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A treatment of breakdowns and near breakdowns in a reduction of a matrix to upper J -Hessenberg form and related topics

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Abstract

The reduction of a matrix to an upper J -Hessenberg form is a crucial step in the SR -algorithm (which is a QR -like algorithm), structure-preserving, for computing eigenvalues and vectors, for a class of structured matrices. This reduction may be handled via the algorithm JHESS or via the recent algorithm JHMSH and its variants.

The main drawback of JHESS (or JHMSH) is that it may suffer from a fatal breakdown, causing a brutal stop of the computations and hence, the SR -algorithm does not run. JHESS may also encounter near-breakdowns, source of serious numerical instability.

In this paper, we focus on these aspects. We first bring light on the necessary and sufficient condition for the existence of the SR -decomposition, which is intimately linked to J -Hessenberg reduction. Then we will derive a strategy for curing fatal breakdowns and also for treating near breakdowns. Hence, the J -Hessenberg form may be obtained. Numerical experiments are given, demonstrating the efficiency of our strategies to cure and treat breakdowns or near breakdowns.

Keywords: SR decomposition, symplectic Householder transformations, upper J -Hessenberg form, breakdowns and near-breakdowns, SR -algorithm.

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1. Introduction

Let A be a $2n \times 2n$ real matrix. The SR factorization consists in writing A as a product SR , where S is symplectic and $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$ is such that R_{11} , R_{12} , R_{22} are upper triangular and R_{21} is strictly upper triangular [3, 4]. The factor R is called J -triangular. This decomposition plays an important role in structure-preserving methods for solving the eigenproblem of a class of structured matrices.

More precisely, the SR decomposition can be interpreted as the analog of the QR decomposition [6], when instead of an Euclidean space, one considers a symplectic space : a linear space, equipped with a skew-symmetric inner product (see for example [8] and the references therein). The orthogonal group with respect to this indefinite inner product, is called the symplectic group and is unbounded (contrasting with the Euclidean case).

In the literature, the SR decomposition is carried out, via the algorithm SRDECO, derived in [2]. SRDECO is based in the use of two kind of both symplectic and orthogonal transformations introduced in [7, 13] and a third symplectic but non-orthogonal transformations, proposed in [2]. In fact, in [3], it has been shown that SR decomposition of a general matrix can not be performed by employing only the above orthogonal and symplectic transformations.

We mention that the above transformations involved in SRDECO algorithm are not elementary rank-one modification of the identity (transvections), see [1, 6].

Recently in [9], an algorithm, SRSB, based on symplectic transformations which are rank-one modification of the identity is derived, for computing the SR decomposition. These transformations are called symplectic Householder transformations. The new algorithm SRSB involves free parameters and advantages may be taken from this fact. An optimal version of SRSB, called SROSB is given in [10]. Error analysis and computational aspects of this algorithm have been studied [11].

In order to build a SR -algorithm (which is a QR -like algorithm) for computing the eigenvalues and eigenvectors of a matrix [14], a reduction of the matrix to an upper J -Hessenberg form is needed and is crucial.

In [2], the algorithm JHESS, for reducing a general matrix to an upper J -Hessenberg form is presented, using to this aim, an adaptation of SRDECO.

In [12], the algorithm JHSH, based on an adaptation of SRSB, is introduced, for reducing a general matrix, to an upper J -Hessenberg form. Variants of JHSH, named JHOSH, JHMSB and JHMSB2 are then derived, motivated by the numerical stability. The algorithms JHESS as well as JHSH and its variants have $O(n^3)$ as complexity.

The algorithm JHESS (and also SRDECO), as described in [2] may be subject of a fatal breakdown, causing a brutal stop of the computations. As consequence, the J -Hessenberg reduction can not be computed and the SR -algorithm does not run. Moreover, we demonstrate that the algorithm JHESS may breaks down while a condensed J Hessenberg form exists. These algorithms also may suffer from severe form of near-breakdowns, source of serious numerical instability.

In this paper, we restrict ourselves to the study of such aspects, bringing significant insights on SRDECO and JHESS algorithms.

We will show derive a strategy for curing fatal breakdowns and treating near breakdowns. To this aim, we first bring light on the SR -decomposition and SRDECO algorithm, in connection with the theory developed by Elsner in [5]. Then, a strategy for remedying to such breakdowns is proposed. The same strategy is used for treating the near-breakdowns. Numerical experiments are given, demonstrating the efficiency of our strategies to cure breakdowns or to treat near breakdowns.

The remainder of this paper is organized as follows. Section 2, is devoted to the necessary preliminaries. In the section 3, the algorithms SRDECO, SRSB or SRMSB are presented. Then, we establish a connection between some coefficients of the current matrix produced by the SRDECO algorithm, when applied to a matrix A and the necessary and sufficient condition for the existence of the SR decomposition of A , as given in [5]. In section 4, we recall the algorithms JHESS and JHMSB. We present then an example, for which a fatal breakdown is meet, in JHESS algorithm (also for JHMSB), for reducing the matrix to an upper J -Hessenberg. And hence, following the description of JHESS in [2], the algorithm stops. No hope to build then a SR -algorithm. We will show on the same example, that such breakdown is curable. We then develop our strategies for curing the fatal breakdowns. The same strategies are applied for near-breakdowns. The Section 5 is devoted to numerical experiments and comparisons. We conclude in the section 6.

2. Preliminaries

Let J_{2n} (or simply J) be the $2n$ -by- $2n$ real matrix

$$J_{2n} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}, \quad (1)$$

where 0_n and I_n stand respectively for n -by- n null and identity matrices. The linear space \mathbb{R}^{2n} with the indefinite skew-symmetric inner product

$$(x, y)_J = x^T J y \quad (2)$$

is called symplectic. For $x, y \in \mathbb{R}^{2n}$, the orthogonality $x \perp' y$ stands for $(x, y)_J = 0$. The symplectic adjoint x^J of a vector x , is defined by

$$x^J = x^T J. \quad (3)$$

The symplectic adjoint of $M \in \mathbb{R}^{2n \times 2k}$ is defined by

$$M^J = J_{2k}^T M^T J_{2n}. \quad (4)$$

A matrix $S \in \mathbb{R}^{2n \times 2k}$ is called symplectic if

$$S^J S = I_{2k}. \quad (5)$$

The symplectic group (multiplicative group of square symplectic matrices) is denoted \mathbb{S} . A transformation T given by

$$T = I + cvv^J \text{ where } c \in \mathbb{R}, \ v \in \mathbb{R}^\nu \text{ (with } \nu \text{ even)}, \quad (6)$$

is called symplectic Householder transformation [9]. It satisfies

$$T^J = I - cvv^J. \quad (7)$$

The vector v is called the direction of T .

For $x, y \in \mathbb{R}^{2n}$, there exists a symplectic Householder transformation T such that $Tx = y$ if $x = y$ or $x^J y \neq 0$. When $x^J y \neq 0$, T is given by

$$T = I - \frac{1}{x^J y} (y - x)(y - x)^J.$$

Moreover, each non null vector x can be mapped onto any non null vector y by a product of at most two symplectic Householder transformations

[9]. Symplectic Householder transformations are rotations, i.e. $\det(T) = 1$ and the symplectic group \mathbb{S} is generated by symplectic Householder transformations. In [7, 13] two orthogonal and symplectic transformations have been introduced. The first, for which we refer as Van Loan's Householder transformation, has the form

$$H(k, w) = \begin{pmatrix} \text{diag}(I_{k-1}, P) & 0 \\ 0 & \text{diag}(I_{k-1}, P) \end{pmatrix}, \quad (8)$$

where

$$P = I - 2ww^T/w^T w, \quad w \in \mathbb{R}^{n-k+1}.$$

The second, for which we refer as Van Loan's Givens transformation, is

$$J(k, \theta) = \begin{pmatrix} C & S \\ -S & C \end{pmatrix}, \quad (9)$$

where

$$C = \text{diag}(I_{k-1}, \cos\theta, I_{n-k}) \text{ and } S = \text{diag}(0_{k-1}, \sin\theta, 0_{n-k}).$$

$J(k, \theta)$ is a Givens symplectic matrix, that is an "ordinary" $2n$ -by- $2n$ Givens rotation that rotates in planes k and $k + n$ [15]. The SR factorization can not be performed for a general matrix by using the sole $H(k, w)$ and $J(k, \theta)$ transformations [2]. A third type, introduced in [2], is given by

$$G(k, \nu) = \begin{pmatrix} D & F \\ 0 & D^{-1} \end{pmatrix}, \quad (10)$$

where $k \in \{2, \dots, n\}$, $\nu \in \mathbb{R}$ and D, F are the $n \times n$ matrices

$$D = I_n + \left(\frac{1}{(1 + \nu^2)^{1/4}} - 1 \right) (e_{k-1}e_{k-1}^T + e_k e_k^T),$$

$$F = \frac{\nu}{(1 + \nu^2)^{1/4}} (e_{k-1}e_k^T + e_k e_{k-1}^T).$$

The matrix $G(k, \nu)$ is symplectic and non-orthogonal. The SRDECO algorithm is then derived for computing SR factorization for a general matrix, based on H, J and G transformations. A reduction of a general matrix to an upper J -Hessenberg, is obtained by using the same transformations involved in SRDECO, giving rise to JHESS algorithm. The breakdown in SR -decomposition via SRDECO or in the reduction to an upper J -Hessenberg form via JHESS, when it occurs, is caused by the latest transformations G .

3. SRDECO, SRSR algorithms

The aim of this work, is to bring significant contributions on the understanding and the behaviour of the algorithms SRDECO, SRSR, JHESS and JHMSH. Also, we propose strategies for curing breakdowns. Similar strategies are applied also for remedying to near breakdowns.

3.1. SR decomposition : SRDECO, SRSR algorithms

We consider the SRDECO algorithm, as introduced in [2]. Given $A \in \mathbb{R}^{2n \times 2n}$, the algorithm determines an SR decomposition of A , using functions vlg, vlh and gal below. The function vlg uses Van Loan's Givens transformation $J(k, c, s)$ as follows : for a given integer $1 \leq k \leq n$ and a vector $a \in \mathbb{R}^{2n}$, it determines coefficients c and s such that the $n + k$ th component of $J(k, c, s)a$ is zero. All components of $J(k, c, s)a$ remain unchanged, except eventually the k th and the $n + k$.

Algorithm 1. *function*[c,s]=vlg(k,a)
 $twon = \text{length}(a)$; $n = twon/2$;
 $r = \sqrt{a(k)^2 + a(n+k)^2}$;
 if $r = 0$ then $c = 1$; $s = 0$;
 else $c = \frac{a(k)}{r}$; $s = \frac{a(n+k)}{r}$;
 end

The function vlh uses Van Loan's Householder transformation $H(k, w)$ as follows : for a given integer $k \leq n$ and a vector $a \in \mathbb{R}^{2n}$, a vector $w = (w_1, \dots, w_{n-k+1})^T$ is determined such that the components $k + 1, \dots, n$ of $H(k, w)a$ are zeros. All components $1, \dots, k - 1$ and $n + 1, \dots, n + k - 1$ remain unchanged.

Algorithm 2. *function*[\beta,w]=vlh(k,a)
 $twon = \text{length}(a)$; $n = twon/2$;
 $\% w = (w_1, \dots, w_{n-k+1})^T$;
 $r1 = \sum_{i=2}^{n-k+1} a(i+k-1)^2$;
 $r = \sqrt{a(k)^2 + r1}$;
 $w_1 = a(k) + \text{sign}(a(k))r$;
 $w_i = a(i+k-1)$ for $i = 2, \dots, n - k + 1$;
 $r = w_1^2 + r1$; $\beta = \frac{2}{r}$;
 $\% P = I - \beta ww^T$; $(H(k, w)a)_i = 0$ for $i = k + 1, \dots, n$.
 end

The function gal uses the transformation $G(k, \nu)$ as follows : for a given integer $k \leq n$ and a vector $a \in \mathbb{R}^{2n}$, satisfying the condition $a_{n+k} = 0$ only if $a_{k+1} = 0$, it determines ν such that the $k + 1$ th of $G(k, \nu)a$ is zero.

Algorithm 3. $function[\nu] = gal(k, a)$
 $twon = length(a); n = twon/2;$
 if $a_k = 0$
 $\nu = 0;$
 else
 $\nu = -\frac{a_{k+1}}{a_{n+k}};$
 end
 end

The algorithm SRDECO is as follows : the matrix A is overwritten by the J -upper triangular matrix. If A has no SR decomposition, the algorithm stops.

Algorithm 4. $function [S,A]=SRDECO(A)$

1. For $j = 1, \dots, n$
2. For $k = n, \dots, j$
3. Zero the entry $(n+k, j)$ of A by running the function $[c, s] = vlg(k, A(:, j))$ and computing $J_{k,j} = J(k, c, s)$.
4. Update $A = J_{k,j}A$ and $S = SJ_{k,j}^T$.
5. End for.
6. Zero the entries $(j+1, j), \dots, (n, j)$ of A by running the function $[\beta, w] = vlh(j, A(:, j))$ and computing $H_j = H(j, w)$.
7. Update $A = H_jA$ and $S = SH_j^T$.
8. If $j \leq n - 1$
9. For $k = n, \dots, j + 1$
10. Zero the entry $(n+k, n+j)$ of A by running the function $[c, s] = vlg(k, A(:, n + j))$ and computing $J_{k,n+j} = J(k, c, s)$.
11. Update $A = J_{k,n+j}A$ and $S = SJ_{k,n+j}^T$.
12. End for.
13. Zero the entries $(j + 2, n + j), \dots, (n, n + j)$ of A by running the function $[\beta, w] = vlh(j + 1, A(:, j))$ and computing $H_{n+j} = H(j + 1, w)$.
14. Update $A = H_{n+j}A$ and $S = SH_{n+j}^T$.
15. If the entry $(j + 1, n + j)$ of A is nonzero and the entry $(n + j, n + j)$ is zero then stop the algorithm,
16. else

17. Zero the entry $(j+1, n+j)$ of A by running the function $[\nu] = gal(j+1, A(:, n+j))$ and computing $G_{j+1} = G(j+1, \nu)$.
18. Update $A = G_{j+1}A$ and $S = SG_{j+1}^{-1}$. % $G_{j+1}^{-1} = G_{j+1}^J$.
19. End if
20. End if
21. End for.

The SR decomposition can be also performed using only the symplectic Householder transformations T of (6), which are rank-one modifications of the identity, giving rise to the algorithm SRSH. More on this can be found in [9, 10, 11]. A modified version of SRSH, numerically more stable, is SRMSH. It turn out that SRMSH shares the same steps (1-16) of SRDECO, but not the remaining ones. In fact, the function gal and the symplectic matrices G_{j+1} in SRDECO are replaced by the function $sh2$ and T_j having the form of (6). The function $sh2$ is as follows

Algorithm 5. function $[c, v] = sh2(a)$

```

%compute c and v such that  $T_2 e_1 = e_1$ , and  $T_2 a = \mu e_1 + \nu e_{n+1}$ ,
% $\mu$  is a free parameter, and  $T_2 = (eye(twon) + c * v * v' * J)$ ;
twon = length(a); n = twon/2;
J = [zeros(n), eye(n); -eye(n), zeros(n)];
If n == 1
    v = zeros(twon, 1); c = 0; %T = eye(twon);
else
    Choose  $\mu$ ;
     $\nu = a(n+1)$ ;
    If  $\nu == 0$ 
        Display('division by zero')
    Return
    else
         $v = \mu e_1 + \nu e_{n+1} - a$ ,  $c = \frac{1}{a(n+1)(a(1) - \mu)}$ ;
    End
End

```

We obtain the algorithm

Algorithm 6. function $[S, A] = SRMSH(A)$

1. Run steps 1.-16. of SRDECO

17. Set $c0 = [j : n, n + j : 2n]$, $c1 = [j + 1 : n, n + j : 2n]$,

- 18. Zero the entry $(j+1, n+j)$ of A by running the function $[c, v] = sh2(A(c0, n + j))$
- 19. compute $T_j = I + cvv^T J$, update $A(:, c1) = T_j A(:, c1)$ and $S(:, c0) = ST_j^J$.
- 20. End if
- 21. End if
- 22. End for.

Remark 1. The function $sh2$ in the body of the algorithm $SRMSH$ may be replaced by the function $osh2$ (see [10, 11]) which presents the best conditioning among all possible choices.

3.2. Discussion : existence of SR decomposition, link with $SRDECO$ and $SRMSH$

In this subsection, we bring light on the connection between the existence of SR decomposition and the algorithm $SRDECO$ or equivalently $SRMSH$. We recall first the following result, given in [5] :

Theorem 7. *Let $A \in \mathbb{R}^{2n \times 2n}$ be nonsingular and P the permutation matrix $P = [e_1, e_{n+1}, e_2, e_{n+2}, \dots, e_n, e_{2n}]$, where e_i denotes the i th canonical vector of \mathbb{R}^{2n} . There exists $S \in \mathbb{R}^{2n \times 2n}$ symplectic and $R \in \mathbb{R}^{2n \times 2n}$ upper J -triangular, such that $A = SR$ if and only if all even leading minors of $P^T A^T J A P$ are nonzero.*

In [2], a comment on $SRDECO$ states : "if at any stage $j \in \{1, \dots, n - 1\}$ the algorithm ends because of the stopping condition, then the $2j$ th leading principal minor of $P^T A^T J A P$ is zero, and A has no SR decomposition (see Theorem 7)." However, a proof of how is connected the stopping condition of the algorithm $SRDECO$ with the condition of Theorem 7 is not given. Remark also that for $SRDECO$ algorithm, the condition A nonsingular is not required, while it is for Theorem 7. Here we give a proof on how this connection is made. Notice first that if $A = SR$, where S is any symplectic matrix and R is any matrix, then $A^T J A = R^T S^T J S R = R^T J R$. Hence a minor of $A^T J A$ is equal to its corresponding one of $R^T J R$. The same equality between minors is valid also for $P^T A^T J A P$ and $P^T R^T J R P$. The following Theorem establishes an explicit relation between the leading $2j$ -by- $2j$ minors of $P^T A^T J A P$ and the computed coefficients which determine the stopping condition of $SRDECO$. For a given matrix M , let us denote by $M_{[j,j]}$ the submatrix obtained from M by deleting all rows and columns except rows and columns $1, \dots, j$. We have

Theorem 8. Let $A \in \mathbb{R}^{2n \times 2n}$ be a matrix (not necessarily nonsingular), and let R be the matrix that one obtains at stage $1 \leq j \leq n-1$ of the algorithm SRDECO, by executing instructions **1.** to **14.** (corresponding to the current updated matrix A in the process, at stage j and until instruction **14.**). Then the leading $2j$ -by- $2j$ minor of $P^T A^T J A P$ satisfies

$$\det((P^T A^T J A P)_{[2j, 2j]}) = [r_{1,1} r_{n+1, n+1} \cdots r_{i,i} r_{n+i, n+i} \cdots r_{j,j} r_{n+j, n+j}]^2. \quad (11)$$

PROOF. Partitioning $R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$, then R_{11} , R_{12} , R_{21} , R_{22} have the form

$$R_{11} = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,j} & r_{1,j+1} & \cdots & r_{1,n} \\ 0 & r_{2,2} & \cdots & r_{2,j} & r_{2,j+1} & \cdots & r_{2,n} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \vdots & & \ddots & r_{j,j} & r_{j,j+1} & \cdots & r_{j,n} \\ 0 & \cdots & \cdots & 0 & r_{j+1,j+1} & \cdots & r_{j+1,n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & r_{n,j+1} & \cdots & r_{n,n} \end{bmatrix},$$

$$R_{12} = \begin{bmatrix} r_{1,n+1} & r_{1,n+2} & \cdots & r_{1,n+j-1} & r_{1,n+j} & r_{1,n+j+1} & \cdots & r_{1,2n} \\ 0 & r_{2,n+2} & \cdots & r_{2,n+j-1} & r_{2,n+j} & r_{2,n+j+1} & \cdots & r_{2,2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & r_{j-1,n+j-1} & r_{j-1,n+j} & r_{j-1,n+j+1} & \cdots & r_{j-1,2n} \\ 0 & \cdots & \cdots & 0 & r_{j,n+j} & r_{j,n+j+1} & \cdots & r_{j,2n} \\ 0 & \cdots & \cdots & 0 & r_{j+1,n+j} & r_{j+1,n+j+1} & \cdots & r_{j+1,2n} \\ 0 & \cdots & \cdots & 0 & 0 & r_{j+2,n+j+1} & \cdots & r_{j+2,2n} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & r_{n,n+j+1} & \cdots & r_{n,2n} \end{bmatrix},$$

$$R_{21} = \begin{bmatrix} 0 & r_{n+1,2} & \cdots & r_{n+1,j} & r_{n+1,j+1} & \cdots & r_{n+1,n} \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & r_{n+j-1,j} & r_{n+j-1,j+1} & \cdots & r_{n+j-1,n} \\ 0 & \cdots & 0 & 0 & r_{n+j,j+1} & \cdots & r_{n+j,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & r_{2n,j+1} & \cdots & r_{2n,n} \end{bmatrix},$$

and

$$R_{22} = \begin{bmatrix} r_{n+1,n+1} & r_{n+1,n+2} & \cdots & r_{n+1,n+j} & r_{n+1,n+j+1} & \cdots & r_{n+1,2n} \\ 0 & r_{n+2,n+2} & \cdots & r_{n+2,n+j} & r_{n+2,n+j+1} & \cdots & r_{n+2,2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \cdots & \ddots & r_{n+j,n+j} & r_{n+j,n+j+1} & \cdots & r_{n+j,2n} \\ 0 & \cdots & \cdots & 0 & r_{n+j+1,n+j+1} & \cdots & r_{n+j+1,2n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & r_{2n,n+j+1} & \cdots & r_{2n,2n} \end{bmatrix}.$$

The stopping condition at this stage j is " $r_{n+j,n+j} = 0$ and $r_{j+1,n+j} \neq 0$ ". We will establish connection between the coefficients $r_{i,i}$, $r_{n+i,n+i}$ of the current matrix R , with $1 \leq i \leq j$ and the leading $2j$ -by- $2j$ minor of $P^T A^T J A P$. Setting $\hat{J} = P^T J P$ and $\hat{R} = P^T R P$, we get $P^T A^T J A P = P^T R^T J R P = (P^T R^T P)(P^T J P)(P^T R P) = \hat{R}^T \hat{J} \hat{R}$. Recall that $\hat{J} = \text{diag}(J_2, \dots, J_2)$. Partitioning $\hat{R} = \begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{pmatrix}$, with the block \hat{R}_{11} is $2j$ -by- $2j$. Then we obtain for \hat{R}_{11} , \hat{R}_{12} , \hat{R}_{21} , \hat{R}_{22} :

$$\hat{R}_{11} = \begin{bmatrix} r_{1,1} & r_{1,n+1} & \cdots & r_{1,i} & r_{1,n+i} & \cdots & r_{1,j} & r_{1,n+j} \\ 0 & r_{n+1,n+1} & \cdots & r_{n+1,i} & r_{n+1,n+i} & \cdots & r_{n+1,j} & r_{n+1,n+j} \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & r_{i,i} & r_{i,n+i} & \cdots & r_{i,j} & r_{i,n+j} \\ \vdots & \vdots & & 0 & r_{n+i,n+i} & \cdots & r_{n+i,j} & r_{n+i,n+j} \\ \vdots & \vdots & & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & 0 & 0 & \ddots & r_{j,j} & r_{j,n+j} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & r_{n+j,n+j} \end{bmatrix}, \quad (12)$$

which is a upper $2j$ -by- $2j$ triangular matrix. The $2(n-j)$ -by- $2j$ block \hat{R}_{21} turn out to have all entries zeros except the entry in position $(1, j)$. More precisely, $\hat{R}_{21} = \begin{pmatrix} 0 & r_{j+1,n+j} \\ 0 & 0 \end{pmatrix}$. Setting $\hat{J}_{2k} = \text{diag}(J_2, \dots, J_2) \in \mathbb{R}^{2k \times 2k}$

for an integer k , and due to the special structures of \hat{J} , we get

$$\begin{aligned} \hat{R}^T \hat{J} \hat{R} &= \begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{pmatrix}^T \hat{J} \begin{pmatrix} \hat{R}_{11} & \hat{R}_{12} \\ \hat{R}_{21} & \hat{R}_{22} \end{pmatrix} \\ &= \begin{pmatrix} \hat{R}_{11}^T & \hat{R}_{21}^T \\ \hat{R}_{12}^T & \hat{R}_{22}^T \end{pmatrix} \begin{pmatrix} \hat{J}_{2j} \hat{R}_{11} & \hat{J}_{2j} \hat{R}_{12} \\ \hat{J}_{2(n-j)} \hat{R}_{21} & \hat{J}_{2(n-j)} \hat{R}_{22} \end{pmatrix}. \end{aligned}$$

Let $(\hat{R}^T \hat{J} \hat{R})_{[2j,2j]}$ denote the leading $2j$ -by- $2j$ block of $\hat{R}^T \hat{J} \hat{R}$, we obtain

$$(\hat{R}^T \hat{J} \hat{R})_{[2j,2j]} = \hat{R}_{11}^T \hat{J}_{2j} \hat{R}_{11} + \hat{R}_{21}^T \hat{J}_{2(n-j)} \hat{R}_{21}.$$

Denoting $\tilde{e}_j = (0, \dots, 0, 1)^T$ the j th canonical vector of \mathbb{R}^j and $e_1 = (1, 0, \dots, 0)^T$, $e_2 = (0, 1, 0, \dots, 0)^T$ respectively the first and the second canonical vectors of $\mathbb{R}^{2(n-j)}$, then \hat{R}_{21} may be expressed as $\hat{R}_{21} = r_{j+1, n+j} e_1 \tilde{e}_{2j}^T$. Hence, $\hat{R}_{21}^T \hat{J}_{2(n-j)} \hat{R}_{21} = r_{j+1, n+j}^2 \tilde{e}_{2j} e_1^T \hat{J}_{2(n-j)} e_1 \tilde{e}_{2j}^T$. As $\hat{J}_{2(n-j)} e_1 = -e_2$, we obtain directly $\hat{R}_{21}^T \hat{J}_{2(n-j)} \hat{R}_{21} = -r_{j+1, n+j}^2 \tilde{e}_{2j} e_1^T e_2 \tilde{e}_{2j}^T = 0$. Thus

$$(\hat{R}^T \hat{J} \hat{R})_{[2j,2j]} = \hat{R}_{11}^T \hat{J}_{2j} \hat{R}_{11}.$$

It follows

$$\det((\hat{R}^T \hat{J} \hat{R})_{[2j,2j]}) = \det(\hat{R}_{11}^T) \det(\hat{J}_{2j}) \det(\hat{R}_{11}).$$

As $\det(\hat{J}_{2j}) = 1$, and $\det(\hat{R}_{11}^T) = \det(\hat{R}_{11})$, we get

$$\det((\hat{R}^T \hat{J} \hat{R})_{[2j,2j]}) = (\det(\hat{R}_{11}))^2.$$

Since $P^T A^T J A P = (\hat{R}^T \hat{J} \hat{R})$, it follows that

$$(P^T A^T J A P)_{[2j,2j]} = (\hat{R}^T \hat{J} \hat{R})_{[2j,2j]},$$

which implies for the $2j$ -by- $2j$ leading minor of $P^T A^T J A P$

$$\det((P^T A^T J A P)_{[2j,2j]}) = \det((\hat{R}^T \hat{J} \hat{R})_{[2j,2j]}) = (\det(\hat{R}_{11}))^2.$$

The matrix \hat{R}_{11} is $2j$ -by- $2j$ upper triangular matrix, and from relation (12), we have

$$\det((P^T A^T J A P)_{[2j,2j]}) = [r_{1,1} r_{n+1, n+1} \cdots r_{i,i} r_{n+i, n+i} \cdots r_{j,j} r_{n+j, n+j}]^2.$$

Corollary 9. *Let $A \in \mathbb{R}^{2n \times 2n}$ be a nonsingular matrix, and let R be the matrix that one obtains at stage $1 \leq j \leq n-1$ of the algorithm SRDECO, by executing instructions **1.** to **14.** (corresponding to the current updated matrix A in the process, at stage j and until instruction **14.**). Then A admits an SR decomposition if and only if $r_{n+j, n+j} \neq 0$, $\forall j \in \{1, \dots, n\}$.*

PROOF. Since A is nonsingular and using Theorem 7 and Theorem 8, we have : A admits a SR decomposition if and only if $r_{1,1} r_{n+1, n+1} \cdots r_{j,j} r_{n+j, n+j} \neq 0, \forall j \in \{1, \dots, n\}$. At the stage j , we have $A = SR$ for some symplectic matrix S . Due to the structure of R , we deduce that the coefficients $r_{1,1}, r_{2,2}, \dots, r_{j,j}$ are automatically all nonzero (otherwise R would be singular and so would be A). The result is then straightforward.

If the condition A nonsingular is not required, one may ask in this case whether the SR -decomposition exists even when a $2j$ -by- $2j$ leading minor $\det((P^T A^T J A P)_{[2j, 2j]})$ is equal zero for some j . We precise this in the following result

Theorem 10. *Let $A \in \mathbb{R}^{2n \times 2n}$ be any matrix, and let R be the matrix that one obtains at stage $1 \leq j \leq n-1$ of the algorithm $SRDECO$ (or $SRMSH$), by executing instructions **1.** to **14.** (corresponding to the current updated matrix A in the process, at stage j and until instruction **14.**) Then A admits an SR decomposition if and only if $(r_{n+j, n+j} \neq 0 \text{ or } r_{j+1, n+j} = 0), \forall j \in \{1, \dots, n\}$.*

PROOF. The condition is sufficient, since if it is satisfied, the stopping condition in $SRDECO$ (or $SRMSH$) is never met and a SR decomposition is furnished at the end of the process. We show now that the condition is necessary, i.e. we show that if there exists an index j such that $r_{n+j, n+j} = 0$ and $r_{j+1, n+j} \neq 0$, then SR decomposition does not exist. In the fact, suppose that there exists an integer $1 \leq j \leq n-1$ such that $r_{n+j, n+j} = 0$ and $r_{j+1, n+j} \neq 0$ and let us seek for a symplectic matrix S_j such that the product $S_j a = a$ for any vector a possessing the same structure of any column $1, \dots, j$ and $n+1, \dots, n+j-1$ of R and transforms the $n+j$ th column $R(:, n+j) = \sum_{i=1}^{j+1} r_{i, n+j} e_i + \sum_{i=1}^{j-1} r_{n+i, n+j} e_{n+i}$ into the desired form

$$S_j R(:, n+j) = \sum_{i=1}^j r'_{i, n+j} e_i + \sum_{i=1}^j r'_{n+i, n+j} e_{n+i}. \quad (13)$$

The matrix S_j has necessarily the form

$$S_j = [e_1, \dots, e_j, s_{j+1}, \dots, s_n, e_{n+1}, \dots, e_{n+j-1}, s_{n+j}, \dots, s_{2n}],$$

where e_k stands for the k th canonical vector of \mathbb{R}^{2n} . Hence we get

$$S_j R(:, n+j) = \sum_{i=1}^j r_{i, n+j} e_i + r_{j+1, n+j} s_{j+1} + \sum_{i=1}^{j-1} r_{n+i, n+j} e_{n+i}. \quad (14)$$

In one hand, from relation (13), we get $e_j^T S_j R(:, n+j) = r'_{n+j, n+j}$. In the other hand, from relation (14), and the fact that S_j is symplectic, we get $e_j^T S_j R(:, n+j) = 0$. Thus, we deduce $r'_{n+j, n+j} = 0$. Therefore, the relations (13 - 14), imply $r_{j+1, n+j} s_{j+1}$ belongs to the space spanned by $\{e_1, \dots, e_j, e_{n+1}, \dots, e_{n+j-1}\}$. Since the vectors of $e_1, \dots, e_j, s_{j+1}, e_{n+1}, \dots, e_{n+j-1}$ are linearly independents, we deduce $r_{j+1, n+j} = 0$, which is absurd. The matrix S_j does not exist and hence SR decomposition does not exist.

Remark 2. Remark that S_j corresponds to the symplectic matrix G_{j+1} for SRDECO and to the symplectic matrix T_j for SRMSH.

4. Curing breakdowns or treating near-breakdowns in JHESS, JHMSH algorithms

4.1. Breakdowns or near-breakdowns in JHESS, JHMSH algorithms

The algorithm SRDECO may be adapted for reducing a matrix to the condensed upper J -Hessenberg form, see [2]. This leads to the algorithm JHESS. In a similar way, the algorithm SRSH or its variant SRMSH may be adapted for handling the reduction of a matrix to J -Hessenberg form, see [12].

We recall that a matrix $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$, is upper J -Hessenberg when H_{11} , H_{21} , H_{22} are upper triangular and H_{12} is upper Hessenberg. H is called unreduced when H_{21} is nonsingular and the Hessenberg H_{12} is unreduced, i.e. the entries of the subdiagonal are all nonzero.

The algorithm JHESS is formulated in [2] as follows : "given a matrix $A \in \mathbb{R}^{2n \times 2n}$ and $S = I_{2n}$, the following algorithm reduces, if it is possible, A to upper J -Hessenberg form $H = \Pi^{-1}A\Pi$, with a symplectic matrix Π whose first column is a multiple of e_1 . A is overwritten by the J -Hessenberg matrix H and S is overwritten by the transforming matrix Π . If this reduction of A does not exist, the algorithm stops".

Algorithm 11. *function* [S,A]=JHESS(A)

1. For $j = 1, \dots, n - 1$
2. For $k = n, \dots, j + 1$
3. Zero the entry $(n+k, j)$ of A by running the function $[c, s] = vlg(k, A(:, j))$ and computing $J_{k,j} = J(k, c, s)$.
4. Update $A = J_{k,j}AJ_{k,j}^T$ and $S = SJ_{k,j}^T$
5. End for.
6. Zero the entries $(j+2, j), \dots, (n, j)$ of A by running the function $[\beta, w] = vlh(j + 1, A(:, j))$ and computing $H_j = H(j + 1, w)$.
7. Update $A = H_jAH_j^T$ and $S = SH_j^T$.
8. If the entry $(j + 1, j)$ of A is nonzero and the entry $(n + j, j)$ is zero then stop the algorithm
9. else
10. Zero the entry $(j + 1, j)$ of A by running the function $[\nu] = gal(A(:, j))$
11. Compute $G_{j+1} = G(j + 1, \nu)$.

12. Update $A = G_{j+1}AG_{j+1}^{-1}$ and $S = SG_{j+1}^{-1}$.
13. End if
14. For $k = n, \dots, j + 1$
15. Zero the entry $(n+k, n+j)$ of A by running the function $[c, s] = vlg(k, A(:, n + j))$ and compute $J_{k,n+j} = J(k, c, s)$.
16. Update $A = J_{k,n+j}AJ_{k,n+j}^T$ and $S = SJ_{k,n+j}^T$.
17. End for.
18. If $j \leq n - 2$
19. Zero the entries $(j + 2, n + j), \dots, (n, n + j)$ of A by running the function $[\beta, w] = vlh(j + 1, A(:, n + j))$ and compute $H_{n+j} = H(j + 1, w)$.
20. Update $A = H_{n+j}AH_{n+j}^T$ and $S = SH_{n+j}^T$.
21. End if
22. End for.

One of the main drawback of JHESS is that a fatal breakdown can be encountered. To illustrate our purpose, we consider the following example. Let A_6 be the 6-by-6 matrix

$$A_6 = \begin{pmatrix} 1 & 0 & 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{pmatrix}. \quad (15)$$

The algorithm JHESS, applied to A_6 , meets a fatal breakdown at the first step : the entry $A_6(2, 1) \neq 0$ and the entry $A_6(4, 1) = 0$, the algorithm stops. In fact, it is impossible to find a symplectic matrix S_1 , with the first column proportional to e_1 such that $SA_6e_1 = \alpha e_1 + \beta e_4$, as showed in the above subsection. Thus, A_6 can not be reduced to an upper J -Hessenberg form, via symplectic similarity transformations, for which the first column is proportional to e_1 . The SR-algorithm as described in [2] , uses first JHESS algorithm for reducing a matrix to the J -Hessenberg form. As consequence, if applied to A_6 , the SR-algorithm stops also at the first step. Let us remark also that the basic SR-algorithm (which can be roughly described as consisting in repeating the factorisation $A = SR$, and the product $A = RS$) works and converges, when applied to the example A_6 . The algorithm JHESS may also suffers from another serious problem : the near-breakdown. The latter occurs when the condition number of the symplectic and non-orthogonal

matrix G_{j+1} at the step **11.** of the algorithm JHESS becomes very large. This causes a dramatic growth of the rounding errors.

The following strategy is proposed in [2] for remedying to such problems : if in the j th iteration, the condition number of the matrix G_{j+1} is larger than a certain tolerance, the iteration is stopped. In the implicit form (which is the useful one) of the algorithm SR, an exceptional similarity transformation is computed, with the symplectic (but non-orthogonal) matrix $S_j = I - ww^T J$, where w is a random vector with $\|w\|_2 = 1$. The algorithm JHESS is then applied to the new similar matrix $S_j^{-1}AS_j$. If the number of encountered near-breakdowns/breakdowns exceeds a given bound, the whole process is definitively stopped. This strategy presents certain serious drawbacks : 1) The condition number of $S_j^{-1}AS_j$ will be worse than the condition number of A . This du to the fact that S_j can never be orthogonal. Hence, numerical instability is expected. 2) The cost of forming the product $S_j^{-1}AS_j$ is $O(n^2)$ where $2n$ is the dimension of A . 3) The product $S_j^{-1}AS_j$ fills-up the matrix and destroys the previous partially created J -Hessenberg form of A . Hence an additional cost of $O(n^3)$ is needed to restore the J -Hessenberg form. To summarize, each application of this strategy creates a current matrix with worse condition number than the previous, and needs an expensive cost of $O(n^3)$ flops.

In the sequel, we propose two alternatives, for which either all or some of the above drawbacks are avoided. The first consists in a careful choice of the random vector w so that the product $S_j^{-1}AS_j$ does not fill-up the matrix and preserves all the created zeros during the previous steps $1, \dots, j - 1$. This diminish considerably the cost. However, the condition number of $S_j^{-1}AS_j$ may become worse than this of A . The second alternative is more attractive since it allows us to avoid all of the above drawbacks. It consists in computing a similarity transformation $S_j^{-1}AS_j$ for which : 1) the proposed matrix S_j is not only symplectic but also orthogonal. Thus, the condition number of $S_j^{-1}AS_j$ remains the same, and the process is numerically as accurate as possible. 2) The cost for computing the product $S_j^{-1}AS_j$ is only $O(n)$. Thus, a gain of an order-of- magnitude is guaranteed. 3) The product $S_j^{-1}AS_j$ does not fills-up the matrix and preserves all the created zeros in previous steps. Also, to restore the J -Hessenberg form of A , only a cheaper additional cost of $O(n^2)$ is needed.

In the sequel, we explain first how one may remedy to the fatal breakdown, encountered by JHESS, when applied to the example A_6 and highlights

the main lines of the method. Then we present a method to cure the fatal breakdown in the general case. The idea is the following : one seeks for a symplectic transforming matrix S so that the similar matrix SA_6S^{-1} , may be reduced by JHESS. The choice of S should be done carefully. A judicious choice of S consists in taken S equal to Van Loan's Householder matrix

$$S = \begin{pmatrix} H_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & H_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (16)$$

or Van Loan's Givens matrix

$$S = \begin{pmatrix} G_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & G_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (17)$$

where H_2 (respectively G_2) is a 2-by-2 Householder matrix (respectively a 2-by-2 Givens matrix) such that $H_2(1, 2)^T = \sqrt{5}(1, 0)^T$ (respectively $G_2(1, 2)^T = \sqrt{5}(1, 0)^T$). If we proceed with choices (16) or (17), we get the first column of SA_6 proportional to e_1 and only rows 1,2 and 4,5 of SA_6 may change. With the choice (17), we obtain $G_2 = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$, with $c = 1/\sqrt{5}$, $s = 2c$, and

$$SA_6 = \begin{pmatrix} \sqrt{5} & 2/\sqrt{5} & 0 & \sqrt{5} & 4/\sqrt{5} & 0 \\ 0 & 1/\sqrt{5} & 0 & 0 & -3/\sqrt{5} & 0 \\ 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 4/\sqrt{5} & 4/\sqrt{5} & 7/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & -3/\sqrt{5} & 2/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 & 0 & 3 & 1 \end{pmatrix}.$$

The multiplication of SA_6 on the left by S^{-1} acts only on the columns 1, 2 and 4, 5 of SA_6 . The other columns remain unchanged. We obtain

$$SA_6S^{-1} = \begin{pmatrix} 9/5 & -8/5 & 0 & 13/5 & -6/5 & 0 \\ 2/5 & 1/5 & 0 & -6/5 & 3/5 & 0 \\ 4/\sqrt{5} & 2/\sqrt{5} & 1 & 4/\sqrt{5} & 2/\sqrt{5} & 1 \\ 8/5 & 4/5 & 4/\sqrt{5} & 11/5 & 12/5 & 0 \\ -6/5 & -3/5 & 2/\sqrt{5} & 3/5 & -1/5 & 0 \\ 0 & 0 & 1 & 6/\sqrt{5} & 3/\sqrt{5} & 1 \end{pmatrix}. \quad (18)$$

We applied JHESS (also JHMSH) to the matrix SA_6S^{-1} of (18). The algorithm run well and the reduction to the J -Hessenberg form is obtained. The SR-algorithm is then applied with explicit and implicit versions, and both converge. Thus the fatal breakdown of JHESS (or similarly JHMSH) is cured. Recall that the algorithm JHMSH as described in [12] is as follows

Algorithm 12. *function [S,H]=JHMSH(A)*
 $twon = size(A(:,1)); n = twon/2; S = eye(twon);$
for $j = 1 : n - 1$
 $J = [zeros(n - j + 1), eye(n - j + 1); -eye(n - j + 1), zeros(n - j + 1)];$
 $ro = [j : n, n + j : 2n]; co = [j : n, n + j : 2n];$
 $[c, v] = osh2(A(ro, j));$
% Updating A :
 $A(ro, co) = A(ro, co) + c * v * (v' * J * A(ro, co));$
 $A(:, co) = A(:, co) - (A(:, co) * (c * v)) * v' * J;$
% Updating S (if needed) :
 $S(:, co) = S(:, co) - c * (v * v') * J * S(:, co);$
for $k = 2n : n + j + 1,$
 $[c, s] = vlg(k, A(:, n + j)),$
%Updating A :

$$\begin{bmatrix} A(k, co) \\ A(n + k, co) \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} A(k, co) \\ A(n + k, co) \end{bmatrix};$$

$$\begin{bmatrix} A(:, k) & A(:, n + k) \end{bmatrix} = \begin{bmatrix} A(:, k) & A(:, n + k) \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix};$$

%Updating S (if needed) :

$$\begin{bmatrix} S(:, k) & S(:, n + k) \end{bmatrix} = \begin{bmatrix} S(:, k) & S(:, n + k) \end{bmatrix} \begin{bmatrix} c & -s \\ s & c \end{bmatrix};$$

end
if $j \leq n - 2$
 $[\beta, w] = vlh(j + 1, A(:, n + j));$
%Updating A :
 $A(j + 1 : n, co) = A(j + 1 : n, co) - \beta * w * w' * A(j + 1 : n, co)$
 $A(j + 1 + n : 2n, co) = A(j + 1 + n : 2n, co) - \beta * w * w' * A(j + 1 + n : 2n, co);$
 $A(:, j + 1 : n) = A(:, j + 1 : n) - \beta * A(:, j + 1 : n) * w * w';$
 $A(:, n + j + 1 : 2n) = A(:, n + j + 1 : 2n) - \beta * A(:, n + j + 1 : 2n) * w * w';$
%Updating S (if needed) :
 $S(:, j + 1 : n) = S(:, j + 1 : n) - \beta * S(:, j + 1 : n) * w * w';$
 $S(:, n + j + 1 : 2n) = S(:, n + j + 1 : 2n) - \beta * S(:, n + j + 1 : 2n) * w * w';$

end
end
end

The breakdown in JHMSH occurs exactly in the same conditions as in JHESS, and is located in the call of the function *osh2*. A slight different version of JHMSH is JHMSH2 (see [12]).

4.2. Curing breakdowns in JHESS, JHMSH algorithms

We present here, in a general manner, the strategy of curing breakdowns or near breakdowns which may occur in JHESS or JHMSH algorithms. Let us remark that breakdowns (or near-breakdowns) in JHESS (respectively in JHMSH) may occur only when the function *gal* (respectively *osh2*) is called, and hence it concerns only columns from the first half of the current matrix.

Let $A \in \mathbb{R}^{2n \times 2n}$ be a matrix and let H be the matrix that one obtains at stage $1 \leq j \leq n - 1$ of the algorithm JHESS, by executing instructions **1.** to **7.** (corresponding to the current updated matrix A in the process, at stage j and until instruction **7.**). Partitioning $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$, then H_{11} , H_{12} , H_{21} , H_{22} have the form

$$H_{11} = \begin{bmatrix} h_{1,1} & h_{1,2} & \dots & h_{1,j} & h_{1,j+1} & \dots & h_{1,n} \\ 0 & h_{2,2} & \dots & h_{2,j} & h_{2,j+1} & \dots & h_{2,n} \\ \vdots & \ddots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & & 0 & h_{j,j} & h_{j,j+1} & \dots & h_{j,n} \\ 0 & \dots & 0 & h_{j+1,j} & h_{j+1,j+1} & \dots & h_{j+1,n} \\ 0 & \dots & 0 & 0 & h_{j+2,j+1} & \dots & h_{j+2,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & 0 & h_{n,j+1} & \dots & h_{n,n} \end{bmatrix},$$

$$\begin{aligned}
H_{12} &= \begin{bmatrix} h_{1,n+1} & h_{1,n+2} & \cdots & h_{1,n+j-1} & h_{1,n+j} & \cdots & h_{1,2n} \\ h_{2,n+1} & h_{2,n+2} & \cdots & h_{2,n+j-1} & h_{2,n+j} & \cdots & h_{2,2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & h_{j-1,n+j-1} & h_{j-1,n+j} & \cdots & h_{j-1,2n} \\ 0 & \cdots & \cdots & h_{j,n+j-1} & h_{j,n+j} & \cdots & h_{j,2n} \\ 0 & \cdots & \cdots & 0 & h_{j+1,n+j} & \cdots & h_{j+1,2n} \\ 0 & \cdots & \cdots & 0 & h_{j+2,n+j} & \cdots & h_{j+2,2n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & & \cdots & 0 & h_{n,n+j} & \cdots & h_{n,2n} \end{bmatrix}, \\
H_{21} &= \begin{bmatrix} h_{n+1,1} & h_{n+1,2} & \cdots & h_{n+1,j} & h_{n+1,j+1} & \cdots & h_{n+1,n} \\ 0 & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & h_{n+j-1,j} & h_{n+j-1,j+1} & \cdots & h_{n+j-1,n} \\ \vdots & & \ddots & h_{n+j,j} & h_{n+j,j+1} & \cdots & h_{n+j,n} \\ 0 & \cdots & \cdots & 0 & h_{n+j+1,j+1} & \cdots & h_{n+j+1,n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & h_{2n,j+1} & \cdots & h_{2n,n} \end{bmatrix}, \\
H_{22} &= \begin{bmatrix} h_{n+1,n+1} & h_{n+1,n+2} & \cdots & h_{n+1,n+j-1} & h_{n+1,n+j} & \cdots & h_{n+1,2n} \\ 0 & h_{n+2,n+2} & \cdots & h_{n+2,n+j-1} & h_{n+2,n+j} & \cdots & h_{n+2,2n} \\ 0 & 0 & \ddots & & \vdots & & \vdots \\ \vdots & \vdots & \ddots & h_{n+j-1,n+j-1} & h_{n+j-1,n+j} & \cdots & h_{n+j-1,2n} \\ 0 & \cdots & \cdots & 0 & h_{n+j,n+j} & \cdots & h_{n+j,2n} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & h_{2n,n+j} & \cdots & h_{2n,2n} \end{bmatrix}.
\end{aligned}$$

The breakdown occurs in JHESS when the coefficient $h_{n+j,j} = 0$ and $h_{j+1,j} \neq 0$. In this case, JHESS stops computations. To overcome this fatal breakdown, we construct the orthogonal matrix $P^{(j)} = \text{diag}(I_{j-1}, H_2^{(j)}, I_{n-j-1})$ where $H_2^{(j)}$ is a 2-by-2 Householder matrix. We set $S^{(j)} = \text{diag}(P^{(j)}, P^{(j)})$. The matrix $S^{(j)}$ is symplectic and orthogonal. The choice of the 2-by-2 Householder matrix $H_2^{(j)}$ is so that $H_2^{(j)} \begin{pmatrix} h_{j,j} \\ h_{j+1,j} \end{pmatrix} = \begin{pmatrix} h'_{j,j} \\ 0 \end{pmatrix}$. Thus, the action $S^{(j)}H$ annihilates the position $(j+1, j)$ of the updated matrix H and keep unchanged all zeros created previously except potentially the position $(j+1, n+j-1)$ (in the block H_{12}). Keep in mind that the action of $S^{(j)}H$ on H affects only

rows $j, j+1, n+j, n+j+1$. The action $S^{(j)}H[S^{(j)}]^{-1}$ on $S^{(j)}H$ affects only the columns $j, j+1, n+j, n+j+1$. Thus, the columns $1, \dots, j-1$ and $n+1, \dots, n+j-1$ remain unchanged (hence all zeros created previously are not affected). The process is pursued as follows : one must annihilate the potentially nonzero entry in position $(j+1, n+j-1)$ and keep unchanged all zeros created previously in columns $1, \dots, j-1$ and $n+1, \dots, n+j-1$. This may be addressed by applying the similarity $T^{(j)}H[T^{(j)}]^{-1}$ to the obtained matrix H , where $T^{(j)} = \text{diag}(Q^{(j)}, Q^{(j)})$ states for Van Loan's Householder matrix, given by $Q^{(j)} = \text{diag}(I_{j-1}, K_2^{(j)}, I_{n-j-1})$ where $K_2^{(j)}$ is a 2-by-2 Householder matrix. The choice of the 2-by-2 Householder matrix $K_2^{(j)}$ is so that $K_2^{(j)} \begin{pmatrix} h_{j,n+j-1} \\ h_{j+1,n+j-1} \end{pmatrix} = \begin{pmatrix} h'_{j,n+j-1} \\ 0 \end{pmatrix}$. The cost of this curing strategy step is $O(n)$. Next, the algorithm is pursued normally by executing again step j .

Remark 3. The 2-by-2 Householder matrix $H_2^{(j)}$ (respectively $K_2^{(j)}$) may be replaced by a 2-by-2 Givens matrix $G_2^{(j)} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$ (respectively by a 2-by-2 Givens $L_2^{(j)} = \begin{pmatrix} c' & s' \\ -s' & c' \end{pmatrix}$) where the coefficient c, s (respectively c', s') are chosen so that $G_2^{(j)} \begin{pmatrix} h_{j,j} \\ h_{j+1,j} \end{pmatrix} = \begin{pmatrix} h'_{j,j} \\ 0 \end{pmatrix}$ (respectively $L_2^{(j)} \begin{pmatrix} h_{j,n+j-1} \\ h_{j+1,n+j-1} \end{pmatrix} = \begin{pmatrix} h'_{j,n+j-1} \\ 0 \end{pmatrix}$).

It is worth noting that one may take an arbitrary k -by- k Householder matrix $H_k^{(j)}$ instead of $H_2^{(j)}$, with $P^{(j)} = \text{diag}(I_{j-1}, H_k^{(j)}, I_{n-j-k+1})$. The action $P^{(j)}H$ on H affects only rows $j, \dots, j+k-1$, rows $n+j, \dots, n+j+k-1$. The action $P^{(j)}H[P^{(j)}]^{-1}$ on $P^{(j)}H$ affects only columns $j, \dots, j+k-1$, and columns $n+j, \dots, n+j+k-1$. Hence, all zeros created previously in columns $1, \dots, j-1$ and columns $n+1, \dots, n+j-1$ remain unchanged, except potentially in positions $(j+1, n+j-1), \dots, (j+k-1, n+j-1)$. To pursue the process, one must annihilate the potentially nonzero entries in position $(j+1, n+j-1), \dots, (j+k-1, n+j-1)$ and keep unchanged all zeros created previously in columns $1, \dots, j-1$ and $n+1, \dots, n+j-1$. This may be addressed by applying the similarity $T^{(j)}H[T^{(j)}]^{-1}$ to the obtained matrix H , where $T^{(j)} = \text{diag}(Q^{(j)}, Q^{(j)})$ states for Van Loan's Householder matrix, given by $Q^{(j)} = \text{diag}(I_{j-1}, K_k^{(j)}, I_{n-j-1})$ where $K_k^{(j)}$ is a k -by- k Householder matrix. The choice of the k -by- k Householder matrix $K_k^{(j)}$ is so that

$K_k^{(j)} \begin{pmatrix} h_{j,n+j-1} \\ \vdots \\ h_{j+k-1,n+j-1} \end{pmatrix} = \begin{pmatrix} h'_{j,n+j-1} \\ 0 \end{pmatrix}$. The cost of this curing strategy step is $O(kn)$.

4.3. Curing near-breakdowns in JHESS, JHMSH algorithms

The near-breakdown occurs in JHESS (or in JHMSH) when the coefficients $h_{n+j,j}$ and $h_{j+1,j}$ are both different from zero but are near to the situation of breakdown. This can be measured by the fact that the ratio $\frac{h_{j+1,j}}{h_{n+j,j}}$ is very large. In this case, the non-orthogonal and symplectic transformations involved in JHESS (respectively JHMSH) become ill-conditioned and numerical instability is encountered reducing the accuracy of the reduction. In order to remedy to a such near breakdown in JHESS (or JHMSH) algorithm, one may proceed exactly as for curing a breakdown, the only difference is that the test $h_{n+j,j} = 0$ and $h_{j+1,j} \neq 0$ (corresponding to a breakdown) is replaced by the $\frac{h_{j+1,j}}{h_{n+j,j}} \geq \tau$ (corresponding to a near-breakdown), where τ is a certain tolerance.

4.4. SR algorithm

The SR algorithm, is a QR like algorithm which can roughly be described as follows. For a given matrix $M \in \mathbb{R}^{2n \times 2n}$, it computes :

1. An upper J -Hessenberg reduction $M_1 = S_0^J M S_0$ where S_0 is symplectic. Set $S = S_0$.
2. Iteration : For $k = 1, \dots$, compute $M_{k+1} = S_k^J M_k S_k$ where S_k stands for the symplectic factor of the SR decomposition $p_k(M_k) = S_k R_k$ of a polynomial p_k of M_k and update $S = S S_k$.

The iterate M_{k+1} remain J -Hessenberg if the matrix M_k is J -Hessenberg. Like the QR algorithm, SR algorithm admits an implicit version : the decompositions $p_k(M_k) = S_k R_k$ are not performed explicitly. Since SR algorithm is based on J -Hessenberg reductions and SR decompositions, breakdowns or near-breakdowns may be encountered both in the explicit or implicit versions of the algorithm. Of course, the implicit form is preferred to the explicit one. In [2] there is no strategy proposed when a breakdown is meet. The algorithm is topped. However, a technique has been proposed in the situation of a near breakdown, occurring at the iteration j .

The proposed technique consists in computing the similarity $M_{j+1} = S_j M_j [S_j]^{-1}$, where $S_j = I - ww^T J$ and $w \in \mathbb{R}^{2n \times 2n}$ is a random vector, with $\|w\|_2 = 1$. The algorithm continue with M_{j+1} . This technique presents several serious drawbacks :

1. The symplectic transformation $(I - ww^T J)$ is never orthogonal (except for $w = 0$) and hence its condition number may be large. The condition number of M_{j+1} could be worse than this of M_j .
2. Computing the similarity $M_{j+1} = S_j M_j [S_j]^{-1}$ costs $O(n^2)$.
3. The similarity $M_{j+1} = S_j M_j [S_j]^{-1}$ destroys the structure J -Hessenberg of M_j . Thus M_j is no longer J -Hessenberg. Moreover, the matrix M_{j+1} fills up. Hence, a reduction to a J -Hessenberg form is needed to restore the previous structure. The cost of this restoration is $O(n^3)$ which is very expensive. Instead of this similarity, when breakdown or near breakdown occurs with respect to the column say i of the matrix M_j , we propose the similarity $M_{j+1} = P^{(j)} M_j [P^{(j)}]^{-1}$, where $P^{(j)} = \text{diag}(I_{i-1}, H_l^{(i)}, I_{n-i-l+1})$ and $H_l^{(i)}$ an arbitrary l -by- l Householder matrix. The action $P^{(j)} M_j$ on M_j affects only rows $i, \dots, i+l-1$, rows $n+i, \dots, n+i+l-1$. The action $P^{(j)} M_j [P^{(j)}]^{-1}$ on $P^{(j)} M_j$ affects only columns $i, \dots, i+l-1$, and columns $n+i, \dots, n+i+l-1$. Hence, all zeros in columns $1, \dots, i-1$ and columns $n+1, \dots, n+i-1$ (because of the form J -Hessenberg of M_j) remain unchanged, except potentially in positions $(i+1, n+i-1), \dots, (i+l-1, n+i-1)$. To pursue the process, one restores the J -Hessenberg form. The advantage of this similarity is that first $P^{(j)}$ is symplectic and orthogonal (hence it is stable), preserves most of the created zeros because of the J -Hessenberg structure of M_j and the cost of restoring the J -Hessenberg form of M_j do not exceed $O(in)$. Unlike QR algorithm, SR algorithm still needs profound investigations. This will be the aim of a forthcoming work.

5. Numerical experiments

To illustrate our purpose, we consider the following numerical example. Let A be the 12-by-12 matrix

$$A = \begin{pmatrix} 1 & 5 & 7 & 9 & 5 & 1 & 1 & 3 & 1 & 3 & 7 & 2 \\ 0 & 1 & 4 & 6 & 1 & 2 & 2 & 1 & 5 & 4 & 3 & 5 \\ 0 & 0 & 1 & 2 & 3 & 2 & 0 & 0 & 1 & 2 & 5 & 3 \\ 0 & 0 & 2 & 1 & 9 & 8 & 0 & 0 & 2 & 1 & 2 & 4 \\ 0 & 0 & 0 & 2 & 1 & 3 & 0 & 0 & 5 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 2 & 1 & 0 & 0 & 4 & 3 & 2 & 1 \\ 1 & 4 & 7 & 2 & 1 & 3 & 1 & 7 & 6 & 1 & 6 & 7 \\ 0 & 1 & 9 & 3 & 5 & 1 & 0 & 1 & 4 & 5 & 8 & 3 \\ 0 & 0 & 0 & 2 & 7 & 9 & 0 & 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 & 8 & 0 & 0 & 3 & 1 & 7 & 3 \\ 0 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 4 & 3 & 1 & 2 \\ 0 & 0 & 0 & 9 & 3 & 1 & 0 & 0 & 1 & 2 & 3 & 1 \end{pmatrix}.$$

Following the steps of the algorithms JHESS, JHMSH and JHMSH2, one remarks that the condition of a breakdown is fulfilled at the beginning of the step $j = 3$ for all of them. Let us call MJHESS (respectively JHM²SH and JHM²SH2) the modified algorithm JHESS (respectively JHMSH and JHMSH2) obtained by applying our strategy for curing breakdowns. We obtain the following numerical results, showing the efficiency of the method. Thus the J -orthogonality is numerically preserved up to the machine preci-

		Loss of J -Orthogonality $\ I - S^J S\ _2$			
$2n$		JHESS	MJHESS	JHM ² SH	JHM ² SH2
12	fails		$1.8553e - 15$	$5.0842e - 15$	$6.6428e - 15$

sion for MJHESS (respectively JHM²SH and JHM²SH2). It is worth noting that preserving the J -orthogonality is crucial for SR-algorithm in order to get accurate eigenvalues and vectors of a matrix.

One observes also that the error in the reduction to J -Hessenberg form is very satisfactory for MJHESS (respectively JHM²SH and JHM²SH2). Notice that the algorithm JHESS as given in [2], applied to the matrix A , without our strategy for curing breakdown, simply fails to perform a reduction to a J -Hessenberg form.

$2n$	Reduction error $\ A - SHS^{-1}\ _2$			
	<i>JHCESS</i>	<i>MJHCESS</i>	<i>JHM²SH</i>	<i>JHM²SH2</i>
12	fails	$3.2709e - 14$	$3.8777e - 13$	$2.7653e - 13$

Let $A = S_i H_i S_i^{-1}$, $i = 1, 2, 3$ be the J -Hessenberg reduction obtained respectively by the algorithms MJHCESS, JHM²SH and JHM²SH2. It is known (see [2]) that there exist D_1, D_2, D_3 such that $S_1 D_1 = S_2$, $D_1^{-1} H_1 D_1 = H_2$, $S_1 D_2 = S_3$, $D_2^{-1} H_1 D_2 = H_3$, $S_2 D_3 = S_3$, and $D_3^{-1} H_2 D_3 = H_3$, with each matrix $D_i = \begin{pmatrix} C_i & F_i \\ 0 & C_i^{-1} \end{pmatrix}$, where C_i and F_i are diagonals.

We obtain numerically $C_1 = \text{diag}(1, 1, 0.4113, 0.3688, 0.7747, 0.6638)$ and $F_1 = \text{diag}(0, 0, -2.3962, -2.3371, 1.2880, -1.9982)$, and

$$\|D_1^{-1} H_1 D_1 - H_2\|_2 = 5.3417e - 13.$$

Also, we have

$C_2 = (1, 1, 0.4113, 0.3688, 0.7747, 0.6638)$, $F_2 = \text{diag}(0, 0, -2.3962, -2.3371, 1.2880, -1.9982)$, and

$$\|D_2^{-1} H_1 D_2 - H_3\|_2 = 3.7881e - 13,$$

and finally

$$D_3 = I_{12},$$

where I_{12} stands for identity matrix of size 12, with

$$\|D_3^{-1} H_2 D_3 - H_3\|_2 = 9.4799e - 13.$$

Thus the matrices D_i have numerically the desired forms and as expected, the algorithms JHM²SH and JHM²SH2 perform quite the same results.

6. Conclusions

In this work, we linked the necessary and sufficient condition of the existence of a SR-decomposition with the computations during the process, of some coefficients of the current matrix. The SR-decomposition is intimately related to the J -Hessenberg reduction via the algorithm JHCESS. The later (also JHMSH and its different variants) may encounter fatal breakdowns or suffer from near-breakdowns. We derive efficient strategies for treating them. The numerical experiments show the efficiency of these strategies.

- [1] E. Artin, *Geometric Algebra*, Interscience Publishers, New York, 1957.
- [2] A. Bunse-Gerstner and V. Mehrmann, A symplectic QR-like algorithm for the solution of the real algebraic Riccati equation, *IEEE Trans. Automat. Control* **AC-31** (1986), 1104–1113.
- [3] A. Bunse-Gerstner, Matrix factorizations for symplectic QR-like methods, *Linear Algebra Appl.* **83** (1986), 49–77.
- [4] J. Della-Dora, Numerical linear algorithms and group theory, *Linear Algebra Appl.* **10** (1975), 267–283.
- [5] L. Elsner, On some algebraic problems in connection with general eigenvalue algorithms, *Linear Alg. Appl.*, 26 :123-38 (1979).
- [6] G. Golub and C. Van Loan, *Matrix Computations*, third ed., The Johns Hopkins U.P., Baltimore, 1996.
- [7] C. Paige and C. Van Loan, A Schur decomposition for Hamiltonian matrices, *Linear Algebra Appl.* **41** (1981), 11–32.
- [8] A. Salam, On theoretical and numerical aspects of symplectic Gram-Schmidt-like algorithms, *Numer. Algo.*, **39** (2005), 237-242.
- [9] A. Salam, A. El Farouk, E. Al-Aidarous, Symplectic Householder Transformations for a QR-like decomposition, a Geometric and Algebraic Approaches, *J. of Comput. and Appl. Math.*, Vol. 214, Issue 2, 1 May 2008, Pages 533-548.
- [10] A. Salam and E. Al-Aidarous and A. Elfarouk, Optimal symplectic Householder transformations for SR-decomposition, *Linear Algebra and Its Appl.*, 429 (2008), no. 5-6, 1334-1353.
- [11] A. Salam, E. Al-Aidarous, Error analysis and computational aspects of SR factorization, via optimal symplectic Householder Transformations, *Electronic Trans. on Numer. Anal.*, Vol. 33, pp. 189-206, 2009.
- [12] A. Salam and H. Ben Kahla, An upper J - Hessenberg reduction of a matrix through symplectic Householder transformations, submitted.
- [13] C. Van Loan, A symplectic method for approximating all the eigenvalues of a Hamiltonian matrix, *Linear Algebra Appl.* **61** (1984), 233–251.

- [14] D.S. Watkins, *The Matrix Eigenvalue Problem : GR and Krylov subspace methods*, SIAM, 2007.
- [15] J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, England.