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On the Optimal Control of Variational-Hemivariational Inequalities

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Abstract. The present paper represents a continuation of [26]. There, a continuous dependence result for the solution of an elliptic variational-hemivariational inequality was obtained and then used to prove the existence of optimal pairs for two associated optimal control problems. In the current paper we complete this study with more general results. Indeed, we prove the continuous dependence of the solution with respect to a parameter which appears in all the data of the problem, including the set of constraints, the nonlinear operator and the two functionals which govern the variational-hemivariational inequality. This allows us to consider a general associated optimal control problem for which we prove the existence of optimal pairs, together with a new convergence result. The mathematical tools developed in this paper are useful in the analysis and control of a large class of boundary value problems which, in a weak formulation, lead to elliptic variational-hemivariational inequalities. To provide an example, we illustrate our results in the study of an inequality which describes the equilibrium of an elastic body in frictional contact with a foundation made of a rigid body covered by a layer of soft material.

Keywords: Variational-hemivariational inequality, Clarke subdifferential, Mosco convergence, optimal pair, optimal control, elastic body, frictional contact problem.

2010 Mathematics Subject Classification: 47J20, 47J22, 49J53, 74M15.

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1 Introduction

Variational and hemivariational inequalities are widely used in the study of many nonlinear boundary value problems and have a large number of applications in Contact Mechanics and Engineering see, for instance, [4, 19, 24, 25, 29]. The theory of variational inequalities was developed in early sixty's, by using arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. In contrast, the analysis of hemivariational inequalities uses as main ingredient the properties of the subdifferential in the sense of Clarke [7], defined for locally Lipschitz functions, which may be nonconvex. Hemivariational inequalities were first introduced in early eighty's by Panagiotopoulos in the context of applications in engineering problems. Studies of variational and hemivariational inequalities can be found in several comprehensive references, e.g., [1, 3, 6, 9, 10, 12, 15, 21, 34, 35, 36, 37].

Variational-hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functions are present. Recent references in the field include the book [28] and the paper [5]. The book [28] deals with existence, uniqueness and convergence results for various classes of variational-hemivariational inequalities. It also contains applications of these inequalities in the study of mathematical models which describe the contact between a deformable body and a foundation. The paper [5] deals with existence, approximation, and regularization results for semicoercive variational-hemivariational inequalities and includes an application to unilateral contact problems with nonmonotone boundary conditions. A similar regularization technique was used in [23] in the study of a variational-hemivariational inequality which describes the delamination of composite structures with a contaminated interface layer.

The optimal control theory deals with the existence and, when possible, the uniqueness of optimal state-control pair. It also deals with the derivation of necessary conditions of optimality or, better, necessary and sufficient conditions of optimality. Optimal control problems for variational and hemivariational inequalities have been discussed in several works, including [2, 8, 14, 16, 17, 22, 30, 31, 32, 33]. Due to the nonsmooth and nonconvex feature of the functionals involved, the treatment of optimal control problems for such inequalities requires the use of their approximation by smooth optimization problems. And, on this matter, establishing convergence results for the optimal pairs represents a topic of major interest.

In [26] we have studied variational-hemivariational inequalities of the form: find $u \in K_g$ such that

$$\langle Au, v - u \rangle + \varphi(u, v) - \varphi(u, u) + j^0(u; v - u) \geq (f, \pi v - \pi u)_Y \quad \forall v \in K_g. \quad (1.1)$$

Here and everywhere in this paper X is a reflexive Banach space, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and its dual X^* , Y is a real Hilbert space endowed with the inner product $(\cdot, \cdot)_Y$ and $\pi : X \rightarrow Y$. Moreover, in (1.1) we supposed that $K_g = gK$ where $K \subset X$ and $g > 0$, $A : X \rightarrow X^*$, $\varphi : X \times X \rightarrow \mathbb{R}$, $j : X \rightarrow \mathbb{R}$ and $f \in Y$. Note that the function $\varphi(u, \cdot)$ is assumed to be convex and the function j is locally Lipschitz and, in general, nonconvex. Therefore, following the terminology in the literature, we see that inequality (1.1) represents a variational-hemivariational inequality.

A short description of the results obtained in [26] is the following. First, the existence and uniqueness of the solution of (1.1) was proved by using a result proved in [18], based on arguments of surjectivity for pseudomonotone operators and the Banach fixed point argument. Then, under specific assumptions on the functions φ and j , the continuous dependence of the solution with respect f and g was studied and a convergence result was proved. Next, two optimal control problems were considered, in which the control were f and g , respectively. The existence of optimal pairs together with some convergence results were proved, for each problem. Finally, these abstract results were used in the study of a one-dimensional mathematical model which describes the equilibrium of an elastic rod in unilateral contact with a foundation, under the action of a body force.

As it results from above, the study in [26] was focused on the dependence of the solution with respect the parameters f and g or, equivalently, with respect to f and the set of constraints K_g . Nevertheless, various examples can be considered in which the solution depends on a number of parameters which appear in the operator A , or in the functions φ and j , as well. All of these parameters could play the role of control in optimal control problems associated to inequality (1.1). For this reason, there is a need to extend the results in [26] to more general cases and this is the aim of this current paper. Considering such extension leads to various mathematical difficulties. To overcome them we use new assumptions and new arguments, different to those used in [26], which represents the trait of novelty of this current paper.

The rest of the manuscript is organized as follows. In Section 2 we review some notation and preliminary results. In Section 3 we introduce a variational-hemivariational inequality in which all the data depend on a parameter p . We state the behavior of the solution of this inequality with respect to p and provide a convergence result. Then, in Section 4 we consider a class of optimal control problems associated to the variational-hemivariational inequality, for which we prove the existence and convergence of the optimal pairs. Finally, in Section 5 we give an example which illustrate a potential application of our abstract study. The example arises from Contact Mechanics and is given by a variational-hemivariational inequality which describes the contact of an elastic body with a foundation made of a rigid body covered by a layer of deformable material.

2 Preliminaries

We use notation $\|\cdot\|_X$ and 0_X for the norm and the zero space element of X , respectively. Unless stated otherwise, all the limits, upper and lower limits below are considered as $n \rightarrow \infty$, even if we do not mention it explicitly. The symbols “ \rightharpoonup ” and “ \rightarrow ” denote the weak and the strong convergence in various spaces which will be specified. Nevertheless, for simplicity, we write $g_n \rightarrow g$ for the convergence in \mathbb{R} .

DEFINITION 1 *An operator $A: X \rightarrow X^*$ is said to be:*

- a) monotone, if for all $u, v \in X$, we have $\langle Au - Av, u - v \rangle \geq 0$;
- b) bounded, if A maps bounded sets of X into bounded sets of X^* ;

c) pseudomonotone, if it is bounded and $u_n \rightarrow u$ weakly in X with

$$\limsup \langle Au_n, u_n - u \rangle \leq 0$$

implies $\liminf \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle$ for all $v \in X$;

d) hemicontinuous, if for all $u, v, w \in X$, the function $\lambda \mapsto \langle A(u + \lambda v), w \rangle$ is continuous on $[0, 1]$.

We now recall the definition of the Clarke subdifferential for a locally Lipschitz function.

DEFINITION 2 A function $j: X \rightarrow \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$ there exists U_x a neighborhood of x and a constant $L_x > 0$ such that

$$|j(y) - j(z)| \leq L_x \|y - z\|_X \quad \text{for all } y, z \in U_x.$$

The generalized (Clarke) directional derivative of j at the point $x \in X$ in the direction $v \in X$ is defined by

$$j^0(x; v) = \limsup_{y \rightarrow x, \lambda \downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The generalized gradient (subdifferential) of j at x is a subset of the dual space X^* given by

$$\partial j(x) = \{ \zeta \in X^* \mid j^0(x; v) \geq \langle \zeta, v \rangle \quad \forall v \in X \}.$$

A locally Lipschitz function j is said to be regular (in the sense of Clarke) at the point $x \in X$ if for all $v \in X$ the one-sided directional derivative $j^0(x; v)$ exists and $j^0(x; v) = j'(x; v)$.

We shall use the following properties of the generalized directional derivative and the generalized gradient.

PROPOSITION 3 Assume that $j: X \rightarrow \mathbb{R}$ is a locally Lipschitz function. Then the following hold:

(i) For every $x \in X$, the function $X \ni v \mapsto h^0(x; v) \in \mathbb{R}$ is positively homogeneous and subadditive, i.e., $j^0(x; \lambda v) = \lambda j^0(x; v)$ for all $\lambda \geq 0$, $v \in X$ and $j^0(x; v_1 + v_2) \leq j^0(x; v_1) + j^0(x; v_2)$ for all $v_1, v_2 \in X$, respectively.

(ii) For every $v \in X$, we have $j^0(x; v) = \max \{ \langle \xi, v \rangle \mid \xi \in \partial j(x) \}$.

We now recall the notion of convergence in the sense of Mosco, denoted by “ \xrightarrow{M} ”, which will be used in the Sections 3–5 of this paper.

DEFINITION 4 Let X be a normed space, $\{K_n\}$ a sequence of nonempty subsets of X and K a nonempty subset of X . We say that the sequence $\{K_n\}$ converges to K in the Mosco sense if the following conditions hold.

(M₁) $\left\{ \begin{array}{l} \text{For each } v \in K \text{ there exists a sequence } \{v_n\} \text{ such that} \\ v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \rightarrow v \text{ in } X. \end{array} \right.$

(M₂) $\left\{ \begin{array}{l} \text{For each sequence } \{v_n\} \text{ such that} \\ v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \rightharpoonup v \text{ in } X \text{ we have } v \in K. \end{array} \right.$

Note that the convergence in the sense of Mosco depends on the topology of the normed space X and, for this reason, we write it explicitly $K_n \xrightarrow{M} K_n$ in X . More details on this topic could be found in [20].

We proceed with the following version of the Weierstrass theorem.

THEOREM 5 *Let X be a reflexive Banach space, K a nonempty weakly closed subset of X and $J : X \rightarrow \mathbb{R}$ a weakly lower semicontinuous function. In addition, assume that either K is bounded or J is coercive, i.e., $J(v) \rightarrow \infty$ as $\|v\|_X \rightarrow \infty$. Then, there exists at least an element u such that*

$$u \in K, \quad J(u) \leq J(v) \quad \forall v \in K. \quad (2.1)$$

The proof of Theorem 5 can be found in many books and survey, see, for instance, [13, 27].

We now introduce the variational-hemivariational inequality we are interested in and state its unique solvability. The functional framework is the following. Let Λ be a set of parameters and, for any $p \in \Lambda$, consider a set $K_p \subset X$, an operator $A_p : X \rightarrow X^*$, the functions $\varphi_p : X \times X \rightarrow \mathbb{R}$ and $j_p : X \rightarrow \mathbb{R}$ and an element $f_p \in Y$. With these data, for $p \in \Lambda$ given, we state the following inequality problem.

Problem \mathcal{P} . *Find an element $u = u(p)$ such that*

$$\begin{aligned} u \in K_p, \quad \langle A_p u, v - u \rangle + \varphi_p(u, v) - \varphi_p(u, u) + j_p^0(u; v - u) \\ \geq \langle f_p, \pi v - \pi u \rangle_Y \quad \forall v \in K_p. \end{aligned} \quad (2.2)$$

In the study of Problem \mathcal{P} , we consider the following hypotheses on the data.

$$K_p \text{ is nonempty, closed and convex subset of } X. \quad (2.3)$$

$$\left\{ \begin{array}{l} A_p : X \rightarrow X^* \text{ is such that} \\ \text{(a) it is pseudomonotone;} \\ \text{(b) it is strongly monotone, i.e., there exists } m_p > 0 \text{ such that} \\ \quad \langle Av_1 - Av_2, v_1 - v_2 \rangle \geq m_p \|v_1 - v_2\|_X^2 \quad \text{for all } v_1, v_2 \in X. \end{array} \right. \quad (2.4)$$

$$\left\{ \begin{array}{l} \varphi_p : X \times X \rightarrow \mathbb{R} \text{ is such that} \\ \text{(a) } \varphi_p(u, \cdot) : X \rightarrow \mathbb{R} \text{ is convex and lower semicontinuous,} \\ \quad \text{for all } u \in X; \\ \text{(b) there exists } \alpha_p > 0 \text{ such that} \\ \quad \varphi_p(u_1, v_2) - \varphi_p(u_1, v_1) + \varphi_p(u_2, v_1) - \varphi_p(u_2, v_2) \\ \quad \leq \alpha_p \|u_1 - u_2\|_X \|v_1 - v_2\|_X \quad \text{for all } u_1, u_2, v_1, v_2 \in X. \end{array} \right. \quad (2.5)$$

$$\left\{ \begin{array}{l} j_p: X \rightarrow \mathbb{R} \text{ is such that} \\ \text{(a) it is locally Lipschitz;} \\ \text{(b) } \|\xi\|_{X^*} \leq c_{0p} + c_{1p} \|v\|_X \text{ for all } v \in X, \xi \in \partial j_p(v), \\ \quad \text{with } c_{0p}, c_{1p} \geq 0; \\ \text{(c) there exists } \beta_p > 0 \text{ such that} \\ \quad j_p^0(v_1; v_2 - v_1) + j_p^0(v_2; v_1 - v_2) \leq \beta_p \|v_1 - v_2\|_X^2 \\ \quad \text{for all } v_1, v_2 \in X. \end{array} \right. \quad (2.6)$$

$$\alpha_p + \beta_p < m_p. \quad (2.7)$$

$$f_p \in Y. \quad (2.8)$$

$$\left\{ \begin{array}{l} \pi: X \rightarrow Y \text{ is a linear continuous operator, i.e.,} \\ \|\pi v\|_Y \leq d_0 \|v\|_X \quad \forall v \in X \text{ with } d_0 > 0. \end{array} \right. \quad (2.9)$$

The unique solvability of the variational-hemivariational inequality (1.1) is provided by the following result.

THEOREM 6 *Assume (2.3)–(2.9). Then, Problem \mathcal{P} has a unique solution $u \in K_p$.*

Theorem 6 represents a slightly modified version of Theorem 18 in [18]. Its proof is carried out in several steps, by using the properties of the subdifferential, both in the sense of Clarke and in the sense of convex analysis, surjectivity result for pseudomonotone multivalued operators, and the Banach fixed point argument. For additional details we also refer the reader to [26] and [28, Remark 13].

We end this section with the remark that everywhere in this paper assumption (2.4) (a) can be dropped if we suppose that the operator A_p is bounded and hemicontinuous. Indeed, it is well known that a bounded monotone hemicontinuous operator defined on X with values in X^* is pseudomonotone.

3 A convergence result

Theorem 6 allows us to define the map $p \mapsto u(p)$ which associates to each $p \in \Lambda$ the solution $u = u(p) \in K_p$ of the variational-hemivariational inequality (2.2). In this section we present an important property of this map, which represents a crucial ingredient in the study of optimal control problems associated to inequality (2.2).

Let $\{p_n\}$ be a sequence of elements in Λ and assume that, for each $n \in \mathbb{N}$, the corresponding set K_{p_n} , operator A_{p_n} , functions φ_{p_n} , j_{p_n} and element f_{p_n} satisfy assumptions (2.3)–(2.8) with constants m_{p_n} , α_{p_n} , c_{0p_n} , c_{1p_n} , β_{p_n} . To avoid any confusion, when used for $p = p_n$, we refer to these assumptions as assumptions (2.3)_n–(2.8)_n.

Then, if condition (2.9) is satisfied, we deduce from Theorem 6 that for each $n \in \mathbb{N}$ there exists a unique solution $u_n = u(p_n)$ for the following problem.

Problem \mathcal{P}_n . Find an element $u_n = u(p_n)$ such that

$$\begin{aligned} u_n \in K_{p_n}, \quad \langle A_{p_n} u_n, v - u_n \rangle + \varphi_{p_n}(u_n, v) - \varphi_{p_n}(u_n, u_n) + j_{p_n}^0(u_n; v - u_n) \quad (3.1) \\ \geq (f_{p_n}, \pi v - \pi u_n)_Y \quad \forall v \in K_{p_n}. \end{aligned}$$

We now consider the following additional assumptions.

$$K_{p_n} \xrightarrow{M} K_p \quad \text{in } X, \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

$$\left\{ \begin{array}{l} \text{For any } n \in \mathbb{N} \text{ there exists } F_n \geq 0 \text{ and } \delta_n \geq 0 \text{ such that} \\ \text{(a) } \|A_{p_n} v - A_p v\|_X \leq F_n(\|v\|_X + \delta_n) \quad \text{for all } v \in X; \\ \text{(b) } \lim_{n \rightarrow \infty} F_n = 0; \\ \text{(c) the sequence } \{\delta_n\} \subset \mathbb{R} \text{ is bounded.} \end{array} \right. \quad (3.3)$$

$$\left\{ \begin{array}{l} \text{For any } n \in \mathbb{N} \text{ there exists a function } c_n : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ such that} \\ \text{(a) } \varphi_{p_n}(u, v_1) - \varphi_{p_n}(u, v_2) \leq c_n(\|u\|_X) \|v_1 - v_2\|_X \\ \quad \forall u, v_1, v_2 \in X; \\ \text{(b) the sequence } \{c_n(\|v_n\|_X)\} \subset \mathbb{R} \text{ is bounded} \\ \quad \text{whenever the sequence } \{v_n\} \subset X \text{ is bounded.} \end{array} \right. \quad (3.4)$$

$$\left\{ \begin{array}{l} \text{For all sequences } \{u_n\}, \{v_n\} \subset X \text{ such that} \\ u_n \rightharpoonup u \text{ in } X, v_n \rightarrow v \text{ in } X, \text{ we have} \\ \limsup (\varphi_{p_n}(u_n, v_n) - \varphi_{p_n}(u_n, u_n)) \leq \varphi_p(u, v) - \varphi_p(u, u). \end{array} \right. \quad (3.5)$$

$$\left\{ \begin{array}{l} \text{For all sequences } \{u_n\}, \{v_n\} \subset X \text{ such that} \\ u_n \rightharpoonup u \text{ in } X, v_n \rightarrow v \text{ in } X, \text{ we have} \\ \limsup j_{p_n}^0(u_n; v_n - u_n) \leq j_p^0(u; v - u). \end{array} \right. \quad (3.6)$$

$$\left\{ \begin{array}{l} \text{For all sequence } \{v_n\} \subset X \text{ such that} \\ v_n \rightharpoonup v \text{ in } X, \text{ we have } \pi v_n \rightarrow \pi v \quad \text{in } Y. \end{array} \right. \quad (3.7)$$

$$f_{p_n} \rightharpoonup f_p \quad \text{in } Y, \quad \text{as } n \rightarrow \infty. \quad (3.8)$$

$$\text{There exists } m_0 > 0 \text{ such that } m_0 + \alpha_{p_n} + \beta_{p_n} \leq m_{p_n} \quad \forall n \in \mathbb{N}. \quad (3.9)$$

$$\text{The sequences } \{c_{0p_n}\}, \{c_{1p_n}\} \subset \mathbb{R} \text{ are bounded.} \quad (3.10)$$

The main result of this section is the following.

THEOREM 7 Assume (2.3)–(2.9) and, for each $n \in \mathbb{N}$, assume (2.3)_n–(2.8)_n. Moreover, assume (3.2)–(3.10) and denote by u_n and u the solutions of Problems \mathcal{P}_n and \mathcal{P} , respectively. Then the sequence $\{u_n\}$ converges strongly to u , i.e.,

$$u_n \rightarrow u \quad \text{in } X. \quad (3.11)$$

Note that in applications, $\Lambda \subset Z$ where Z is a normed space and conditions (3.2)–(3.6) and (3.8)–(3.10) are satisfied when $p_n \rightarrow p$ in Z . In this case Theorem 7 implies that

$$p_n \rightarrow p \quad \text{in } Z \implies u(p_n) \rightarrow u(p) \quad \text{in } X, \quad (3.12)$$

which shows that the operator $p \mapsto u(p) : \Lambda \rightarrow X$ is weakly-strongly continuous.

The proof of Theorem 7 will be carried out in several steps that we present in what follows. Everywhere below we assume that the hypotheses of Theorem 7 hold. The first step of the proof is the following.

LEMMA 8 *There is an element $\tilde{u} \in K_p$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \rightarrow \tilde{u}$ in X , as $n \rightarrow \infty$.*

Proof. We first establish the boundedness of $\{u_n\}$ in X . Let v be a given element in K_p . We use assumption (3.2) and consider a sequence $\{v_n\}$ such that $v_n \in K_{p_n}$ for all $n \in \mathbb{N}$ and

$$v_n \rightarrow v \quad \text{in } X. \quad (3.13)$$

Let $n \in \mathbb{N}$. We test with $v_n \in K_{p_n}$ in (3.1) to obtain

$$\begin{aligned} \langle A_{p_n} u_n, u_n - v_n \rangle &\leq \\ &+ \varphi_{p_n}(u_n, v_n) - \varphi_{p_n}(u_n, u_n) + j_{p_n}^0(u_n; v_n - u_n) + (f_{p_n}, \pi u_n - \pi v_n)_Y. \end{aligned} \quad (3.14)$$

Moreover, using the strongly monotonicity of the operator A_{p_n} we deduce that

$$\langle A_{p_n} u_n, u_n - v_n \rangle \geq m_{p_n} \|u_n - v_n\|_X^2 + \langle A_{p_n} v_n, u_n - v_n \rangle. \quad (3.15)$$

Next, we write

$$\begin{aligned} &\varphi_{p_n}(u_n, v_n) - \varphi_{p_n}(u_n, u_n) \\ &= \varphi_{p_n}(u_n, v_n) - \varphi_{p_n}(u_n, u_n) + \varphi_{p_n}(v_n, u_n) - \varphi_{p_n}(v_n, v_n) \\ &\quad + \varphi_{p_n}(v_n, v_n) - \varphi_{p_n}(v_n, u_n), \end{aligned}$$

then we use assumptions (2.5)_n(b), (3.4) to see that

$$\begin{aligned} &\varphi_{p_n}(u_n, v_n) - \varphi_{p_n}(u_n, u_n) \\ &\leq \alpha_{p_n} \|u_n - v_n\|_X^2 + c_n (\|v_n\|_X) \|u_n - v_n\|_X. \end{aligned} \quad (3.16)$$

On the other hand, by (2.6)_n(b) and Proposition 3(ii), we have

$$\begin{aligned}
& j_{p_n}^0(u_n; v_n - u_n) \\
&= j_{p_n}^0(u_n; v_n - u_n) + j_{p_n}^0(v_n; u_n - v_n) - j_{p_n}^0(v_n; u_n - v_n) \\
&\leq j_{p_n}^0(u_n; v_n - u_n) + j_{p_n}^0(v_n; u_n - v_n) + |j_{p_n}^0(v_n; u_n - v_n)| \\
&\leq \beta_{p_n} \|u_n - v_n\|_X^2 + |\max\{\langle \zeta, u_n - v_n \rangle \mid \zeta \in \partial j_{p_n}(v_n)\}|
\end{aligned}$$

and, using assumption (2.6)_n(b) we find that

$$j_{p_n}^0(u_n; v_n - u_n) \leq \beta_{p_n} \|u_n - v_n\|_X^2 + (c_{0p_n} + c_{1p_n} \|v_n\|_X) \|u_n - v_n\|_X. \quad (3.17)$$

Finally, assumption (2.9) yields

$$(f_{p_n}, \pi u_n - \pi v_n)_Y \leq d_0 \|f_{p_n}\|_Y \|u_n - v_n\|_X. \quad (3.18)$$

We now combine inequalities (3.14)–(3.18) to see that

$$\begin{aligned}
& m_{p_n} \|u_n - v_n\|_X^2 \leq \langle A_{p_n} v_n, v_n - u_n \rangle \\
&+ \alpha_{p_n} \|u_n - v_n\|_X^2 + c_n (\|v_n\|_X) \|u_n - v_n\|_X \\
&+ \beta_{p_n} \|u_n - v_n\|_X^2 + (c_{0p_n} + c_{1p_n} \|v_n\|_X) \|u_n - v_n\|_X + d_0 \|f_{p_n}\|_Y \|u_n - v_n\|_X,
\end{aligned}$$

which implies that

$$\begin{aligned}
& (m_{p_n} - \alpha_{p_n} - \beta_{p_n}) \|u_n - v_n\|_X \leq \|A_{p_n} v_n\|_X \\
&+ c_n (\|v_n\|_X) + (c_{0p_n} + c_{1p_n} \|v_n\|_X) + d_0 \|f_{p_n}\|_Y.
\end{aligned} \quad (3.19)$$

Note that assumptions (3.9) and (3.3) imply that

$$m_0 \leq m_{p_n} - \alpha_{p_n} - \beta_{p_n} \quad \text{and} \quad \|A_{p_n} v_n\|_X \leq \|A_p v_n\|_X + F_n (\|v_n\|_X + \delta_n),$$

respectively. Therefore, using these inequalities in (3.19) we deduce that

$$\begin{aligned}
& m_0 \|u_n - v_n\|_X \leq \|A_p v_n\|_X + F_n (\|v_n\|_X + \delta_n) \\
&+ c_n (\|v_n\|_X) + (c_{0p_n} + c_{1p_n} \|v_n\|_X) + d_0 \|f_{p_n}\|_Y.
\end{aligned} \quad (3.20)$$

Next, by (3.13) and (3.8) we know that the sequences $\{v_n\}$ and $\{f_{p_n}\}$ are bounded in X and Y , respectively. Therefore, using assumptions (2.4)(a), (3.3)(b), (c), (3.4)(b), (3.10) and inequality (3.20), we deduce that there is a constant $C > 0$ independent of n such that $\|u_n - v_n\|_X \leq C$. This implies that $\{u_n\}$ is a bounded sequence in X and, from the reflexivity of X , by passing to a subsequence, if necessary, we deduce that

$$u_n \rightharpoonup \tilde{u} \quad \text{in } X, \quad \text{as } n \rightarrow \infty \quad (3.21)$$

with some $\tilde{u} \in X$. Finally, recall that $u_n \in K_{p_n}$. Therefore, the convergence (3.21) combined with assumption (3.2) shows that $\tilde{u} \in K_p$ which concludes the proof. \square

The second step in the proof is as follows.

LEMMA 9 *The element $\tilde{u} \in K_p$ is a solution of Problem \mathcal{P} .*

Proof. Let v be a given element in K_p . We use assumption (3.2) and consider a sequence $\{v_n\}$ such that $v_n \in K_{p_n}$ for all $n \in \mathbb{N}$ and (3.13) holds. Let $n \in \mathbb{N}$. Then inequality (3.14) holds and, passing to the upper limit in this inequality, we find that

$$\begin{aligned} \limsup \langle A_{p_n} u_n, u_n - v_n \rangle &\leq \limsup [\varphi_{p_n}(u_n, v_n) - \varphi_{p_n}(u_n, u_n)] \\ &+ \limsup j_{p_n}^0(u_n; v_n - u_n) + \limsup (f_{p_n}, \pi u_n - \pi v_n)_Y. \end{aligned} \quad (3.22)$$

We now use the convergences (3.21), (3.13) and assumptions (3.5), (3.6), to deduce that

$$\limsup [\varphi_{p_n}(u_n, v_n) - \varphi_{p_n}(u_n, u_n)] \leq \varphi_p(\tilde{u}, v) - \varphi_p(\tilde{u}, \tilde{u}), \quad (3.23)$$

$$\limsup j_{p_n}^0(u_n; v_n - u_n) \leq j_p^0(\tilde{u}; v - \tilde{u}). \quad (3.24)$$

On the other hand, the convergences (3.21), (3.13) and assumptions (3.7), (3.8) yield

$$\lim (f_{p_n}, \pi u_n - \pi v_n)_Y = (f_p, \pi \tilde{u} - \pi v)_Y. \quad (3.25)$$

We now combine relations (3.22)–(3.25) to deduce that

$$\begin{aligned} \limsup \langle A_{p_n} u_n, u_n - v_n \rangle &\leq \varphi_p(\tilde{u}, v) - \varphi_p(\tilde{u}, \tilde{u}) \\ &+ j_p^0(\tilde{u}; v - \tilde{u}) + (f_p, \pi \tilde{u} - \pi v)_Y. \end{aligned} \quad (3.26)$$

Next, we write

$$\langle A_{p_n} u_n, u_n - v_n \rangle = \langle A_p u_n, u_n - v \rangle + \langle A_p u_n, v - v_n \rangle + \langle A_{p_n} u_n - A_p u_n, u_n - v_n \rangle.$$

Then we use we assumptions (2.4)(a), (3.3) and the convergence (3.13) to see that

$$\begin{aligned} \langle A_p u_n, v - v_n \rangle &\rightarrow 0, \\ |\langle A_{p_n} u_n - A_p u_n, u_n - v_n \rangle| &\leq F_n(\|u_n\|_X + \delta_n) \|u_n - v_n\|_X \rightarrow 0, \end{aligned}$$

which imply that

$$\limsup \langle A_{p_n} u_n, u_n - v_n \rangle = \limsup \langle A_p u_n, u_n - v \rangle. \quad (3.27)$$

We now combine inequality (3.26) with (3.27) to obtain that

$$\begin{aligned} \limsup \langle A_p u_n, u_n - v \rangle &\leq \varphi_p(\tilde{u}, v) - \varphi_p(\tilde{u}, \tilde{u}) \\ &+ j_p^0(\tilde{u}; v - \tilde{u}) + (f_p, \pi \tilde{u} - \pi v)_Y. \end{aligned} \quad (3.28)$$

Next, we take $v = \tilde{u}$ in (3.28) and use Proposition 3(i) to deduce that

$$\limsup \langle A_p u_n, u_n - \tilde{u} \rangle \leq 0. \quad (3.29)$$

Exploiting now the pseudomonotonicity of A_p , from (3.21) and (3.29), we have

$$\langle A_p \tilde{u}, \tilde{u} - v \rangle \leq \liminf \langle A_p u_n, u_n - v \rangle \quad \text{for all } v \in X. \quad (3.30)$$

Next, from (3.30) and (3.28) we obtain that \tilde{u} satisfies the inequality (2.2), which concludes the proof. \square

We are now in position to provide the proof of Theorem 7.

Proof. Since Problem \mathcal{P} has a unique solution, denoted u , it follows from Lemma 9 that $\tilde{u} = u$. This implies that every subsequence of $\{u_n\}$ which converges weakly in X has the same limit and, therefore, it follows that the whole sequence $\{u_n\}$ converges weakly in X to u , as $n \rightarrow \infty$.

We now prove that $u_n \rightarrow u$ in X , as $n \rightarrow \infty$. To this end, we take $v = \tilde{u} \in K_p$ in both (3.30) and (3.28), then we use equality $\tilde{u} = u$ to obtain

$$0 \leq \liminf \langle A_p u_n, u_n - u \rangle \leq \limsup \langle A_p u_n, u_n - u \rangle \leq 0,$$

which shows that $\langle A_p u_n, u_n - u \rangle \rightarrow 0$, as $n \rightarrow \infty$. Therefore, using the strong monotonicity of the operator A_p and the convergence $u_n \rightarrow u$ in X , we have

$$m_A \|u_n - u\|_X^2 \leq \langle A_p u_n - A_p u, u_n - u \rangle = \langle A_p u_n, u_n - u \rangle - \langle A_p u, u_n - u \rangle \rightarrow 0,$$

as $n \rightarrow \infty$. Hence, it follows that $u_n \rightarrow u$ in X , which concludes the proof of the theorem. \square

4 An optimal control problem

In this section we assume that the set of parameters Λ has a special structure that we describe in what follows. Let Q be a reflexive Banach space with the norm $\|\cdot\|_Q$, Θ a normed space endowed with the norm $\|\cdot\|_\Theta$ and denote by Z their product space, i.e., $Z = Q \times \Theta$. A generic element of Z will be denoted by $p = (q, \eta)$. We endow Z with the norm

$$\|p\|_Z = \|q\|_Q + \|\eta\|_\Theta \quad \forall p = (q, \eta) \in Z.$$

Let $U \subset Q$ and $\Sigma \subset \Theta$ be given nonempty subsets assumed to be weakly closed in Q and Θ , respectively, and let $\Lambda = U \times \Sigma$. Then, for each $q \in U$ and $\eta \in \Sigma$ we have $p = (q, \eta) \in \Lambda$ and, under the assumption of Theorem 6, we denote by $u = u(p) = u(q, \eta)$ the solution of Problem \mathcal{P} . Moreover, for each $\eta \in \Sigma$, let $F(\eta)$ denote a subset of U which depends on η . With these notation define the set of admissible pairs for Problem \mathcal{P} by equality

$$\mathcal{V}_{ad}(\eta) = \{(u, q) : q \in F(\eta), u = u(q, \eta)\}. \quad (4.1)$$

In other words, a pair (u, q) belongs to $\mathcal{V}_{ad}(\eta)$ if and only if $q \in F(\eta)$ and, moreover, u is the solution of Problem \mathcal{P} with $p = (q, \eta)$. Consider also a cost functional $\mathcal{L} : X \times U \rightarrow \mathbb{R}$. Then, the optimal control problem we are interested in is the following.

Problem Q. Given $\eta \in \Sigma$, find $(u^*, q^*) \in \mathcal{V}_{ad}(\eta)$ such that

$$\mathcal{L}(u^*, q^*) = \min_{(u, q) \in \mathcal{V}_{ad}(\eta)} \mathcal{L}(u, q). \quad (4.2)$$

To solve Problem Q we consider the following assumptions.

$$F(\eta) \text{ is a nonempty weakly closed subset of } Q. \quad (4.3)$$

$$\left\{ \begin{array}{l} \text{For all sequences } \{u_k\} \subset X \text{ and } \{q_k\} \subset U \text{ such that} \\ u_k \rightarrow u \text{ in } X, \quad q_k \rightharpoonup q \text{ in } Q, \text{ we have} \\ \text{(a) } \liminf_{k \rightarrow \infty} \mathcal{L}(u_k, q_k) \geq \mathcal{L}(u, q), \\ \text{(b) } \lim_{k \rightarrow \infty} [\mathcal{L}(u_k, q_k) - \mathcal{L}(u, q_k)] = 0. \end{array} \right. \quad (4.4)$$

$$\left\{ \begin{array}{l} \text{There exists } h : U \rightarrow \mathbb{R} \text{ such that} \\ \text{(a) } \mathcal{L}(u, q) \geq h(q) \quad \forall u \in X, \quad q \in U, \\ \text{(b) } \|q_k\|_Q \rightarrow +\infty \implies h(q_k) \rightarrow \infty. \end{array} \right. \quad (4.5)$$

$$U \text{ is a bounded subset of } Q. \quad (4.6)$$

EXAMPLE 10 A typical example of function \mathcal{L} which satisfies conditions (4.4) and (4.5) is obtained by taking

$$\mathcal{L}(u, p) = g(u) + h(q) \quad \forall u \in X, \quad q \in U,$$

where $g : X \rightarrow \mathbb{R}$ is a continuous positive function and $h : U \rightarrow \mathbb{R}$ is a weakly lower semicontinuous coercive function, i.e., it satisfies condition (4.5)(b).

Our first result in this section is the following existence result.

THEOREM 11 Assume (2.3)–(2.8), for any $p \in \Lambda$. Moreover, assume (2.9), (3.12), (4.3), (4.4) and, in addition, assume that either (4.5) or (4.6) hold. Then Problem Q has at least one solution (u^*, q^*) .

The proof will be carried out in several steps that we present in what follows. To this end, everywhere below we assume that the hypotheses of Theorem 11 hold and, given $\eta \in \Sigma$, we consider the function $J(\cdot, \eta) : U \rightarrow \mathbb{R}$ defined by

$$J(q, \eta) = \mathcal{L}(u(q, \eta), q) \quad \forall q \in U. \quad (4.7)$$

The first step of the proof is as follows.

LEMMA 12 For all sequences $\{q_k\} \subset U$ and $\{\eta_k\} \subset \Sigma$ such that $q_k \rightharpoonup q$ in Q , $\eta_k \rightharpoonup \eta$ in Θ and for all $s \in U$, the inequality below holds:

$$\limsup_{k \rightarrow \infty} [J(s, \eta_k) - J(q_k, \eta_k)] \leq J(s, \eta) - J(q, \eta). \quad (4.8)$$

Proof. Assume that $\{q_k\} \subset U$ and $\{\eta_k\} \subset \Lambda$ are two sequences such that $q_k \rightharpoonup q$ in Q , $\eta_k \rightharpoonup \eta$ in Θ and let $s \in U$. Then, using (4.7) we have

$$J(s, \eta_k) - J(q_k, \eta_k) = \mathcal{L}(u(s, \eta_k), s) - \mathcal{L}(u(q_k, \eta_k), q_k). \quad (4.9)$$

Moreover, assumption (3.12) implies that $u(s, \eta_k) \rightarrow u(s, \eta)$, $u(q_k, \eta_k) \rightarrow u(q, \eta)$, both in X . Therefore, assumption (4.4) and definition (4.7) imply that

$$\lim_{k \rightarrow \infty} \mathcal{L}(u(s, \eta_k), s) = \mathcal{L}(u(s, \eta), s) = J(s, \eta), \quad (4.10)$$

$$\limsup_{k \rightarrow \infty} [-\mathcal{L}(u(q_k, \eta_k), q_k)] \leq -\mathcal{L}(u(q, \eta), q) = -J(q, \eta). \quad (4.11)$$

We now pass to the upper limit in (4.9) and use relations (4.10), (4.11) to deduce that (4.8) holds. \square

We now consider the following auxiliary problem.

Problem \mathcal{R} . Given $\eta \in \Sigma$, find $q^* \in F(\eta)$ such that

$$J(q^*, \eta) = \min_{q \in F(\eta)} J(q, \eta). \quad (4.12)$$

The solvability of Problem \mathcal{R} is provided by the following result.

LEMMA 13 *Problem \mathcal{R} has at least one solution q^* .*

Proof. Let $\eta \in \Sigma$. We take $\eta_k = \eta$ in the statement of Lemma 12 to see that for all sequences $\{q_k\} \subset U$ such that $q_k \rightharpoonup q$ and for all $s \in U$ we have

$$\limsup_{k \rightarrow \infty} [J(s, \eta) - J(q_k, \eta)] \leq J(s, \eta) - J(q, \eta),$$

which implies that

$$\liminf_{k \rightarrow \infty} J(q_k, \eta) \geq J(q, \eta).$$

We conclude from here that the function $J(\cdot, \eta) : U \rightarrow \mathbb{R}$ is lower semicontinuous.

Assume now that (4.5) holds. Then, for any sequence $\{q_k\} \subset U$, we have

$$J(q_k, \eta) = \mathcal{L}(u(q_k, \eta), q_k) \geq h(q_k).$$

Therefore, if $\|q_k\|_Q \rightarrow \infty$ we deduce that $J(q_k, \eta) \rightarrow \infty$ which shows that the function $J(\cdot, \eta)$ is coercive. Recall also the assumption (4.3) and the reflexivity of the space Q . The existence of at least one solution to Problem \mathcal{R} is now a direct consequence of Theorem 5. On the other hand, if (4.6) holds we are still in position to apply Theorem 5. We deduce from here that, if either (4.5) or (4.6) hold, then Problem \mathcal{R} has at least a solution, which concludes the proof. \square

We are now in position to provide the proof of Theorem 11.

Proof. Using the definitions (4.7) and (4.1) it is easy to see that

$$\begin{cases} (u^*, q^*) \text{ is a solution of Problem } \mathcal{Q} \text{ if and only if} \\ q^* \text{ is a solution of Problem } \mathcal{R} \text{ and } u^* = u(q^*, \eta). \end{cases} \quad (4.13)$$

Theorem 11 is a direct consequence of the equivalence (4.13) and Lemma 13. \square

Let $\eta \in \Sigma$ and, for each $n \in \mathbb{N}$, consider a perturbation $\eta_n \in \Sigma$ of η together with the set of admissible pairs defined by

$$\mathcal{V}_{ad}(\eta_n) = \{ (u, q) : q \in F(\eta_n), u = u(q, \eta_n) \}. \quad (4.14)$$

With these data we consider the following perturbation of Problem \mathcal{Q} .

Problem \mathcal{Q}_n . Given $\eta_n \in \Sigma$, find $(u_n^*, q_n^*) \in \mathcal{V}_{ad}(\eta_n)$ such that

$$\mathcal{L}(u_n^*, q_n^*) = \min_{(u, q) \in \mathcal{V}_{ad}(\eta_n)} \mathcal{L}(u, q). \quad (4.15)$$

We also consider the following auxiliary problem.

Problem \mathcal{R}_n . Given $\eta_n \in \Sigma$, find $q_n^* \in F(\eta_n)$ such that

$$J(q_n^*, \eta_n) = \min_{q \in F(\eta_n)} J(q, \eta_n). \quad (4.16)$$

The proof of Lemma 13 shows that Problem \mathcal{R}_n has at least one solution q_n^* , for each $n \in \mathbb{N}$. Moreover, using (4.7) and definition (4.14) it is easy to see that

$$\begin{cases} (u_n^*, q_n^*) \text{ is a solution of Problem } \mathcal{Q}_n \text{ if and only if} \\ q_n^* \text{ is a solution of Problem } \mathcal{R}_n \text{ and } u_n^* = u(q_n^*, \eta_n). \end{cases} \quad (4.17)$$

This implies that for each $n \in \mathbb{N}$, Problem \mathcal{Q}_n has at least one solution (u_n^*, q_n^*) .

In order to study the link between the solutions to Problems \mathcal{Q}_n and Problem \mathcal{Q} we consider the following assumptions.

$$\eta_n \rightharpoonup \eta \quad \text{in } \Theta. \quad (4.18)$$

$$F(\eta_n) \xrightarrow{M} F(\eta) \quad \text{in } Q. \quad (4.19)$$

$$\begin{cases} \text{For all sequences } \{u_k\} \subset X \text{ and } \{q_k\} \subset U \text{ such that} \\ u_k \rightarrow u \text{ in } X, \quad q_k \rightharpoonup q \text{ in } Q, \text{ we have} \\ \lim_{k \rightarrow \infty} \mathcal{L}(u_k, q_k) = \mathcal{L}(u, q). \end{cases} \quad (4.20)$$

Then, we have the following convergence result.

THEOREM 14 Assume (2.3)–(2.8), for any $p \in \Lambda$. Moreover, assume (2.9), (3.12), (4.3), (4.4) and, in addition, assume that either (4.5) or (4.6) holds. For each $n \in \mathbb{N}$, let (u_n^*, q_n^*) be a solution of Problem \mathcal{Q}_n . Then, if (4.18)–(4.20) hold, there exists a subsequence of the sequence $\{(u_n^*, q_n^*)\}$, again denoted by $\{(u_n^*, q_n^*)\}$, and an element $(u^*, q^*) \in X \times Q$, such that

$$u_n^* \rightarrow u^* \quad \text{in } X, \quad (4.21)$$

$$q_n^* \rightharpoonup q^* \quad \text{in } Q. \quad (4.22)$$

Moreover, (u^*, q^*) is a solution of Problem \mathcal{Q} .

The proof of Theorem 14 will be carried out in several steps that we present in what follows. To this end, everywhere below we assume that the hypotheses of Theorem 14 are satisfied. The first step of the proof concerns the function (4.7) and is as follows.

LEMMA 15 (i) *For all sequence $\{\eta_k\} \subset \Sigma$ such that $\eta_k \rightarrow \eta$ in Θ and for all $q \in U$, the equality below holds:*

$$\lim_{k \rightarrow \infty} J(q, \eta_k) = J(q, \eta). \quad (4.23)$$

(ii) *For all sequences $\{q_k\} \subset U$ and $\{\eta_k\} \subset \Sigma$ such that $q_k \rightarrow q$ in Q , $\eta_k \rightarrow \eta$ in Θ the equality below holds:*

$$\lim_{k \rightarrow \infty} [J(q_k, \eta_k) - J(q, \eta_k)] = 0. \quad (4.24)$$

Proof. (i) Let $\{\eta_k\} \subset \Sigma$ be a sequence such that $\eta_k \rightarrow \eta$ in Θ and let $q \in U$. We use the definition (4.7) to see that

$$J(q, \eta_k) - J(q, \eta) = \mathcal{L}(u(q, \eta_k), q) - \mathcal{L}(u(q, \eta), q). \quad (4.25)$$

On the other hand, using the convergence $u(q, \eta_k) \rightarrow u(q, \eta)$ in X , guaranteed by assumption (3.12), we deduce from (4.20) that

$$\lim_{k \rightarrow \infty} [\mathcal{L}(u(q, \eta_k), q) - \mathcal{L}(u(q, \eta), q)] = 0. \quad (4.26)$$

We now combine relations (4.25) and (4.26) to see that the equality (4.23) holds.

(ii) Assume now that $\{q_k\} \subset U$ and $\{\eta_k\} \subset \Sigma$ are two sequences such that $q_k \rightarrow q$ in Q and $\eta_k \rightarrow \eta$ in Θ . We write

$$\begin{aligned} J(q_k, \eta_k) - J(q, \eta_k) &= \mathcal{L}(u(q_k, \eta_k), q_k) - \mathcal{L}(u(q, \eta_k), q) \\ &= \mathcal{L}(u(q_k, \eta_k), q_k) - \mathcal{L}(u(q, \eta), q_k) \\ &\quad + \mathcal{L}(u(q, \eta), q_k) - \mathcal{L}(u(q, \eta), q) \\ &\quad + \mathcal{L}(u(q, \eta), q) - \mathcal{L}(u(q, \eta_k), q) \end{aligned} \quad (4.27)$$

and, since $u(q_k, \eta_k) \rightarrow u(q, \eta)$, $u(q, \eta_k) \rightarrow u(q, \eta)$, both in X , assumptions (4.4)(b) and (4.20) imply that

$$\begin{aligned} \mathcal{L}(u(q_k, \eta_k), q_k) - \mathcal{L}(u(q, \eta), q_k) &\rightarrow 0, \\ \mathcal{L}(u(q, \eta), q_k) - \mathcal{L}(u(q, \eta), q) &\rightarrow 0, \\ \mathcal{L}(u(q, \eta), q) - \mathcal{L}(u(q, \eta_k), q) &\rightarrow 0. \end{aligned}$$

We now use these convergences in equality (4.27) to deduce that (4.24) holds, which concludes the proof \square

In the next step we prove the following convergence result concerning Problems \mathcal{R}_n and \mathcal{R} .

LEMMA 16 For each $n \in \mathbb{N}$, let q_n^* be a solution of Problem \mathcal{R}_n . Then, there exists a subsequence of the sequence $\{q_n^*\}$, again denoted by $\{q_n^*\}$, and an element $q^* \in Q$, such that (4.22) holds. Moreover, q^* is a solution of Problem \mathcal{R} .

Proof. We claim that the sequence $\{q_n^*\}$ is bounded in Q . This claim is obviously satisfied if we assume that (4.6) holds. Assume in what follows that (4.5) holds. If $\{q_n^*\}$ is not bounded in Q , then we can find a subsequence of the sequence $\{q_n^*\}$, again denoted by $\{q_n^*\}$, such that $\|q_n^*\|_Q \rightarrow \infty$. Therefore, using definition (4.7) and condition (4.5) we deduce that

$$J(q_n^*, \eta_n) = \mathcal{L}(u(q_n^*, \eta_n), q_n^*) \geq h(q_n^*) \rightarrow \infty,$$

which implies that

$$J(q_n^*, \eta_n) \rightarrow \infty. \quad (4.28)$$

Let s be a given element in $F(\eta)$ and note that assumption (4.19) and condition (M_1) in Definition 4 imply that there exists a sequence $\{s_n\}$ such that $s_n \in F(\eta_n)$ for each $n \in \mathbb{N}$ and

$$s_n \rightarrow s \quad \text{in } Q. \quad (4.29)$$

Moreover, since q_n^* is a solution of Problem \mathcal{R}_n we have $J(q_n^*, \eta_n) \leq J(s_n, \eta_n)$ and, therefore,

$$J(q_n^*, \eta_n) \leq [J(s_n, \eta_n) - J(s, \eta_n)] + [J(s, \eta_n) - J(s, \eta)] + J(s, \eta) \quad \forall n \in \mathbb{N}. \quad (4.30)$$

On the other hand, the convergences (4.29) and (4.18) allow us to use equality (4.24) in Lemma 15(ii) to find that $J(s_n, \eta_n) - J(s, \eta_n) \rightarrow 0$ and, in addition, equality (4.23) in Lemma 15(i) shows that $J(s, \eta_n) - J(s, \eta) \rightarrow 0$. Thus, (4.30) implies that the sequence $\{J(q_n^*, \eta_n)\}$ is bounded, which contradicts (4.28). We conclude from above that the sequence $\{q_n^*\}$ is bounded in Q and, therefore, there exists a subsequence of the sequence $\{q_n^*\}$, again denoted by $\{q_n^*\}$, and an element $q^* \in Q$, such that (4.22) holds.

We now prove that q^* is a solution of Problem \mathcal{R} . To this end we recall that $q_n^* \in F(\eta_n)$, for all $n \in \mathbb{N}$. Therefore, using (4.22) and condition (M_2) in Definition 4, guaranteed by assumption (4.19), we deduce that $q^* \in F(\eta)$. Next, we consider an arbitrary element $s \in F(\eta)$ and, using condition (M_1) , we know that there exists a sequence $\{s_n\}$ such that $s_n \in F(\eta_n)$ for each $n \in \mathbb{N}$ and (4.29) holds. Since q_n^* is the solution to Problem \mathcal{R}_n we have $J(q_n^*, \eta_n) \leq J(s_n, \eta_n)$ which implies that

$$0 \leq [J(s, \eta_n) - J(q_n^*, \eta_n)] + [J(s_n, \eta_n) - J(s, \eta_n)]. \quad (4.31)$$

We now use the convergences (4.18), (4.22), (4.29), Lemma 12 and Lemma 15(ii) to see that

$$\limsup_{n \rightarrow \infty} [J(s, \eta_n) - J(q_n^*, \eta_n)] \leq J(s, \eta) - J(q^*, \eta), \quad (4.32)$$

$$\lim_{n \rightarrow \infty} [J(s_n, \eta_n) - J(s, \eta_n)] = 0, \quad (4.33)$$

respectively. We now pass to the upper limit (4.31) and use (4.32), (4.33) to deduce that

$$0 \leq J(s, \eta) - J(q^*, \eta)$$

and, since $q^* \in F(\eta)$, we deduce that q^* is a solution of Problem \mathcal{R} , which concludes the proof. \square

We now have all the ingredients to provide the proof of Theorem 14.

Proof. We first use (4.17) to see that, for each $n \in \mathbb{N}$, q_n^* is a solution to Problem \mathcal{R}_n . Then, we use Lemma 16 to see that there exists a subsequence of the sequence $\{q_n^*\}$, again denoted by $\{q_n^*\}$, and an element $q^* \in Q$, such that (4.22) holds. Moreover, q^* is a solution of Problem \mathcal{R} . Let $u^* = u(q^*, \eta)$. Then, equivalence (4.13) shows that (u^*, q^*) is a solution of Problem \mathcal{Q} . On the other hand, (4.17) implies that $u_n^* = u(q_n^*, \eta_n)$ and the convergences (4.22), (4.18) show that $p_n^* = (q_n^*, \eta_n) \rightharpoonup p^* = (q^*, \eta)$ in Z . It follows now from assumption (3.12) that (4.21) holds, which concludes the proof. \square

5 An elastic contact problem

The abstract results in Sections 3–4 are useful in the study of various mathematical models which describe the equilibrium of elastic bodies in frictional contact with a foundation. In this section we provide an example of such model and, to this end, we need some notations and preliminaries. For details on the material below we refer the reader to [11, 27], for instance.

Let $d \in \{2, 3\}$. We denote by \mathbb{S}^d the space of second order symmetric tensors on \mathbb{R}^d or, equivalently, the space of symmetric matrices of order d . We recall that inner product and norm on \mathbb{R}^d and \mathbb{S}^d are defined by

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i, & \|\mathbf{v}\| &= (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}} & \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\tau} &= \sigma_{ij} \tau_{ij}, & \|\boldsymbol{\tau}\| &= (\boldsymbol{\tau} \cdot \boldsymbol{\tau})^{\frac{1}{2}} & \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, \end{aligned}$$

where the indices i, j run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. The zero element of the spaces \mathbb{R}^d and \mathbb{S}^d will be denoted by $\mathbf{0}$.

Let $\Omega \subset \mathbb{R}^d$ be a domain with smooth boundary Γ . The boundary Γ is divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $meas(\Gamma_1) > 0$. A generic point in $\Omega \cup \Gamma$ will be denoted by $\mathbf{x} = (x_i)$. We use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ . In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$, $L^2(\Gamma_3)$ and $H^1(\Omega)^d$, endowed with their canonical inner products and associated norms. Moreover, for an element $\mathbf{v} \in H^1(\Omega)^d$ we still write \mathbf{v} for the trace of \mathbf{v} to Γ . In addition, we consider the space

$$V = \{ \mathbf{v} \in H^1(\Omega)^d : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1 \},$$

which is a real Hilbert space endowed with the canonical inner product

$$(\mathbf{u}, \mathbf{v})_V = \int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \tag{5.1}$$

and the associated norm $\|\cdot\|_V$. Here and below $\boldsymbol{\varepsilon}$ represents the deformation operator, i.e.,

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} (u_{i,j} + u_{j,i}),$$

where an index that follows a comma denotes the partial derivative with respect to the corresponding component of \mathbf{x} , e.g., $u_{i,j} = \frac{\partial u_i}{\partial x_j}$. The completeness of the space V follows from the assumption $meas(\Gamma_1) > 0$ which allows the use of Korn's inequality. We denote by $\mathbf{0}_V$ the zero element of V and we recall that, for an element $\mathbf{v} \in V$, the normal and tangential components on Γ are given by $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$, respectively. Finally, V^* represents the dual of V and $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V^* and V . We also denote by $\|\gamma\|$ the norm of the trace operator $\gamma : V \rightarrow L^2(\Gamma_3)^d$ and we recall the inequality

$$\|\mathbf{v}\|_{L^2(\Gamma)^d} \leq \|\gamma\| \|\mathbf{v}\|_V \quad \forall \mathbf{v} \in V. \quad (5.2)$$

In addition, we use the space $Y = L^2(\Omega)^d \times L^2(\Gamma_3)^d$ equipped with the canonical product topology and the operator $\pi : V \rightarrow Y$ defined by

$$\pi \mathbf{v} = (\mathbf{v}, \gamma \mathbf{v}) \quad \forall \mathbf{v} \in V. \quad (5.3)$$

The contact model we consider in this section is constructed by using a function \mathcal{F} and a set B which satisfy the following conditions.

$$\left\{ \begin{array}{l} \mathcal{F} : \Omega \times \mathbb{S}^d \rightarrow \mathbb{S}^d \text{ is such that} \\ \text{(a) there exists } L_{\mathcal{F}} > 0 \text{ such that} \\ \quad \|\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega, \\ \text{(b) there exists } m_{\mathcal{F}} > 0 \text{ such that} \\ \quad (\mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_1) - \mathcal{F}(\mathbf{x}, \boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{F}} \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \quad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \mathbf{x} \in \Omega, \\ \text{(c) } \mathcal{F}(\cdot, \boldsymbol{\varepsilon}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \text{(d) } \mathcal{F}(\mathbf{x}, \mathbf{0}) = \mathbf{0} \text{ for a.e. } \mathbf{x} \in \Omega. \end{array} \right. \quad (5.4)$$

$$m_{\mathcal{F}} > \|\gamma\|^2. \quad (5.5)$$

$$B \text{ is a closed convex subset of } \mathbb{S}^d \text{ such that } \mathbf{0} \in B. \quad (5.6)$$

Note that assumption (5.5) allows us to find a constant $\tilde{m}_0 \in \mathbb{R}$ such that

$$m_{\mathcal{F}} - \|\gamma\|^2 > \tilde{m}_0 > 0. \quad (5.7)$$

Moreover, we use (5.6) and denote in what follows by $P_B : \mathbb{S}^d \rightarrow B$ the projection operator on the set B .

To complete the model we consider the data ω , μ , \mathbf{f}_0 , \mathbf{f}_2 , g and ρ assumed to satisfy the following conditions.

$$\omega \in L^\infty(\Omega), \quad \omega(\mathbf{x}) \geq 0 \text{ for a.e. } \mathbf{x} \in \Omega. \quad (5.8)$$

$$\mu \in L^\infty(\Gamma_3), \quad \mu(\mathbf{x}) \geq 0 \text{ for a.e. } \mathbf{x} \in \Gamma_3, \quad \|\mu\|_{L^\infty(\Gamma_3)} \|\gamma\|^2 \leq \tilde{m}_0. \quad (5.9)$$

$$\mathbf{f}_0 \in L^2(\Omega)^d, \quad \mathbf{f}_2 \in L^2(\Gamma_2)^d. \quad (5.10)$$

$$g \geq 0, \quad \rho > 0. \quad (5.11)$$

Let Λ be the set of elements $p = (\omega, \mu, \mathbf{f}_0, \mathbf{f}_2, g, \rho)$ such that (5.8)–(5.11) hold. Note that Λ is a subset of the product space $L^\infty(\Omega) \times L^\infty(\Gamma_3) \times L^2(\Omega) \times L^2(\Gamma_2) \times \mathbb{R}^2$ and, moreover, inequality $\tilde{m}_0 > 0$ in (5.7) guarantees that Λ is not empty. Next, for any $p = (\omega, \mu, \mathbf{f}_0, \mathbf{f}_2, g, \rho) \in \Lambda$ we define the set K_p , the operator A_p , the functions φ_p, j_p and the element \mathbf{f}_p , by equalities

$$K_p = \{ \mathbf{v} \in V \mid v_\nu \leq g \text{ a.e. on } \Gamma_3 \}, \quad (5.12)$$

$$\begin{aligned} A_p: V \rightarrow V^*, \quad \langle A_p \mathbf{u}, \mathbf{v} \rangle &= \int_\Omega \mathcal{F} \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx \\ &+ \int_\Omega \omega(\boldsymbol{\varepsilon}(\mathbf{u}) - P_B \boldsymbol{\varepsilon}(\mathbf{u})) \cdot \boldsymbol{\varepsilon}(\mathbf{v}) \, dx, \end{aligned} \quad (5.13)$$

$$\varphi_p: V \times V \rightarrow \mathbb{R}, \quad \varphi_p(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \mu u_\nu^+ \|\mathbf{v}_\tau\| \, da, \quad (5.14)$$

$$j_p: V \rightarrow \mathbb{R}, \quad j_p(\mathbf{v}) = \int_{\Gamma_3} j_\rho(v_\nu) \, da, \quad (5.15)$$

$$\mathbf{f}_p \in Y, \quad (\mathbf{f}_p, \boldsymbol{\pi} \mathbf{v})_Y = \int_\Omega \mathbf{f}_0 \cdot \mathbf{v} \, dx + \int_{\Gamma_3} \mathbf{f}_2 \cdot \boldsymbol{\gamma} \mathbf{v} \, da, \quad (5.16)$$

for all $\mathbf{u}, \mathbf{v} \in V$. Here and below r^+ represents the positive part of r , i.e., $r^+ = \max\{r, 0\}$. Moreover, $j_\rho: \mathbb{R} \rightarrow \mathbb{R}$ is the function defined by

$$j_\rho(r) = \int_0^r k_\rho(s) \, ds \quad \forall r \in \mathbb{R}, \quad (5.17)$$

where

$$k_\rho(r) = \begin{cases} 0 & \text{if } r < 0, \\ r & \text{if } 0 \leq r < \rho, \\ 2\rho - r & \text{if } \rho \leq r < 2\rho, \\ r - 2\rho & \text{if } r \geq 2\rho. \end{cases} \quad (5.18)$$

It is easy to see that the function $k_\rho: \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous, yet not monotone. As a result the function j_ρ is not convex.

With these notation, for $p \in \Lambda$ given, we consider the following problem.

Problem \mathcal{S} . Find a displacement field $\mathbf{u} = \mathbf{u}(p)$ such that

$$\begin{aligned} \mathbf{u} \in K_p, \quad & \langle A_p \mathbf{u}, \mathbf{v} - \mathbf{u} \rangle + \varphi_p(\mathbf{u}, \mathbf{v}) - \varphi_p(\mathbf{u}, \mathbf{u}) + j_p^0(\mathbf{u}; \mathbf{v} - \mathbf{u}) \quad (5.19) \\ & \geq (\mathbf{f}_p, \pi \mathbf{v} - \pi \mathbf{u})_Y \quad \text{for all } \mathbf{v} \in V. \end{aligned}$$

Following the arguments in [28], it can be shown that Problem \mathcal{S} represents the variational formulation of a mathematical model which describes the equilibrium of an elastic body in frictional contact with a foundation, under the action of external forces. It is assumed that the foundation is made of a rigid material covered by a layer of deformable material. The data \mathcal{F} , ω and B are related to the constitutive law, while \mathbf{f}_0 and \mathbf{f}_2 denote the density of body forces and applied tractions which act on the body and the surface Γ_2 , respectively. In addition, μ represents the coefficient of friction and ρ is a given stiffness coefficient. Finally, g represents the thickness of the deformable layer.

Contact models which lead to inequality problems of the form (5.19) have been considered in the books [19, 28]. There, the reader can find the classical formulation of the models, including the mechanical assumptions which lead to their construction, as well as their variational analysis, based on arguments similar to that we briefly resume below in this section.

Our first result in the study of Problem \mathcal{S} is the following.

THEOREM 17 *Assume (5.4)–(5.6). Then, for each $p = (\omega, \mu, \mathbf{f}_0, \mathbf{f}_2, g, \rho) \in \Lambda$ there exists a unique solution $\mathbf{u} = \mathbf{u}(p)$ to the Problem \mathcal{S} . Moreover, if the sequence $\{p_n\}$ with $p_n = (\omega_n, \mu_n, \rho_n, g_n, \mathbf{f}_{0n}, \mathbf{f}_{2n}) \in \Lambda$ is such that*

$$\omega_n \rightarrow \omega \quad \text{in } L^\infty(\Omega), \quad (5.20)$$

$$\mu_n \rightarrow \mu \quad \text{in } L^\infty(\Gamma_3), \quad (5.21)$$

$$\mathbf{f}_{0n} \rightharpoonup \mathbf{f}_0 \quad \text{in } L^2(\Omega)^d, \quad \mathbf{f}_{2n} \rightharpoonup \mathbf{f}_2 \quad \text{in } L^2(\Gamma_2)^d, \quad (5.22)$$

$$g_n \rightarrow g, \quad (5.23)$$

$$\rho_n \rightarrow \rho, \quad (5.24)$$

then $\mathbf{u}(p_n) \rightarrow \mathbf{u}(p)$ in V .

Proof. Let $p = (\omega, \mu, \mathbf{f}_0, \mathbf{f}_2, g, \rho) \in \Lambda$. For the existence and uniqueness part we apply Theorem 6 on the space $X = V$. To this end, we remark that, obviously, condition (2.3) is satisfied. Moreover, the operator A_p defined by (5.13) satisfies condition (2.4). Indeed, for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, by assumption (5.4)(a) and the properties of the projection operator P_B , we have

$$\langle A_p \mathbf{u} - A_p \mathbf{v}, \mathbf{w} \rangle_{V^* \times V} \leq (L_{\mathcal{F}} + 2\|\omega\|_{L^\infty(\Omega)}) \|\mathbf{u} - \mathbf{v}\|_V \|\mathbf{w}\|_V.$$

This proves that

$$\|A_p \mathbf{u} - A_p \mathbf{v}\|_{V^*} \leq (L_{\mathcal{F}} + 2\|\omega\|_{L^\infty(\Omega)}) \|\mathbf{u} - \mathbf{v}\|_V,$$

for all $\mathbf{u}, \mathbf{v} \in V$, which implies that A_p is Lipschitz continuous. On the other hand, using assumption (5.4)(b) and the nonexpansivity of the projector operator yields

$$\langle A_p \mathbf{u} - A_p \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle_{V^* \times V} \geq m_{\mathcal{F}} \|\mathbf{u} - \mathbf{v}\|_V^2,$$

for all $\mathbf{u}, \mathbf{v} \in V$. This shows that condition (2.4)(b) is satisfied with $m_p = m_{\mathcal{F}}$. Since A_p is Lipschitz continuous and monotone, it follows that A_p is pseudomonotone and, therefore, (2.4)(a) holds.

Next, for φ_p defined by (5.14), we use the trace inequality (5.2) to see that (2.5) holds with $\alpha_p = \|\mu\|_{L^\infty(\Gamma_3)} \|\gamma\|^2$.

On the other hand, since $j'_\rho(r) = k_\rho(r)$ for all $r \in \mathbb{R}$, it follows that j_ρ is a C^1 function and, therefore, is a locally Lipschitz function. Moreover, it is easy to see that $|k_\rho(r)| \leq 2\rho + |r|$ for $r \in \mathbb{R}$ and the function $r \mapsto k_\rho(r) + r$ is nondecreasing. We use these properties and equality $j_\rho^0(r; s) = k_\rho(r)s$, valid for all $r, s \in \mathbb{R}$, to see that the function j_ρ satisfies condition (2.6) on $X = \mathbb{R}$ with $c_{0\rho} = 2\rho$, $c_{1\rho} = 1$ and $\beta_\rho = 1$. Therefore, following the arguments in [28, p.219] we deduce that the function j_p given by (5.15) satisfies (2.6) with $c_{0p} = 2\rho\sqrt{2\text{meas}(\Gamma_3)} \|\gamma\|$, $c_{1p} = \sqrt{2} \|\gamma\|^2$ and $\beta_p = \|\gamma\|^2$.

It follows from above that $\alpha_p + \beta_p = (\|\mu\|_{L^\infty(\Gamma_3)} + 1)\|\gamma\|^2$ and, using assumption (5.9) and inequality (5.7), we deduce that $\alpha_p + \beta_p < m_{\mathcal{F}} = m_p$ which shows that the smallness condition (2.7) is satisfied. We also note that conditions (2.8) and (2.9) are obviously satisfied.

Therefore, we are in position to use Theorem 6. In this way we deduce that there exists a unique element $\mathbf{u} \in V$ such that (5.19) holds, which concludes the existence and uniqueness part of the theorem.

For the continuous dependence part we use Theorem 7. Assume that (5.20)–(5.24) hold. We start with the remark that (5.23) implies (3.2). Moreover, using the convergence (5.20) it is easy to see that condition (3.3) is satisfied with $F_n = 2 \|\omega_n - \omega\|_{L^\infty(\Omega)}$ and $\delta_n = 0$. Next, (3.4)(a) holds with $c_n(r) = \|\mu_n\|_{L^\infty(\Gamma_3)} \|\gamma\|^2 r$ and, using convergence (5.21) combined with the compactness of the trace, we see that (3.4)(b) and (3.5) hold, too.

On the other hand, since the function j_ρ is regular, Lemma 8 in [28] guarantees that

$$j_p^0(\mathbf{u}; \mathbf{v}) = \int_{\Gamma_3} j_\rho^0(u_\nu; v_\nu) da = \int_{\Gamma_3} k_\rho(u_\nu) v_\nu da \quad \forall \mathbf{u}, \mathbf{v} \in V, p \in \Lambda. \quad (5.25)$$

Moreover, an elementary argument shows that

$$|k_{\rho_n}(r) - k_\rho(r)| \leq 2|\rho_n - \rho| \quad \forall r \in \mathbb{R}. \quad (5.26)$$

Therefore, if $\mathbf{u}_n \rightharpoonup \mathbf{u}$ and $\mathbf{v}_n \rightarrow \mathbf{v}$ in V , using (5.25) and (5.26) we deduce that

$$\begin{aligned} j_{p_n}^0(\mathbf{u}_n; \mathbf{v}_n - \mathbf{u}_n) &= \int_{\Gamma_3} k_{\rho_n}(u_{n\nu})(v_{n\nu} - u_{n\nu}) da \\ &\leq 2|\rho_n - \rho| \int_{\Gamma_3} |v_{n\nu} - u_{n\nu}| da + \int_{\Gamma_3} k_\rho(u_{n\nu})(v_{n\nu} - u_{n\nu}) da. \end{aligned}$$

Then, the convergence (5.24), the compactness of the trace and the properties of the function k_ρ imply that

$$\limsup j_{p_n}^0(\mathbf{u}_n; \mathbf{v}_n - \mathbf{u}_n) \leq \int_{\Gamma_3} k_\rho(u_\nu)(v_\nu - u_\nu) da = j_p^0(\mathbf{u}; \mathbf{v} - \mathbf{u})$$

which shows that condition (3.6) holds.

Next, by standard compactness arguments it follows that the operator (5.3) satisfies condition (3.7). Moreover, the convergences (5.22) show that (3.8) holds, too. Finally, we recall that $\alpha_{p_n} = \|\mu_n\|_{L^\infty(\Gamma_3)}\|\gamma\|^2$, $c_{0p_n} = 2\rho_n\sqrt{2\text{meas}(\Gamma_3)}\|\gamma\|$, $c_{1p_n} = \sqrt{2}\|\gamma\|^2$, $\beta_{p_n} = \|\gamma\|^2$ and $m_{p_n} = m_{\mathcal{F}}$. Therefore, using (5.9) and (5.7) we deduce that condition (3.9) is satisfied with $m_0 = m_{\mathcal{F}} - \tilde{m}_0 - \|\gamma\|^2$ and, obviously, (3.10) holds.

It follows from above that we are in position to use Theorem 7 in order to deduce that $\mathbf{u}(p_n) \rightarrow \mathbf{u}(p)$ in V , which concludes the proof. \square

We now associate to Problem \mathcal{S} an optimal control problem, in the framework described in Section 4. To this end we consider the space $Z = Q \times \Theta$, where $Q = \mathbb{R} \times L^2(\Gamma_2)^d$ and $\Theta = \mathbb{R}^3 \times L^2(\Omega)^d$. The spaces Q and Θ are equipped with their canonical product norms, denoted by $\|\cdot\|_Q$ and $\|\cdot\|_\Theta$, respectively. We endow Z with the norm

$$\|p\|_Z = \|q\|_Q + \|\eta\|_\Theta \quad \forall p = (q, \eta) \in Z.$$

A generic element of Z will be denoted by $p = (q, \eta)$ where $q = (g, \mathbf{f}_2) \in Q$ and $\eta = (\omega, \mu, \rho, \mathbf{f}_0) \in \Theta$.

Let $U \subset Q$, $\Sigma \subset \Theta$, $\Lambda \subset Z$ be the sets given by

$$\begin{aligned} U &= \{q = (g, \mathbf{f}_2) \in Q : 0 \leq g \leq g_0, \|\mathbf{f}_2\|_{L^2(\Gamma_2)^d} \leq h_0\}, \\ \Sigma &= \{\eta = (\omega, \mu, \rho, \mathbf{f}_0) \in \Theta : \omega, \mu \geq 0, \rho > \rho_0, \mu\|\gamma\|^2 \leq \tilde{m}_0\}, \\ \Lambda &= U \times \Sigma, \end{aligned}$$

where $g_0 > 0$, $h_0 > 0$ and $\rho_0 > 0$ are given, $\rho_0 < g_0$. Moreover, for each $\eta = (\omega, \mu, \rho, \mathbf{f}_0) \in \Sigma$, let $F(\eta) \subset U$ be the subset given by

$$F(\eta) = \{q = (g, \mathbf{f}_2) \in U : \rho \leq g \leq g_0\}. \quad (5.27)$$

Then, under the assumptions (5.4)–(5.6), it follows from Theorem 17 that for each $p \in \Lambda$ Problem \mathcal{S} has a unique solution $\mathbf{u} = \mathbf{u}(p) = \mathbf{u}(q, \eta)$. For each $\eta \in \Sigma$ we define the set of admissible pairs for Problem \mathcal{S} by equality

$$\mathcal{V}_{ad}(\eta) = \{(\mathbf{u}, q) : q \in F(\eta), \mathbf{u} = \mathbf{u}(q, \eta)\}.$$

Consider also the cost functional $\mathcal{L} : V \times U \rightarrow \mathbb{R}$ given by

$$\mathcal{L}(\mathbf{u}, q) = \int_{\Gamma_3} (u_\nu - \phi)^2 da \quad \forall \mathbf{u} \in V, \quad q \in U, \quad (5.28)$$

where $\phi \in L^2(\Gamma_3)$ is given. Then, the optimal control problem we are interested in is the following.

Problem \mathcal{T} . Given $\eta = (\omega, \mu, \rho, \mathbf{f}_0) \in \Sigma$, find $(\mathbf{u}^*, q^*) \in \mathcal{V}_{ad}(\eta)$ such that

$$\mathcal{L}(\mathbf{u}^*, q^*) = \min_{(u, q) \in \mathcal{V}_{ad}(\eta)} \mathcal{L}(u, q).$$

With this choice, the mechanical interpretation of Problem \mathcal{T} is the following: given a contact process described by the variational-hemivariational inequality (5.19) with the data $q = (g, \mathbf{f}_2) \in U$ and $\eta = (\omega, \mu, \rho, \mathbf{f}_0) \in \Sigma$, we are looking for a thickness $g \in [\rho, g_0]$ and a density of surface tractions $\mathbf{f}_2 \in L^2(\Gamma_2)$ with $\|\mathbf{f}_2\|_{L^2(\Gamma_2)^d} \leq h_0$ such that the normal component of the corresponding solution is as close as possible, on Γ_3 , to the “desired normal displacement” ϕ .

Note that, in this case assumptions (3.12), (4.3), (4.4) and (4.6) are satisfied. Therefore Theorem 11 guarantees the existence of the solutions of the optimal control problem \mathcal{T} . Moreover, note that assumption (4.18) implies the convergence (4.19) and, in addition, (4.20) holds. Therefore, the convergence result stated in Theorem 14 can be applied in the study of Problem \mathcal{T} .

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