMI-LINEAR HEAT EQUATION WITH AN IRRATIONAL POWER

Giao Ky Duong ¹

Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), F-93430, Villetaneuse, France.

February 18, 2019

ABSTRACT. In this paper, we consider the following semi-linear complex heat equation

$$\partial_t u = \Delta u + u^p, u \in \mathbb{C}$$

in \mathbb{R}^n , with an arbitrary power $p, p > 1$. We construct for this equation a complex solution $u = u_1 + iu_2$, which blows up in finite time T and only at one blowup point a . Moreover, we also describe the asymptotics of the solution by the following final profiles:

$$u(x, T) \sim \left[\frac{(p-1)^2 |x-a|^2}{8p |\ln|x-a||} \right]^{-\frac{1}{p-1}},$$
$$u_2(x, T) \sim \frac{2p}{(p-1)^2} \left[\frac{(p-1)^2 |x-a|^2}{8p |\ln|x-a||} \right]^{-\frac{1}{p-1}} \frac{1}{|\ln|x-a||} > 0, \text{ as } x \rightarrow a.$$

In addition to that, since we also have $u_1(0, t) \rightarrow +\infty$ and $u_2(0, t) \rightarrow -\infty$ as $t \rightarrow T$, the blowup in the imaginary part shows a new phenomenon unknown for the standard heat equation in the real case: a non constant sign near the singularity, with the existence of a vanishing surface for the imaginary part, shrinking to the origin. In our work, we have succeeded to extend for any power p where the non linear term u^p is not continuous if p is not integer. In particular, the solution which we have constructed has a positive real part. We study our equation as a system of the real part and the imaginary part u_1 and u_2 . Our work relies on two main arguments: the reduction of the problem to a finite dimensional one and a topological argument based on the index theory to get the conclusion.

1. Introduction

1.1. Ealier work

In this work, we are interested in the following complex-valued semilinear heat equation

$$\begin{cases} \partial_t u &= \Delta u + F(u), t \in [0, T), \\ u(0) &= u_0 \in L^\infty(\mathbb{R}^n), \end{cases} \quad (1.1)$$

where $F(u) = u^p$ and $u(t) : \mathbb{R}^n \rightarrow \mathbb{C}$, $L^\infty := L^\infty(\mathbb{R}^n, \mathbb{C})$, $p > 1$.

Typically, when $p = 2$, model (1.1) becomes the following

$$\begin{cases} \partial_t u &= \Delta u + u^2, t \in [0, T), \\ u(0) &= u_0 \in L^\infty. \end{cases} \quad (1.2)$$

This model is connected to the viscous Constantin-Lax-Majda equation with a viscosity term, which is a one dimensional model for the vorticity equation in fluids. For more details, the readers are addressed to the following works: Constantin, Lax, Majda [2], Guo, Ninomiya and Yanagida in [8], Okamoto, Sakajo and Wunsch [25], Sakajo in [26] and [27], Schochet [28]. The local Cauchy problem for model (1.1) can be solved in $L^\infty(\mathbb{R}^n, \mathbb{C})$ when p is integer, thanks to a fixed-point argument. However, if p is not an integer

2010 *Mathematics Subject Classification*. Primary: 35K50, 35B40; Secondary: 35K55, 35K57.

Key words and phrases. Blowup solution, Blowup profile, Stability, Semilinear complex heat equation, non variation heat equation.

February 18, 2019.

¹ G. K. Duong is supported by the project INSPIRE. This project has received funding from the European Unions Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 665850.

number, then, the local Cauchy problem has not been solved yet, up to our knowledge. In my point of view, this probably comes from the discontinuity of $F(u)$ on $\{u \in \mathbb{R}_+^*\}$ and this challenge is also one of the main difficulties of the paper. As a matter of fact, we solve the Cauchy problem in Appendix A for data $u_0 \in L^\infty$, with $\operatorname{Re}(u_0) \geq \lambda$, for some $\lambda > 0$. Accordingly, a maximal solution may be global in time or may exist only for $t \in [0, T)$, for some $T > 0$. In that case, we have two options:

- (i) Either $\|u(t)\|_{L^\infty(\mathbb{R}^n)} \rightarrow +\infty$ as $t \rightarrow T$.
- (ii) Or $\min_{x \in \mathbb{R}^n} \operatorname{Re}(u(x, t)) \rightarrow 0$ as $t \rightarrow T$.

In this paper, we are interested in the case (i), which is referred to as *finite-time blow-up* in the sequel. A blowup solution u is called *Type I* if

$$\limsup_{t \rightarrow T} (T - t)^{\frac{1}{p-1}} \|u(\cdot, t)\|_{L^\infty} < +\infty.$$

Otherwise, the solution u is called *Type II*.

In addition to that, T is called the blowup time of u and a point $a \in \mathbb{R}^n$ is called a blowup point if and only if there exists a sequence $\{(a_j, t_j)\} \rightarrow (a, T)$ as $j \rightarrow +\infty$ such that

$$|u_1(a_j, t_j)| + |u_2(a_j, t_j)| \rightarrow +\infty \text{ as } j \rightarrow +\infty.$$

In our work, we are interested in constructing a blowup solution of (1.1) which is *Type I*. Let us quickly mention some typical works for this situation (for more details, see the introduction in [5], treated the integer case).

(i) **For the real case:** Bricmont and Kupiainen [1] constructed a real positive solution to the following equation

$$\partial_t u = \Delta u + |u|^{p-1} u, p > 1, \quad (1.3)$$

which blows up in finite time T , only at the origin and they have derived the profile of the solution such that

$$\left\| (T - t)^{\frac{1}{p-1}} u(\cdot, t) - f_0 \left(\frac{\cdot}{\sqrt{(T - t)|\ln(T - t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{1 + \sqrt{|\ln(T - t)|}},$$

where the profile f_0 is defined as follows

$$f_0(x) = \left(p - 1 + \frac{(p - 1)^2 |x|^2}{4p} \right)^{-\frac{1}{p-1}}. \quad (1.4)$$

In addition to that, in [13], Herrero and Velázquez derived the same result with a different method. Particularly, in [17], Merle and Zaag gave a proof which is simpler than the one in [1] and proposed the following two-step method (see also the note [15]):

- Reduction of the infinite dimensional problem to a finite dimensional one.
- Solution of the finite dimensional problem thanks to a topological argument based on Index theory.

Moreover, they also proved the stability of the blowup profile for (1.3). In addition to that, we would like to mention that this method has been successful in various situations such as the work of Ebde and Zaag [6], Tayachi and Zaag [29], and also the works of Ghoul, Nguyen and Zaag in [9], [10] (with a gradient term) and [11]. We would like to mention also the work of Nguyen and Zaag in [23], who considered the following quasi-critical double source equation

$$\partial_t u = \Delta u + |u|^{p-1} u + \frac{\mu |u|^{p-1} u}{\ln^a(2 + u^2)},$$

and also the work of Duong, Nguyen and Zaag in [4], who considered the following non scale invariant equation

$$\partial_t u = \Delta u + |u|^{p-1} u \ln^\alpha(2 + u^2).$$

(ii) **For the complex case:** The blowup problem for the complex-valued parabolic equations has been studied intensively by many authors, in particular for the Complex Ginzburg Landau (CGL) equation

$$\partial_t u = (1 + i\beta)\Delta u + (1 + i\delta)|u|^{p-1} u + \gamma u. \quad (1.5)$$

This is the case of an earlier work of Zaag in [30] for equation (1.5) when $\beta = 0$ and δ small enough. Later, Masmoudi and Zaag in [18] generalized the result of [30] and constructed a blowup solution for (1.5) with a super critical condition $p - \delta^2 - \beta\delta - \beta\delta p > 0$. Recently, Nouaili and Zaag in [24] has constructed a blowup solution for (1.5), for a critical case where $\beta = 0$ and $p = \delta^2$.

In addition to that, there are many works for equation (1.1) or (1.2), such as the work of Nouaili and Zaag in [21] for equation (1.2), who constructed a complex solution $u = u_1 + iu_2$, which blows up in finite time T only at the origin. Note that the real and the imaginary parts blow up simultaneously. Note also that [21] leaves unanswered the question of the derivation of the profile of the imaginary part, and this is precisely our aim in this paper, not only for equation (1.2), but also for equation (1.1) with $p > 1$. We would like to mention also some classification results, proven by Harada in [12], for blowup solutions of (1.2) which satisfy some reasonable assumptions. In particular, in that works, we are able to derive a sharp blowup profile for the imaginary part of the solution. Recently, the author handled in [5] the case p is an arbitrary integer.

1.2. Statement of the result

In [5], we only treated the case, $p \in \mathbb{N}, p > 1$ which the handling of the nonlinear term is much easier. In the present paper, we do better, and give a proof which holds also in the case $p \notin \mathbb{N}$. We believe we made an important achievement, we acknowledge that we left unanswered the case where $p > 1$ and $p \notin \mathbb{N}$. From the limitation of the above works, it motivates us to study model (1.1) in general even for irrational p . The following theorem is our main result:

Theorem 1.1 (Existence of a blowup solution for (1.1) and a sharp description of its profile). *For each $p > 1$ and $p_1 \in (0, \min(\frac{p-1}{4}, \frac{1}{2}))$, there exists $T_1(p, p_1) > 0$ such that for all $T \leq T_1$, there exist initial data $u(0) = u_1(0) + iu_2(0)$, such that equation (1.1) has a unique solution u on $\mathbb{R}^n \times [0, T)$ satisfying the following:*

- i) *The solution u blows up in finite time T only at the origin and $\text{Re}(u) > 0$ on $\mathbb{R}^n \times [0, T)$. Moreover, it satisfies the following*

$$\left\| (T-t)^{\frac{1}{p-1}} u(\cdot, t) - f_0 \left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{1 + \sqrt{|\ln(T-t)|}}, \quad (1.6)$$

and

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_2(\cdot, t) - g_0 \left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{1 + |\ln(T-t)|^{\frac{p-1}{2}}}, \quad (1.7)$$

where f_0 is defined in (1.4) and $g_0(x)$ is defined as follows

$$g_0(x) = \frac{|x|^2}{\left(p-1 + \frac{(p-1)^2}{4p}|x|^2\right)^{\frac{p-1}{2}}}. \quad (1.8)$$

- ii) *There exists a complex function u^* in $C^2(\mathbb{R}^n \setminus \{0\})$ such that $u(t) \rightarrow u^* = u_1^* + iu_2^*$ as $t \rightarrow T$, uniformly on compact sets of $\mathbb{R}^n \setminus \{0\}$, and we have the following asymptotic expansions:*

$$u^*(x) \sim \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||} \right]^{-\frac{1}{p-1}}, \quad \text{as } x \rightarrow 0, \quad (1.9)$$

and

$$u_2^*(x) \sim \frac{2p}{(p-1)^2} \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||} \right]^{-\frac{1}{p-1}} \frac{1}{|\ln|x||}, \quad \text{as } x \rightarrow 0. \quad (1.10)$$

Remark 1.2. We remark that the condition on the parameter $p_1 < \min(\frac{p-1}{4}, \frac{1}{2})$ comes from the definition of the set $V_A(s)$ (see in item (i) of Definition 3.1), Proposition 4.1 and Lemma B.3. Indeed, this condition ensures that the projections of the quadratic term $B_2(q_2, q_2)$ on the negative and outer parts are smaller than the conditions in $V_A(s)$. Then, we can conclude (4.6) and (4.8) by using Lemma B.3 and definition of $V_A(s)$.

Remark 1.3. We can show that the constructed solution in the above Theorem satisfies the following asymptotics:

$$u(0, t) \sim \kappa(T-t)^{-\frac{1}{p-1}}, \quad (1.11)$$

$$u_2(0, t) \sim -\frac{2n\kappa}{(p-1)} \frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|^2}, \quad (1.12)$$

as $t \rightarrow T$, (see (3.37) and (3.38)). Therefore, we deduce that u blows up at time T only at 0. Note that, the real and imaginary parts simultaneously blow up. Moreover, from item (ii) of Theorem 1.1, the blowup speed of u_2 is softer than u_1 because of the quantity $\frac{1}{|\ln|x||}$ (see (1.9) and (1.10)).

Remark 1.4 (A strong singularity of the imaginary part). We observe from (1.10) and (1.12) that there is a strong singularity at the neighborhood of a as $t \rightarrow T$; when x close to 0, we have $u_2(x, t)$ which becomes large and positive as $t \rightarrow T$, however, we always have $u_2(0, t) \rightarrow -\infty$ as $t \rightarrow T$. Thus the imaginary part has no constant sign near the singularity. In particular, if t is near T , there exists $b(t) > 0$ in \mathbb{R}^n and $b(t) \rightarrow 0$ as $t \rightarrow T$ such that at time t , $u_2(\cdot, t)$ vanishes on some surface close to the sphere of center 0 and radius $b(t)$. Therefore, we don't have $|u_2(x, t)| \rightarrow +\infty$ as $(x, t) \rightarrow (0, T)$. This non constant property for the imaginary part is very surprising to us. In the frame work of semilinear heat equation, such a property can be encountered for phase invariant complex equations, such as the Complex Ginzburg-Landau (CGL) equation (see Zaag in [30], Masmoudi and Zaag in [18], Nouaili-Zaag [24]). As for complex parabolic equation with no phase invariance, this is the first time such a sign change is available, up to our knowledge. We would like to mention that such a sign change near the singularity was already observed for the semilinear wave equation non characteristic blowup point (see Merle and Zaag in [19], [20]) and Côte and Zaag in [3].

Remark 1.5. For each $a \in \mathbb{R}^n$, by using the translation $u_a(\cdot, t) = u(\cdot - a, t)$, we can prove that u_a also satisfies equation (1.1) and the solution blows up at time T only at the point a . We can derive that u_a satisfies all estimates (1.6) - (1.10) by replacing x by $x - a$.

Remark 1.6. In Theorem (1.1), the initial data $u(0)$ is given exactly as follows

$$u(0) = u_1(0) + iu_2(0),$$

where

$$\begin{aligned} u_1(x, 0) &= T^{-\frac{1}{p-1}} \left\{ \left(p-1 + \frac{(p-1)^2|x|^2}{4pT|\ln T|} \right)^{-\frac{1}{p-1}} + \frac{n\kappa}{2p|\ln T|} \right. \\ &+ \frac{A}{|\ln T|^2} \left(d_{1,0} + d_{1,1} \cdot \frac{x}{\sqrt{T}} \right) \chi_0 \left(\frac{16|x|}{K_0\sqrt{T}|\ln T|} \right) \left. \right\} \chi_0 \left(\frac{|x|}{\sqrt{T}|\ln T|} \right) + U^*(x) \left(1 - \chi_0 \left(\frac{|x|}{\sqrt{T}|\ln T|} \right) \right), \\ &+ 1, \\ u_2(x, 0) &= T^{-\frac{1}{p-1}} \chi_0 \left(\frac{|x|}{\sqrt{T}|\ln T|} \right) \left\{ \frac{|x|^2}{T|\ln T|^2} \left(p-1 + \frac{(p-1)^2|x|^2}{4pT|\ln T|} \right)^{-\frac{p}{p-1}} - \frac{2n\kappa}{(p-1)|\ln T|^2} \right. \\ &+ \left. \left[\frac{A^2}{|\ln T|^{p_1+2}} \left(d_{2,0} + d_{2,1} \cdot \frac{x}{\sqrt{T}} \right) + \frac{A^5 \ln(|\ln(T)|)}{|\ln T|^{p_1+2}} \left(\frac{1}{2} \frac{x^T}{\sqrt{T}} \cdot d_{2,2} \cdot \frac{x}{\sqrt{T}} - \text{Tr}(d_{2,2}) \right) \right] \chi_0 \left(\frac{2x}{K_0\sqrt{T}|\ln T|} \right) \right\}. \end{aligned}$$

with $\kappa = (p-1)^{-\frac{1}{p-1}}$, K_0, A are positive constants fixed large enough, $d_1 = (d_{1,0}, d_{1,1})$, $d_2 = (d_{2,0}, d_{2,1}, d_{2,2})$ are parameters we fine tune in our proof, and $\chi_0 \in C_0^\infty[0, +\infty)$, $\|\chi_0\|_{L^\infty} \leq 1$, $\text{supp}\chi_0 \subset [0, 2]$ and $\chi_0(x) = 1$ for all $|x| \leq 1$, and U^* is given in (3.32) and related to the final profile given in item (ii) of Theorem 1.1. Note that when $p \in \mathbb{N}$, we took in [5] a simpler expression for initial data, not involving the final profile U^* , nor the (+1) term in $u_1(0)$. In particular, adding this (+1) term in our idea to ensure that the real part of the solution stays positive.

Remark 1.7. We see in (2.3) that the equation satisfied by u_2 is almost 'linear' in u_2 . Hence, given an arbitrary $c_0 \neq 0$, we can change a little in our proof to construct a solution $u_{c_0} = u_{1,c_0} + iu_{2,c_0}$ in $t \in [0, T)$, which blows up in finite time T only at the origin such that (1.6) and (1.9) hold and the following holds

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_{2,c_0}(\cdot, t) - c_0 g_0 \left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{|\ln(T-t)|^{\frac{p_1}{2}}}, \quad (1.13)$$

and

$$u_2^*(x) \sim \frac{2pc_0}{(p-1)^2} \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||} \right]^{-\frac{1}{p-1}} \frac{1}{|\ln|x||}, \text{ as } x \rightarrow 0, \tag{1.14}$$

Remark 1.8. As in the case $p = 2$ treated by Nouaïli and Zaag [21], and we also mentioned we suspect the behavior in Theorem 1.1 to be unstable. This is due to the fact that the number of parameters in the initial data we consider below in Definition 3.4 (see also Remark 1.6 above) is higher than the dimension of the blowup parameters which is $n + 1$ (n for the blowup points and 1 for the blowup time).

Besides that, we can use the technique of Merle [14] to construct a solution which blows up at arbitrary given points. More precisely, we have the following Corollary:

Corollary 1.9 (Blowing up at k distinct points). *For any given points, x_1, \dots, x_k , there exists a solution of (1.1) which blows up exactly at x_1, \dots, x_k . Moreover, the local behavior at each blowup point x_j is also given by (1.6), (1.7), (1.9), (1.10) by replacing x by $x - x_j$ and $L^\infty(\mathbb{R}^n)$ by $L^\infty(|x - x_j| \leq \epsilon_0)$, for some $\epsilon_0 > 0$.*

1.3. The strategy of the proof

From the singularity of the nonlinear term (u^p) when $p \notin \mathbb{N}$, we can not apply the techniques we used in [5] when $p \in \mathbb{N}$ (also used in [17], [22], ...). We need to modify this method. We see that, although our nonlinear term is not continuous in general, it is continuous in the following half plane

$$\{u \mid \operatorname{Re}(u) > 0\}.$$

Relying on this property, our problem will be derived by using the techniques which were used in [5] and the fine control of the positivity of the real part. We treat this challenge by relying on the ideas of the work of Merle and Zaag in [16] (or the work of Ghoul, Nguyen and Zaag in [10] with inherited ideas from [16]) for the construction of the initial data. We define a shrinking set $S(t)$ (see in Definition 3.1) which allows a very fine control of the positivity of the real part. More precisely, it is proceed to control our solution on three regions $P_1(t), P_2(t)$ and $P_3(t)$ which are given in subsection 3.2 and which we recall here:

- $P_1(t)$, called the blowup region, i.e $|x| \leq K_0\sqrt{(T-t)|\ln(T-t)|}$: We control our solution as a perturbation of the intermediate blowup profiles (for $t \in [0, T)$) f_0 and g_0 given in (1.6) and (1.7).

- $P_2(t)$, called the intermediate region, i.e $\frac{K_0}{4}\sqrt{(T-t)|\ln(T-t)|} \leq |x| \leq \epsilon_0$: In this region, we will control our solution by control the rescaled function U of u (see more (3.20)) to approach $\hat{U}_{K_0}(\tau)$ (see in (3.25)), by using a classical parabolic estimates. Roughly speaking, we control our solution as a perturbation of the final profiles for $t = T$ given in (1.9) and (1.10).

- $P_3(t)$, called the regular region, i.e $|x| \geq \frac{\epsilon_0}{4}$: In this region, we control the solution as a perturbation of initial data ($t = 0$). Indeed, T will be chosen small by the end of the proof.

Fixing some constants involved in the definition $S(t)$, we can prove that our problem will be solved by the control of the solution in $S(t)$. Moreover, we prove via a priori estimates in the different regions P_1, P_2, P_3 that the control is reduced to the control of a finite dimensional component of the solution. Finally, we may apply the techniques in [5] to get our conclusion.

We will organize our paper as follows:

- In Section 2: We give a formal approach to explain how the profiles we have in Theorem 1.1 appear naturally. Moreover, we also approach our problem through two independant directions: *Inner expansion* and *Outer expansion*, in order to show that our profiles are reasonable.

- In Section 3: We give a formulation for our problem (see equation (3.2)) and, step by step we give the rigorous proof for Theorem 1.1, assuming some technical estimates.

- In Section 4, we prove the techical estimates assumed in Section 3.

2. Derivation of the profile (formal approach)

In this section, we would like to give a formal approach to our problem which explains how we derive the profiles for the solution of equation (1.1) given in Theorem (1.1), as well the asymptotics of the solution. In particular, we would like to mention that the main difference between the case $p \in \mathbb{N}$ and $p \notin \mathbb{N}$ resides in the way we handle the nonlinear term u^p . For that reason, we will give a lot of care for the estimates involving the nonlinear term, and go quickly while giving estimates related to other terms, kindly refering the reader to [5] where the case $p \in \mathbb{N}$ was treated.

2.1. Modeling the problem

In this part, we will give definitions and special symbols important for our work and explain how the functions f_0, g_0 arise as blowup profiles for the solution of equation (1.1) as stated in (1.6) and (1.7). Our aim in this section is to give solid (though formal) hints for the existence of a solution $u(t) = u_1(t) + iu_2(t)$ to equation (1.1) such that

$$\lim_{t \rightarrow T} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty, \quad (2.1)$$

and u obeys the profiles in (1.6) and (1.7), for some $T > 0$. As we have pointed out in the introduction, we are interested in the case where

$$p \notin \mathbb{N},$$

noting that in this case, we already have a difficulty to properly define the nonlinear term u^p as a continuous term. In order to overcome this difficulty, we will restrict ourselves to the case where

$$\operatorname{Re}(u) > 0. \quad (2.2)$$

Our main challenge in this work will be to show that (2.2) is propagated by the flow, at least for the initial data we are suggesting (see Definition 3.4 below). Therefore, under the condition (2.2), by using equation (1.1), we deduce that u_1, u_2 solve:

$$\begin{cases} \partial_t u_1 &= \Delta u_1 + F_1(u_1, u_2), \\ \partial_t u_2 &= \Delta u_2 + F_2(u_1, u_2). \end{cases} \quad (2.3)$$

where $F_1(0, 0) = F_2(0, 0) = 0$ and for all $(u_1, u_2) \neq 0$ we have

$$\begin{cases} F_1(u_1, u_2) &= \operatorname{Re}[(u_1 + iu_2)^p] = |u|^p \cos[p \operatorname{Arg}(u_1, u_2)], \\ F_2(u_1, u_2) &= \operatorname{Im}[(u_1 + iu_2)^p] = |u|^p \sin[p \operatorname{Arg}(u_1, u_2)], \end{cases} \quad (2.4)$$

with $|u| = (u_1^2 + u_2^2)^{\frac{1}{2}}$ and $\operatorname{Arg}(u_1, u_2), u_1 > 0$ is defined as follows:

$$\operatorname{Arg}(u_1, u_2) = \arcsin \left[\frac{u_2}{\sqrt{u_1^2 + u_2^2}} \right]. \quad (2.5)$$

Note that, in the case where $p \in \mathbb{N}$, we had the following simple expressions for F_1, F_2

$$\begin{cases} F_1(u_1, u_2) &= \operatorname{Re}[(u_1 + iu_2)^p] = \sum_{j=0}^{\lfloor \frac{p}{2} \rfloor} C_p^{2j} (-1)^j u_1^{p-2j} u_2^{2j}, \\ F_2(u_1, u_2) &= \operatorname{Im}[(u_1 + iu_2)^p] = \sum_{j=0}^{\lfloor \frac{p-1}{2} \rfloor} C_p^{2j+1} (-1)^j u_1^{p-2j-1} u_2^{2j+1}. \end{cases} \quad (2.6)$$

Of course, both expressions (2.4) and (2.6) coincide when $p \in \mathbb{N}$. In fact, we will follow our strategy in [5] for $p \in \mathbb{N}$ and focus mainly on how we handle the nonlinear terms, since we have a different expression when $p \notin \mathbb{N}$.

Let us introduce *the similarity-variables* for $u = u_1 + iu_2$ as follows:

$$w_1(y, s) = (T - t)^{\frac{1}{p-1}} u_1(x, t), w_2(y, s) = (T - t)^{\frac{1}{p-1}} u_2(x, t), y = \frac{x}{\sqrt{T-t}}, s = -\ln(T-t). \quad (2.7)$$

By using (2.3), we obtain a system satisfied by (w_1, w_2) , for all $y \in \mathbb{R}^n$ and $s \geq -\ln T$ as follows:

$$\begin{cases} \partial_s w_1 &= \Delta w_1 - \frac{1}{2} y \cdot \nabla w_1 - \frac{w_1}{p-1} + F_1(w_1, w_2), \\ \partial_s w_2 &= \Delta w_2 - \frac{1}{2} y \cdot \nabla w_2 - \frac{w_2}{p-1} + F_2(w_1, w_2). \end{cases} \quad (2.8)$$

Then note that studying the asymptotics of $u_1 + iu_2$ as $t \rightarrow T$ is equivalent to studying the asymptotics of $w_1 + iw_2$ in long time. We are first interested in the set of constant solutions of (2.8), denoted by

$$\mathcal{S} = \{(0, 0)\} \cup \left\{ \kappa \left(\cos \left(\frac{2k\pi}{p-1} \right), \sin \left(\frac{2k\pi}{p-1} \right) \right) \text{ where } \kappa = (p-1)^{-\frac{1}{p-1}}, \text{ and } k \in \mathbb{N} \right\}.$$

We remark that \mathcal{S} is infinity if p is not integer. However, from the transformation (2.7), we slightly precise our goal in (2.1) by requiring in addition that

$$(w_1, w_2) \rightarrow (\kappa, 0) \text{ as } s \rightarrow +\infty.$$

Introducing $w_1 = \kappa + \bar{w}_1$, our goal because to get

$$(\bar{w}_1, w_2) \rightarrow (0, 0) \text{ as } s \rightarrow +\infty.$$

From (2.8), we deduce that \bar{w}_1, w_2 satisfy the following system

$$\begin{cases} \partial_s \bar{w}_1 &= \mathcal{L}\bar{w}_1 + \bar{B}_1(\bar{w}_1, w_2), \\ \partial_s w_2 &= \mathcal{L}w_2 + \bar{B}_2(\bar{w}_1, w_2). \end{cases} \quad (2.9)$$

where

$$\mathcal{L} = \Delta - \frac{1}{2}y \cdot \nabla + Id, \quad (2.10)$$

$$\bar{B}_1(\bar{w}_1, w_2) = F_1(\kappa + \bar{w}_1, w_2) - \kappa^p - \frac{p}{p-1}\bar{w}_1, \quad (2.11)$$

$$\bar{B}_2(\bar{w}_1, w_2) = F_2(\kappa + \bar{w}_1, w_2) - \frac{p}{p-1}w_2. \quad (2.12)$$

It is important to study the linear operator \mathcal{L} and the asymptotics of \bar{B}_1, \bar{B}_2 as $(\bar{w}_1, w_2) \rightarrow (0, 0)$ which will appear as quadratic.

- *The properties of \mathcal{L} :*

We observe that the operator \mathcal{L} plays an important role in our analysis. It is easy to find an analysis space such that \mathcal{L} is self-adjoint. Indeed, \mathcal{L} is self-adjoint in $L_\rho^2(\mathbb{R}^n)$, where L_ρ^2 is the weighted space associated to the weight ρ defined by

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{n}{2}}} = \prod_{j=1}^n \rho(y_j), \quad \text{with } \rho(\xi) = \frac{e^{-\frac{|\xi|^2}{4}}}{(4\pi)^{\frac{1}{2}}}, \quad (2.13)$$

and the spectrum set of \mathcal{L}

$$\text{spec}(\mathcal{L}) = \left\{ 1 - \frac{m}{2}, m \in \mathbb{N} \right\}. \quad (2.14)$$

Moreover, we can find eigenfunctions which correspond to each eigenvalue $1 - \frac{m}{2}, m \in \mathbb{N}$:

- The one space dimensional case: the eigenfunction corresponding to the eigenvalue $1 - \frac{m}{2}$ is h_m , the rescaled Hermite polynomial given as follows

$$h_m(y) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^j m! y^{m-2j}}{j!(m-2j)!}. \quad (2.15)$$

In particular, we have the following orthogonality property:

$$\int_{\mathbb{R}} h_i h_j \rho dy = i! 2^i \delta_{i,j}, \quad \forall (i, j) \in \mathbb{N}^2.$$

- The higher dimensional case: $n \geq 2$, the eigenspace \mathcal{E}_m , corresponding to the eigenvalue $1 - \frac{m}{2}$ is defined as follows:

$$\mathcal{E}_m = \{h_\beta = h_{\beta_1} \cdots h_{\beta_n}, \text{ for all } \beta \in \mathbb{N}^n, |\beta| = m, |\beta| = \beta_1 + \cdots + \beta_n\}. \quad (2.16)$$

Accordingly, we can represent an arbitrary function $r \in L_\rho^2$ as follows

$$r = \sum_{\beta, \beta \in \mathbb{N}^n} r_\beta h_\beta(y),$$

where: r_β is the projection of r on h_β for any $\beta \in \mathbb{R}^n$ which is defined as follows:

$$r_\beta = \mathbb{P}_\beta(r) = \int r k_\beta \rho dy, \quad \forall \beta \in \mathbb{N}^n, \quad (2.17)$$

with

$$k_\beta(y) = \frac{h_\beta}{\|h_\beta\|_{L_\rho^2}^2}. \quad (2.18)$$

- *The asymptotics of $\bar{B}_1(\bar{w}_1, w_2), \bar{B}_2(\bar{w}_1, w_2)$:* The following asymptotics hold:

$$\bar{B}_1(\bar{w}_1, w_2) = \frac{p}{2\kappa} \bar{w}_1^2 + O(|\bar{w}_1|^3 + |w_2|^2), \quad (2.19)$$

$$\bar{B}_2(\bar{w}_1, w_2) = \frac{p}{\kappa} \bar{w}_1 w_2 + O(|\bar{w}_1|^2 |w_2|) + O(|w_2|^3), \quad (2.20)$$

as $(\bar{w}_1, w_2) \rightarrow (0, 0)$. Note that although we have here the expressions of the nonlinear terms F_1, F_2 which are different from the case $p \in \mathbb{N}$ (see (2.4) and (2.6)), the expressions coincide, since we have $u \sim \kappa = (p-1)^{-\frac{1}{p-1}}$ in all case (see Lemma B.1 below).

2.2. Inner expansion

In this part, we study the asymptotics of the solution in $L_\rho^2(\mathbb{R}^n)$. Moreover, for simplicity we suppose that $n = 1$, and we recall that we aim at constructing a solution of (2.9) such that $(\bar{w}_1, w_2) \rightarrow (0, 0)$. Note first that the spectrum of \mathcal{L} contains two positive eigenvalues $1, \frac{1}{2}$, a neutral eigenvalue 0 and all the other ones are strictly negative. So, in the representation of the solution in L_ρ^2 , it is reasonable to think that the part corresponding to the negative spectrum is easily controlled. Imposing a symmetry condition on the solution with respect of y , it is reasonable to look for a solution \bar{w}_1, w_2 of the form:

$$\begin{aligned}\bar{w}_1 &= \bar{w}_{1,0}h_0 + \bar{w}_{1,2}h_2, \\ w_2 &= w_{2,0}h_0 + w_{2,2}h_2.\end{aligned}$$

From the assumption that $(\bar{w}_1, w_2) \rightarrow (0, 0)$, we see that $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2} \rightarrow 0$ as $s \rightarrow +\infty$. We see also that we can understand the asymptotics of the solution \bar{w}_1, w_2 in L_ρ^2 from the study of the asymptotics of $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2}$. We now project equations (2.9) on h_0 and h_2 . Using the asymptotics of \bar{B}_1, \bar{B}_2 in (2.19) and (2.20), we get the following ODEs for $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2}$:

$$\partial_s \bar{w}_{1,0} = \bar{w}_{1,0} + \frac{p}{2\kappa} (\bar{w}_{1,0}^2 + 8\bar{w}_{1,2}^2) + O(|\bar{w}_{1,0}|^3 + |\bar{w}_{1,2}|^3) + O(|w_{2,0}|^2 + |w_{2,2}|^2), \quad (2.21)$$

$$\partial_s \bar{w}_{1,2} = \frac{p}{\kappa} (\bar{w}_{1,0}\bar{w}_{1,2} + 4\bar{w}_{1,2}^2) + O(|\bar{w}_{1,0}|^3 + |\bar{w}_{1,2}|^3) + O(|w_{2,0}|^2 + |w_{2,2}|^2), \quad (2.22)$$

$$\begin{aligned}\partial_s w_{2,0} &= w_{2,0} + \frac{p}{\kappa} [\bar{w}_{1,0}w_{2,0} + 8\bar{w}_{1,2}w_{2,2}] + O((|\bar{w}_{1,0}|^2 + |\bar{w}_{1,2}|^2)(|w_{2,0}| + |w_{2,2}|)) \\ &\quad + O(|w_{2,0}|^3 + |w_{2,2}|^3),\end{aligned} \quad (2.23)$$

$$\begin{aligned}\partial_s w_{2,2} &= \frac{p}{\kappa} [\bar{w}_{1,0}w_{2,2} + \bar{w}_{1,2}w_{2,0} + 8\bar{w}_{1,2}w_{2,2}] + O((|\bar{w}_{1,0}|^2 + |\bar{w}_{1,2}|^2)(|w_{2,0}| + |w_{2,2}|)) \\ &\quad + O(|w_{2,0}|^3 + |w_{2,2}|^3).\end{aligned} \quad (2.24)$$

Assuming that

$$\bar{w}_{1,0}, w_{2,0}, w_{2,2} \ll \bar{w}_{1,2}, \quad (2.25)$$

and

$$\bar{w}_{1,0}, w_{2,0}, w_{2,2} \lesssim \frac{1}{s^2}, \quad (2.26)$$

as $s \rightarrow +\infty$. Similarly as in [5], where we have $p \in \mathbb{N}$, we obtain the following asymptotics of $\bar{w}_{1,0}, \bar{w}_{1,2}, w_{2,0}, w_{2,2}$:

$$\begin{aligned}\bar{w}_{1,0} &= O\left(\frac{1}{s^2}\right), \\ \bar{w}_{1,2} &= -\frac{\kappa}{4ps} + O\left(\frac{\ln s}{s^2}\right), \\ w_{2,0} &= O\left(\frac{1}{s^3}\right), \\ w_{2,2} &= \frac{c_{2,2}}{s^2} + O\left(\frac{\ln s}{s^3}\right), c_{2,2} \neq 0,\end{aligned}$$

as $s \rightarrow +\infty$ which satisfy the assumption in (2.25) and (2.26). Then, we have

$$w_1 = \kappa - \frac{\kappa}{4ps}(y^2 - 2) + O\left(\frac{1}{s^2}\right), \quad (2.27)$$

$$w_2 = \frac{c_{2,2}}{s^2}(y^2 - 2) + O\left(\frac{\ln s}{s^3}\right), \quad (2.28)$$

in $L_\rho^2(\mathbb{R})$ for some $c_{2,2}$ in \mathbb{R}^* . Note that, by using parabolic regularity, we can derive that the asymptotics (2.27), (2.28) also hold for all $|y| \leq K$, where K is an arbitrary positive constant.

2.3. Outer expansion

As for the inner expansion, we here assume that $n = 1$. We see that asymptotics (2.27) and (2.28) can not give us a shape, since they hold uniformly on compact sets (where we only see the constant solution $(\kappa, 0)$) and not in larger sets. Fortunately, we observe from (2.27) and (2.28) that the profile may be based on the following variable:

$$z = \frac{y}{\sqrt{s}}. \quad (2.29)$$

This motivates us to look for solutions of the form:

$$\begin{aligned} w_1(y, s) &= \sum_{j=0}^{\infty} \frac{R_{1,j}(z)}{s^j}, \\ w_2(y, s) &= \sum_{j=1}^{\infty} \frac{R_{2,j}(z)}{s^j}. \end{aligned}$$

Note that, our purpose is to construct a solution where the real part is positive. So, it is reasonable to assume that $w_1 > 0$ and $R_{1,0}(z) > 0$ for all $z \in \mathbb{R}$. Besides that, we also assume that $R_{1,j}, R_{2,j}$ are smooth and have bounded derivatives. From the definitions of F_1, F_2 , given in (2.4), we have the following

$$\begin{aligned} \left| F_1 \left(\sum_{j=0}^{\infty} \frac{R_{1,j}(z)}{s^j}, \sum_{j=1}^{\infty} \frac{R_{2,j}(z)}{s^j} \right) - R_{1,0}^p(z) - \frac{pR_{1,0}^{p-1}(z)R_{1,1}(z)}{s} \right| &\leq \frac{C(z)}{s^2}, \\ \left| F_2 \left(\sum_{j=0}^{\infty} \frac{R_{1,j}(z)}{s^j}, \sum_{j=1}^{\infty} \frac{R_{2,j}(z)}{s^j} \right) - \frac{pR_{1,0}^{p-1}(z)R_{2,1}(z)}{s} \right. \\ \left. - \frac{1}{s^2} \left(pR_{1,0}^{p-1}(z)R_{2,2} + p(p-1)R_{1,0}^{p-2}(z)R_{1,1}(z)R_{2,1}(z) \right) \right| &\leq \frac{C(z)}{s^3}. \end{aligned}$$

Thus, for each $z \in \mathbb{R}$, by using system (2.8), taking $s \rightarrow +\infty$, we obtain the following system:

$$0 = -\frac{1}{2}R'_{1,0}(z) \cdot z - \frac{R_{1,0}(z)}{p-1} + R_{1,0}^p(z), \quad (2.30)$$

$$0 = -\frac{1}{2}zR'_{1,1}(z) - \frac{R_{1,1}(z)}{p-1} + pR_{1,0}^{p-1}(z)R_{1,1}(z) + R''_{1,0}(z) + \frac{zR'_{1,0}(z)}{2}, \quad (2.31)$$

$$0 = -\frac{1}{2}R'_{2,1}(z) \cdot z - \frac{R_{2,1}(z)}{p-1} + pR_{1,0}^{p-1}(z)R_{2,1}(z), \quad (2.32)$$

$$\begin{aligned} 0 &= -\frac{1}{2}R'_{2,2}(z) \cdot z - \frac{R_{2,2}(z)}{p-1} + pR_{1,0}^{p-1}(z)R_{2,2}(z) + R''_{2,1}(z) + R_{2,1}(z) + \frac{1}{2}R'_{2,1}(z) \cdot z \\ &+ p(p-1)R_{1,0}^{p-2}(z)R_{1,1}(z)R_{2,1}(z). \end{aligned} \quad (2.33)$$

This system is quite similar to [5] (where $p \in \mathbb{N}$), and we can find the formulas of $R_{1,0}, R_{1,1}, R_{2,1}, R_{2,2}$ as follows:

$$R_{1,0}(z) = (p-1 + b|z|^2)^{-\frac{1}{p-1}}, \quad (2.34)$$

$$R_{1,1}(z) = \frac{(p-1)}{2p}(p-1 + bz^2)^{-\frac{p}{p-1}} - \frac{p-1}{4p}z^2 \ln(p-1 + bz^2)(p-1 + bz^2)^{-\frac{p}{p-1}}, \quad (2.35)$$

$$R_{2,1}(z) = \frac{z^2}{(p-1 + bz^2)^{\frac{p}{p-1}}}, \quad (2.36)$$

$$R_{2,2}(z) = -2(p-1 + bz^2)^{-\frac{p}{p-1}} + H_{2,2}(z), \quad (2.37)$$

where $b = \frac{(p-1)^2}{4p}$ and

$$\begin{aligned} H_{2,2}(z) &= C_{2,1}(p)z^2(p-1 + bz^2)^{-\frac{2p-1}{p-1}} + C_{2,3}(p)z^2 \ln(p-1 + bz^2)(p-1 + bz^2)^{-\frac{p}{p-1}} \\ &+ C_{2,3}(p)z^2 \ln(p-1 + bz^2)(p-1 + bz^2)^{-\frac{2p-1}{p-1}}. \end{aligned}$$

2.4. Matching asymptotics

By comparing the inner expansion and the outer expansion, then fixing several constants, we have the following profiles for w_1 and w_2

$$\begin{cases} w_1(y, s) & \sim \Phi_1(y, s), \\ w_2(y, s) & \sim \Phi_2(y, s), \end{cases} \quad (2.38)$$

where

$$\Phi_1(y, s) = \left(p - 1 + \frac{(p-1)^2 |y|^2}{4p s} \right)^{-\frac{1}{p-1}} + \frac{n\kappa}{2ps}, \quad (2.39)$$

$$\Phi_2(y, s) = \frac{|y|^2}{s^2} \left(p - 1 + \frac{(p-1)^2 |y|^2}{4p s} \right)^{-\frac{p}{p-1}} - \frac{2n\kappa}{(p-1)s^2}, \quad (2.40)$$

for all $(y, s) \in \mathbb{R}^n \times (0, +\infty)$. In this section, we will give a rigorous proof for the existence of a solution (w_1, w_2) of equation (2.8) where (2.38) holds.

3. Existence of a blowup solution in Theorem 1.1

In Section 2, we adopted a formal approach in order to justify how the profiles f_0, g_0 arise as blowup profiles for the solution of equation (1.1), given in Theorem 1.1. In this section, we give a rigorous proof to justify the existence of a solution approaching those profiles.

3.1. Formulation of the problem

In this subsection, we aim at giving a complete formulation of our problem in order to justify the formal approach which is given in the previous section. We introduce

$$\begin{cases} w_1 & = \Phi_1 + q_1, \\ w_2 & = \Phi_2 + q_2, \end{cases} \quad (3.1)$$

where Φ_1, Φ_2 are defined in (2.39) and (2.40) respectively. Then, by using (2.8), we derive the following system, satisfied by (q_1, q_2) :

$$\partial_s \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L} + V & 0 \\ 0 & \mathcal{L} + V \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} V_{1,1} & V_{1,2} \\ V_{2,1} & V_{2,2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1(q_1, q_2) \\ B_2(q_1, q_2) \end{pmatrix} + \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \quad (3.2)$$

where linear operator \mathcal{L} is defined in (2.10) and:

- The potential functions $V, V_{1,1}, V_{1,2}, V_{2,1}, V_{2,2}$ are defined as follows

$$V(y, s) = p \left(\Phi_1^{p-1} - \frac{1}{p-1} \right), \quad (3.3)$$

$$V_{1,1}(y, s) = \partial_{u_1} F_1(u_1, u_2)|_{(u_1, u_2)=(\Phi_1, \Phi_2)} - p\Phi_1^{p-1}, \quad (3.4)$$

$$V_{1,2}(y, s) = \partial_{u_2} F_1(u_1, u_2)|_{(u_1, u_2)=(\Phi_1, \Phi_2)}, \quad (3.5)$$

$$V_{2,1}(y, s) = \partial_{u_1} F_2(u_1, u_2)|_{(u_1, u_2)=(\Phi_1, \Phi_2)}, \quad (3.6)$$

$$V_{2,2}(y, s) = \partial_{u_2} F_2(u_1, u_2)|_{(u_1, u_2)=(\Phi_1, \Phi_2)} - p\Phi_1^{p-1}. \quad (3.7)$$

- The quadratic terms $B_1(q_1, q_2), B_2(q_1, q_2)$ are defined as follows:

$$\begin{aligned} B_1(q_1, q_2) &= F_1(\Phi_1 + q_1, \Phi_2 + q_2) - F_1(\Phi_1, \Phi_2) - \partial_{u_1} F_1(u_1, u_2)|_{(u_1, u_2)=(\Phi_1, \Phi_2)} q_1 \\ &\quad - \partial_{u_2} F_1(u_1, u_2)|_{(u_1, u_2)=(\Phi_1, \Phi_2)} q_2, \end{aligned} \quad (3.8)$$

$$\begin{aligned} B_2(q_1, q_2) &= F_2(\Phi_1 + q_1, \Phi_2 + q_2) - F_2(\Phi_1, \Phi_2) - \partial_{u_1} F_2(u_1, u_2)|_{(u_1, u_2)=(\Phi_1, \Phi_2)} q_1 \\ &\quad - \partial_{u_2} F_2(u_1, u_2)|_{(u_1, u_2)=(\Phi_1, \Phi_2)} q_2. \end{aligned} \quad (3.9)$$

- The rest terms $R_1(y, s), R_2(y, s)$ are defined as follows:

$$R_1(y, s) = \Delta \Phi_1 - \frac{1}{2} y \cdot \nabla \Phi_1 - \frac{\Phi_1}{p-1} + F_1(\Phi_1, \Phi_2) - \partial_s \Phi_1, \quad (3.10)$$

$$R_2(y, s) = \Delta \Phi_2 - \frac{1}{2} y \cdot \nabla \Phi_2 - \frac{\Phi_2}{p-1} + F_2(\Phi_1, \Phi_2) - \partial_s \Phi_2, \quad (3.11)$$

where F_1, F_2 are defined in (2.4).

By the linearization around Φ_1, Φ_2 , our problem is reduced to constructing a solution (q_1, q_2) of system (3.2), satisfying

$$\|q_1\|_{L^\infty(\mathbb{R}^n)} + \|q_2\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0 \text{ as } s \rightarrow +\infty.$$

Looking at system (3.2), we already know some of the main properties of the linear operator \mathcal{L} (see page 7). As for the potentials $V_{j,k}$ where $j, k \in \{1, 2\}$, they admit the following asymptotics:

$$\begin{aligned} \|V_{1,1}(\cdot, s)\|_{L^\infty} + \|V_{2,2}(\cdot, s)\|_{L^\infty} &\leq \frac{C}{s^2}, \\ \|V_{1,2}(\cdot, s)\|_{L^\infty} + \|V_{2,1}(\cdot, s)\|_{L^\infty} &\leq \frac{C}{s}, \forall s \geq 1, \end{aligned}$$

(see Lemma B.2 below).

Regarding the terms B_1, B_2 which are quadratic, we have these estimates

$$\begin{aligned} \|B_1(q_1, q_2)\|_{L^\infty} &\leq \frac{CA^4}{s^{\frac{p}{2}}}, \\ \|B_2(q_1, q_2)\|_{L^\infty} &\leq \frac{CA^2}{s^{1+\min(\frac{p-1}{4}, \frac{1}{2})}}, \end{aligned}$$

if q_1, q_2 are small in some sense (see Lemma B.3 below).

In addition to that, the rest terms R_1, R_2 satisfy the following asymptotics

$$\begin{aligned} \|R_1(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} &\leq \frac{C}{s}, \\ \|R_2(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} &\leq \frac{C}{s^2}, \end{aligned}$$

(see Lemma B.4 below).

As a matter of fact, the dynamics of equation (3.2) will mainly depend on the main linear operator

$$\begin{pmatrix} \mathcal{L} + V & 0 \\ 0 & \mathcal{L} + V \end{pmatrix},$$

and the effects of the orther terms will be less important except on the zero mode of this equation. For that reason, we need to understand the dynamics of $\mathcal{L} + V$. Since the spectral properties of \mathcal{L} were already introduced in Section 2.1, we will focus here on the effect of V .

i) Effect of V inside the blowup region $\{|y| \leq K_0\sqrt{s}\}$ with $K_0 > 0$: It satisfies the following estimate:

$$V \rightarrow 0 \text{ in } L^2_\rho(|y| \leq K_0\sqrt{s}) \text{ as } s \rightarrow +\infty,$$

which means that the effect of V will be negligible with respect of the effect of \mathcal{L} , except perhaps on the null mode of \mathcal{L} (see item *ii*) of Proposition 4.1 below).

ii) Effect of V outside the blowup region: For each $\epsilon > 0$, there exist $K_\epsilon > 0$ and $s_\epsilon > 0$ such that

$$\sup_{\frac{y}{\sqrt{s}} \geq K_\epsilon, s \geq s_\epsilon} \left| V(y, s) - \left(-\frac{p}{p-1} \right) \right| \leq \epsilon.$$

Since 1 is the biggest eigenvalue of \mathcal{L} (see (2.14)), the operator $\mathcal{L} + V$ behaves as one with with a fully negative spectrum outside blowup region $\{|y| \geq K_\epsilon\sqrt{s}\}$, which makes the control of the solution in this region easy.

Since the behavior of the potential V inside and outside the blowup region is different, we will consider the dynamics of the solution for $|y| \leq 2K_0\sqrt{s}$ and for $|y| \geq K_0\sqrt{s}$ separately for some K_0 to be fixed large. For that purpose, we introduce the following cut-off function

$$\chi(y, s) = \chi_0 \left(\frac{|y|}{K_0\sqrt{s}} \right), \quad (3.12)$$

where χ_0 is defined as a cut-off function:

$$\chi_0 \in C_0^\infty[0, +\infty), \chi_0(x) = \begin{cases} 1 & \text{for } x \leq 1, \\ 0 & \text{for } x \geq 2, \end{cases} \text{ and } \|\chi_0\|_{L^\infty} \leq 1. \quad (3.13)$$

Hence, it is reasonable to consider separately the solution in the blowup region $\{|y| \leq 2K_0\sqrt{s}\}$ and in the regular region $\{|y| \geq K_0\sqrt{s}\}$. More precisely, let us define the following notation for all functions r in L^∞ as follows

$$r = r_b + r_e \text{ with } r_b = \chi r \text{ and } r_e = (1 - \chi)r. \quad (3.14)$$

Note in particular that $\text{supp}(r_b) \subset \mathbb{B}(0, 2K_0\sqrt{s})$ and $\text{supp}(r_e) \subset \mathbb{R}^n \setminus \mathbb{B}(0, K_0\sqrt{s})$. Besides that, we also expand r_b in L^2_ρ according to the spectrum of \mathcal{L} (see Section 2.1 above):

$$r_b(y) = r_0 + r_1 \cdot y + \frac{1}{2}y^\mathcal{T} \cdot r_2 \cdot y - \text{Tr}(r_2) + r_-(y), \quad (3.15)$$

where r_0 is a scalar, r_1 is a vector in \mathbb{R}^n and r_2 is a $n \times n$ matrix defined by

$$\begin{aligned} r_0 &= \int_{\mathbb{R}^n} r_b \rho(y) dy, \\ r_1 &= \int_{\mathbb{R}^n} r_b \frac{y}{2} \rho(y) dy, \\ r_2 &= \left(\int_{\mathbb{R}^n} r_b \left(\frac{1}{4} y_j y_k - \frac{1}{2} \delta_{j,k} \right) \rho(y) dy \right)_{1 \leq j, k \leq n}, \end{aligned}$$

with $\text{Tr}(r_2)$ being the trace of matrix r_2 . The reader should keep in mind that r_0, r_1, r_2 are only the coordinates of r_b , not for r . Note that r_m is the projection of r_b on the eigenspace of \mathcal{L} corresponding to the eigenvalue $\lambda = 1 - \frac{m}{2}$. Accordingly, r_- is the projection of r_b on the negative part of the spectrum of \mathcal{L} . As a consequence of (3.14) and (3.15), we see that every $r \in L^\infty(\mathbb{R}^n)$ can be decomposed into 5 components as follows:

$$r = r_b + r_e = r_0 + r_1 \cdot y + \frac{1}{2}y^\mathcal{T} \cdot r_2 \cdot y - \text{Tr}(r_2) + r_- + r_e. \quad (3.16)$$

3.2. The shrinking set

According to (2.7) and (3.1), our goal is to construct a solution (q_1, q_2) of system (3.2) such that they satisfy the following estimates:

$$\|q_1(\cdot, s)\|_{L^\infty} + \|q_2(\cdot, s)\|_{L^\infty} \rightarrow 0 \text{ as } s \rightarrow +\infty. \quad (3.17)$$

Here, we aim at constructing a shrinking set to 0. Then, the control of $(q_1, q_2) \rightarrow 0$, will be a consequence of the control of (q_1, q_2) in this shrinking set. In addition to that, we have to control the solution q_1 so that

$$w_1 = q_1 + \Phi_1 > 0, \quad (3.18)$$

(this is equivalent to have $u_1 > 0$) and it is one of the main difficulties in our analysis. As a matter of fact, the shrinking sets which were constructed in [17] by Merle and Zaag or even in [5], are not sharp enough to ensure (3.18). In other words, our set has to shrink to 0 as $s \rightarrow +\infty$ and ensure that the real part of the solution to (2.8) is always positive. In fact, the positivity is the first thing to be solved. For the control of the positivity of the real part, we rely on the ideas, given by Merle and Zaag in [16] for the control of the solution of the following equation:

$$\partial_t u = \Delta u - \eta \frac{|\nabla u|^2}{u} + |u|^{p-1} u, u \in \mathbb{R}. \quad (3.19)$$

In [16], the authors needed a sharp control of u and $|\nabla u|$ near zero, in order to bound the term $\frac{|\nabla u|^2}{u}$. Here, we will use their ideas in order to control u_1 near zero and ensure its positivity. As in [16], we will control the solution differently in 3 overlapping regions defined as follows:

For $K_0 > 0, \alpha_0 > 0, \epsilon_0 > 0, t \in [0, T], s \in [-\ln T, +\infty), s = -\ln(T - t)$, we introduce a cover of \mathbb{R}^n as follows

$$\mathbb{R}^n \subset P_1(t) \cup P_2(t) \cup P_3(t),$$

where

$$\begin{aligned}
 P_1(t) &= \{x \mid |x| \leq K_0 \sqrt{(T-t)|\ln(T-t)|}\} = \{x \mid |y| \leq K_0 \sqrt{s}\} = \{x \mid |z| \leq K_0\}, \\
 P_2(t) &= \left\{x \mid \frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|} \leq |x| \leq \epsilon_0\right\} = \left\{x \mid \frac{K_0}{4} \sqrt{s} \leq |y| \leq \epsilon_0 e^{\frac{s}{2}}\right\} \\
 &= \left\{x \mid \frac{K_0}{4} \leq |z| \leq \frac{\epsilon_0}{\sqrt{s}} e^{\frac{s}{2}}\right\}, \\
 P_3(t) &= \left\{x \mid |x| \geq \frac{\epsilon_0}{4}\right\} = \left\{x \mid |y| \geq \frac{\epsilon_0 e^{\frac{s}{2}}}{4}\right\} = \left\{x \mid |z| \geq \frac{\epsilon_0}{4\sqrt{s}} e^{\frac{s}{2}}\right\},
 \end{aligned}$$

with

$$y = \frac{x}{\sqrt{T-t}} \quad \text{and} \quad z = \frac{y}{\sqrt{s}} = \frac{x}{\sqrt{(T-t)|\ln(T-t)|}}.$$

In the following, let us explain how we derive the positivity condition from the various estimate we impose on the solution in the 3 regions. Then

- a) In $P_1(t)$, the *blowup region*: In this region, we control the positivity of u_1 by controlling the positivity of w_1 (see the similarity variables given in (2.7)). More precisely, as we mentioned in Subsection 1.3, w will be controlled as a perturbation of the profiles Φ_1, Φ_2 ((2.39) and (2.40)). By using the positivity of Φ_1 and a good estimate of the distance of w_1 to these profiles, we may deduce the positivity of w_1 , which leads to the positivity of u_1 .
- b) In $P_2(t)$, the *intermediate region*: In this region, we control u via a rescaled function U of u as follows:

$$U(x, \xi, \tau) = (T - t(x))^{-\frac{1}{p-1}} u(x + \xi \sqrt{T - t(x)}, t(x) + \tau(T - t(x))), \quad (3.20)$$

where $t(x)$ is uniquely defined for $|x|$ small enough by

$$|x| = \frac{K_0}{4} \sqrt{(T - t(x)) |\ln(T - t(x))|}. \quad (3.21)$$

We also introduce

$$\theta(x) = T - t(x). \quad (3.22)$$

We see that, on the domain $(\xi, \tau) \in \mathbb{R}^n \times \left[-\frac{t(x)}{T-t(x)}, 1\right)$, U satisfies the following equation:

$$\partial_\tau U = \Delta_\xi U + U^p. \quad (3.23)$$

By using classical parabolic estimates on U , we can prove the following the rescaled U at time $\tau(x, t)$, has a behavior similar to $\hat{U}_{K_0}(\tau(x, t))$, for all $|\xi| \leq \alpha_0 \sqrt{|\ln(T - t(x))|}$ where

$$\tau(x, t) = \frac{t - t(x)}{T - t(x)},$$

and $\hat{U}_{K_0}(\tau)$ is unique solution of the following ODE

$$\begin{cases} \partial_\tau \hat{U}_{K_0} &= \hat{U}_{K_0}^p(\tau), \\ \hat{U}_{K_0}(0) &= \left(p - 1 + \frac{(p-1)^2 K_0^2}{64p}\right)^{-\frac{1}{p-1}}. \end{cases} \quad (3.24)$$

In particular, we can solve (3.24) with an explicit solution:

$$\hat{U}_{K_0}(\tau) = \left((p-1)(1-\tau) + \frac{(p-1)^2 K_0^2}{64p}\right)^{-\frac{1}{p-1}}, \quad \forall \tau \in [0, 1). \quad (3.25)$$

Then, by using the positivity of \hat{U}_{K_0} , we derive that $u_1 > 0$, in this region.

- c) In $P_3(t)$, the *regular region*: We control the solution in this region as a perturbation of the initial data, thanks to the well-posedness property of the Cauchy problem for equation (1.1), to derive that our solution is close to the initial data, (in fact, T will be taken small enough). Therefore, if the initial data is strictly larger than some constant, we will derive the positivity of u_1 .

The above strategy makes the real part of our solution becomes positive. Therefore, it remains to control the solution in order to get

$$\|q_1(\cdot, s)\|_{L^\infty} + \|q_2(\cdot, s)\|_{L^\infty} \rightarrow +\infty,$$

(see (3.1)). This part is in fact quite similar to the integer case, done in [5].

From the above arguments, we give in the following our definition of the shrinking set.

Definition 3.1 (A shrinking set to 0). *For all $T > 0, K_0 > 0, \alpha_0 > 0, \epsilon_0 > 0, A > 0, \delta_0 > 0, \eta_0 > 0, p_1 \in (0, \min(\frac{p-1}{4}, \frac{1}{2}))$ for all $t \in [0, T)$, we define the set $S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t) \subset C([0, t], L^\infty(\mathbb{R}^n, \mathbb{C}))$ (or $S(t)$ for short) as follows: $u = u_1 + iu_2 \in S(t)$ if the following condition hold:*

- (i) *Control in the blowup region $P_1(t)$: We have $(q_1, q_2)(s) \in V_{p_1, K_0, A}(s)$ where $s = -\ln(T - t)$, (q_1, q_2) is defined as in (3.1) and $V_{p_1, K_0, A}(s) = V_A(s) \in (L^\infty(\mathbb{R}^n))^2$ is the set of all function $(q_1, q_2) \in (L^\infty)^2$ such that the following holds:*

$$\begin{aligned} |q_{1,0}(s)| &\leq \frac{A}{s^2} & \text{and} & & |q_{2,0}(s)| &\leq \frac{A^2}{s^{p_1+2}}, \\ |q_{1,j}(s)| &\leq \frac{A}{s^2} & \text{and} & & |q_{2,j}(s)| &\leq \frac{A^2}{s^{p_1+2}}, \forall j \leq n, \\ |q_{1,j,k}(s)| &\leq \frac{A^2 \ln s}{s^2} & \text{and} & & |q_{2,j,k}(s)| &\leq \frac{A^5 \ln s}{s^{p_1+2}}, \forall j, k \leq n, \\ \left\| \frac{q_{1,-}(y, s)}{1 + |y|^3} \right\|_{L^\infty} &\leq \frac{A}{s^2} & \text{and} & & \left\| \frac{q_{2,-}(y, s)}{1 + |y|^3} \right\|_{L^\infty} &\leq \frac{A^2}{s^{\frac{p_1+5}{2}}}, \\ \|q_{1,e}(\cdot, s)\|_{L^\infty} &\leq \frac{A^2}{\sqrt{s}} & \text{and} & & \|q_{2,e}(\cdot, s)\|_{L^\infty} &\leq \frac{A^3}{s^{\frac{p_1+2}{2}}}, \end{aligned}$$

where the coordinates of q_1 and q_2 are introduced in (3.16) with $r = q_1$ or $r = q_2$.

- (ii) *Control in the intermediate region $P_2(t)$: For all $|x| \in \left[\frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}, \epsilon_0 \right]$, $\tau(x, t) = \frac{t-t(x)}{T-t(x)}$ and $|\xi| \leq \alpha_0 \sqrt{|\ln(T-t(x))|}$, we have*

$$\left| U(x, \xi, \tau(x, t)) - \hat{U}(\tau(x, t)) \right| \leq \delta_0,$$

where \hat{U}_{K_0} defined in (3.25).

- iii *Control in the regular region $P_3(t)$: For all $|x| \geq \frac{\epsilon_0}{4}$,*

$$|u(x, t) - u(x, 0)| \leq \eta_0, \forall i = 0, 1.$$

Finally, we also define the set $S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0) \subset C([0, T], L^\infty(\mathbb{R}^n, \mathbb{C}))$ as the set of all $u \in C([0, T], L^\infty(\mathbb{R}^n, \mathbb{C}))$ such that

$$u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t), \forall t \in [0, T).$$

The following lemma, we show the estimates of the fuction being in $V_A(s)$ and this lemma is given in [5]:

Lemma 3.2. *For all $A \geq 1, s \geq 1$, if we have $(q_1, q_2) \in V_A(s)$, then the following estimates hold:*

- (i) $\|q_1\|_{L^\infty(\mathbb{R}^n)} \leq \frac{CA^2}{\sqrt{s}}$ and $\|q_2\|_{L^\infty(\mathbb{R}^n)} \leq \frac{CA^3}{s^{\frac{p_1+2}{2}}}$.

(ii)

$$|q_{1,b}(y)| \leq \frac{CA^2 \ln s}{s^2} (1 + |y|^3), \quad |q_{1,e}(y)| \leq \frac{CA^2}{s^2} (1 + |y|^3) \quad \text{and} \quad |q_1| \leq \frac{CA^2 \ln s}{s^2} (1 + |y|^3),$$

and

$$|q_{2,b}(y)| \leq \frac{CA}{s^{\frac{p_1+5}{2}}} (1 + |y|^3), \quad |q_{2,e}(y)| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}} (1 + |y|^3) \quad \text{and} \quad |q_2| \leq \frac{CA^3}{s^{\frac{p_1+5}{2}}} (1 + |y|^3).$$

- (iii) *For all $y \in \mathbb{R}^n$ we have*

$$|q_1| \leq C \left[\frac{A}{s^2} (1 + |y|) + \frac{A^2 \ln s}{s^2} (1 + |y|^2) + \frac{A^2}{s^2} (1 + |y|^3) \right],$$

and

$$|q_2| \leq C \left[\frac{A^2}{s^{p_1+2}}(1 + |y|) + \frac{A^5 \ln s}{s^{p_1+2}}(1 + |y|^2) + \frac{A^3}{s^{\frac{p_1+5}{2}}}(1 + |y|^3) \right].$$

where C will henceforth be an constant which depends only on K_0 .

Proof. See Lemma 3.2, given in [5]. \square

As matter of fact, if $u \in S_A(t)$ then, from item (i) of Lemma 3.2, the similarity variables (2.7) and (3.1), we derive the following

$$\left\| (T-t)^{\frac{1}{p-1}} u(\cdot, t) - f_0 \left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{CA^2}{1 + \sqrt{|\ln(T-t)|}}, \quad (3.26)$$

$$\left\| (T-t)^{\frac{1}{p-1}} |\ln(T-t)| u_2(\cdot, t) - g_0 \left(\frac{\cdot}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right\|_{L^\infty(\mathbb{R}^n)} \leq \frac{CA^3}{1 + |\ln(T-t)|^{\frac{p_1}{2}}}. \quad (3.27)$$

We see in the definition of $S(t)$ that there are many parameters, so the dependence of the constants on them is very important in our analysis. We would like to mention that, we use the notation C for these constants which depend at most on K_0 . Otherwise, if the constant depends on K_0, A_1, A_2, \dots we will write $C(A_1, A_2, \dots)$.

We now prove in the following lemma the positivity of $\text{Re}(u)$ at time t if u belongs to $S(t)$ (this is a crucial estimate in our argument):

Lemma 3.3 (The positivity of the real part of functions trapped in $S(t)$). *For all $K_0, A \geq 1, \alpha_0 > 0, \delta_0 < \frac{\tilde{U}(0)}{2}, \eta_0 < \frac{1}{2}$, there exists $\epsilon_1(K_0) > 0$ such that for all $\epsilon_0 \leq \epsilon_1$ there exists $T_1(A, K_0, \epsilon_0)$ such that for all $T \leq T_1$ the following holds: if $u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_1]$ for some $t_1 \in [0, T)$, and $\text{Re}(u(0)) \geq 1$ for all $|x| \geq \frac{\epsilon_0}{4}$, then*

$$\text{Re}(u)(x, t) \geq \frac{1}{2}, \forall x \in \mathbb{R}^n, \forall t \in [0, t_1].$$

Proof. We write that $u = u_1 + iu_2$, with $\text{Re}(u) = u_1$. Then, we estimate u_1 on the 3 regions $P_1(t), P_2(t)$ and $P_3(t)$.

+ *The estimate in $P_1(t)$:* We use the fact that $(q_1, q_2) \in V_A(s)$ together with item (i) in Lemma 3.2, and the definition (3.1) of q_1 and the definition of Φ_1 given in (2.39), to derive the following: for all $|y| \leq K_0\sqrt{s}$,

$$\left| w_1(y, s) - f_0 \left(\frac{y}{\sqrt{s}} \right) \right| \leq \frac{CA^2}{\sqrt{s}}.$$

Using the definition (2.39) of Φ_1 , we write for all $|y| \leq K_0\sqrt{s}$

$$\begin{aligned} w_1(y, s) &\geq f_0 \left(\frac{y}{\sqrt{s}} \right) - \frac{CA^2}{\sqrt{s}} \\ &\geq \left(p-1 + \frac{(p-1)^2}{4p} K_0^2 \right)^{-\frac{1}{p-1}} - \frac{CA^2}{\sqrt{s}}, \end{aligned}$$

By definition (2.7) of the similarity variables, we implies that: for all $|x| \leq K_0\sqrt{(T-t)|\ln(T-t)|}$,

$$(T-t)^{\frac{1}{p-1}} u_1(x, t) \geq \left(p-1 + \frac{(p-1)^2}{4p} K_0^2 \right)^{-\frac{1}{p-1}} - \frac{CA^2}{\sqrt{|\ln(T-t)|}}.$$

Therefore,

$$u_1(x, t) \geq (T-t)^{-\frac{1}{p-1}} \left[\left(p-1 + \frac{(p-1)^2}{4p} K_0^2 \right)^{-\frac{1}{p-1}} - \frac{CA^2}{\sqrt{|\ln(T-t)|}} \right] \geq \frac{1}{2},$$

provided that $T \leq T_{1,1}(K_0, A)$.

+ *The estimate in $P_2(t)$:* Since we have $u \in S(t)$, using item (ii) in the Definition 3.1, we derive that: for all $x \in \left[\frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}, \epsilon_0 \right]$

$$\left| U(x, 0, \tau(x, t)) - \hat{U}_{K_0}(\tau(x, t)) \right| \leq \delta_0,$$

where $\tau(x, t) = \frac{t-t(x)}{T-t(x)}$. In particular, by using the definition of $t(x)$ given in (3.21) and the fact that

$$|x| \geq \frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|},$$

we have $\tau(x, t) \in [0, 1)$. Therefore,

$$\begin{aligned} U_1(x, 0, \tau(x, t)) &\geq \hat{U}_{K_0}(\tau(x, t)) - \delta_0 \\ &\geq \hat{U}_{K_0}(0) - \delta_0 \\ &\geq \frac{1}{2} \hat{U}_{K_0}(0) = \frac{1}{2} \left(p-1 + \frac{(p-1)^2 K_0^2}{4p \cdot 16} \right)^{-\frac{1}{p-1}}, \end{aligned}$$

provided that $\delta_0 \leq \frac{1}{2} \hat{U}_{K_0}(0)$. By definition (3.20) of U , this implies that

$$(T-t(x))^{\frac{1}{p-1}} u_1(x, t) = U_1(x, 0, \tau(x, t)) \geq \frac{1}{2} \left(p-1 + \frac{(p-1)^2 K_0^2}{4p \cdot 16} \right)^{-\frac{1}{p-1}}.$$

Using the definition of $t(x)$ in (3.21) we write

$$T-t(x) \sim \frac{8}{K_0^2} \frac{|x|^2}{|\ln|x||}, \text{ as } |x| \rightarrow 0.$$

Therefore, there exists $\epsilon_{1,1}(K_0) > 0$ such that for all $\epsilon_0 \leq \epsilon_{1,1}$, and for all $|x| \leq \epsilon_0$, we have

$$(T-t(x))^{-\frac{1}{p-1}} \frac{1}{2} \left(p-1 + \frac{(p-1)^2 K_0^2}{4p \cdot 16} \right)^{-\frac{1}{p-1}} \geq \frac{1}{2}.$$

Then, we conclude that for all $|x| \in \left[\frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}, \epsilon_0 \right]$, we have

$$u_1(x, t) \geq \frac{1}{2},$$

provided that $T \leq T_{2,1}(\epsilon_0)$.

+ *The estimate in $P_3(t)$* : This is very easy to derive. Indeed, item (iii) of Definition 3.1, we have for all $|x| \geq \frac{\epsilon_0}{4}$

$$u_1(x, t) \geq \operatorname{Re}(u)(x, 0) - \eta_0 \geq 1 - \frac{1}{2} = \frac{1}{2},$$

provided that $\eta_0 \leq \frac{1}{2}$. This concludes the proof of Lemma 3.3. \square

Thanks to Lemma 3.3, we can handle the singularity of the nonlinear term u^p when our solution is in $S(T, A, \alpha_0, \epsilon_0, A, \delta_0, \eta_0)$. In addition to that, from item (i) of Lemma 3.3, (3.26) and (3.27) our problem is reduced to finding parameters $T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0$, and constructing initial data $u(0) \in L^\infty(\mathbb{R}^n, \mathbb{C})$ such that the solution u of equation (1.1), exists on $[0, T)$ and satisfies

$$u \in S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0). \quad (3.28)$$

3.3. Preparing initial data and the existence of a solution trapped in $S(t)$

In this subsection, we would like to define initial data $u(0)$, which depend on some parameters to be fine-tuned in order to get a good solution. The following is our definition:

Definition 3.4 (Preparing of initial data). *For each $A \geq 1, T > 0, d_1 = (d_{1,0}, d_{1,1}) \in \mathbb{R}^1 \times \mathbb{R}^n$, and $d_2 = (d_{2,0}, d_{2,1}, d_{2,1}) \in \mathbb{R}^{1+n} \times \mathbb{R}^{\frac{n(n+1)}{2}}$, we introduce the following functions defined at $s_0 = -\ln T$:*

$$\begin{aligned} \phi_{1, K_0, A, d_1}(y, s_0) &= \frac{A}{s_0^2} (d_{1,0} + d_{1,1} \cdot y) \chi_0 \left(\frac{16|y|}{K_0 \sqrt{s_0}} \right), \\ \phi_{2, K_0, A, d_2}(y, s_0) &= \left(\frac{A^2}{s_0^{p_1+2}} (d_{2,0} + d_{2,1} \cdot y) + \frac{A^5 \ln s_0}{s_0^{p_1+2}} (y^\mathcal{T} \cdot d_{2,2} \cdot y - \operatorname{Tr}(d_{2,2})) \right) \chi_0 \left(\frac{16|y|}{K_0 \sqrt{s_0}} \right). \end{aligned}$$

We also define initial data $u_{K_0,A,d_1,d_2}(0) = u_{1,K_0,A,d_1}(0) + iu_{2,K_0,A,d_2}(0)$ for equation (1.1) as follows:

$$u_{1,K_0,A,d_1}(x,0) = T^{-\frac{1}{p-1}} \left\{ \phi_{1,K_0,A,d_1} \left(\frac{x}{\sqrt{T}}, -\ln T \right) + \Phi_1 \left(\frac{x}{\sqrt{T}}, -\ln T \right) \right\} \chi_1(x) + U^*(x)(1 - \chi_1(x)) + 1, \quad (3.29)$$

$$u_{2,K_0,A,d_2}(x,0) = T^{-\frac{1}{p-1}} \left\{ \phi_{2,K_0,A,d_2} \left(\frac{x}{\sqrt{T}}, -\ln T \right) + \Phi_2 \left(\frac{x}{\sqrt{T}}, -\ln T \right) \right\} \chi_1(x), \quad (3.30)$$

where Φ_1, Φ_2 are defined in (2.39), (2.40) and $\chi_1(x)$ is defined as follows

$$\chi_1(x) = \chi_0 \left(\frac{|x|}{\sqrt{T} |\ln T|} \right), \quad (3.31)$$

with χ_0 defined in (3.13), and $U^* \in C^1(\mathbb{R}^n \setminus \{0\}, \mathbb{R})$ is defined for all $x \in \mathbb{R}^n, x \neq 0$

$$U^*(x) = \begin{cases} \left[\frac{(p-1)^2 |x|^2}{8p |\ln |x||} \right]^{-\frac{1}{p-1}} & \text{if } |x| \leq C^*, \\ \frac{1}{1+|x|^2} & \text{if } |x| \geq 1, \\ U^*(x) > 0 & \text{for all } x \neq 0, \end{cases} \quad (3.32)$$

where C^* is a fixed constant strictly less than 1 enough, and U^* satisfies the following property: for each $\epsilon_0 \leq \frac{C^*}{2}$ we have

$$U^*(x) \leq U^*(\epsilon_0), \text{ for all } |x| \geq \epsilon_0. \quad (3.33)$$

Remark 3.5. Roughly speaking, the critical data we done here are superposition of two items:

- $T^{-\frac{1}{p-1}} \{\phi_1 + \Phi_1\}$ in $P_1(0)$
- U^* in $P_2(0)$.

The first form is well-known in previous construction problems. As for the second, we borrowed it from Merle and Zaag in [16]. Note that U^* is the candidate for the final profile of the real part, as we can see from our main result in Theorem 1.1. More crucially, we draw your attention to the fact that in comparison with [16], we add here $+1$ to the expression in (3.29), and this term will allow us to have the initial condition

$$\operatorname{Re}(u(0)) \geq 1,$$

which is essential to make the nonlinear term u^p well-defined, and the Cauchy problem solvable (see Appendix A). This is an important idea of ours.

From the above definition, we show in the following lemma some rough properties of the initial data.

Lemma 3.6. *For all $K_0 \geq 1, A \geq 1, |d_1|_\infty \leq 2, |d_2|_\infty \leq 2$, and for all $\epsilon_0 \leq \frac{C^*}{2}$ (where C^* is introduced in (3.33)), there exists $T_2(\epsilon_0, K_0, A) > 0$ such that for all $T \leq T_2$, if $u(0) = u_{K_0,A,d_1,d_2}(0)$ is defined as in Definition 3.4, then the following holds:*

- (i) *The initial data belongs to L^∞ and satisfies the following*

$$\|u(\cdot, 0)\|_{L^\infty(|x| \geq \epsilon_0)} \leq 1 + \left(\frac{(p-1)^2 |\epsilon_0|^2}{8p |\ln \epsilon_0|} \right)^{-\frac{1}{p-1}}.$$

- (ii) *The real part of the initial data, $\operatorname{Re}(u(0))$ is positive. In particular,*

$$\operatorname{Re}(u(x, 0)) \geq 1, \forall x \in \mathbb{R}^n.$$

Proof.

- (i) It is obvious to see that the initial data belongs to L^∞ with the assumptions in this Lemma. It remains to prove the estimate in item (i). We now take $\epsilon_0 \leq \frac{C^*}{2}$, and we use definition of χ_1 in (3.31) to deduce that $\operatorname{supp}(\chi_1) \subset \{|x| \leq 2\sqrt{T} |\ln T|\}$. Moreover, we have

$$\sqrt{T} |\ln T| \rightarrow 0 \text{ as } T \rightarrow 0.$$

Then, we have

$$\sqrt{T} |\ln T| \leq \frac{\epsilon_0}{4},$$

provided that $T \leq T_{2,1}(\epsilon_0)$. Hence,

$$\text{supp}(\chi_1) \subset \{|x| \leq \frac{\epsilon_0}{2}\},$$

Hence, it follows the definition of $u(0)$ that: for all $|x| \geq \epsilon_0$, we have

$$u(x, 0) = U^*(x) + 1,$$

Using (3.33), our result follows.

(ii) We see in the definition of $u(0)$ that we have $\text{supp}(\phi_{1,K_0,A,d_1}) \subset \{|y| \leq \frac{K_0}{8}\sqrt{s_0}\}$ and we have the following

$$|\phi_{1,K_0,A,d_1}\left(\frac{x}{\sqrt{T}}, -\ln T\right)|_{L^\infty} \leq \frac{CA}{|\ln T|^{\frac{3}{2}}}.$$

In addition to that, in the region $\{|x| \leq \frac{K_0}{8}\sqrt{T|\ln T|}\}$, the function $\Phi_1\left(\frac{x}{\sqrt{T}}, -\ln T\right)$ is bounded from below by a positive constant which depends only on K_0 . Therefore, there exists $T_{2,2}(A, K_0) > 0$ such that for all $T \leq T_{2,2}$ for all $|x| \leq \frac{K_0}{8}\sqrt{T|\ln T|}$ we have

$$\phi_{1,K_0,A,d_1}\left(\frac{x}{\sqrt{T}}, -\ln T\right) + \Phi_1\left(\frac{x}{\sqrt{T}}, -\ln T\right) > 0.$$

Therefore: for all $|x| \leq \frac{K_0}{8}\sqrt{T|\ln T|}$, we have

$$\text{Re}(u(x, 0)) \geq 1.$$

Now, if $|x| \geq \frac{K_0}{8}\sqrt{T|\ln T|}$, then we have $\phi_{1,K_0,A,d_1}(y, s_0) = 0$. Since $\Phi_1(y, s_0) > 0$ from (2.39) and $U^*(x) > 0$ from (3.33), we directly see from the definition (3.29) for $\text{Re}(u(0))$ that

$$\text{Re}(u(x, 0)) \geq 1.$$

This concludes the proof of Lemma 3.6. \square

Following the above lemma, we will prove that there exists a domain \mathcal{D}_{K_0,A,s_0} , with $s_0 = -\ln T$ such that for all $(d_1, d_2) \in \mathcal{D}_{K_0,A,s_0}$, the initial $u_{K_0,A,d_1,d_2}(0)$ is trapped in

$$S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, 0) = S(0).$$

In particular, we show that the initial data strictly satisfies almost the conditions of $S(0)$ except a few of the conditions in item (i) of Definition 3.1. More precisely, these conditions concern the following modes

$$(q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s_0).$$

The following is our lemma:

Lemma 3.7 (Control of initial data). *There exists $K_3 \geq 1$ such that for all each $K_0 \geq K_3, A \geq 1$ and $\delta_1 > 0$, there exists $\alpha_3(K_0, \delta_1)$ such that for all $\alpha_0 \leq \alpha_3$, there exists $\epsilon_3(K_0, \alpha_0, \delta_1) > 0$ such that for all $\epsilon_0 \leq \epsilon_3, \eta_0 > 0$, there exists $T_3(K_0, \alpha_0, \epsilon_1, A, \delta_1, \eta_1) > 0$ such that for all $T \leq T_3$ and $s_0 = -\ln T$, there exists $\mathcal{D}_{K_0,A,s_0} \subset [-2, 2]^{1+n} \times [-2, 2]^{1+n} \times [-2, 2]^{\frac{n(n+1)}{2}}$ such that the following holds: if $u(0) = u_{K_0,A_0,d_1,d_2}(0)$ (see Definition 3.4), then*

(I) *For all $(d_1, d_2) \in \mathcal{D}_{K_0,A,s_0}$, we have $u(0) \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_1, \eta_0, 0)$. In particular, we have:*

(i) *Estimates in $P_1(0)$: We have $(q_1, q_2)(s_0) \in V_A(s_0)$ where $(q_1, q_2)(s_0)$ are defined in (2.7) and (3.1), satisfy the following estimates:*

$$\begin{aligned} |q_{1,j,k}(s_0)| &\leq \frac{A^2 \ln s_0}{2s_0^2}, \forall 1 \leq j, k \leq n \\ \left\| \frac{q_{1,-}(\cdot, s_0)}{1 + |y|^3} \right\|_{L^\infty} &\leq \frac{A}{2s_0^2} \quad \text{and} \quad \left\| \frac{q_{2,-}(\cdot, s_0)}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{A^2}{2s_0^{\frac{p_1+5}{2}}}, \\ \|q_{1,e}(\cdot, s_0)\|_{L^\infty} &\leq \frac{A^2}{2\sqrt{s_0}} \quad \text{and} \quad \|q_{2,e}(\cdot, s_0)\|_{L^\infty} \leq \frac{A^3}{2s_0^{\frac{p_1+2}{2}}}. \end{aligned}$$

(ii) *Estimates in $P_2(0)$: For all $|x| \in \left[\frac{K_0}{4} \sqrt{T |\ln T|}, \epsilon_0 \right]$, $\tau_0(x) = \frac{-t(x)}{\theta(x)}$ with $\theta(x) = T - t(x)$ and $|\xi| \leq \alpha_0 \sqrt{|\ln(T - t(x))|}$, we have*

$$|U(x, \xi, \tau_0(x)) - \hat{U}_{K_0}(\tau_0(x))| \leq \delta_1,$$

where $U(x, \xi, \tau)$ is defined in (3.20) and $\hat{U}_{K_0}(\tau)$ is defined in (3.25).

(II) *There exists a mapping Ψ_1 such that*

$$\begin{aligned} \Psi_1 : \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \times \mathbb{R}^{\frac{n(n+1)}{2}} &\rightarrow \mathbb{R}^{1+n} \times \mathbb{R}^{1+n} \times \mathbb{R}^{\frac{n(n+1)}{2}} \\ (d_1, d_2) &\mapsto (q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s_0) \end{aligned}$$

is linear, one to one from $\mathcal{D}_{K_0, A, s_0}$ to $\hat{V}_A(s_0)$, where

$$\hat{V}_A(s) = \left[-\frac{A}{s^2}, \frac{A}{s^2} \right]^{1+n} \times \left[-\frac{A^2}{s^{p_1+2}}, \frac{A^2}{s^{p_1+2}} \right]^{1+n} \times \left[-\frac{A^5 \ln s}{s^{p_1+2}}, \frac{A^5 \ln s}{s^{p_1+2}} \right]^{\frac{n(n+1)}{2}}. \quad (3.34)$$

Moreover,

$$\Psi_1(\partial \mathcal{D}_{K_0, A, s_0}) \subset \partial \hat{V}_A(s_0),$$

and

$$\deg(\Psi_1|_{\mathcal{D}_{K_0, A, s_0}}) \neq 0. \quad (3.35)$$

Proof. If we forget about the terms involving U^* and the +1 term in our definition (3.29) - (3.30) of initial data, then we are exactly in the framework of the case p integer treated in [5] (see Lemma 3.4 in [5]). Therefore, when p is not integer, we only need to understand the effect of U^* and the +1 term in order to complete the proof. The argument is only technical. For that reason, we leave it to Appendix C. \square

Now, we give a key-proposition for our argument. More precisely, in the following proposition, we prove the existence of a solution of equation (3.2) trapped in the shrinking set:

Proposition 3.8 (Existence of a solution trapped in $S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0)$). *We can chose the parameters $T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0$ such that there exist (d_1, d_2) such that the solution u of equation (1.1) with initial data given in Definition 3.4, exists on $[0, T)$ and satisfies*

$$u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0).$$

Proof. The proof of this Proposition is given 2 steps:

- The first step: We reduce our problem to a finite dimensional one. In other words, we aim at proving that the control of $u(t)$ in the shrinking set $S(t)$ reduces to the control of the components

$$(q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s)$$

in $\hat{V}_A(s)$, defined in (3.34).

- The second step: We get the conclusion of Proposition 3.8 by using a topological argument in finite dimension.

- *Step 1: Reduction to a finite dimensional problem:* Using *a priori estimates*, our problem will be reduced to the control of a finite number of components.

Proposition 3.9 (Reduction to a finite dimensional problem). *There exist parameters $K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0$ and $T > 0$ such that the following holds:*

- Assume that initial data $u(0) = u_{K_0, A, d_1, d_2}(0)$ is given in Definition 3.4 with $(d_1, d_2) \in \mathcal{D}_{K_0, A, s_0}$
- Assume furthermore that the solution u of equation (1.1) satisfies: $u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_*]$, for some $t_* \in [0, T)$ and

$$u \in \partial S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t_*).$$

Then, we have:

- (Reduction to finite dimensions): It holds that $(q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s_*) \in \partial \hat{V}_A(s_*)$, where $(q_1, q_2)(s)$ are defined in (2.7) and (3.1), $\hat{V}_A(s)$ is defined as in (3.34), and $s_* = -\ln(T - t_*)$.

(ii) (*Transverse outgoing crossing*): There exists $\nu_0 > 0$ such that

$$\forall \nu \in (0, \nu_0), (q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s_* + \nu) \notin \hat{V}_A(s_* + \nu), \quad (3.36)$$

which implies that there exists $\nu_1 > 0$ such that u exists on $[0, t_* + \nu_1)$ and for all $\nu \in (0, \nu_1)$

$$u(t_* + \nu) \notin S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t_* + \nu).$$

The proof of this Lemma uses techniques given in [16] which were developed from [1] and [17] in the real case. However, it is true that our shrinking set involves more conditions than the shrinking set used in [1], [17], [5]. In fact, the additional conditions are useful to ensure that our solution always stays positive. In particular, the set $V_A(s)$ plays an important role. Indeed, as for the integer case in [5], only the nonnegative modes $(q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s_*)$ may touch the boundary of $\hat{V}_A(s_*)$ and leave in short time later. However, the control of the solution with the positive real part is also our highlight and of course it is the main difficulty in our work. This proposition makes the heart of the paper and needs many steps to be proved. For that reason, we dedicate a whole section to its proof (Section 4 below). Let us admit it here, and get to the conclusion of Proposition 3.8 in the second step.

- *Step 2: Conclusion of Proposition 3.8 by a topological argument.* In this step, we give the proof of Proposition 3.8 assuming that Proposition 3.9 holds. In fact, we aim at proving the existence of a parameter $(d_1, d_2) \in \mathcal{D}_{K_0, A, s_0}$ such that the solution u of equation (1.1) with initial data $u_{K_0, A, d_1, d_2}(0)$ (given in Definition 3.4), exists on $[0, T)$ and satisfies

$$u \in S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0),$$

where the parameters will be suitably chosen. Our argument is analogous to the argument of Merle and Zaag in [17]. For that reason, we only give a brief proof. Let us fix $T, K_0, \delta_0, \alpha_0, \epsilon_0, A, \alpha_0, \eta_0$ such that Lemma 3.7, Proposition 3.9 and Lemma 3.3 hold. Then, for all $(d_1, d_2) \in \mathcal{D}_{K_0, A, s_0}$ and from Lemma 3.7 we have the initial data

$$u_{K_0, A, d_1, d_2}(0) \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, 0).$$

Thanks to Lemmas 3.3 and 3.7, for each $(d_1, d_2) \in \mathcal{D}_{K_0, A, s_0}$ we can define $t_*(d_1, d_2) \in [0, T)$ as the maximum time such that the solution u_{d_1, d_2} of equation (1.1), with initial data $u_{K_0, A, d_1, d_2}(0)$ trapped in $S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_*(d_1, d_2))$. We have the two following cases:

- + Case 1: If there exists (d_1, d_2) such that $t_*(d_1, d_2) = T$ then our problem is solved
- + Case 2: For all $(d_1, d_2) \in \mathcal{D}_{K_0, A, s_0}$, we have

$$t_*(d_1, d_2) < T.$$

By contradiction, we can prove that the second case can not occur. Indeed, if it is true, by using the continuity of the solution u in time and the definition of $t_* = t_*(d_1, d_2)$, we can deduce that $u \in \partial S(t_*)$. Using item (i) of Proposition 3.9, we derive

$$(q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s_*) \in \partial \hat{V}_A(s_*),$$

where $s_* = -\ln(T - t_*)$. Then, the following mapping Γ is well-defined:

$$\begin{aligned} \Gamma : \mathcal{D}_{K_0, A, s_0} &\rightarrow \partial \left([-1, 1]^{1+n} \times [-1, 1]^{1+n} \times [-1, 1]^{\frac{n(n+1)}{2}} \right) \\ (d_1, d_2) &\mapsto \left(\frac{s_*^2}{A} (q_{1,0}, (q_{1,j})_{j \leq n})(s_*), \frac{s_*^{p_1+2}}{A^2} (q_{2,0}, (q_{2,j})_{j \leq n})(s_*), \frac{s_*^{p_1+2}}{A^5 \ln s_*} (q_{2,j,k})_{j,k \leq n}(s_*) \right). \end{aligned}$$

Moreover, it satisfies the two following properties:

- (i) Γ is continuous from $\mathcal{D}_{K_0, A, s_0}$ to $\partial \left([-1, 1]^{1+n} \times [-1, 1]^{1+n} \times [-1, 1]^{\frac{n(n+1)}{2}} \right)$. This is a consequence of item (ii) in Proposition (3.9).
- (ii) The degree of the restriction $\Gamma|_{\partial \mathcal{D}_{A, s_0}}$ is non zero. Indeed, again by item (ii) in Proposition 3.9, we have

$$s^*(d_1, d_2) = s_0,$$

in this case. Applying (3.35), we get the conclusion.

In fact, such a mapping Γ can not exist by Index theorem and this is a contradiction. Thus, Proposition 3.8 follows, assuming that Proposition 3.9 holds (see Section 4 for the proof of latter). \square

3.4. The proof of Theorem 1.1

In this section, we aim at giving the proof of Theorem 1.1 by using Proposition 3.8.

The proof of Theorem 1.1: Except for the treatment of the nonlinear term, this part is quite similar to what we did in [5] when p is integer. Nevertheless, for the reader's convenience, we give the proof here, insisting on the way we handle the nonlinear term.

+ *The proof of item (i) of Theorem 1.1:* Using Proposition 3.8, there exist (d_1, d_2) such that the solution u of equation (1.1) with initial data $u_{K_0, A, d_1, d_2}(0)$ (given in Definition 3.4), exists on $[0, T)$ and satisfies:

$$u \in S^*(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0).$$

Thanks to item (i) in Definition 3.1, item (i) of Lemma 3.2, and definition (2.7) and definition (3.1) of (w_1, w_2) and (q_1, q_2) we conclude (1.6) and (1.7). In addition to that we have $\text{Re}(u) > 0$. Moreover, we use again the definition of $V_A(s)$ to conclude the following asymptotics:

$$u(0, t) \sim \kappa(T-t)^{-\frac{1}{p-1}}, \quad (3.37)$$

$$u_2(0, t) \sim -\frac{2n\kappa}{(p-1)} \frac{(T-t)^{-\frac{1}{p-1}}}{|\ln(T-t)|^2}, \quad (3.38)$$

as $t \rightarrow T$, which means that u blows up at time T and the origin is a blowup point. Moreover, the real and imaginary parts simultaneously blow up. It remains to prove that for all $x \neq 0$, x is not a blowup point of u . The following Lemma allows us to conclude.

Lemma 3.10 (No blow-up under some threshold; Giga and Kohn [7]). *For all $C_0 > 0, 0 \leq T_1 < T$ and $\sigma > 0$ small enough, there exists $\epsilon_0(C_0, T, \sigma) > 0$ such that if $u(\xi, \tau)$ satisfies the following estimates for all $|\xi| \leq \sigma, \tau \in [T_1, T)$:*

$$|\partial_\tau u - \Delta u| \leq C_0 |u|^p,$$

and

$$|u(\xi, \tau)| \leq \epsilon_0 (1 - \tau)^{-\frac{1}{p-1}}.$$

Then, u does not blow up at $\xi = 0, \tau = T$.

Proof. See Theorem 2.1 in Giga and Kohn [7]. Although the proof of [7] was given in the real case, it extends naturally to the complex valued case. \square

We next use Lemma 3.10 to conclude that u does not blow up at $x_0 \neq 0$. Since from (1.7), we have

$$(T-t)^{-\frac{1}{p-1}} \|u_2(\cdot, t)\|_{L^\infty} \leq \frac{C}{|\ln(T-t)|},$$

if $x_0 \neq 0$ we use (1.6) to deduce the following:

$$\sup_{|x-x_0| \leq \frac{|x_0|}{2}} (T-t)^{\frac{1}{p-1}} |u(x, t)| \leq \left| f_0 \left(\frac{\frac{|x_0|}{2}}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right| + \frac{C}{\sqrt{|\ln(T-t)|}} \rightarrow 0, \text{ as } t \rightarrow T. \quad (3.39)$$

Applying Lemma 3.10 to $u(x-x_0, t)$, with some σ small enough such that $\sigma \leq \frac{|x_0|}{2}$, and T_1 close enough to T , we see that $u(x-x_0, t)$ does not blow up at time T and $x=0$. Hence, x_0 is not a blow-up point of u . This concludes the proof of item (i) in Theorem 1.1.

+ *The proof of item (ii) of Theorem 1.1:* Here, we use the argument of Merle in [14] to deduce the existence of $u^* = u_1^* + iu_2^*$ such that $u(t) \rightarrow u^*$ as $t \rightarrow T$ uniformly on compact sets of $\mathbb{R}^n \setminus \{0\}$. In addition to that, we use the techniques in Zaag [31], Masmoudi and Zaag [18], Tayachi and Zaag [29] for the proofs of (1.9) and (1.10).

Indeed, for all $x_0 \in \mathbb{R}^n, x_0 \neq 0$, we deduce from (1.6), (1.7) that not only (3.39) holds but also the following is satisfied:

$$\begin{aligned} \sup_{|x-x_0| \leq \frac{|x_0|}{2}} (T-t)^{\frac{1}{p-1}} |\ln(T-t)| |u_2(x, t)| &\leq \left| \frac{9|x_0|^2}{4(T-t)|\ln(T-t)|} f_0^p \left(\frac{\frac{|x_0|}{2}}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right| \\ &+ \frac{C}{|\ln(T-t)|^{\frac{p-1}{2}}} \rightarrow 0, \text{ as } t \rightarrow T. \end{aligned} \quad (3.40)$$

We now consider x_0 such that $|x_0|$ is small enough, and K to be fixed later. We define $t_0(x_0)$ by

$$|x_0| = K \sqrt{(T - t_0(x_0)) |\ln(T - t_0(x_0))|}. \quad (3.41)$$

Note that $t_0(x_0)$ is unique when $|x_0|$ is small enough and $t_0(x_0) \rightarrow T$ as $x_0 \rightarrow 0$. We introduce the rescaled functions $U(x_0, \xi, \tau)$ and $V_2(x_0, \xi, \tau)$ as follows:

$$U(x_0, \xi, \tau) = (T - t_0(x_0))^{\frac{1}{p-1}} u(x, t). \quad (3.42)$$

and

$$V_2(x_0, \xi, \tau) = |\ln(T - t_0(x_0))| U_2(x_0, \xi, \tau), \quad (3.43)$$

where $U_2(x_0, \xi, \tau)$ is defined by

$$U(x_0, \xi, \tau) = U_1(x_0, \xi, \tau) + iU_2(x_0, \xi, \tau),$$

and

$$(x, t) = (x_0 + \xi \sqrt{T - t_0(x_0)}, t_0(x_0) + \tau(T - t_0(x_0))), \text{ and } (\xi, \tau) \in \mathbb{R}^n \times \left[-\frac{t_0(x_0)}{T - t_0(x_0)}, 1 \right). \quad (3.44)$$

We can see that with these notations, we derive from item (i) in Theorem 1.1 the following estimates for initial data at $\tau = 0$ of U and V_2

$$\sup_{|\xi| \leq |\ln(T - t_0(x_0))|^{\frac{1}{4}}} |U(x_0, \xi, 0) - f_0(K_0)| \leq \frac{C}{1 + (|\ln(T - t_0(x_0))|^{\frac{1}{4}})} \rightarrow 0 \quad \text{as } x_0 \rightarrow 0, \quad (3.45)$$

$$\sup_{|\xi| \leq |\ln(T - t_0(x_0))|^{\frac{1}{4}}} |V_2(x_0, \xi, 0) - g_0(K_0)| \leq \frac{C}{1 + (|\ln(T - t_0(x_0))|^{\gamma_1})} \rightarrow 0 \quad \text{as } x_0 \rightarrow 0. \quad (3.46)$$

where $f_0(x), g_0(x)$ are defined as in (1.4) and (1.8) respectively, and $\gamma_1 = \min(\frac{1}{4}, \frac{p_1}{2})$. Moreover, using equations (2.3), we derive the following equations for U, V_2 : for all $\xi \in \mathbb{R}^n, \tau \in [0, 1)$

$$\partial_\tau U = \Delta_\xi U + U^p, \quad (3.47)$$

$$\partial_\tau V_2 = \Delta_\xi V_2 + |\ln(T - t_0(x_0))| F_2(U_1, U_2), \quad (3.48)$$

where F_2 is defined in (2.4).

Besides that, from (3.39) and (3.47), we can apply Lemma 3.10 to U when $|\xi| \leq |\ln(T - t_0(x_0))|^{\frac{1}{4}}$ and obtain:

$$\sup_{|\xi| \leq \frac{1}{2} |\ln(T - t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |U(x_0, \xi, \tau)| \leq C. \quad (3.49)$$

and we aim at proving for $V_2(x_0, \xi, \tau)$ that

$$\sup_{|\xi| \leq \frac{1}{16} |\ln(T - t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |V_2(x_0, \xi, \tau)| \leq C. \quad (3.50)$$

+ *The proof for (3.50)*: We first use (3.49) to derive the following rough estimate:

$$\sup_{|\xi| \leq \frac{1}{2} |\ln(T - t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |V_2(x_0, \xi, \tau)| \leq C |\ln(T - t_0(x_0))|. \quad (3.51)$$

We first introduce ψ a cut-off function $\psi \in C_0^\infty(\mathbb{R}^n), 0 \leq \psi \leq 1, \text{supp}(\psi) \subset B(0, 1), \psi = 1$ on $B(0, \frac{1}{2})$. Introducing

$$\psi_1(\xi) = \psi \left(\frac{2\xi}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}} \right) \text{ and } V_{2,1}(x_0, \xi, \tau) = \psi_1(\xi) V_2(x_0, \xi, \tau). \quad (3.52)$$

Then, we deduce from (3.48) an equation satisfied by $V_{2,1}$

$$\partial_\tau V_{2,1} = \Delta_\xi V_{2,1} - 2 \operatorname{div}(V_2 \nabla \psi_1) + V_2 \Delta \psi_1 + |\ln(T - t_0(x_0))| \psi_1 F_2(U_1, U_2). \quad (3.53)$$

Hence, we can write $V_{2,1}$ with a integral equation as follows

$$V_{2,1}(\tau) = e^{\Delta \tau} (V_{2,1}(0)) + \int_0^\tau e^{(\tau - \tau') \Delta} (-2 \operatorname{div}(V_2 \nabla \psi_1) + V_2 \Delta \psi_1 + |\ln(T - t_0(x_0))| \psi_1 F_2(U_1, U_2))(\tau') d\tau'. \quad (3.54)$$

Besides that, using (3.49) and (3.51) and the fact that

$$|\nabla\psi_1| \leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}}, |\Delta\psi_1| \leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}},$$

we deduce that

$$\begin{aligned} \left| \int_0^\tau e^{(\tau-\tau')\Delta} (-2 \operatorname{div} (V_2 \nabla \psi_1)) d\tau' \right| &\leq C \int_0^\tau \frac{\|V_2 \nabla \psi_1\|_{L^\infty}(\tau')}{\sqrt{\tau-\tau'}} d\tau' \leq C |\ln(T-t_0(x_0))|^{\frac{3}{4}}, \\ \left| \int_0^\tau e^{(\tau-\tau')\Delta} (V_2(\tau') \Delta \psi_1) d\tau' \right| &\leq C \int_0^\tau \|V_2 \Delta \psi_1\|_{L^\infty}(\tau') d\tau' \leq C |\ln(T-t_0(x_0))|^{\frac{1}{2}}, \\ \left| \int_0^\tau e^{(\tau-\tau')\Delta} (\psi_1 |\ln(T-t_0(x_0))| F_2(U_1, U_2)(\tau')) d\tau' \right| &\leq C \int_0^\tau \| |\ln(T-t_0(x_0))| \psi_1 F_2(U_1, U_2) \|_{L^\infty}(\tau') d\tau'. \end{aligned}$$

Since the last term in (3.54) involves the nonlinear term $F_2(U_1, U_2)$, we need to handle it differently from the case where p is integer: using the definition (2.4) of F_2 , and (3.49) and the fact that U_1 is positive, we write from for all $|\xi| \leq \frac{1}{2} |\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)$ we have

$$|\psi_1 \ln(T-t_0(x_0)) F_2(U_1, U_2)(\tau)| \leq C (U_1^2 + U_2^2)^{\frac{p-1}{2}} |\psi_1 \ln(T-t_0(x_0)) U_2(\tau)| \leq C \|V_{2,1}(\tau)\|_{L^\infty}.$$

Hence, from (3.54) and the above estimates, we derive

$$\|V_{2,1}(\tau)\|_{L^\infty} \leq C |\ln(T-t_0(x_0))|^{\frac{3}{4}} + C \int_0^\tau \|V_{2,1}(\tau')\|_{L^\infty} d\tau'.$$

Thanks to Gronwall Lemma, we deduce that

$$\|V_{2,1}(\tau)\|_{L^\infty} \leq C |\ln(T-t_0(x_0))|^{\frac{3}{4}}, \forall \tau \in [0, 1),$$

which yields

$$\sup_{|\xi| \leq \frac{1}{4} |\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |V_2(x_0, \xi, \tau)| \leq C |\ln(T-t_0(x_0))|^{\frac{3}{4}}. \quad (3.55)$$

We apply iteratively for

$$V_{2,2}(x_0, \xi, \tau) = \psi_2(\xi) V_2(x_0, \xi, \tau) \text{ where } \psi_2(\xi) = \left(\frac{4\xi}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}} \right).$$

Similarly, we deduce that

$$\sup_{|\xi| \leq \frac{1}{8} |\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |V_2(x_0, \xi, \tau)| \leq C |\ln(T-t_0(x_0))|^{\frac{1}{2}}.$$

We apply this process a finite number of steps to obtain (3.50). We now come back to our problem, and aim at proving that:

$$\sup_{|\xi| \leq \frac{1}{16} |\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} \left| U(x_0, \xi, \tau) - \hat{U}_{K_0}(\tau) \right| \leq \frac{C}{1 + |\ln(T-t_0(x_0))|^{\gamma_2}}, \quad (3.56)$$

$$\sup_{|\xi| \leq \frac{1}{32} |\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} \left| V_2(x_0, \xi, \tau) - \hat{V}_{2, K_0}(\tau) \right| \leq \frac{C}{1 + |\ln(T-t_0(x_0))|^{\gamma_3}}, \quad (3.57)$$

where γ_2, γ_3 are positive small enough and $(\hat{U}_{K_0}, \hat{V}_{2, K_0})(\tau)$ is the solution of the following system:

$$\partial_\tau \hat{U}_{K_0} = \hat{U}_{K_0}^p, \quad (3.58)$$

$$\partial_\tau \hat{V}_{2, K_0} = p \hat{U}_{K_0}^{p-1} \hat{V}_{2, K_0}. \quad (3.59)$$

with initial data at $\tau = 0$

$$\begin{aligned} \hat{U}_{K_0}(0) &= f_0(K_0), \\ \hat{V}_{2, K_0}(0) &= g_0(K_0). \end{aligned}$$

given by

$$\hat{U}_{K_0}(\tau) = \left((p-1)(1-\tau) + \frac{(p-1)^2 K_0^2}{4p} \right)^{-\frac{1}{p-1}}, \quad (3.60)$$

$$\hat{V}_{2,K_0}(\tau) = K_0^2 \left((p-1)(1-\tau) + \frac{(p-1)^2 K_0^2}{4p} \right)^{-\frac{p}{p-1}}. \quad (3.61)$$

for all $\tau \in [0, 1)$. The proof of is cited to Section 5 of Tayachi and Zaag [29] and, here we will use (3.56) to prove (3.57). For the reader's convenience, we give it here. Let us consider

$$\mathcal{V}_2 = V_2 - \hat{V}_{2,K_0}(\tau). \quad (3.62)$$

Using (3.50), we deduce the following

$$\sup_{|\xi| \leq \frac{1}{16} |\ln(T-t_0(x_0))|^{\frac{1}{4}}, \tau \in [0, 1)} |\mathcal{V}_2| \leq C. \quad (3.63)$$

In addition to that, from (3.48) we write an equation on \mathcal{V}_2 as follows:

$$\partial_\tau \mathcal{V}_2 = \Delta \mathcal{V}_2 + p \hat{U}_{K_0}^{p-1} \mathcal{V}_2 + p(U_1^{p-1} - \hat{U}_{K_0}^{p-1}) V_2 + \mathcal{G}_2(x_0, \xi, \tau), \quad (3.64)$$

where

$$\mathcal{G}_2(x_0, \xi, \tau) = |\ln(T - t_0(x_0))| \left(F_2(U_1, U_2) - p U_1^{p-1} U_2 \right).$$

As for the last term in (3.64), we need here to carefully handle this expression, sine it involves a nonlinear term, which needs a treatment different from the case where p is integer. From the definition (2.4) of F_2 , we have

$$\begin{aligned} \left| F_2(U_1, U_2) - p U_1^{p-1} U_2 \right| &\leq \left| p U_2 \left((U_1^2 + U_2^2)^{\frac{p-1}{2}} - U_1^{p-1} \right) \right| \\ &+ \left| (U_1^2 + U_2^2)^{\frac{p}{2}} \left\{ \sin \left(p \arcsin \left(\frac{U_2}{\sqrt{U_1^2 + U_2^2}} \right) \right) - \frac{p U_2}{\sqrt{U_1^2 + U_2^2}} \right\} \right|. \end{aligned}$$

And we deduce from (3.50) and (3.56) with $\epsilon_0 > 0$ small enough that

$$\left| F_2(U_1, U_2) - p U_1^{p-1} U_2 \right| \leq C |U_2|^3,$$

Plugging the above estimate and using (3.43) and (3.50), we have the following

$$\sup_{|\xi| \leq \frac{1}{16} |\ln(T-t_0)|^{\frac{1}{4}}, \tau \in [0, 1)} |\mathcal{G}_2(x_0, \xi, \tau)| \leq \frac{C}{|\ln(T-t_0(x_0))|^2}. \quad (3.65)$$

Introducing

$$\bar{\mathcal{V}}_2 = \psi_*(\xi) \mathcal{V}_2,$$

where

$$\psi_* = \psi \left(\frac{16\xi}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}} \right),$$

and ψ is the cut-off function which has been introduced above. We also note that $\nabla \psi_*, \Delta \psi_*$ satisfy the following estimates

$$\|\nabla_\xi \psi_*\|_{L^\infty} \leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{4}}} \text{ and } \|\Delta_\xi \psi_*\|_{L^\infty} \leq \frac{C}{|\ln(T-t_0(x_0))|^{\frac{1}{2}}}. \quad (3.66)$$

In particular, $\bar{\mathcal{V}}_2$ satisfies

$$\partial_\tau \bar{\mathcal{V}}_2 = \Delta \bar{\mathcal{V}}_2 + p \hat{U}_{K_0}^{p-1}(\tau) \bar{\mathcal{V}}_2 - 2 \operatorname{div}(\mathcal{V}_2 \nabla \psi_*) + \mathcal{V}_2 \Delta \psi_* + p(U_1^{p-1} - \hat{U}_{K_0}^{p-1}) \psi_* V_2 + \psi_* \mathcal{G}_2, \quad (3.67)$$

By Duhamel principal, we derive the following integral equation

$$\bar{\mathcal{V}}_2(\tau) = e^{\tau \Delta}(\bar{\mathcal{V}}_2(\tau)) + \int_0^\tau e^{(\tau-\tau') \Delta} \left(p \hat{U}_{K_0}^{p-1} \bar{\mathcal{V}}_2 - 2 \operatorname{div}(\mathcal{V}_2 \nabla \psi_*) + \mathcal{V}_2 \Delta \psi_* + p(U_1^{p-1} - \hat{U}_{K_0}^{p-1}) \psi_* V_2 + \psi_* \mathcal{G}_2 \right) (\tau') d\tau'. \quad (3.68)$$

Besides that, we use (3.56), (3.60), (3.63), (3.66), (3.65) to derive the following estimates: for all $\tau \in [0, 1]$

$$\begin{aligned} |\hat{U}_{K_0}(\tau)| &\leq C, \\ \|\mathcal{V}_2 \nabla \psi_*\|_{L^\infty}(\tau) &\leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}}, \\ \|\mathcal{V}_2 \Delta \psi_*\|_{L^\infty}(\tau) &\leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}}, \\ \left\| \left(U_1^{p-1} - \hat{U}_{K_0}^{p-1} \right) \psi_* \right\|_{L^\infty}(\tau) &\leq \frac{C}{|\ln(T - t_0(x_0))|^{\gamma_2}}, \\ \|\mathcal{G}_2 \psi_*\|_{L^\infty} &\leq \frac{C}{|\ln(T - t_0(x_0))|^2}. \end{aligned}$$

where γ_2 given in (3.56). Hence, we derive from the above estimates that: for all $0 \leq \tau' < \tau < 1$

$$\begin{aligned} |e^{(\tau-\tau')\Delta} p \hat{U}_{K_0}^{p-1} \bar{\mathcal{V}}_2(\tau')| &\leq C \|\bar{\mathcal{V}}_2(\tau')\|, \\ |e^{(\tau-\tau')\Delta} (\operatorname{div}(\mathcal{V}_2 \nabla \psi_*))| &\leq C \frac{1}{\sqrt{\tau - \tau'}} \frac{1}{|\ln(T - t_0(x_0))|^{\frac{1}{4}}}, \\ |e^{(\tau-\tau')\Delta} (\mathcal{V}_2 \Delta \psi_*)| &\leq \frac{C}{|\ln(T - t_0(x_0))|^{\frac{1}{2}}}, \\ |e^{(\tau-\tau')\Delta} (p(U_1^{p-1} - \hat{U}_{K_0}^{p-1}) \psi_* V_2)(\tau')| &\leq \frac{C}{|\ln(T - t_0(x_0))|^{\gamma_2}}, \\ |e^{(\tau-\tau')\Delta} (\psi_* \mathcal{G}_2)(\tau')| &\leq \frac{C}{|\ln(T - t_0(x_0))|}. \end{aligned}$$

Plugging into (3.68), we obtain

$$\|\bar{\mathcal{V}}_2(\tau)\|_{L^\infty} \leq \frac{C}{|\ln(T - t_0(x_0))|^{\gamma_3}} + C \int_0^\tau \|\bar{\mathcal{V}}_2(\tau')\|_{L^\infty} d\tau',$$

where $\gamma_3 = \min(\frac{1}{4}, \gamma_2)$. Then, thanks to Gronwall inequality, we get

$$\|\bar{\mathcal{V}}_2\|_{L^\infty} \leq \frac{C}{|\ln(T - t_0(x_0))|^{\gamma_3}}.$$

Hence, (3.57) follows. Finally, we easily find the asymptotics of u^* and u_2^* as follows, thanks to the definition of U and V_2 and to estimates (3.56) and (3.57):

$$u^*(x_0) = \lim_{t \rightarrow T} u(x_0, t) = (T - t_0(x_0))^{-\frac{1}{p-1}} \lim_{\tau \rightarrow 1} U(x_0, 0, \tau) \sim (T - t_0(x_0))^{-\frac{1}{p-1}} \left(\frac{(p-1)^2}{4p} K_0^2 \right)^{-\frac{1}{p-1}}, \quad (3.69)$$

and

$$u_2^*(0) = \lim_{t \rightarrow T} u_2(x_0, t) = \frac{(T - t_0(x_0))^{-\frac{1}{p-1}}}{|\ln(T - t_0(x_0))|} \lim_{\tau \rightarrow 1} V_2(x_0, 0, \tau) \sim \frac{(T - t_0(x_0))^{-\frac{1}{p-1}}}{|\ln(T - t_0(x_0))|} \left(\frac{(p-1)^2}{4p} \right)^{-\frac{p}{p-1}} (K_0^2)^{-\frac{1}{p-1}}. \quad (3.70)$$

Using the relation (3.41), we find that

$$T - t_0(x_0) \sim \frac{|x_0|^2}{2K_0^2 |\ln|x_0||} \quad \text{and} \quad \ln(T - t_0(x_0)) \sim 2 \ln(|x_0|), \quad \text{as } x_0 \rightarrow 0. \quad (3.71)$$

Plugging (3.71) into (3.69) and (3.70), we get the conclusion of item (ii) of Theorem 1.1.

This concludes the proof of Theorem 1.1 assuming that Proposition 3.9 holds. Naturally, we need to prove this proposition on order to finish the argument. This will be done in the next section.

4. The proof of Proposition 3.9

This section is devoted to the proof of Proposition 3.9, which is considered as central in our analysis. We would like to proceed into two parts:

+ In the first part, we derive *a priori estimates* on u in every component $P_j(t)$ where $j = 1, 2$ or 3 .

+ In the second part, we use the priori estimates to derive new bounds which improve all the bounds in Definition 3.1, except for the non-negative modes $(q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})$. This means that the problem is reduced to the control of these components, which is the conclusion of item (i) of Proposition 3.9. As for item (ii) of Proposition 3.9 is just a direct consequence of the dynamics of these modes.

4.1. A priori estimates in $P_1(t), P_2(t)$ and $P_3(t)$

In this section, we aim at giving a *priori estimates* to the solution $u(t)$ on $P_1(t), P_2(t)$ and $P_3(t)$ which are important to get the conclusion of Proposition 3.9:

+ *A priori estimates in $P_1(t)$* : Here we give in the following proposition some estimates relevant to the region $P_1(t)$:

Proposition 4.1. *For all $A, K_0 \geq 1$ and $\epsilon_0 > 0, \alpha_0 > 0, \delta_0 > 0, \eta_0 > 0$, there exists $T_4(K_0, A, \epsilon_0)$ such that for all $T \leq T_4$, if u is a solution of equation (1.1) on $[0, t_1]$ for some $t_1 \in [0, T)$ and $u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_1]$, then, the following holds: for all $s_0 \leq \tau \leq s \leq s_1$ with $s_1 = \ln(T - t_1)$, we have:*

(i) *(ODE satisfied by the positive modes) For all $j \in \{1, n\}$ we have*

$$\left| q'_{1,0}(s) - q_{1,0}(s) \right| + \left| q'_{1,j}(s) - \frac{1}{2} q_{1,j}(s) \right| \leq \frac{C}{s^2}, \forall j \leq n. \quad (4.1)$$

$$\left| q'_{2,0}(s) - q_{2,0}(s) \right| + \left| q'_{2,j}(s) - \frac{1}{2} q_{2,j}(s) \right| \leq \frac{C}{s^{p_1+2}}, \forall j \leq n. \quad (4.2)$$

(ii) *(ODE satisfied by the null modes) For all $j, k \leq n$*

$$\left| q'_{1,j,k}(s) + \frac{2}{s} q_{1,j,k}(s) \right| \leq \frac{CA}{s^3}, \quad (4.3)$$

$$\left| q'_{2,j,k}(s) + \frac{2}{s} q_{2,j,k}(s) \right| \leq \frac{CA^2 \ln s}{s^{p_1+3}}. \quad (4.4)$$

(iii) *(Control of the negative part)*

$$\left\| \frac{q_{1,-}(\cdot, s)}{1 + |y|^3} \right\|_{L^\infty} \leq C e^{-\frac{s-\tau}{2}} \left\| \frac{q_{1,-}(\cdot, \tau)}{1 + |y|^3} \right\|_{L^\infty} + C \frac{e^{-(s-\tau)^2}}{s^{\frac{3}{2}}} \|q_{1,e}(\cdot, \tau)\|_{L^\infty} + \frac{C(1+s-\tau)}{s^2}, \quad (4.5)$$

$$\left\| \frac{q_{2,-}(\cdot, s)}{1 + |y|^3} \right\|_{L^\infty} \leq C e^{-\frac{s-\tau}{2}} \left\| \frac{q_{2,-}(\cdot, \tau)}{1 + |y|^3} \right\|_{L^\infty} + C \frac{e^{-(s-\tau)^2}}{s^{\frac{3}{2}}} \|q_{2,e}(\cdot, \tau)\|_{L^\infty} + \frac{C(1+s-\tau)}{s^{\frac{p_1+5}{2}}}. \quad (4.6)$$

(v) *(Control of the outer part)*

$$\|q_{1,e}(\cdot, s)\|_{L^\infty} \leq C e^{-\frac{(s-\tau)}{p}} \|q_{1,e}(\cdot, \tau)\|_{L^\infty} + C e^{s-\tau} s^{\frac{3}{2}} \left\| \frac{q_{1,-}(\cdot, \tau)}{1 + |y|^3} \right\|_{L^\infty} + \frac{C(1+s-\tau)e^{s-\tau}}{\sqrt{s}}, \quad (4.7)$$

$$\|q_{2,e}(\cdot, s)\|_{L^\infty} \leq C e^{-\frac{(s-\tau)}{p}} \|q_{2,e}(\cdot, \tau)\|_{L^\infty} + C e^{s-\tau} s^{\frac{3}{2}} \left\| \frac{q_{2,-}(\cdot, \tau)}{1 + |y|^3} \right\|_{L^\infty} + \frac{C(1+s-\tau)e^{s-\tau}}{s^{\frac{p_1+2}{2}}}. \quad (4.8)$$

Proof. By using the fact that $u(t) \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_1]$, we derive by the definition that $(q_1, q_2)(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ and $(q_1, q_2)(s)$ satisfies equation (3.2). In addition to that, we deduce also the fact that $q_1(s) + \Phi_1(s) \geq \frac{e^{-\frac{s}{p-1}}}{2}$ for all $s \in [s_0, s_1]$ (see Lemma 3.3). Although the potential terms $V_{j,k}$, the quadratic terms B_1, B_2 and the rest terms R_1, R_2 (see equation (3.2)) are different from the case where p is integer, they behavior as in that case (see Lemmas B.2, B.3, B.4 below). Hence, the result is derived from the projection of equation (3.2) and the dynamics of the operator $\mathcal{L} + V$. For that reason, we kindly refer the reader to the proof of Lemma 4.2 given in [5] for the case where p is integer. \square

+ *A priori estimates in $P_2(t)$* :

In this step, we aim at proving the following lemma which gives a priori estimates on u in $P_2(t)$. The following is our main result:

Lemma 4.2. *For all $K_0 \geq 1, \delta_1 \leq 1, \xi_0 \geq 1, \Lambda_5 > 0, \lambda_5 > 0$, the following holds: If $U(\xi, \tau)$ a solution of equation (3.47), for all ξ and $\tau \in [\tau_1, \tau_2]$ with $0 \leq \tau_1 \leq \tau_2 \leq 1$, such that for all $\tau \in [\tau_1, \tau_2]$ and for all $\xi \in [-2\xi_0, 2\xi_0]$, we have*

$$|U(\xi, \tau)| \leq \Lambda_5 \text{ and } \operatorname{Re}(U(\xi, \tau)) \geq \lambda_5 \text{ and } \left| U(\xi, \tau_1) - \hat{U}_{K_0}(\tau_1) \right| \leq \delta_1, \quad (4.9)$$

then, there exists $\epsilon = \epsilon(K_0, \Lambda_5, \lambda_5, \delta_1, \xi_0)$ such that for all $\xi \in [-\xi_0, \xi_0]$ and for all $\tau \in [\tau_1, \tau_2]$ we have

$$\left| U(\xi, \tau) - \hat{U}(\tau) \right| \leq \epsilon,$$

where $\hat{U}_{K_0}(\tau)$ is given (3.25). in particular, $\epsilon(K_0, \Lambda_5, \lambda_5, \delta_1, \xi_0) \rightarrow 0$ as $(\delta_1, \xi_0) \rightarrow (0, +\infty)$.

Proof. We introduce ψ as a cut-off function in $C_0^\infty(\mathbb{R})$ which satisfies the following:

$$\psi(x) = 0 \text{ if } |x| \geq 2, |\psi(x)| \leq 1 \text{ for all } x \text{ and } \psi(x) = 1 \text{ for all } |x| \leq 1,$$

and we also define ψ_1 as follows

$$\psi_1(\xi) = \psi\left(\frac{|\xi|}{\xi_0}\right).$$

Then, we have $\psi_1 \in C_0^\infty(\mathbb{R}^n)$, and $\operatorname{supp}(\psi_1) \subset \{|\xi| \text{ such that } |\xi| \leq 2\xi_0\}$ and $\psi_1(\xi) = 1$ for all $|\xi| \leq \xi_0$. In addition to that, we let

$$V_1(\xi, \tau) = \psi_1(\xi) \left(U(\xi, \tau) - \hat{U}_{K_0}(\tau) \right), \forall \tau \in [\tau_1, \tau_2], \xi \in \mathbb{R}^n.$$

Thanks to equation (3.47), we derive that V_1 satisfies the following equation:

$$\partial_\tau V_1 = \Delta_\xi V_1 - 2 \operatorname{div} (U \nabla \psi_1) + U \Delta \psi_1 + \psi_1(\xi) \left(U^p - \hat{U}^p \right). \quad (4.10)$$

Therefore, we can write $V_1(\xi, \tau)$ under the following intergral equation

$$V_1(\tau) = e^{(\tau-\tau_1)\Delta} (V_1(\tau_1)) + \int_{\tau_1}^{\tau} e^{(\tau-\tau')\Delta} \left(-2 \operatorname{div} (U \nabla \psi_1) + U \Delta \psi_1 + \psi_1 \left(U^p - \hat{U}^p \right) \right) (\tau') d\tau'. \quad (4.11)$$

In addition to that, we have the following fact from (4.9) (in particular the estimate $\operatorname{Re}(U(\xi, \tau)) \geq \lambda_5$ in (4.9) is crucial for the 4th term in (4.11)): for all $\tau \in [\tau_1, \tau_2]$

$$\begin{aligned} \|V_1(\tau_1)\|_{L^\infty} &\leq \delta_1, \\ \|U \nabla \psi_1\|_{L^\infty}(\tau) &\leq \frac{C(\Lambda_5)}{\xi_0}, \\ \|U \Delta \psi_1\|_{L^\infty}(\tau) &\leq \frac{C(\Lambda_5)}{\xi_0^2}, \\ \left\| \psi_1(U^p - \hat{U}^p) \right\|_{L^\infty}(\tau) &\leq C(K_0, \Lambda_5, \lambda_5) \|V_1\|_{L^\infty}(\tau), \end{aligned}$$

which yields when $\tau_1 \leq \tau' < \tau \leq \tau_2$,

$$\begin{aligned} \left\| e^{(\tau-\tau_1)\Delta} (V_1(\tau_1)) \right\| &\leq \delta_1, \\ \left\| e^{(\tau-\tau')\Delta} (\operatorname{div} (U \nabla \psi_1)(\tau')) \right\|_{L^\infty} &\leq \frac{C(\Lambda_5)}{\xi_0} \frac{1}{\sqrt{\tau-\tau'}}, \\ \left\| e^{(\tau-\tau')\Delta} (U \Delta \psi_1(\tau')) \right\|_{L^\infty} &\leq \frac{C(\Lambda_5)}{\xi_0^2}, \\ \left\| e^{(\tau-\tau')\Delta} (\psi_1(U^p - \hat{U}^p)(\tau')) \right\|_{L^\infty} &\leq C(K_0, \Lambda_5, \lambda_5) \|V_1\|_{L^\infty}(\tau'). \end{aligned}$$

Plugging into (4.11), we have for all $\tau \in [\tau_1, \tau_2]$

$$\|V_1(\tau)\|_{L^\infty} \leq C(K_0, \Lambda_5, \lambda_5) \left(\delta_1 + \frac{1}{\xi_0} \right) + C(K_0, \Lambda_5, \lambda_5) \int_{\tau_1}^{\tau} \|V_1(\tau')\|_{L^\infty} d\tau'.$$

Thanks to Gronwall lemma, we obtain the following

$$\|V_1(\tau)\|_{L^\infty} \leq C(K_0, \Lambda_5, \lambda_5) \left(\delta_1 + \frac{1}{\xi_0} \right), \forall \tau \in [\tau_1, \tau_2].$$

Since $V_1(\tau) = U(\tau) - \hat{U}(\tau)$ for all $\xi \in [-\xi_0, \xi_0]$ and for all $\tau \in [\tau_1, \tau_2]$, this concludes our lemma. \square

+ *A priori estimates in $P_3(t)$* : We aim at proving the following lemma which gives a priori estimates on u in $P_3(t)$.

Lemma 4.3 (A priori estimates in $P_3(t)$). *For all $K_0 \geq 1, A \geq 1, \eta > 0, \epsilon_0 > 0, \sigma \geq 1$ and $|d_1|_\infty, |d_2|_\infty \leq 2$, there exists $T_6(K_0, A, \epsilon_0, \eta, \sigma) > 0$, such that for all $T \leq T_6$ the following holds: if u is a solution of equation (1.1) for all $t \in [0, t_*]$ for some $t_* \in [0, T]$ with the initial data $u(0) = u_{K_0, A, d_1, d_2}(0)$ (see Definition 3.4) and*

$$|u(x, t)| \leq \sigma, \forall |x| \in \left[\frac{\epsilon_0}{8}, +\infty \right), t \in [0, t_*], \quad (4.12)$$

then,

$$|u(x, t) - u(x, 0)| \leq \eta, \forall |x| \geq \frac{\epsilon_0}{4}, t \in [0, t_*].$$

Proof. We introduce ψ , a cut-off function in $C^\infty(\mathbb{R})$ defined as follows

$$\psi(r) = 0 \text{ if } |r| \leq \frac{1}{2}, \quad \psi(r) = 1 \text{ for all } |r| \geq 1 \text{ and } |\psi(r)| \leq 1 \text{ for all } r,$$

and we also introduce $\psi_{\epsilon_0} \in C^\infty(\mathbb{R}^n)$ as follows

$$\psi_{\epsilon_0}(x) = \psi\left(\frac{4|x|}{\epsilon_0}\right).$$

Then, $\psi_{\epsilon_0} \in C^\infty(\mathbb{R}^n)$, and $\psi_{\epsilon_0}(x) = 1$ for all $|x| \geq \frac{\epsilon_0}{4}$ and $\psi_{\epsilon_0} = 0$ for all $|x| \leq \frac{\epsilon_0}{8}$. We define as well

$$v = \psi_{\epsilon_0} u.$$

Thanks to equation (1.1), we derive an equation satisfied by v

$$\partial_t v = \Delta v - 2 \operatorname{div}(u \nabla \psi_{\epsilon_0}) + u \Delta \psi_{\epsilon_0} + \psi_{\epsilon_0} u^p = \Delta v - 2 \operatorname{div}(u \nabla \psi_{\epsilon_0}) + G(u), \quad (4.13)$$

where

$$G(u) = u \Delta \psi_{\epsilon_0} + \psi_{\epsilon_0} u^p.$$

Using (4.12), we get

$$\|G(t, u(t))\|_{L^\infty(\mathbb{R}^n)} \leq C(\sigma, \epsilon_0), \forall t \in [0, t_*].$$

By Duhamel formula, we derive

$$v(t) = e^{t\Delta}(v(0)) + \int_0^t e^{(t-s)\Delta}(G(s, u(s))) ds, \quad (4.14)$$

which yields

$$v(t) - v(0) = e^{t\Delta}(v(0)) - v(0) + \int_0^t e^{(t-s)\Delta}(G(s, u(s))) ds.$$

Thus,

$$\|v(t) - v(0)\|_{L^\infty(\mathbb{R}^n)} \leq \|e^{t\Delta}(v(0)) - v(0)\|_{L^\infty} + \left\| \int_0^t e^{(t-s)\Delta}(G(s, u(s))) ds \right\|_{L^\infty}.$$

In addition to that, if $T \leq T_{6,1}(\epsilon_0)$, we have $\chi_1(x) = 0$, for all $|x| \geq \frac{\epsilon_0}{8}$, where χ_1 defined in (3.33) is involved in Definition 3.1 of initial data $u(0)$. As a matter of fact, from the definition of $u(0)$, we deduce from this fact that

$$v(0) = \psi_{\epsilon_0}(U^* + 1).$$

Since $\Delta v(0) \in L^\infty(\mathbb{R}^n)$, it follows that

$$\|e^{t\Delta}(v(0)) - v(0)\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0 \text{ as } t \rightarrow 0.$$

Besides that, we have also

$$\left\| \int_0^t e^{(t-s)\Delta}(G(s, u(s))) ds \right\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0 \text{ as } t \rightarrow 0.$$

Therefore, for all $t \in [t_0, t_*]$ we have

$$\|v(t) - v(0)\|_{L^\infty(\mathbb{R}^n)} \leq \eta,$$

provided that $T \leq T_{6,2}(K_0, A, \epsilon_0, \eta, \sigma)$. This concludes our lemma. \square

Finally, we need the following Lemma to get the conclusion of our proof:

Lemma 4.4. *There exists $K_7 \geq 1$ such that for all $K_0 \geq K_7, A \geq 1$, and $\delta_1 > 0$, there exists $\alpha_7(K_0, A, \delta_1) > 0$ such that for all $\alpha_0 \leq \alpha_7$, there exists $\epsilon_7(K_0, \alpha_0, A, \delta_1) > 0$ such that for all $\epsilon_0 \leq \epsilon_7$ there exist $\delta_7(\delta_1) > 0, T_7(K_0, \epsilon_0, A, \delta_1) > 0, \eta_7(K_0, \epsilon_0, A) > 0$ such that for all $\delta_0 \leq \delta_7, \eta_0 \leq \eta_7$ and for all $T \leq T_7$ if $u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t)$ for all $t \in [0, t_*]$, for some $t_* \in [0, T)$, then the following holds:*

$$\text{whenever } |x| \in \left[\frac{K_0}{4} \sqrt{(T-t_*)|\ln(T-t_*)|}, \epsilon_0 \right]$$

(i) For all $|\xi| \leq 2\alpha_0 \sqrt{|\ln(T-t(x))|}$ and for all

$$\tau \in \left[\max \left(0, \frac{-t(x)}{T-t(x)} \right), \frac{t_* - t(x)}{T-t(x)} \right],$$

if $U(x, \xi, \tau)$ satisfies equation (3.47), then

$$|U(x, \xi, \tau)| \leq C_7^*(p) \text{ and } \operatorname{Re}(U(\xi, \tau)) \geq C_7^{**}(K_0, p),$$

where $U(\xi, \tau)$ is defined as in (3.20), $t(x)$ is defined in (3.21), and C_7^* depends only on the parameter p and $C_7^{**}(K_0, p)$ depends on the parameters K_0 and p .

(ii) For all $|\xi| \leq 2\alpha_0 \sqrt{|\ln(T-t(x))|}$, if we define

$$\tau_0(x) = \max \left(0, \frac{-t(x)}{T-t(x)} \right), \quad (4.15)$$

then, we have

$$|U(x, \xi, \tau_0) - \hat{U}_{K_0}(\tau_0)| \leq \delta_1.$$

Proof. The idea of the proof relies on the argument in Lemma 2.6, given in [16].

+ *The proof of item (i):* We aim at proving that for all $|x| \in \left[\frac{K_0}{4} \sqrt{(T-t_*)|\ln(T-t_*)|}, \epsilon_0 \right]$, $|\xi| \leq 2\alpha_0 \sqrt{|\ln(T-t(x))|}$ and $t \in [\max(0, t(x)), t_*]$, we have

$$|U(x, \xi, \tau(x, t))| \leq C_7^*, \quad (4.16)$$

and

$$\operatorname{Re}(U(\xi, \tau)) \geq C_7^{**}, \quad (4.17)$$

where $\tau(x, t) = \frac{t-t(x)}{T-t(x)}$ and $C_7^*, C_7^{**} > 0$. Let us introduce a parameter $\delta > 0$ to be fixed later in our proof, small enough (note that δ has nothing to do with the parameters δ_0, δ_1 in the statement of our lemma). We observe that if we have $\alpha_0 \leq \alpha_{1,7}(K_0, \delta)$ for some $\alpha_{1,7} > 0$ and small enough, then for all $|\xi| \leq 2\alpha_0 \sqrt{|\ln(T-t(x))|}$, we have

$$(1-\delta)|x| \leq |x + \xi \sqrt{T-t(x)}| \leq (1+\delta)|x|. \quad (4.18)$$

We also recall the definition of rescaled function $U(x, \xi, \tau(x, t))$ as follows

$$U(x, \xi, \tau) = (T-t(x))^{\frac{1}{p-1}} u(x + \xi \sqrt{T-t(x)}, t(x) + \tau(T-t(x))).$$

Introducing $X = x + \xi \sqrt{T-t(x)}$, we write

$$U(x, \xi, \tau(x, t)) = (T-t(x))^{\frac{1}{p-1}} u(X, t).$$

We here consider 3 cases:

+ *Case 1:* We consider the case where

$$|X| \leq \frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}.$$

Using the fact that $u \in S(t)$, in particular item (i) of Definition 3.1, we see that Lemma 3.2 and (3.26) hold, hence

$$\left| (T-t)^{\frac{1}{p-1}} u(X, t) - f_0 \left(\frac{X}{\sqrt{(T-t)|\ln(T-t)|}} \right) \right| \leq \frac{CA^3}{\sqrt{1+|\ln(T-t)|}}.$$

Then, we derive the following

$$\begin{aligned} |U(x, \xi, \tau(x, t))| &\leq \left(\frac{T-t}{T-t(x)} \right)^{-\frac{1}{p-1}} \left(f_0(0) + \frac{CA^3}{\sqrt{1+|\ln(T-t)|}} \right) \\ &= \left(\frac{T-t}{T-t(x)} \right)^{-\frac{1}{p-1}} \left(\kappa + \frac{CA^3}{\sqrt{1+|\ln(T-t)|}} \right), \end{aligned} \quad (4.19)$$

$$\begin{aligned} \operatorname{Re}(U(x, \xi, \tau(x, t))) &\geq \left(\frac{T-t}{T-t(x)} \right)^{-\frac{1}{p-1}} \left(f_0(0) - \frac{CA^3}{\sqrt{1+|\ln(T-t)|}} \right) \\ &= \left(\frac{T-t}{T-t(x)} \right)^{-\frac{1}{p-1}} \left(\kappa - \frac{CA^3}{\sqrt{1+|\ln(T-t)|}} \right). \end{aligned} \quad (4.20)$$

Besides that, we deduce the following from (4.18) and the fact that $|X| \leq \frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}$:

$$|x| \leq \frac{K_0}{4(1-\delta)} \sqrt{(T-t)|\ln(T-t)|}.$$

In addition to that, we have that the function $T-t(x)$ is an increasing function if $|x|$ small enough. Therefore,

$$T-t(x) \leq T-t \left(\frac{K_0}{4(1-\delta)} \sqrt{(T-t)|\ln(T-t)|} \right). \quad (4.21)$$

As a matter of fact, we have the following asymptotics of function $\theta(x) = T-t(x)$,

$$\ln \theta(x) \sim 2 \ln |x| \text{ and } \theta(x) \sim \frac{8}{K_0^2} \frac{|x|^2}{|\ln |x||} \text{ as } |x| \rightarrow 0. \quad (4.22)$$

Plugging (4.22) in (4.21), we obtain the following

$$T-t(x) \leq T-t \left(\frac{K_0}{4(1-\delta)} \sqrt{(T-t)|\ln(T-t)|} \right) \sim \frac{8K_0^2(T-t)|\ln(T-t)|}{K_0^2 16(1-\delta)^2 \frac{1}{2} |\ln(T-t)|} = \frac{(T-t)}{(1-\delta)^2}.$$

In particular, from $t \in [\max(0, t(x)), t_*]$, we have the following

$$T-t(x) \geq T-t.$$

Plugging into (4.19) and (4.20), we obtain

$$|U(x, \xi, \tau)| \leq C_{1,7}^*(p, \delta),$$

and

$$\operatorname{Re}(U(x, \xi, \tau(x, t))) \geq C_{1,7}^{**}(p, \delta),$$

provided that δ is small enough, $K_0 \geq K_{1,7}(\delta)$ which is large enough and $T \leq T_{1,7}(K_0, A)$. Note that $C_{1,7}^*(p, \delta)$ and $C_{1,7}^{**}(p, \delta)$ depend on δ and p , in particular, $C_{1,7}^*(\delta, p)$ is bounded when $\delta \rightarrow 0$.

+ *The second case:* We consider the case where

$$|X| \in \left[\frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}, \epsilon_0 \right].$$

By using the definition of $U(x, \xi, \tau(x, t))$, we deduce that

$$U(x, \xi, \tau(x, t)) = \left(\frac{T-t(x)}{T-t(X)} \right)^{\frac{1}{p-1}} U(X, 0, \tau(X, t)).$$

However, using the fact that $u \in S(t)$, in particular item (ii) of Definition 3.1, we have

$$|U(X, 0, \tau(X, t))| \leq \delta_0 + \hat{U}(1).$$

In addition to that, we use (4.18), the definition of $t(x)$ and the fact that $|X| \geq \frac{K_0}{4} \sqrt{(T-t)|\ln(T-t)|}$ to derive the following

$$1 \leq \frac{T-t(x)}{T-t(X)} \leq 2,$$

provided that δ small enough. Therefore, we have

$$|U(x, \xi, \tau(x, t))| \leq 2^{\frac{1}{p-1}} \left(\delta_0 + \hat{U}_{K_0}(1) \right) \leq \frac{1}{2},$$

and

$$\operatorname{Re}(U(x, \xi, \tau(x, t))) \geq \hat{U}_{K_0}(0) - \delta_0 \geq \frac{1}{2} \hat{U}_{K_0}(0),$$

provided that $\delta_0 \leq \frac{1}{2} \hat{U}_{K_0}(0)$ and $K_0 \geq K_{2,7}$.

+ *The third case:* We consider the case where $|X| \geq \epsilon_0$. Using the fact that $u \in S(t)$, in particular item (iii) of Definition 3.1, we have

$$\begin{aligned} |U(x, \xi, \tau(x, t))| &= (T - t(x))^{\frac{1}{p-1}} |u(X, t)| \leq (T - t(x))^{\frac{1}{p-1}} (|u(X, 0)| + \eta_0), \\ \operatorname{Re}(U(x, \xi, \tau(x, t))) &= (T - t(x))^{\frac{1}{p-1}} \operatorname{Re}(u(X, t)) \geq (T - t(x))^{\frac{1}{p-1}} (\operatorname{Re}(u(X, 0)) - \eta_0). \end{aligned}$$

Using the definition (3.29), we have for all $|X| \geq \epsilon_0$

$$u(X, 0) = U^*(X) + 1,$$

provided that $T \leq T_{2,7}(\epsilon_0)$. In addition to that, we have the following fact

$$\begin{aligned} T - t(x) &\sim \frac{16|x|^2}{K_0^2 |\ln|x||}, \\ u(X, 0) &\sim U^*(X) = \left[\frac{(p-1)^2|x|^2}{8p|\ln|x||} \right]^{-\frac{1}{p-1}}, \end{aligned}$$

as $(X, x) \rightarrow (0, 0)$, and in particular, from (4.18), we have

$$(1 - \delta)|x| \leq |X| \leq (1 + \delta)|x|.$$

Therefore, we have

$$\begin{aligned} |U(x, \xi, \tau(x, t))| &\leq C_{2,7}^*(\delta), \\ \operatorname{Re}(U(x, \xi, \tau(x, t))) &\geq C_{2,7}^{**}(K_0, \delta), \end{aligned}$$

provided that $K_0 \geq K_{3,7}$, $\eta_0 \leq \eta_{1,7}(\delta)$ and δ is small. We conclude item (i).

The proof of item (ii): We aim at proving that for all $|\xi| \leq 2\alpha_0 \sqrt{|\ln \theta(x)|}$ and $\tau_0(x) = \max\left(0, -\frac{t(x)}{\theta(x)}\right)$, we have

$$\left| U(x, \xi, \tau_0(x)) - \hat{U}_{K_0}(\tau_0(x)) \right| \leq \delta_1. \quad (4.23)$$

Considering 2 cases for the proof of (4.23):

+ Case 1: We consider the case where

$$|x| \leq \frac{K_0}{4} \sqrt{T |\ln T|},$$

then, we deduce from the definition of $t(x)$ given by (3.21) that $t(x) \leq 0$. Thus, by definition (4.15), we have

$$\tau_0(x) = \frac{-t(x)}{\theta(x)}.$$

Therefore, (4.23) directly follows item (ii) of Lemma 3.7 with $K_0 \geq K_{4,7}$, $\alpha_0 \leq \alpha_{3,7}$, $\epsilon_0 \leq \epsilon_{3,7}$ (see in Lemma 3.7)

+ Case 2: We consider the case where

$$|x| \geq \frac{K_0}{4} \sqrt{T |\ln T|},$$

which yields $t(x) \geq 0$. Thus, by definition (4.15), we have

$$\tau_0(x) = 0.$$

We let $X = x + \xi\sqrt{\theta(x)}$. According to the definitions of U, \hat{U}_{K_0} which are given by (3.20) and (3.25), we write

$$\begin{aligned} |U(x, \xi, 0) - \hat{U}_{K_0}(0)| &= \left| \theta^{-\frac{1}{p-1}}(x)u(X, t(x)) - \left((p-1) + \frac{(p-1)^2 K_0^2}{4p \cdot 16} \right)^{-\frac{1}{p-1}} \right| \\ &= \left| \theta^{-\frac{1}{p-1}}(x)u(X, t(x)) - \left((p-1) + \frac{(p-1)^2}{4p} \frac{|X|^2}{\theta(x)|\ln \theta(x)|} \right)^{-\frac{1}{p-1}} \right| \\ &\quad + \left| \left((p-1) + \frac{(p-1)^2}{4p} \frac{|X|^2}{\theta(x)|\ln \theta(x)|} \right)^{-\frac{1}{p-1}} - \left((p-1) + \frac{(p-1)^2 K_0^2}{4p \cdot 16} \right)^{-\frac{1}{p-1}} \right| \\ &\leq (I) + (II), \end{aligned}$$

where $\theta(x) = T - t(x)$, and

$$\begin{aligned} (I) &= \left| \theta^{-\frac{1}{p-1}}(x)u(X, t(x)) - \left((p-1) + \frac{(p-1)^2}{4p} \frac{|X|^2}{\theta(X)|\ln \theta(X)|} \right)^{-\frac{1}{p-1}} \right|, \\ (II) &= \left| \left((p-1) + \frac{(p-1)^2}{4p} \frac{|X|^2}{\theta(X)|\ln \theta(X)|} \right)^{-\frac{1}{p-1}} - \left((p-1) + \frac{(p-1)^2 K_0^2}{4p \cdot 16} \right)^{-\frac{1}{p-1}} \right|. \end{aligned}$$

Since

$$|X| \leq (1 + \delta)|x| \leq \frac{(1 + \delta)K_0}{4} \sqrt{(T - t(x))|\ln(T - t(x))|} \leq K_0 \sqrt{(T - t(x))|\ln(T - t(x))|},$$

Using item (i) of Definition 3.1, taking $t = t(x)$, we write

$$(I) \leq \frac{C(K_0)A^2}{\sqrt{|\ln(T - t(x))|}} \leq \frac{C(K_0)A^2}{\sqrt{|\ln T|}} \leq \frac{\delta_1}{2},$$

provided that $T \leq T_{4,7}(K_0, A, \delta_1)$. Besides that, from (4.18) we have

$$(1 - \delta)^2 \frac{K_0^2}{16} \leq \frac{|X|^2}{\theta(X)|\ln \theta(X)|} \leq (1 + \delta)^2 \frac{K_0^2}{16}.$$

This yields

$$(II) \leq \frac{\delta_1}{2},$$

provided that δ is small enough. Then, (4.23) follows. Finally, we fix $\delta > 0$ small enough and we conclude our lemma. \square

4.2. The conclusion of Proposition 3.9

In this subsection, we would like to conclude the proof of Proposition 3.9. As we mentioned earlier, in the analysis of the shrinking set $S(t)$, the heart is the set $V_A(s)$ (see item (i) of Definition 3.1 of $S(t)$). So, let us first give an important argument related the analysis of $V_A(s)$; the reduction to finite dimensions. More precisely, we prove that if the solution (q_1, q_2) of equation (3.2) satisfies $(q_1, q_2)(s) \in V_A(s)$ for all $s \in [s_0, s_*]$ and $(q_1, q_2)(s_*) \in \partial V_A(s_*)$ for some $s_* \in [s_0, +\infty)$ with $s_0 = -\ln T$, then, we can directly derive that

$$(q_{1,0}, (q_{1,j})_{j \leq n}, q_{2,0}, (q_{2,j})_{j \leq n}, (q_{2,j,k})_{j,k \leq n})(s_*) \in \hat{\partial} V_A(s_*),$$

where $\hat{V}_A(s_*)$ is defined in (3.34). After that, we will use the dynamic of these modes to derive that they will leave \hat{V}_A after that. The following is our statement

Proposition 4.5 (A reduction to finite dimensional problem). *There exists $A_8 \geq 1, K_8 \geq 1$ such that for all $A \geq A_8, K_0 \geq K_8$, there exists $s_8(A, K_0) \geq 1$ such that for all $s_0 \geq s_8(A, K_0)$, we have the following properties: If the following conditions hold:*

- We take the initial data $(q_1, q_2)(s_0)$ are defined by $u_{A, K_0, d_1, d_2}(0)$ with $s_0 = -\ln T$ (see Definition 3.4, (2.7) and (3.1)) and $(d_0, d_1) \in \mathcal{D}_{K_0, A, s_0}$ (see in Lemma (3.7)).
- For all $s \in [s_0, s_1]$, the solution (q_1, q_2) of equation (3.2) satisfies: $(q_1, q_2)(s) \in V_A(s)$ and $q_1(s) + \Phi_1(s) \geq \frac{1}{2}e^{-\frac{s}{p-1}}$.

Then, for all $s \in [s_0, s_1]$, we have

$$\forall i, j \in \{1, \dots, n\}, \quad |q_{2,i,j}(s)| \leq \frac{A^2 \ln s}{2s^2}, \quad (4.24)$$

$$\left\| \frac{q_{1,-}(\cdot, s)}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{A}{2s^2}, \quad \|q_{1,e}(s)\|_{L^\infty} \leq \frac{A^2}{2\sqrt{s}}, \quad (4.25)$$

$$\left\| \frac{q_{2,-}(\cdot, s)}{1 + |y|^3} \right\|_{L^\infty} \leq \frac{A^2}{2s^{\frac{p_1+5}{2}}}, \quad \|q_{2,e}(s)\|_{L^\infty} \leq \frac{A^3}{2s^{\frac{p_1+2}{2}}}. \quad (4.26)$$

Proof. The proof is quite similar to Proposition 4.4 in [5]. Indeed, the proof is a consequence of Proposition 4.1, exactly as in [5]. Thus, we omit the proof and refer the reader to [5]. \square

Here, we give the conclusion of the proof of Proposition 3.9:

Conclusion of the proof of Proposition 3.9: We first choose the parameters $K_0, A, \alpha_0, \epsilon_0, \delta_0, \delta_1, \eta_0, \eta$ and $T > 0$ such that all the above Lemmas and Propositions which are necessary to the proof, are satisfied. In particular, we also note that the parameters δ_1 and η which are introduced in Lemma 3.7 and Lemma 4.3, will be small enough ($\delta_1 \ll \delta_0$ and $\eta \ll \eta_0$). Finally, we fix the constant T small enough, depending on all the above parameters, then we conclude our Proposition. We now assume the solution u of equation (1.1) with initial data $u_{K_0, A, d_1, d_2}(0)$, defined in Definition 3.4, satisfies the following

$$u \in S(T, K_0, \alpha_0, \epsilon_0, A, \delta_0, \eta_0, t) = S(t),$$

for all $t \in [0, t_*]$ for some $t_* \in [0, T)$ and

$$u \in \partial S(t_*).$$

We aim at proving that

$$(q_1, q_2)(s_*) \in \partial V_A(s_*), \quad (4.27)$$

where $s_* = \ln(T - t_*)$. Indeed, by contradiction, we suppose that (4.27) is not true, then, by using Definition 3.1 of $S(t)$, we derive the following:

(I) Either, there exist x_*, ξ_* which satisfy

$$\begin{aligned} |x_*| &\in \left[\frac{K_0}{4} \sqrt{(T - t_*) |\ln(T - t_*)|}, \epsilon_0 \right], \\ |\xi_*| &\leq \alpha_0 \sqrt{|\ln(T - t(x_*))|}. \end{aligned}$$

and

$$|U(x_*, \xi_*, \tau(x_*, t_*)) - \hat{U}(\tau(x_*, \tau_*))| = \delta_0.$$

(II) Or, there exists x^* such that $|x^*| \geq \frac{\epsilon_0}{4}$ and

$$|u(x^*, t_*) - u(x^*, 0)| = \eta_0.$$

We would like to prove that (I) and (II) can not occur. Indeed, if the first case occurs, then, letting $\tau_0(x_*) = \max\left(-\frac{t(x_*)}{\theta(x_*)}, 0\right)$, it follows from Lemma 4.4 that: For all $|\xi| \leq 2\alpha_0 \sqrt{|\ln(T - t(x_*))|}$, we have

$$\left| U(x_*, \xi, \tau_0(x_*)) - \hat{U}(\tau_0(x_*)) \right| \leq \delta_1,$$

and for all $\tau \in \left[\max\left(-\frac{t(x_*)}{T-t(x_*)}, \frac{t_*-t(x_*)}{T-t(x_*)}\right) \right]$, we have

$$\begin{aligned} |U(x_*, \xi, \tau(x_*))| &\leq C_7^*, \\ \operatorname{Re}(U(x_*, \xi, \tau(x_*))) &\geq C_7^{**}, \end{aligned}$$

where C_7^*, C_7^{**} are given in Lemma 4.4. Then, we apply Lemma 4.2, with $\xi_0 = \alpha_0 \sqrt{|\ln(T - t(x_*))|}$, $\tau_1 = \tau_0(x_*)$, $\tau_2 = \frac{t_*-t(x_*)}{T-t(x_*)}$, $\lambda_5 = C_7^{**}$ and $\Lambda_5 = C_7^*$, to derive that: for all $\xi \in [-\xi_0, \xi_0]$

$$\left| U(x_*, \xi, \tau(x_*, t_*)) - \hat{U}(\tau(x_*, t_*)) \right| \leq C(K_0, \Lambda_5 \lambda_5, \delta_1, \xi_0),$$

where $C(K_0, \Lambda_5, \lambda_5, \delta_1, \xi_0) \rightarrow 0$ as $(\delta_1, \xi_0) \rightarrow (0, +\infty)$. Taking $(\delta_1, \xi_0) \rightarrow (0, +\infty)$, (note that $\xi_0 \rightarrow +\infty$ as $\epsilon_0 \rightarrow 0$), we write

$$\left| U(x_*, \xi_*, \tau(x_*, t_*)) - \hat{U}(\tau(x_*, t_*)) \right| \leq \frac{\delta_0}{2},$$

this is a contradiction.

If (II) occurs, we have for all $|x| \in [\frac{\epsilon_0}{8}, +\infty)$

$$|u(x, t)| \leq C(\epsilon_0, A, \delta_0, \eta_0), \forall t \in [0, t_*].$$

Indeed, we consider the two following cases:

+ The case where $|x| \geq \frac{\epsilon_0}{4}$, using item (iii) if the definition of $S(t)$, we derive the following

$$|u(x, t)| \leq |u(x, 0)| + \eta_0 \leq C(A, \eta_0), \forall t \in [0, t_*].$$

+ The case where $|x| \in [\frac{\epsilon_0}{8}, \frac{\epsilon_0}{4}]$, using item (ii) in the definition of $S(t)$, we have

$$|u(x, t)| \leq C(\delta_0) (T - t(x))^{-\frac{1}{p-1}} \leq C(\epsilon_0, \delta_0), \forall t \in [0, t_*].$$

Then, we apply Lemma 4.3 with $\eta \leq \frac{\eta_0}{2}$ and $\sigma = C(\epsilon_0, A, \delta_0, \eta_0)$, to derive the following

$$|u(x^*, t_*) - u(x^*, 0)| \leq \frac{\eta_0}{2}.$$

Therefore, (II) can not occurs. Thus, (4.27) follows. In addition to that, from (4.27), Proposition 4.1 and Lemma 4.5, we conclude the proof of item (i) of Proposition 3.9. Since, item (ii) follows from item (i) (see for instance the proof of Proposition 3.6, given in [5]). This concludes the proof of Proposition 3.9.

A. Cauchy problem for equation (1.1)

In this section, we give a proof to a local Cauchy problem in time.

Lemma A.1 (A local Cauchy problem for a complex heat equation). *Let u_0 be any function in $L^\infty(\mathbb{R}^n, \mathbb{C})$ such that*

$$\operatorname{Re}(u_0(x)) \geq \lambda, \forall x \in \mathbb{R}^n, \quad (\text{A.1})$$

for some constant $\lambda > 0$. Then, there exists $T_1 > 0$ such that equation (1.1) with initial data u_0 , has a unique solution on $(0, T_1]$. Moreover, $u \in C((0, T_1], L^\infty(\mathbb{R}^n))$ and

$$\operatorname{Re}(u(t)) \geq \frac{\lambda}{2}, \forall (t, x) \in [0, T_1] \times \mathbb{R}^n.$$

Proof. The proof relies on a fixed-point argument. Indeed, we consider the space

$$X = C((0, T_1], L^\infty(\mathbb{R}^n, \mathbb{C})).$$

It is easy to check that X is an Banach space with the following norm

$$\|u\|_X = \sup_{t \in (0, T_1]} \|u(t)\|_{L^\infty}, \forall u = (u(t))_{t \in (0, T_1]} \in X.$$

We also introduce the closed set $B_\lambda^+(0, 2\|u_0\|_{L^\infty}) \subset X$ defined as follows

$$B_\lambda^+(0, 2\|u_0\|_{L^\infty}) = \{u \in X \text{ such that } \|u\|_X \leq 2\|u_0\|_{L^\infty}\} \cap \left\{u \in X \mid \forall t \in (0, T_1], \operatorname{Re}(u(t, x)) \geq \frac{\lambda}{2} \text{ a. e. } \right\}$$

Let \mathcal{Y} be the following mapping

$$\mathcal{Y} : B_\lambda^+(0, 2\|u_0\|_{L^\infty}) \rightarrow X,$$

where $\mathcal{Y}(u) = (\mathcal{Y}(u)(t))_{t \in (0, T_1]}$ is defined by

$$\mathcal{Y}(u)(t) = e^{t\Delta}(u_0) + \int_0^t e^{(t-s)\Delta}(u^p(s))ds. \quad (\text{A.2})$$

Note that, when $u \in B_\lambda^+(0, 2\|u_0\|_{L^\infty})$, u^p is well defined as in (2.4) and (2.5). We claim that there exists $T^* = T^*(\|u_0\|_{L^\infty}, \lambda) > 0$ such that for all $0 < T_1 \leq T^*$, the following assertion hold:

(i) The mapping is reflexive on $B_\lambda^+(0, 2\|u_0\|_{L^\infty})$, meaning that

$$\mathcal{Y} : B_\lambda^+(0, 2\|u_0\|_{L^\infty}) \rightarrow B_\lambda^+(0, 2\|u_0\|_{L^\infty}).$$

(ii) The mapping \mathcal{Y} is a contraction mapping on $B_\lambda^+(0, 2\|u_0\|_{L^\infty})$:

$$\|\mathcal{Y}(u_1) - \mathcal{Y}(u_2)\|_X \leq \frac{1}{2}\|u_1 - u_2\|_X,$$

for all $u_1, u_2 \in B_\lambda^+(0, 2\|u_0\|_{L^\infty})$.

The proof for (i): By observe that, by using the regular property of operator $e^{t\Delta}$, we conclude that $\mathcal{Y}(u) \in C((0, T_1], L^\infty(\mathbb{R}^n, \mathbb{C}) \cap C(\mathbb{R}^n, \mathbb{C}))$. Besides that, for all $u \in B_\lambda^+(0, 2\|u_0\|_{L^\infty})$ we derive from (A.2) that for all $t \in (0, T_1]$

$$\begin{aligned} \|\mathcal{Y}(u)(t)\|_{L^\infty} &= \left\| e^{t\Delta}(u_0) + \int_0^t e^{(t-s)\Delta}(u^p(s))ds \right\|_{L^\infty} \\ &\leq \|e^{t\Delta}(u_0)\|_{L^\infty} + \left\| \int_0^t e^{(t-s)\Delta}(u^p(s))ds \right\|_{L^\infty} \\ &\leq \|u_0\|_{L^\infty} + t2^p\|u_0\|_{L^\infty}^p. \end{aligned}$$

Hence, if we take $T_1 \leq \frac{1}{2^p\|u_0\|_{L^\infty}^{p-1}}$ then we have

$$\|\mathcal{Y}(u)\|_X = \sup_{t \in (0, T_1]} \|\mathcal{Y}(u)\|_{L^\infty} \leq 2\|u_0\|_{L^\infty}.$$

Now, let us note from (A.1) that

$$\operatorname{Re}(e^{t\Delta}(u_0)) = e^{t\Delta}(\operatorname{Re}(u_0)) \geq e^{t\Delta}(\lambda) = \lambda.$$

Therefore, from (A.2) for all $(t, x) \in (0, T_1] \times \mathbb{R}^n$

$$\operatorname{Re}(\mathcal{Y}(u)(t, x)) \geq \lambda - \left\| \int_0^t e^{(t-\tau)\Delta}(u^p)(\tau)d\tau \right\|_{L^\infty}.$$

Note that,

$$\left\| \int_0^t e^{(t-\tau)\Delta}(u^p)(\tau)d\tau \right\|_{L^\infty} \leq t2^p\|u_0\|_{L^\infty}^p.$$

So, if $T_1 \leq \frac{\lambda}{2^{p+1}\|u_0\|_{L^\infty}}$, then for all $t \in (0, T_1] \times \mathbb{R}^n$

$$\operatorname{Re}(\mathcal{Y}(u)(t, x)) \geq \frac{\lambda}{2}.$$

Therefore,

$$\mathcal{Y}(u) \in B_\lambda^+(0, 2\|u_0\|_{L^\infty}).$$

The proof of (ii): We first recall that the function $G(u) = u^p$, $u \in \mathbb{C}$ is analytic on

$$\left\{ u \in \mathbb{C} \text{ such that } \operatorname{Re}(u) \geq \frac{\lambda}{2} \right\}.$$

Then, there exists $C_2(\|u_0\|_{L^\infty}, \lambda) > 0$ such that

$$\begin{aligned} \|\mathcal{Y}(u_1) - \mathcal{Y}(u_2)\|_X &= \sup_{t \in (0, T_1]} \left\| \int_0^t e^{(t-s)\Delta}(u_1^p - u_2^p)(s)ds \right\|_{L^\infty} \\ &\leq T_1 C_2 \sup_{t \in (0, T_1]} \|u_1 - u_2\|_{L^\infty}. \end{aligned}$$

Then, if we impose

$$T_1 \leq \frac{1}{2C_2},$$

(ii) follows.

We now choose $T^* = \min\left(\frac{1}{2^p\|u_0\|_{L^\infty}^{p-1}}, \frac{\lambda}{2^{p+1}\|u_0\|_{L^\infty}}, \frac{1}{2C_2}\right)$. Then, for all $T_1 \leq T^*$, item (i) and (ii) hold. Thanks to a Banach fixed-point argument, there exists a unique $u \in B_\lambda^+(0, 2\|u_0\|_{L^\infty})$ such that

$$\mathcal{Y}(u)(t) = u(t), \forall t \in (0, T_1],$$

and we easily check that $u(t)$ satisfies equation (1.1) for all $(0, T_1]$ with $u(0) = u_0$. Moreover, from the definition of $B_\lambda^+(0, 2\|u_0\|_{L^\infty})$ we have

$$\operatorname{Re}(u)(t, x) \geq \frac{\lambda}{2}.$$

This concludes the proof of Lemma A.1. □

B. Some Taylor expansions

In this section appendix, we state and prove several technical and straightforward results needed in our paper.

Lemma B.1 (Asymptotics of \bar{B}_1, \bar{B}_2). *We consider $\bar{B}_1(\bar{w}_1, w_2)$ as in (2.11), (2.12). Then, the following holds:*

$$\bar{B}_1(\bar{w}_1, w_2) = \frac{p}{2\kappa} \bar{w}_1^2 + O(|\bar{w}_1|^3 + |w_2|^2), \quad (\text{B.1})$$

$$\bar{B}_2(\bar{w}_1, w_2) = \frac{p}{\kappa} \bar{w}_1 w_2 + O(|\bar{w}_1|^2 |w_2|) + O(|w_2|^3). \quad (\text{B.2})$$

as $(\bar{w}_1, w_2) \rightarrow (0, 0)$.

Proof. The proof of (B.1) is quite the same as the proof of (B.2). So, we only prove (B.2), hoping the reader will have no problem to check (B.1) if necessary. Since, $\kappa = (p-1)^{-\frac{1}{p-1}} > 0$, we derive $\kappa + \bar{w}_1 > 0$ when \bar{w}_1 is near 0, so we can write $B_2(\bar{w}_1, w_2)$ as follows

$$\bar{B}_2(\bar{w}_1, w_2) = ((\kappa + \bar{w}_1)^2 + w_2^2)^{\frac{p}{2}} \sin \left[p \arcsin \left(\frac{w_2}{\sqrt{(\kappa + \bar{w}_1)^2 + w_2^2}} \right) \right] - \frac{p}{p-1} w_2,$$

as $\bar{w}_1 \rightarrow 0$. Thus,

$$\begin{aligned} B_2(\bar{w}_1, w_2) &= ((\kappa + \bar{w}_1)^2 + w_2^2)^{\frac{p}{2}} \frac{p w_2}{\sqrt{(\kappa + \bar{w}_1)^2 + w_2^2}} - \frac{p}{p-1} w_2 \\ &+ ((\kappa + \bar{w}_1)^2 + w_2^2)^{\frac{p}{2}} \left\{ \sin \left[p \arcsin \left(\frac{w_2}{\sqrt{(\kappa + \bar{w}_1)^2 + w_2^2}} \right) \right] - \frac{p w_2}{\sqrt{(\kappa + \bar{w}_1)^2 + w_2^2}} \right\} \\ &= ((\kappa + \bar{w}_1)^2 + w_2^2)^{\frac{p-1}{2}} p w_2 - \frac{p}{p-1} w_2 \\ &+ ((\kappa + \bar{w}_1)^2 + w_2^2)^{\frac{p}{2}} \left\{ \sin \left[p \arcsin \left(\frac{w_2}{\sqrt{(\kappa + \bar{w}_1)^2 + w_2^2}} \right) \right] - \frac{p w_2}{\sqrt{(\kappa + \bar{w}_1)^2 + w_2^2}} \right\} \\ &= (I) + (II). \end{aligned}$$

In addition to that, we have the fact

$$\begin{aligned} \sin(px) - px &= O(|x|^3), \\ \frac{w_2}{\sqrt{(\kappa + \bar{w}_1)^2 + w_2^2}} &= O(|w_2|), \end{aligned}$$

as $x \rightarrow 0$ and $(\bar{w}_1, w_2) \rightarrow (0, 0)$. Plugging these estimates in (II), we obtain

$$(II) = O(|w_2|^3).$$

as $(\bar{w}_1, w_2) \rightarrow (0, 0)$. For (I), we use a Taylor expansion for $((\kappa + \bar{w}_1)^2 + w_2^2)$, around $(\bar{w}_1, w_2) = (0, 0)$:

$$((\kappa + \bar{w}_1)^2 + w_2^2)^{\frac{p}{2}} = \frac{1}{p-1} + \frac{(p-1)}{\kappa(p-1)} \bar{w}_1 + O(|\bar{w}_1|^2) + O(|w_2|^2).$$

Plugging this in (I), we derive the following:

$$(I) = \frac{p}{\kappa} \bar{w}_1 w_2 + O(|\bar{w}_1|^2 w_2) + O(|w_2|^3),$$

as $(\bar{w}_1, w_2) \rightarrow (0, 0)$. From the estimates of (I) and (II), we conclude the Lemma. \square

IN the following lemma, we aim at giving bounds on the principal potential V and the potentials $V_{i,j}$:

Lemma B.2 (The potential functions V and $V_{j,k}$ with $j, k \in \{1, n\}$). *We consider $V, V_{1,1}, V_{1,2}, V_{2,1}$ and $V_{2,2}$ defined in (3.3) and (3.4) - (3.7). Then, the following holds:*

(i) *For all $s \geq 1$ and $y \in \mathbb{R}^n$, we have $|V(y, s)| \leq C$,*

$$|V(y, s)| \leq \frac{C(1 + |y|^2)}{s}, \quad (\text{B.3})$$

and

$$V(y, s) = -\frac{(|y|^2 - 2n)}{4s} + \tilde{V}(y, s), \quad (\text{B.4})$$

where \tilde{V} satisfies

$$|\tilde{V}(y, s)| \leq C \frac{(1 + |y|^4)}{s^2}, \forall s \geq 1, |y| \leq 2K_0\sqrt{s}. \quad (\text{B.5})$$

(ii) The potential functions $V_{j,k}$ with $j, k \in \{1, 2\}$ satisfy the following

$$\begin{aligned} \|V_{1,1}\|_{L^\infty} + \|V_{2,2}\|_{L^\infty} &\leq \frac{C}{s^2}, \\ \|V_{1,2}\|_{L^\infty} + \|V_{2,1}\|_{L^\infty} &\leq \frac{C}{s}, \\ |V_{1,1}(y, s)| + |V_{2,2}(y, s)| &\leq \frac{C(1 + |y|^4)}{s^4}, \\ |V_{1,2}(y, s)| + |V_{2,1}(y, s)| &\leq \frac{C(1 + |y|^2)}{s^2}, \end{aligned}$$

for all $s \geq 1$ and $y \in \mathbb{R}^n$.

Proof. We note that the proof of (i) was given in Lemma B.1, page 1270 in [23]. So, it remains to prove item (ii). Moreover, the technique for these estimates is the same, so we only give the proof to the following estimates:

$$\|V_{1,1}\|_{L^\infty} + \|V_{2,2}\|_{L^\infty} \leq \frac{C}{s^2}, \quad (\text{B.6})$$

$$|V_{1,1}(y, s)| + |V_{2,2}(y, s)| \leq \frac{C(1 + |y|^4)}{s^4}. \quad (\text{B.7})$$

+ *The proof of (B.6):* We recall the expression of $V_{1,1}$ and $V_{2,2}$:

$$\begin{aligned} V_{1,1} &= \partial_{u_1} F_1(u_1, u_2)|_{(u_1, u_2) = (\Phi_1, \Phi_2)} - p\Phi_1^{p-1}, \\ V_{2,2} &= \partial_{u_2} F_2(u_1, u_2)|_{(u_1, u_2) = (\Phi_1, \Phi_2)} - p\Phi_1^{p-1}, \end{aligned}$$

where Φ_1, Φ_2 are given by (3.4) and (3.7). Hence, we can rewrite $V_{1,1}$ and $V_{2,2}$ as follows

$$\begin{aligned} V_{1,1} &= p(u_1^2 + u_2^2)^{\frac{p}{2}} \left(u_1 \cos \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] - u_2 \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \right) \\ &\quad - p\Phi_1^{p-1}, \\ V_{2,2} &= p(u_1^2 + u_2^2)^{\frac{p}{2}} \left(u_1 \cos \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \right) + u_2 \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \\ &\quad - p\Phi_1^{p-1}, \end{aligned}$$

We first estimate to $V_{1,1}$, from the above equalities, we decompose $V_{1,1}$ into the following

$$V_{1,1} = V_{1,1,1} + V_{1,1,2} + V_{1,1,3}, \quad (\text{B.8})$$

where

$$\begin{aligned} V_{1,1,1} &= p(\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_1 - p\Phi_1^{p-1}, \\ V_{1,1,2} &= p(\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_1 \left(\cos \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] - 1 \right), \\ V_{1,1,3} &= -p(\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_2 \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right]. \end{aligned}$$

As matter of fact, from the definitions of Φ_1, Φ_2 , we have the following

$$\left\| \frac{\Phi_2(\cdot, s)}{\Phi_1(\cdot, s)} \right\|_{L^\infty} \leq \frac{C}{s}, \quad (\text{B.9})$$

$$\|\Phi_1(\cdot, s)\|_{L^\infty} \leq C, \quad (\text{B.10})$$

$$\|\Phi_2(\cdot, s)\|_{L^\infty} \leq \frac{C}{s}, \quad (\text{B.11})$$

for all $s \geq 1$ and

$$|\cos(p \arcsin x) - 1| \leq C|x|^2, \quad (\text{B.12})$$

$$|\sin(p \arcsin x) - x| \leq C|x|^3, \quad (\text{B.13})$$

for all $|x| \leq 1$. By using (B.9), (B.10), (B.11), (B.12) and (B.13), we get the following bound for $V_{1,1,2}$ and $V_{1,1,3}$

$$\|V_{1,1,2}(\cdot, s)\|_{L^\infty} + \|V_{1,1,3}(\cdot, s)\|_{L^\infty} \leq \frac{C}{s^2}. \quad (\text{B.14})$$

For $V_{1,1,1}$, using (B.9), we derive

$$|V_{1,1,1}| = \left| p\Phi_1^{p-1} \left(\left(1 + \frac{\Phi_2^2}{\Phi_1^2} \right)^{\frac{p-2}{2}} - 1 \right) \right| \leq \frac{C}{s^2}.$$

This gives the following

$$\|V_{1,1}(\cdot, s)\|_{L^\infty} \leq \frac{C}{s^2}.$$

We can apply the technique to $V_{2,2}$ to get a similar estimate as follows

$$\|V_{2,2}(\cdot, s)\|_{L^\infty} \leq \frac{C}{s^2}.$$

Then, (B.6) follows.

+ *The proof of (B.7):* We can see that on the domain $\{|y| \geq K_0\sqrt{s}\}$ we have

$$\frac{1 + |y|^4}{s^4} \geq \frac{C}{s^2},$$

and in particular, we have (B.6). Thus, for all $|y| \geq K_0\sqrt{s}$.

$$|V_{1,1}(y, s)| + |V_{2,2}(y, s)| \leq \frac{C(|y|^4 + 1)}{s^4}.$$

Therefore, it is sufficient to give the estimate on the domain $\{|y| \leq 2K_0\sqrt{s}\}$. On this domain, we have the following: there exists $C(K_0) > 0$ such that

$$\frac{1}{C} \leq \Phi_1(y, s) \leq C.$$

In addition to that, using the definition of Φ_2 given by (2.39), we derive the following

$$|\Phi_2(y, s)| \leq C \frac{(|y|^2 + 1)}{s^2}, \forall (y, s) \in \mathbb{R}^n \times [1, +\infty). \quad (\text{B.15})$$

Then, from (B.8) we have

$$|V_{1,1,2}(y, s)| \leq |\Phi_2(y, s)|^2 \leq C \frac{(1 + |y|^4)}{s^4},$$

$$|V_{1,1,3}(y, s)| \leq |\Phi_2(y, s)|^2 \leq C \frac{(1 + |y|^4)}{s^4}.$$

We now estimate $V_{1,1,1}$, thanks to a Taylor expansion of $(\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}}$, around Φ_2

$$\left| (\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} - \Phi_1^{p-2} \right| \leq C|\Phi_2|^2.$$

This directly yields

$$|V_{1,1,1}(y, s)| \leq C(K_0)|\Phi_2|^2 \leq C \frac{(1 + |y|^4)}{s^4}.$$

So,

$$|V_{1,1}(y, s)| \leq C \frac{(1 + |y|^4)}{s^4}, \forall y \in \mathbb{R}^n.$$

Moreover, we can proceed similarly for $V_{2,2}$, and get

$$|V_{2,2}(y, s)| \leq C \frac{(1 + |y|^4)}{s^4} \forall y \in \mathbb{R}^n.$$

Thus, (B.7) follows. \square

Now, we give some estimates on the nonlinear terms $B_1(q_1, q_2)$ and $B_2(q_1, q_2)$

Lemma B.3 (The terms $B_1(q_1, q_2)$ and $B_2(q_1, q_2)$). *We consider $B_1(q_1, q_2)$, $B_2(q_1, q_2)$ as defined in (3.8) and (3.9), respectively. For all $A \geq 1$, there exists $s_9(A) \geq 1$ such that for all $s_0 \geq s_9(A)$, if $(q_1, q_2)(s) \in V_A(s)$ and $q_1(s) + \Phi_1(s) \geq \frac{1}{2}e^{-\frac{s}{p-1}}$ for all $s \in [s_0, s_1]$, then, the following holds: for all $s \in [s_0, s_1]$,*

$$|\chi(y, s)B_1(q_1, q_2)| \leq C (|q_1|^2 + |q_2|^2), \quad (\text{B.16})$$

$$|\chi(y, s)B_2(q_1, q_2)| \leq C \left(\frac{|q_1|^2}{s} + |q_1||q_2| + |q_2|^2 \right), \quad (\text{B.17})$$

$$\|B_1(q_1, q_2)\|_{L^\infty} \leq \frac{CA^4}{s^{\frac{p}{2}}}, \quad (\text{B.18})$$

$$\|B_2(q_1, q_2)\|_{L^\infty} \leq \frac{CA^2}{s^{1+\min(\frac{p-1}{4}, \frac{1}{2})}}, \quad (\text{B.19})$$

where $\chi(y, s)$ is defined as in (3.12).

Proof. We first would like to note that the condition $q_1(s) + \Phi_1(s) \geq \frac{1}{2}e^{-\frac{s}{p-1}}$ is to ensure that the real part $w_1 = q_1(s) + \Phi_1(s) > 0$. Then, (2.2) holds and F_1, F_2 which iare involved in the definition of B_1, B_2 , are well-defined (see (2.4)). For the proof of Lemma B.3, we only prove for (B.17) and (B.19), because the other ones follow similarly.

+ *The proof for (B.17):* Using the fact that the support of $\chi(y, s)$ is $\{|y| \leq 2K_0\sqrt{s}\}$, it is enough to prove (B.17) for all $\{|y| \leq 2K_0\sqrt{s}\}$. Since we have $(q_1, q_2) \in V_A(s)$, we derive from item (ii) of Lemma 3.2 and the definition of Φ_1, Φ_2 that

$$\frac{1}{C} \leq q_1 + \Phi_1 \leq C, \quad |q_2 + \Phi_2| \leq \frac{C}{s}.$$

and

$$|q_1| \leq \frac{CA}{\sqrt{s}}, \quad |q_2| \leq \frac{CA^2}{s^{\frac{p_1+2}{2}}}, \forall |y| \leq 2K_0\sqrt{s}. \quad (\text{B.20})$$

In addition to that, we write $B_2(q_1, q_2)$ as follows:

$$\begin{aligned} B_2(q_1, q_2) &= F_2(\Phi_1 + q_1, \Phi_2 + q_2) - F_2(\Phi_1, \Phi_2) - \partial_{u_1} F_2(q_1 + \Phi_1, q_2 + \Phi_2)q_1 \\ &\quad - \partial_{u_2} F_2(q_1 + \Phi_1, q_2 + \Phi_2)q_2. \end{aligned}$$

where

$$F_2(u_1, u_2) = (u_1^2 + u_2^2)^{\frac{p}{2}} \sin \left[p \arcsin \left(\frac{u_2}{\sqrt{u_1^2 + u_2^2}} \right) \right].$$

Using a Taylor expansion for the function $F_2(q_1 + \Phi_1, q_2 + \Phi_2)$ at $(q_1, q_2) = (0, 0)$, we derive the following

$$\begin{aligned} F_2(q_1 + \Phi_1, q_2 + \Phi_2) &= \sum_{j+k \leq 4} \frac{1}{j!k!} \partial_{q_1^j q_2^k}^{j+k} (F_2(q_1 + \Phi_1, q_2 + \Phi_2)) \Big|_{(q_1, q_2) = (0, 0)} q_1^j q_2^k + \\ &\quad + \sum_{j+k=5} G_{j,k}(q_1, q_2) q_1^j q_2^k, \end{aligned}$$

where

$$G_{j,k}(q_1, q_2) = \frac{5}{j!k!} \int_0^1 (1-t)^4 \partial_{q_1^j q_2^k}^5 (F_2(\Phi_1 + tq_1, \Phi_2 + tq_2)) dt.$$

In particular, we have

$$|G_{j,k}(q_1, q_2)| \leq C, \forall j+k=4.$$

As a matter of fact, we have

$$\partial_{q_1^j q_2^k}^{j+k} (F_2(q_1 + \Phi_1, q_2 + \Phi_2)) \Big|_{(q_1, q_2)=(0,0)} = \partial_{u_1^j u_2^k}^{j+k} F_2(u_1, u_2) \Big|_{(u_1, u_2)=(0,0)} \quad (\text{B.21})$$

Therefore, from (B.20), we have

$$\begin{aligned} & \left| F_2(q_1 + \Phi_1, q_2 + \Phi_2) - \sum_{j+k \leq 3} \frac{1}{j!k!} \partial_{u_1^j u_2^k}^{j+k} F_2(u_1, u_2) \Big|_{(u_1, u_2)=(\Phi_1, \Phi_2)} q_1^j q_2^k \right| \\ & \leq C \sum_{j=0}^5 |q_1^j q_2^{5-j}| \leq C \left(\frac{|q_1|^2}{s} + |q_1||q_2| + |q_2|^2 \right). \end{aligned}$$

In addition to that, we have the following fact,

$$\left| \partial_{u_1^j u_2^k}^{j+k} F_2(u_1, u_2) \Big|_{(u_1, u_2)=(\Phi_1, \Phi_2)} \right| \leq C, \forall j + k \leq 3,$$

and for all $1 \leq j \leq 4$, we have

$$\left| \partial_{u_1^j} F_2(u_1, u_2) \Big|_{(u_1, u_2)=(\Phi_1, \Phi_2)} \right| \leq \frac{C}{s}.$$

This concludes (B.17).

The proof of (B.19): We rewrite $B_2(q_1, q_2)$ explicitly as follows:

$$\begin{aligned} B_2(q_1, q_2) &= ((q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2)^{\frac{p}{2}} \sin \left[p \arcsin \left(\frac{q_2 + \Phi_2}{\sqrt{(q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2}} \right) \right] \\ &- (\Phi_1^2 + \Phi_2^2)^{\frac{p}{2}} \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \\ &- p (\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \left(\Phi_1 \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] - \Phi_2 \cos \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \right) q_1 \\ &- p (\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \left(\Phi_2 \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] + \Phi_1 \cos \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \right) q_2. \end{aligned}$$

Then, we decompose $B_2(q_1, q_2)$ as follows:

$$B_2(q_1, q_2) = B_{2,1}(q_1, q_2) + B_{2,2}(q_1, q_2) + B_{2,3}(q_1, q_2) + B_{2,4}(q_1, q_2) + B_{2,5}(q_1, q_2) + B_{2,6}(q_1, q_2),$$

where

$$\begin{aligned} B_{2,1}(q_1, q_2) &= p(q_2 + \Phi_2) \left((q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2 \right)^{\frac{p-1}{2}} - p(\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} \Phi_2 \\ &\quad - p(\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_1 q_2, \end{aligned} \quad (\text{B.22})$$

$$\begin{aligned} B_{2,2}(q_1, q_2) &= \left((q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2 \right)^{\frac{p}{2}} \left\{ \sin \left[p \arcsin \left(\frac{q_2 + \Phi_2}{\sqrt{(q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2}} \right) \right] \right. \\ &\quad \left. - \frac{p(q_2 + \Phi_2)}{\sqrt{(q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2}} \right\}, \end{aligned} \quad (\text{B.23})$$

$$B_{2,3}(q_1, q_2) = (\Phi_1^2 + \Phi_2^2)^{\frac{p}{2}} \left(\frac{p\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} - \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \right), \quad (\text{B.24})$$

$$B_{2,4}(q_1, q_2) = p(\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_1 \left(1 - \cos \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \right) q_2, \quad (\text{B.25})$$

$$\begin{aligned} B_{2,5}(q_1, q_2) &= p(\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \left\{ \Phi_2 \cos \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \right. \\ &\quad \left. - \Phi_1 \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] \right\} q_1, \end{aligned} \quad (\text{B.26})$$

$$B_{2,6}(q_1, q_2) = -p(\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_2 \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] q_2. \quad (\text{B.27})$$

we prove that: for all $y \in \mathbb{R}^n$:

$$|B_{2,j}(q_1, q_2)| \leq \frac{CA^2}{s^{1+\min(\frac{p-1}{4}, \frac{1}{2})}}, \forall j = 1, \dots, 6.$$

We now aim at an estimate on $B_{2,1}(q_1, q_2)$: We first need to prove the following:

$$\left| \left((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2 \right)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} \right| \leq C |Z|^{\min(\frac{p-1}{2}, 1)}, \quad (\text{B.28})$$

where

$$|Z| = 2q_1\Phi_1 + 2q_2\Phi_2 + q_1^2 + q_2^2.$$

Note that Z is bounded. On the other hand, we have $(\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2)^{\frac{p-1}{2}} = (\Phi_1^2 + \Phi_2^2 + Z)^{\frac{p-1}{2}}$. Then, if $\frac{p-1}{2} \geq 1$, using a Taylor expansion of the function $(\Phi_1^2 + \Phi_2^2 + Z)^{\frac{p-1}{2}}$ around $Z_0 = 0$ (note that $\Phi_1^2 + \Phi_2^2$ is uniformly bounded), we obtain the following:

$$\left| \left((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2 \right)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} \right| \leq C |Z|,$$

which yields (B.28). If $\frac{p-1}{2} < 1$, then, we have

$$\left| \left((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2 \right)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} \right| = (\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} \left| (1 + \xi)^{\frac{p-1}{2}} - 1 \right|,$$

where

$$\xi = \frac{Z}{\Phi_1^2 + \Phi_2^2}.$$

In particular, we have $\xi \geq -1$. In addition to that, we have the following fact: for all $\xi \geq -1$

$$\left| (1 + \xi)^{\frac{p-1}{2}} - 1 \right| \leq C |\xi|^{\frac{p-1}{2}} \quad (\text{B.29})$$

Therefore, (B.29) gives the following

$$\left| \left((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2 \right)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} \right| \leq C (\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} \left| \frac{Z}{\Phi_1^2 + \Phi_2^2} \right|^{\frac{p-1}{2}} \leq C |Z|^{\frac{p-1}{2}}.$$

Then, (B.28) follows. Using $(q_1, q_2)(s) \in V_A(s)$ and $Z = 2\Phi_1 q_1 + 2\Phi_2 q_2 + q_1^2 + q_2^2$, we write

$$\|Z\|_{L^\infty} \leq \frac{CA^2}{\sqrt{s}}, \forall s \geq 1.$$

So, we deduce from (B.28) that

$$\|p\Phi_2 \left(((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2)^{\frac{p-1}{2}} \right) - p\Phi_2(\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}}\|_{L^\infty} \leq \frac{CA^2}{s^{1+\min(\frac{p-1}{4}, \frac{1}{2})}}. \quad (\text{B.30})$$

Using (B.28), we have the following

$$\left\| \left(((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_1 \right) \right\|_{L^\infty} \leq \frac{CA^2}{s^{\min(\frac{p-1}{4}, \frac{1}{2})}}. \quad (\text{B.31})$$

Indeed, we have

$$\begin{aligned} \left| \left(((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_1 \right) \right| &\leq \left| \left(((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} \right) \right| \\ &\quad + \left| (\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_1 \right| \\ &\leq \frac{CA^2}{s^{\frac{\min(\frac{p-1}{2}, 1)}}} + \frac{C}{s^2}. \end{aligned}$$

Then, (B.31) holds.

On the other hand, using (B.31) and the following

$$\|q_2(\cdot, s)\|_{L^\infty} \leq \frac{CA^3}{s^{\frac{p_1+2}{2}}}, p_1 > 0,$$

we conclude that

$$\left\| pq_2 \left(((\Phi_1 + q_1)^2 + (\Phi_2 + q_2)^2)^{\frac{p-1}{2}} - (\Phi_1^2 + \Phi_2^2)^{\frac{p-2}{2}} \Phi_1 \right) \right\|_{L^\infty} \leq \frac{CA^2}{s^{1+\min(\frac{p-1}{4}, \frac{1}{2})}}, \quad (\text{B.32})$$

provided that $s \geq s_{1,9}(A)$. From (B.30) and (B.32), we have

$$\|B_{2,1}(q_1, q_2)\|_{L^\infty} \leq \frac{CA^2}{s^{1+\min(\frac{p-1}{4}, \frac{1}{2})}}. \quad (\text{B.33})$$

We next give a bound to $B_{2,2}(q_1, q_2)$: Using the following fact

$$|\sin(p \arcsin x) - x| \leq C|x|^3, \forall |x| \leq 1,$$

we derive the following

$$\begin{aligned} &\left| \sin \left[p \arcsin \left(\frac{q_2 + \Phi_2}{\sqrt{(q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2}} \right) \right] - \frac{p(q_2 + \Phi_2)}{\sqrt{(q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2}} \right| \\ &\leq C \frac{|(q_2 + \Phi_2)|^3}{((q_1 + \Phi_1)^2 + (q_2 + \Phi_2)^2)^{\frac{3}{2}}}. \end{aligned}$$

Plugging the above estimate into $B_2(q_1, q_2)$, we deduce the following

$$|B_{2,2}(q_1, q_2)| \leq C \left((q_1 + \Phi_1)^2 + (\Phi_2 + q_2)^2 \right)^{\frac{p-3}{2}} |q_2 + \Phi_2|^3,$$

which yields

$$|B_{2,2}(q_1, q_2)| \leq C |q_2 + \Phi_2|^{\min(p,3)},$$

Using $(q_1, q_2) \in V_A(s)$, it gives the following

$$|q_2 + \Phi_2| \leq \frac{C}{s},$$

provided that $s \geq s_{2,9}(A)$. Then,

$$\|B_{2,2}(q_1, q_2)\|_{L^\infty} \leq \frac{C}{s^{\min(p,3)}}. \quad (\text{B.34})$$

It is similar to estimate to $B_{2,3}(q_1, q_2)$

$$\|B_{2,3}(q_1, q_2)\|_{L^\infty} \leq \frac{C}{s^3}. \quad (\text{B.35})$$

We estimate to $B_{2,4}(q_1, q_2)$, using the following

$$|1 - \cos(p \arcsin x)| \leq C|x|^2, \forall |x| \leq 1,$$

we write

$$|B_{2,4}(q_1, q_2)| \leq C \left\| \frac{\Phi_2}{\Phi_1} \right\|_{L^\infty}^2 \|q_2\|_{L^\infty} \leq \frac{CA^3}{s^3}.$$

Then, we derive that

$$\|B_{2,4}(q_1, q_2)\|_{L^\infty} \leq \frac{CA^3}{s^3}. \quad (\text{B.36})$$

We also estimate to $B_{2,5}, B_{2,6}$ as follows:

$$\|B_{2,5}(q_1, q_2)\|_{L^\infty} \leq \frac{CA^2}{s^{\frac{3}{2}}}, \quad (\text{B.37})$$

$$\|B_{2,6}(q_1, q_2)\|_{L^\infty} \leq \frac{CA^3}{s^2}. \quad (\text{B.38})$$

Thus, from (B.33), (B.34), (B.35), (B.36), (B.37) and (B.38), we conclude (B.19), provided that $s \geq s_{3,9}(A)$. \square

In the following Lemma, we aim at giving estimates to the rest terms R_1, R_2 :

Lemma B.4 (The rest terms R_1, R_2). *For all $s \geq 1$, we consider R_1, R_2 defined in (3.10) and (3.11). Then,*

(i) *For all $s \geq 1$ and $y \in \mathbb{R}^n$*

$$\begin{aligned} R_1(y, s) &= \frac{c_{1,p}}{s^2} + \tilde{R}_1(y, s), \\ R_2(y, s) &= \frac{c_{2,p}}{s^3} + \tilde{R}_2(y, s), \end{aligned}$$

where $c_{1,p}$ and $c_{2,p}$ are constants depended on p and \tilde{R}_1, \tilde{R}_2 satisfy

$$\begin{aligned} |\tilde{R}_1(y, s)| &\leq \frac{C(1 + |y|^4)}{s^3}, \\ |\tilde{R}_2(y, s)| &\leq \frac{C(1 + |y|^6)}{s^4}, \end{aligned}$$

for all $|y| \leq 2K_0\sqrt{s}$.

(ii) *Moreover, we have for all $s \geq 1$*

$$\begin{aligned} \|R_1(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} &\leq \frac{C}{s}, \\ \|R_2(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} &\leq \frac{C}{s^2}. \end{aligned}$$

Proof. The proof for R_1 is quite the same as the proof for R_2 . For that reason, we only give the proof of the estimates on R_2 . This means that, we need to prove the following estimates:

$$R_2(y, s) = -\frac{n(n+4)\kappa}{(p-1)s^3} + \tilde{R}_2(y, s), \quad (\text{B.39})$$

with

$$|\tilde{R}_2(y, s)| \leq \frac{C(1 + |y|^6)}{s^4}, \forall |y| \leq 2K_0\sqrt{s}.$$

and

$$\|R_2(\cdot, s)\|_{L^\infty} \leq \frac{C}{s^2}. \quad (\text{B.40})$$

We recall the definition of $R_2(y, s)$:

$$R_2(y, s) = \Delta\Phi_2 - \frac{1}{2}y \cdot \nabla\Phi_2 - \frac{\Phi_2}{p-1} + F_2(\Phi_1, \Phi_2) - \partial_s\Phi_2,$$

Then, we can rewrite R_2 as follows

$$R_2(y, s) = \Delta\Phi_2 - \frac{1}{2}y \cdot \nabla\Phi_2 - \frac{\Phi_2}{p-1} + p\Phi_1^{p-1}\Phi_2 - \partial_s\Phi_2 + R_2^*(y, s),$$

where

$$R_2^*(y, s) = (\Phi_1^2 + \Phi_2^2)^{\frac{p}{2}} \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] - p\Phi_1^{p-1}\Phi_2.$$

Using the definitions of Φ_1, Φ_2 given in (2.39) and (2.40), we obtain the following:

$$\begin{aligned} |R_2^*(y, s)| &\leq \left| (\Phi_1^2 + \Phi_2^2)^{\frac{p}{2}} \left\{ \sin \left[p \arcsin \left(\frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right) \right] - p \frac{\Phi_2}{\sqrt{\Phi_1^2 + \Phi_2^2}} \right\} \right| \\ &\quad + \left| p\Phi_2((\Phi_1^2 + \Phi_2^2)^{\frac{p-1}{2}} - \Phi_1^{p-1}) \right|. \end{aligned}$$

It is similar to the proofs of estimations given in the proof of Lemma B.3, we can prove the following

$$|R_2^*(y, s)| \leq \frac{C(1 + |y|^6)}{s^4}, \quad \forall |y| \leq 2K_0\sqrt{s},$$

and

$$\|R_2^*(\cdot, s)\|_{L^\infty} \leq \frac{C}{s^2}.$$

In addition to that, we introduce \bar{R}_2 as follows:

$$\bar{R}_2(y, s) = \Delta\Phi_2 - \frac{1}{2}y \cdot \nabla\Phi_2 - \frac{\Phi_2}{p-1} + p\Phi_1^{p-1}\Phi_2 - \partial_s\Phi_2.$$

Then, we aim at proving the following:

$$\left| \bar{R}_2(y, s) + \frac{n(n+4)\kappa}{(p-1)s^3} \right| \leq \frac{C(1 + |y|^6)}{s^4}, \quad \text{for all } |y| \leq 2K_0\sqrt{s} \quad (\text{B.41})$$

$$\|\bar{R}_2(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{s^2}. \quad (\text{B.42})$$

+ *The proof of (B.41):* We first aim at expanding $\Delta\Phi_2$ in a polynomial in y of order less than 4 via the Taylor expansion. Indeed, $\Delta\Phi_2$ is given by

$$\begin{aligned} \Delta\Phi_2 &= \frac{2n}{s^2} \left(p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{p}{p-1}} - \frac{(p-1)|y|^2}{s^3} \left(p-1 + \frac{(p-1)^2|y|^2}{4p} \frac{|y|^2}{s} \right)^{-\frac{2p-1}{p-1}} \\ &\quad - \frac{(n+2)(p-1)|y|^2}{2s^3} \left(p-1 + \frac{(p-1)^2|y|^2}{4p} \frac{|y|^2}{s} \right)^{-\frac{2p-1}{p-1}} + \frac{(2p-1)(p-1)^2|y|^4}{4ps^4} \left(p-1 + \frac{(p-1)^2|y|^2}{4p} \frac{|y|^2}{s} \right)^{-\frac{3p-2}{p-1}}. \end{aligned}$$

Besides that, we make a Taylor expansion in the variable $z = \frac{|y|}{\sqrt{s}}$ for $\left(p-1 + \frac{(p-1)^2|y|^2}{4p} \frac{|y|^2}{s} \right)^{-\frac{p}{p-1}}$ when $|z| \leq 2K$, and we get

$$\left| \left(p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{p}{p-1}} - \frac{\kappa}{p-1} + \frac{\kappa}{4(p-1)} \frac{|y|^2}{s} \right| \leq \frac{C(1 + |y|^4)}{s^2}, \quad \forall |y| \leq 2K\sqrt{s}.$$

which yields

$$\left| \frac{2n}{s^2} \left(p-1 + \frac{(p-1)^2|y|^2}{4ps} \right)^{-\frac{p}{p-1}} - \frac{2n\kappa}{(p-1)s^2} + \frac{n\kappa|y|^2}{2(p-1)s^3} \right| \leq \frac{C(1 + |y|^4)}{s^4} \leq \frac{C(1 + |y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s}.$$

It is similar to estimate the other terms in $\Delta\Phi_2$ as the above. Finally, we obtain

$$\left| \Delta\Phi_2 - \frac{2n\kappa}{(p-1)s^2} + \frac{n\kappa|y|^2}{(p-1)s^3} + 2\frac{\kappa|y|^2}{(p-1)s^3} \right| \leq \frac{C(1 + |y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s}. \quad (\text{B.43})$$

As we did for $\Delta\Phi_2$, we estimate similarly the other terms in \bar{R}_2 : for all $|y| \leq 2K\sqrt{s}$

$$\left| -\frac{1}{2}y \cdot \nabla\Phi_2 + \frac{\kappa|y|^2}{(p-1)s^2} - \frac{\kappa|y|^4}{4(p-1)s^3} - \frac{\kappa|y|^4}{4(p-1)s^3} \right| \leq \frac{C(1+|y|^6)}{s^4} \quad (\text{B.44})$$

$$\left| -\frac{\Phi_2}{p-1} + \frac{\kappa|y|^2}{(p-1)^2s^2} - \frac{\kappa|y|^4}{4(p-1)^2s^3} - \frac{2n\kappa}{(p-1)^2s^2} \right| \leq \frac{C(1+|y|^6)}{s^4} \quad (\text{B.45})$$

$$\left| p\Phi_1^{p-1}\Phi_2 - \frac{p\kappa|y|^2}{(p-1)^2s^2} + \frac{(2p-1)\kappa|y|^4}{4(p-1)^2s^3} - \frac{n\kappa|y|^2}{(p-1)s^3} + \frac{2pn\kappa}{(p-1)^2s^2} + \frac{n^2\kappa}{(p-1)s^3} \right| \leq \frac{C(1+|y|^6)}{s^4} \quad (\text{B.46})$$

$$\left| -\partial_s\Phi_2 - \frac{2\kappa|y|^2}{(p-1)s^3} + \frac{4n\kappa}{(p-1)s^3} \right| \leq \frac{C(1+|y|^6)}{s^4} \quad (\text{B.47})$$

Thus, we use (B.43), (B.44), (B.45), (B.46) and (B.47) to deduce the following

$$\left| \bar{R}_2(y, s) + \frac{n(n+4)\kappa}{(p-1)s^3} \right| \leq \frac{C(1+|y|^6)}{s^4}, \quad \forall |y| \leq 2K\sqrt{s},$$

and (B.41) follows

+ *The proof (B.42)*: We rewrite Φ_1, Φ_2 as follows

$$\Phi_1(y, s) = R_{1,0}(z) + \frac{n\kappa}{2ps} \quad \text{and} \quad \Phi_2(y, s) = \frac{1}{s}R_{2,1}(z) - \frac{2n\kappa}{(p-1)s^2} \quad \text{where } z = \frac{y}{\sqrt{s}},$$

where $R_{1,0}$ and $R_{2,1}$ are defined in (2.34) and (2.36), respectively. In addition to that, we rewrite \bar{R}_2 in terms of $R_{1,0}$ and $R_{2,1}$, and we note that $R_{1,0}$ and $R_{2,1}$ satisfy (2.30) and (2.32). Then, it follows that

$$|\bar{R}_2(y, s)| \leq \frac{C}{s^2}, \quad \forall y \in \mathbb{R}^n.$$

Hence, (B.42) follows. This concludes the proof of this Lemma. \square

C. Preparation of initial data

Here, we here give the proof of Lemma 3.7. We can see that part (II) directly follows from item (i) of part (II). The techniques of the proof are given in [16] and [29]. Although those papers are written in the real-valued case, unlike ours, where we handle the complex-valued case, we reduce in fact to the real case, for the real and the imaginary parts. In addition to that, the set $\mathcal{D}_{K_0, A, s_0}$ is the product of two parts, the first one depends only on d_1 , and the other one depends only on d_2 . Moreover, the real part is almost the same as the initial data in the Vortex model in [16], except for the new term 1, but this term is very small after changing to similarity variable: $e^{-\frac{s}{p-1}}$. In fact, handling the imaginary part is easier than handling the real part. For those reasons, we kindly refer the reader to Lemma 2.4 in [16] and Proposition 4.5 in [29] for the proof of item (i) of (I) and (II). So, we only prove that the initial data satisfies item (ii) in definition of $S(0)$ (the item (iii) is obvious).

Let us consider $T > 0, K_0, \alpha_0, \epsilon_0, \delta_1$ which will be suitably chosen later, then we will prove that for all $|x| \in \left[\frac{K_0}{4}\sqrt{T|\ln T|}, \epsilon_0 \right]$ and $|\xi| \leq 2\alpha_0\sqrt{|\ln(T-t(x))|}$ and $\tau_0(x) = -\frac{t(x)}{T-t(x)}$, we have

$$\left| U(x, \xi, \tau_0(x)) - \hat{U}(\tau_0(x)) \right| \leq \delta_1. \quad (\text{C.1})$$

We now introduce some necessary notations for our proof,

$$\theta_0 = T, \quad r(0) = \frac{K_0}{4}\sqrt{\theta_0|\ln(\theta_0)|} \quad \text{and} \quad R(0) = \theta_0^{\frac{1}{2}}|\ln \theta_0|^{\frac{p}{2}}. \quad (\text{C.2})$$

Then, we have the following asymptotics:

$$\theta(r(0)) \sim \theta_0, \quad \theta(R(0)) \sim \frac{16}{K_0^2}\theta_0|\ln \theta_0|, \quad \theta(2R(0)) \sim \frac{64}{K_0^2}\theta_0|\ln \theta_0|^{p-1}, \quad (\text{C.3})$$

$$\ln \theta(r(0)) \sim \ln \theta(R(0)) \sim \ln \theta(2R(T)). \quad (\text{C.4})$$

In addition to that, if $\alpha_0 \leq \frac{K_0}{16}$ and $\epsilon_0 \leq \frac{2}{3}C^*$, where C^* is introduced in (3.33), then, from the definition (3.21) and $|x| \in [r(0), \epsilon_0]$, and for all $|\xi| \leq 2\alpha_0\sqrt{|\ln \theta(x)|}$, with $\theta(x) = T - t(x)$, we have

$$\left| \xi\sqrt{\theta(x)} \right| \leq \frac{1}{2}|x|,$$

which yields

$$\frac{r(0)}{2} \leq \frac{|x|}{2} = |x| - \frac{|x|}{2} \leq |x + \xi\sqrt{\theta(x)}| \leq \frac{3}{2}|x| \leq \frac{3}{2}\epsilon_0 \leq C^*. \quad (\text{C.5})$$

Hence, using (3.20), (3.4) and definition of χ_1 and $|\xi| \leq 2\alpha_0\sqrt{\theta(x)}$ we have

$$U(x, \xi, \tau_0) = U_1(x, \xi, \tau_0) + iU_2(x, \xi, \tau_0),$$

where

$$\begin{aligned} U_1(x, \xi, \tau_0) &= (I)\chi_1(x + \xi\sqrt{\theta(x)}) + (II)(1 - \chi_1(x + \xi\sqrt{\theta(x)})) + (III), \\ (I) &= \left(\frac{\theta(x)}{\theta_0}\right)^{\frac{1}{p-1}} \Phi_1\left(\frac{x + \xi\sqrt{\theta(x)}}{\sqrt{T}}, |\ln(T)|\right), \\ (II) &= (\theta(x))^{\frac{1}{p-1}} U^*(x + \xi\sqrt{\theta(x)}), \\ (III) &= (\theta(x))^{\frac{1}{p-1}} \\ U_2(x, \tau, \tau_0) &= \left(\frac{\theta(x)}{\theta_0}\right)^{\frac{1}{p-1}} \Phi_2\left(\frac{x + \xi\sqrt{\theta(x)}}{\sqrt{T-t_0}}, |\ln(T-t_0)|\right), \end{aligned}$$

The conclusion of (C.1) follows from the 4 following estimates:

$$|(I) - \hat{U}(\tau_0)| \leq \frac{\delta_1}{4}, \text{ for all } |x| \in \left[r(0), 2\frac{100}{99}R(0)\right] \text{ and for all } |\xi| \leq 2\alpha_0\sqrt{\theta(x)}, \quad (\text{C.6})$$

$$|(II) - \hat{U}(\tau_0)| \leq \frac{\delta_1}{4}, \text{ for all } |x| \in \left[\frac{99}{100}r(0), \epsilon_0\right] \text{ and for all } |\xi| \leq 2\alpha_0\sqrt{\theta(x)}, \quad (\text{C.7})$$

$$|(III)| \leq \frac{\delta_1}{4}, \text{ for all } |x| \in [r(0), \epsilon_0] \text{ and for all } |\xi| \leq 2\alpha_0\sqrt{\theta(x)}, \quad (\text{C.8})$$

$$|U_2(x, \xi, \tau_0)| \leq \frac{\delta_1}{4}, \text{ for all } |x| \in \left[r(0), 2\frac{100}{99}R(0)\right] \text{ and for all } |\xi| \leq 2\alpha_0\sqrt{\theta(x)}. \quad (\text{C.9})$$

It is very easy to estimate for (C.8) for ϵ_0 small enough.

We now estimate (C.9): We rewrite $U_2(x, \xi, \tau_0)$ by using (3.30) as follows:

$$\begin{aligned} |U_2(x, \xi, \tau_0)| &= U_2\left(x, \xi, \frac{-t(x)}{T-t(x)}\right) \\ &= \left(\frac{\theta_0}{\theta(x)}\right)^{-\frac{1}{p-1}} \frac{|x + \xi\sqrt{\theta(x)}|^2}{T|\ln T|} \left(p-1 + \frac{|x + \xi\sqrt{\theta(x)}|^2}{T|\ln T|}\right)^{-\frac{p}{p-1}} \frac{1}{|\ln T|} \\ &\leq \frac{C}{|\ln T|} \left((p-1)\frac{\theta_0}{\theta(x)} + \frac{(p-1)^2}{4p} \frac{|x + \xi\sqrt{x}|^2}{\theta(x)|\ln(\theta_0)}\right)^{-\frac{1}{p-1}}. \end{aligned}$$

In addition to that, for all $|x| \in [r(0), 2\frac{100}{99}R(0)]$ and $|\xi| \leq 2\alpha_0\sqrt{\theta(x)}$, we have

$$\frac{|x + \xi\sqrt{x}|^2}{\theta(x)|\ln(\theta_0)} \geq \frac{1}{CK_0^2},$$

which yields

$$|U_2(x, \xi, \tau_0)| \leq \frac{CK_0^{\frac{2}{p-1}}}{|\ln T|} \leq \frac{\delta_1}{4},$$

if $T \leq T_{1,3}(K_0, \delta_1, \alpha_0)$ and for all $|x| \in [r(0), 2\frac{100}{99}R(0)]$.

The estimate of (C.6): We derive from the definition of Φ_1 in (2.39) and the definition of $\hat{U}(\tau)$ in (3.59) that

$$\begin{aligned} \left| (I) - \hat{U} \left(\frac{-t(x)}{\theta(x)} \right) \right| &= \left| \left((p-1) \left(\frac{\theta_0}{\theta(x)} \right) + \frac{(p-1)^2}{4p} \frac{|x + \xi \sqrt{\theta(x)}|^2}{\theta(x) |\ln \theta_0|} \right)^{-\frac{1}{p-1}} \right. \\ &\quad \left. - \left((p-1) \left(\frac{\theta_0}{\theta(x)} \right) + \frac{(p-1)^2}{4p} \frac{K_0^2}{16} \right)^{-\frac{1}{p-1}} \right| \end{aligned}$$

In addition to that, from (3.21), we have

$$(1 - 2\alpha_0)^2 \frac{K_0^2}{16} \frac{|\ln \theta(x)|}{|\ln \theta_0|} \leq \frac{|x + \xi \sqrt{\theta(x)}|^2}{\theta(x) |\ln \theta_0|} \leq (1 + 2\alpha_0)^2 \frac{K_0^2}{16} \frac{|\ln \theta(x)|}{|\ln \theta_0|}, \forall |\xi| \leq 2\alpha_0 \sqrt{\theta(x)}. \quad (\text{C.10})$$

Using the monotonicity of $\theta(x)$, we have for all $|x| \in [r(0), 2\frac{100}{99}R(0)]$

$$\frac{|\ln r(0)|}{|\ln \theta_0|} \leq \frac{|\ln \theta(x)|}{|\ln \theta_0|} \leq \frac{|\ln R(0)|}{|\ln \theta_0|}.$$

Thanks to (C.3), we have

$$\frac{|\ln \theta(x)|}{|\ln \theta_0|} \sim 1 \text{ as } T \rightarrow 0. \quad (\text{C.11})$$

This yields

$$\left| (I) - \hat{U} \left(\frac{-t(x)}{\theta(x)} \right) \right| \leq C(K_0) \left| \frac{|x + \xi \sqrt{\theta(x)}|^2}{\theta(x) |\ln \theta_0|} - \frac{K_0^2}{16} \right| \rightarrow 0$$

uniformly for all $|x| \in [r(0), 2\frac{100}{99}R(0)]$, $|\xi| \leq 2\alpha_0 \sqrt{\theta(x)}$ as $\alpha_0 \rightarrow 0$ and $T \rightarrow 0$. Hence, there exists $\alpha_{2,3}(K_0, \delta_1)$ and $T_{2,3}(K_0, \delta_1)$ such that

$$\left| (I) - \hat{U} \left(\frac{-t(x)}{\theta(x)} \right) \right| \leq \frac{\delta_1}{4},$$

for all $|x| \in [r(0), 2\frac{100}{99}R(0)]$, $|\xi| \leq 2\alpha_0 \sqrt{\theta(x)}$ provided that $\alpha_0 \leq \alpha_{2,3}$ and $T \leq T_{2,3}$. This concludes the proof of (C.6).

The estimate of (C.7): Let $|x| \in [\frac{99}{100}R(0), \epsilon_0]$. We use the definition of U^* to rewrite (II) as follows

$$\begin{aligned} (II) &= \left(\frac{(p-1)^2}{8p} \frac{|x + \xi \sqrt{\theta(x)}|^2}{\theta(x) |\ln(x + \xi \sqrt{\theta(x)})|} \right)^{-\frac{1}{p-1}} = \left(\frac{(p-1)^2}{8p} \frac{\left| \frac{K_0}{4} \sqrt{|\ln \theta(x)|} + \xi \right|^2}{|\ln(x + \xi \sqrt{\theta(x)})|} \right)^{-\frac{1}{p-1}} \\ &= \left(\frac{(p-1)^2 K_0^2}{64} + \frac{(p-1)^2}{8p} \left(\frac{\left| \frac{K_0}{4} \sqrt{|\ln \theta(x)|} + \xi \right|^2}{|\ln(x + \xi \sqrt{\theta(x)})|} - \frac{K_0^2}{8} \right) \right)^{-\frac{1}{p-1}}. \end{aligned}$$

Then,

$$\begin{aligned} \left| (II) - \hat{U} \left(\frac{t_0 - t(x)}{\theta(x)} \right) \right| &= \left| \left(\frac{(p-1)^2 K_0^2}{64} + \frac{(p-1)^2}{8p} \left(\frac{\left| \frac{K_0}{4} \sqrt{|\ln \theta(x)|} + \xi \right|^2}{|\ln(x + \xi \sqrt{\theta(x)})|} - \frac{K_0^2}{8} \right) \right)^{-\frac{1}{p-1}} \right. \\ &\quad \left. - \left(\frac{(p-1)^2 K_0^2}{64p} + (p-1) \frac{\theta_0}{\theta(x)} \right)^{-\frac{1}{p-1}} \right| \\ &\leq C(K_0)((II_1) + (II_2)), \end{aligned}$$

where

$$(II_1) = \left| \frac{\left| \frac{K_0}{4} \sqrt{|\ln \theta(x)|} + \xi \right|^2}{|\ln(x + \xi \sqrt{\theta(x)})|} - \frac{K_0^2}{8} \right|,$$

$$(II_2) = (p-1) \frac{\theta_0}{\theta(x)}.$$

Let us give a bound to (II_1) : Because $|\xi| \leq 2\alpha_0 \sqrt{\ln \theta(x)}$, we have

$$\begin{aligned} |(II_1)| &\leq \left| \frac{\left| \frac{K_0}{4} \sqrt{|\ln \theta(x)|} + 2\alpha_0 \sqrt{\ln \theta(x)} \right|^2}{|\ln |x + 2\alpha_0 \sqrt{\theta(x) \ln \theta(x)}||} - \frac{K_0^2}{8} \right| \\ &= \left| \frac{\ln \theta(x)}{|\ln |x + \frac{\alpha_0 K_0 |x|}{2}||} \left(\frac{K_0}{4} + 2\alpha_0 \right)^2 - \frac{K_0^2}{8} \right|. \end{aligned}$$

Using the fact that

$$\ln \theta(x) = \ln(T - t(x)) \sim 2 \ln |x|,$$

and

$$|\ln |x + 2\alpha_0 \sqrt{\theta(x) \ln \theta(x)}|| = \left| \ln |x + \frac{K_0}{2} |x|| \right| \sim |\ln |x||,$$

as $|x| \rightarrow 0$, we derive that, there exists $\alpha_{3,3}(K_0, \delta_1)$ such that for all $\alpha_0 \leq \alpha_{3,3}$, there exists $\epsilon_{3,3}(K_0, \alpha_0, \delta_1)$ such that for all $\epsilon_0 \leq \epsilon_{3,3}$, for all $x \in [\frac{99}{100}R(0), \epsilon_0]$ and for all $|\xi| \leq 2\alpha_0 \sqrt{|\ln \theta(x)|}$, we obtain

$$|(II_1)| \leq \frac{\delta_1}{2}.$$

It remains to bound (II_2) . From (C.3), the fact that $|x| \geq \frac{99}{100}R(0)$ and the monotonicity of $\theta(x)$, we have

$$|(II_2)| \leq \left| \frac{\theta(0)}{\theta(R(0))} \right| \leq C |\ln \theta(0)|^{-(p-1)} \leq \frac{\delta_1}{2},$$

provided that $T \leq T_{4,3}(K_0, \delta_1)$. This gives (C.1), and concludes the proof of Lemma 3.7.

REFERENCES

- [1] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.
- [2] P. Constantin, P. D. Lax, and A. Majda. A simple one-dimensional model for the three-dimensional vorticity equation. *Comm. Pure Appl. Math.*, 38(6):715–724, 1985.
- [3] R. Côte and H. Zaag. Construction of a multisoliton blowup solution to the semilinear wave equation in one space dimension. *Comm. Pure Appl. Math.*, 66(10):1541–1581, 2013.
- [4] G. K. Duong, V. T. Nguyen, and H. Zaag. Construction of a stable blowup solution with a prescribed behavior for a non-scaling invariant semilinear heat equation. *Tunisian J. Math.*, to appear, 2018.
- [5] G. K. Duong. Profile for the imaginary part of a blowup solution for a complex-valued semilinear heat equation. *arXiv:1704.08580*, page 32, 2018. submitted.
- [6] M. A. Ebde and H. Zaag. Construction and stability of a blow up solution for a nonlinear heat equation with a gradient term. *SĕMA J.*, (55):5–21, 2011.
- [7] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.
- [8] J. S. Guo, H. Ninomiya, M. Shimojo, and E. Yanagida. Convergence and blow-up of solutions for a complex-valued heat equation with a quadratic nonlinearity. *Trans. Amer. Math. Soc.*, 365(5):2447–2467, 2013.
- [9] T. Ghoul, V. T. Nguyen, and H. Zaag. Construction and stability of blowup solutions for a non-variational semilinear parabolic system. *submitted*., 2016.
- [10] T. Ghoul, V. T. Nguyen, and H. Zaag. Construction of blowup solutions for an exponential reaction-diffusion equation involving a critical power nonlinear gradient term. *preprint*, 2016.
- [11] T. Ghoul, V. T. Nguyen, and H. Zaag. Refined regularity of the blow-up set linked to refined asymptotic behavior for the semilinear heat equation. *preprint*, 2016.
- [12] J. Harada. Blowup profile for a complex valued semilinear heat equation. *J. Funct. Anal.*, 270(11):4213–4255, 2016.
- [13] M. A. Herrero and J. J. L. Velázquez. Blow-up profiles in one-dimensional, semilinear parabolic problems. *Comm. Partial Differential Equations*, 17(1-2):205–219, 1992.

- [14] F. Merle. Solution of a nonlinear heat equation with arbitrarily given blow-up points. *Comm. Pure Appl. Math.*, 45(3):263–300, 1992.
- [15] F. Merle and H. Zaag. Stabilité du profil à l’explosion pour les équations du type $u_t = \Delta u + |u|^{p-1}u$. *C. R. Acad. Sci. Paris Sér. I Math.*, 322(4):345–350, 1996.
- [16] F. Merle and H. Zaag. Reconnection of vortex with the boundary and finite time quenching. *Nonlinearity*, 10(6):1497–1550, 1997.
- [17] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.
- [18] N. Masmoudi and H. Zaag. Blow-up profile for the complex Ginzburg-Landau equation. *J. Funct. Anal.*, 255(7):1613–1666, 2008.
- [19] F. Merle and H. Zaag. Existence and classification of characteristic points at blow-up for a semilinear wave equation in one space dimension. *Amer. J. Math.*, 134(3):581–648, 2012.
- [20] F. Merle and H. Zaag. Isolatedness of characteristic points at blowup for a 1-dimensional semilinear wave equation. *Duke Math. J.*, 161(15):2837–2908, 2012.
- [21] N. Nouaïli and H. Zaag. Profile for a simultaneously blowing up solution to a complex valued semilinear heat equation. *Comm. Partial Differential Equations*, 40(7):1197–1217, 2015.
- [22] N. Nouaïli and H. Zaag. Profile for a simultaneously blowing up solution to a complex valued semilinear heat equation. *Comm. Partial Differential Equations*, 40(7):1197–1217, 2015.
- [23] V. T. Nguyen and H. Zaag. Finite degrees of freedom for the refined blow-up profile for a semilinear heat equation. *Ann. Scient. Éc. Norm. Sup*, 50:5:1241–1282, 2017.
- [24] N. Nouaïli and H. Zaag. Construction of a blow-up solution for the complex ginzburg-landau equation in some critical case. *Arch. Rat. Mech. Anal*, 2018. to appear.
- [25] H. Okamoto, T. Sakajo, and M. Wunsch. On a generalization of the Constantin-Lax-Majda equation. *Nonlinearity*, 21(10):2447–2461, 2008.
- [26] T. Sakajo. Blow-up solutions of the Constantin-Lax-Majda equation with a generalized viscosity term. *J. Math. Sci. Univ. Tokyo*, 10(1):187–207, 2003.
- [27] T. Sakajo. On global solutions for the Constantin-Lax-Majda equation with a generalized viscosity term. *Nonlinearity*, 16(4):1319–1328, 2003.
- [28] S. Schochet. Explicit solutions of the viscous model vorticity equation. *Comm. Pure Appl. Math.*, 39(4):531–537, 1986.
- [29] S. Tayachi and H. Zaag. Existence of a stable blow-up profile for the nonlinear heat equation with a critical power nonlinear gradient term. *preprint*, 2015.
- [30] H. Zaag. Blow-up results for vector-valued nonlinear heat equations with no gradient structure. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 15(5):581–622, 1998.
- [31] H. Zaag. A Liouville theorem and blowup behavior for a vector-valued nonlinear heat equation with no gradient structure. *Comm. Pure Appl. Math.*, 54(1):107–133, 2001.

Address Paris 13 University, Institute Galilée, Laboratory of Analysis, Geometry and Applications, CNRS UMR 7539, 95302, 99 avenue J.B Clément, 93430 Villetaneuse, France
e-mail: duong@math.univ-paris13.fr