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## Estimating contact forces and pressure in a dense crowd: microscopic and macroscopic models

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### Abstract

This paper deals with the estimation of pressure at collisions times during the movement of a dense crowd. Through the **non-smooth** contact dynamics approach for rigid and deformable solids, proposed by Frémond and his collaborators, the value of pressure and contact forces at collisions points, generated through congestion or panic situation **are estimated**. Firstly, we propose a second-order microscopic model, in which the crowd is treated as a system of rigid solids. Contact forces are rigorously defined by taking into account multiple simultaneous contacts and the non-overlapping condition between pedestrians. We show that for a dense crowd, percussions can be seen as contact forces. Secondly, in order to overcome the restrictive hypothesis related to the geometric form adapted to model the pedestrian, a continuous equivalent approach is proposed where the crowd is modeled as a deformable solid, the pressure is then defined by the divergence of the stress tensor and calculated according to volume and surface constraints. This approach makes it possible to retain an admissible right-velocity, **including both the non-local interactions between non-neighbor pedestrians and the choice of displacement strategy of each pedestrian**.

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Finally, the comparison between the two proposed approaches and some other existing approaches are presented on several illustrative examples to estimate the contact forces between pedestrians.

*Keywords:* Crowd modeling, contact forces, non-smooth contact dynamics, non-local interactions, simultaneous collisions

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## 1. Introduction

The modeling of crowd dynamics has been deeply studied in the 25 past years. **Particularly**, approaches drawn from granular modeling [1, 2, 3] offer both a rich mathematical and physical framework to simulate and comprehend the dynamics of a dense crowd. Predicting and anticipating the different scenarios, that can occur when a crowd moves, is a challenge, **which** must be taken into account for infrastructure planning (**airport**, concert hall, **train station ...**), **it becomes necessary to avoid past tragic dramas of a dense crowd in panic**, and an essential tool to achieve this goal is the prediction of the pressure and the contact forces generated **during collisions of pedestrians** in a moving crowd. **Two major existing approaches that have dealt with this issue can be distinguished:**

- A microscopic approach proposed in [4, 3], where pedestrians are identified with rigid disks that must obey the non-overlapping constraint. The right-velocity after collision is the projection of the left-velocity, corresponding to the desired velocity, over the set of admissible velocities with regard to the non-overlapping constraint. This constrained optimization problem is rewritten in a saddle-point form, and the corresponding Lagrange multipliers are interpreted as the pressure between pedestrians in contact. In this first-order microscopic model, pedestrian mass is not taken into account and the dimension of the Lagrange multipliers is incompatible with that of a pressure. To overcome these problems, we propose a second-order model mainly developed by Frémond[5, 6] **for modeling granular media**. The application of this approach for a crowd has been the subject of our previous work [2, 7]. By adopting this approach, we will rigorously define

the contact forces from the notion of percussion. A comparison with the first-order microscopic model will be presented. We cite also the social force model [8, 9, 10], in which repulsive forces can be considered as contact forces. This force is introduced primarily as a penalty term to keep a certain distance and, consequently, prevent inter-penetration between pedestrians.

- Macroscopic approaches: Macroscopic models have been developed, to describe the evolution of the crowd as a continuum medium, **it can be mentioned among them**, a classical fluid approach [11] where the motion of a crowd and the generated pressure are described by Navier-Stokes equations. Important approaches are developed in [12], [13] and [14], which the pressure is described by constitutive laws, which involves the speed and density of the crowd. A first order macroscopic model is developed by authors in [3], this approach makes it possible to take into account the non-local character of the pressure, however, it has been shown [15] that this model in its current version, reproduces a contradictory effect to that observed experimentally; congestion accelerates the evacuation of a convex room through a single door. Note also that several recently developed approaches like [16] are using kinetic model [17] **did not yet deal with the** pressure issue.

### **Contribution and paper content:**

In this work, **our interest is devoted** to the estimation of contact forces and pressure, in a moving dense crowd, the theory is inspired from the non-smooth approach developed by Frémond and his collaborators both for the evolution of rigid and deformable bodies. To fix the ideas, we consider the configuration of a dense crowd at a given moment (*i.e.* a stationary problem), and we ask how to estimate the contact forces or the generated pressure. Therefore, two classes of models will be developed: In section 2, a microscopic second-order model, in which the crowd is identified with a system of rigid bodies is presented.

In previous works [18, 1], we have evoked this question, and we have arrived at an estimation of the percussion which is the integral of the contact force **during the collision**. Now, we propose to define the contact forces rigorously and with regard to physical proprieties. We start with a brief overview on the mathematical and physical framework of the non-smooth dynamics approach, particularly, the collision equation, then, we show that in the case of a dense crowd where the contacts remain maintained that percussions (jumps of the momentum) qualify as contact forces, ie, they will have the dimension of a force. **As a validation test**, the analytical solution of the contact equation is developed in the case of a constrained pedestrian chain to study the shock wave effect. The contact forces obtained by the proposed approach are compared with the results given by a microscopic model of the first order, and also with those obtained by models of the social force type. In order to overcome the restrictive hypothesis related to the choice of the geometric form of a pedestrian ( rigid disk), and as we are interested in the case of a dense crowd, we model the crowd as a continuous medium, which is the object of section 3. This approach makes it possible to take into account the deformability property by modeling a dense crowd as a deformable solid. Contact forces are deduced from the volume and surface stress tensor which is defined by a constitutive laws. The proposed approach is compared to three other macroscopic approaches.

## **2. A second-order microscopic approach: Crowd is modeled as a system of rigid bodies**

### *2.1. The contact equation*

We consider a crowd as a system of rigid bodies, using the theory of non-smooth contact dynamics by Frémond [5, 6], we obtain the contact equation given by:

$$\mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) + \partial\phi\left(\mathbf{D}\frac{(\mathbf{u}^+ + \mathbf{u}^-)}{2}\right) + \mathbf{N}_{\mathcal{U}_q}(\mathbf{u}^+) \ni 0, \quad (1)$$

where:

- $\mathbf{M}$  is the inertia matrix.
- $\mathbf{u}^+$  (resp  $\mathbf{u}^-$ ) Right-velocity or so-called after impact velocity (resp left-velocity).
- $\phi$  is a pseudo-potential of dissipation,  $\phi = \phi^{i,j}$ ,  $\phi^{i,j}(\mathbf{x}) = k(\mathbf{x})^2$ ,  $k$  is a coefficient of dissipation. This choice allows one to find the classical results when the coefficient of restitution is used.
- $\mathcal{U}_q$  the set of admissible velocities defined by:  
 $\mathcal{U}_q = \{\mathbf{u} \in \mathbf{R}^{2n_p}, D^{i,j}(\mathbf{u}) \cdot \mathbf{n}_q^{i,j} \leq 0, 1 \leq i < j \leq n_p\}$ ,  $D^{i,j}(\mathbf{u})$  is the velocity of deformation corresponding to contact of two pedestrians  $p^i$  and  $p^j$ , it is given by  $D^{i,j}(\mathbf{u}) = \mathbf{u}^i(A) - \mathbf{u}^j(A)$ . In 2D representation, the velocity of deformation at the the point  $A$  is  $D^{i,j}(\mathbf{u}) = \mathbf{u}^i(\mathbf{q}^i) + w^i \times G^i A - \mathbf{u}^j(\mathbf{q}^j) - w^j \times G^j A$ , where  $w^i$  is the rotation velocity of pedestrian  $p^i$ . Finally,  $\mathbf{D}(\mathbf{u}) = \{\mathbf{D}^{i,j}(\mathbf{u}), 1 \leq i < j \leq n_p\}$  is a column vector of size  $n_c$ , where  $n_c$  is the number of contact.
- $\mathbf{N}_{\mathcal{U}_q}(\mathbf{u})$  the normal cone of  $\mathcal{U}_q$  at point  $\mathbf{u}$ .

Non local interactions are introduced by percussion  $\mathbf{p}^{i,j}, 1 \leq i < j \leq n_p$ , with:

$$\mathbf{p}_{reac}^{i,j} \in \partial \mathbb{1}_{\mathbb{R}^+} (D^{i,j}(\mathbf{u}^+) \cdot \mathbf{n}_q^{i,j}),$$

where  $\mathbf{n}_q^{i,j}$  is the normal oriented from  $p^i$  to  $p^j$ , and  $\mathbb{1}_{\mathbb{R}^+}(x) = 0$  if  $x \geq 0, +\infty$  otherwise.

Total percussion of reaction  $\mathbf{p}_{reac} = \{\mathbf{p}_{reac}^{i,j}, 1 \leq i < j \leq n_p\}$  has the form:

$$\mathbf{p}_{reac} \in \partial \mathbb{1}_{\mathcal{U}} (\mathbf{u}^+) = \mathbf{N}_{\mathcal{U}}(\mathbf{u}^+)$$

This approach makes it possible to manage multiple and simultaneous contact and to take into account the corresponding local and non-local (at a distance) interactions. Moreover, the capacity to use different constitutive laws make it possible to account for diverse experimental results.

After reformulation of the the contact equation in the **saddle** point form, the solution  $\mathbf{u}^+$  and the percussions of reaction are characterized by:

$$\begin{cases} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) + k\mathbf{D}^T\mathbf{D}(\mathbf{u}^+ + \mathbf{u}^-) + \mathbf{C}^T\lambda = 0 \\ \mathbf{C}\mathbf{u}^+.\lambda = 0 \end{cases}$$

where  $\mathbf{C}$  the matrix defining the constraints on the normal deformation velocities, and  $\lambda^{i,j}$  the corresponding Lagrange multiplier to the contact between pedestrians  $p^i$  and  $p^j$ . We note that  $p_{reac}^{i,j} = \lambda^{i,j} \mathbf{n}^{i,j}$ .

For a dense crowd, contacts are maintained, the constraint of non-interpenetration is saturated, **i.e.** :  $\mathbf{C}\mathbf{u}^+ = 0$ , which means that contact is purely inelastic, and there is no dissipation ( $k = 0$ ). Consequently,  $\lambda$  is a solution of:

$$\mathbf{W}\lambda = \mathbf{C}\mathbf{u}^-, \quad \mathbf{W} = \mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T, \quad (2)$$

$\mathbf{W}$  is the Delassus matrix.

According to the collision equation Eq.(1), the percussion has the dimension of momentum, and hence it is perfectly for an instantaneous collision. For a dense crowd which is the case we are interested in, it does not remain true, indeed: the contact is maintained and the duration is important, because of these considerations, the motion of the crowd is considered smooth and Eq.(1) can be rewritten for a purely inelastic contact as a second-order spewing process [19, 20]:

$$-\mathbf{M}\frac{d\mathbf{u}}{dt} \in \mathbf{N}_{\tilde{\mathcal{U}}_q}(\mathbf{u}),$$

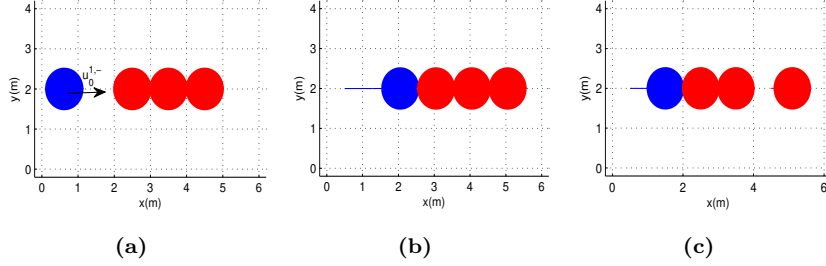
where  $\tilde{\mathcal{U}}_q$  is the moving set of admissible velocities. The contact forces  $\mathbf{f}_{contact}$  correspond to a selector in the outward normal cone, **i.e.** :

$$-\mathbf{M}\frac{d\mathbf{u}}{dt} = \mathbf{f}_{contact}, \text{ where } \mathbf{f}_{contact} \in \mathbf{N}_{\mathcal{V}_{ad}}(\mathbf{u}).$$

In this formalism the acceleration has a sens and consequently the contact force  $\mathbf{f}_{contact}$  which approached numerically by  $\mathbf{C}^T\lambda$ .

## 2.2. Shock wave and non-local interactions

We consider two examples: the first one corresponds to the crashing of a moving disk with three aligned disks, having the same mass  $m = 75kg$  and at



**Figure 1:** Simultaneous contact of four aligned disks. (a) The initial state, (b) The state after impact for an inelastic contact ( $k=0$ ), (c) the state after impact for an elastic contact ( $k=10$ )

rest. Here, the pedestrians are modeled by rigid disks see Fig. 1, generated contact force between the blue and the neighbored red disks is given by the formulate 3, this shows that the movement of each pedestrian is influenced by the movement of others.

$$\mathbf{p}_{react}^{1,2} = -m^1 \frac{m^2(\mathbf{u}^{2,-} - \mathbf{u}^{1,-}) + m^3(\mathbf{u}^{3,-} - \mathbf{u}^{1,-}) + m^4(\mathbf{u}^{4,-} - \mathbf{u}^{1,-})}{m^1 + m^2 + m^3 + m^4}, \quad (3)$$

As one can see the percussion  $\mathbf{p}_{react}^{1,2}$  involves  $\mathbf{u}^{3,-}$  and  $\mathbf{u}^{4,-}$  besides  $\mathbf{u}^{1,-}$  and  $\mathbf{u}^{2,-}$ , we obtain the same for  $\mathbf{p}_{react}^{2,3}$  and  $\mathbf{p}_{react}^{3,4}$ . The effect of the shock wave is responsible for the non-local interaction ( or the so-called indirect interactions ) between non-neighboring pedestrians [21].

The second example, see Fig. 2, it shows the displacement of 53 pedestrians of the same mass  $m = 75kg$ , they are supposed initially at rest in a corridor after being pushed by three moving pedestrians. Fig. 2c illustrates the transmission of contact force by local and non-local interaction in a crowd.

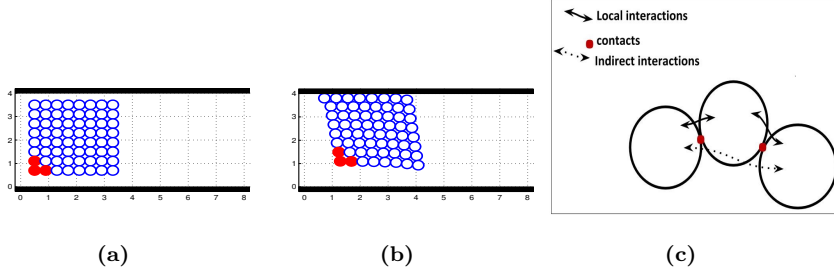
### 2.3. Estimation of contact forces: Comparison of the proposed discrete approach to other microscopic models

#### 2.3.1. The proposed discrete approach versus the first-order microscopic model

In [4], contact forces are characterized by the following discrete Poisson equation:

$$\mathbf{C}\mathbf{C}^T\lambda = \mathbf{C}\mathbf{u}^-, \quad (4)$$





**Figure 2:** Effect of the shock wave for a group of 56 pedestrians in a corridor, blue discs: pedestrians supposed to rest at the initial state while the three pedestrians in red push with an initial velocity  $v = (2, 2)$ . (a) initial stat, (b) final state, (c) schematization of local and non-local interactions in the discrete model.

where  $\mathbf{C}\mathbf{C}^T$  is considered as the discrete Laplacian matrix. According to this equation Eq 4, we see clearly that the authors assumed that all pedestrians have the same mass  $m = 1kg$ , this leads to the equilibrium situations and produces undesirable scenarios like the complete congestion [4], which is specially related to the geometry of pedestrians (disks) and their size (radius of disks). By choosing the matrix inertia equals to the identity matrix, the equation Eq 4 is a special case of the proposed approach given by Eq..

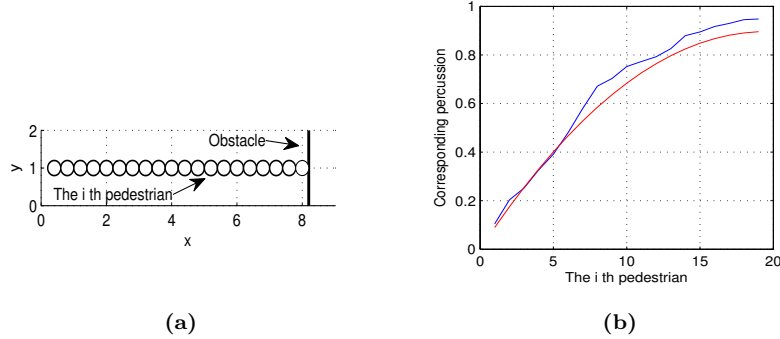
In order to show that the hypothesis "Pedestrians have the same mass  $m = 1kg$ " made by the authors is restrictive, we start by a one-dimensional case of a pedestrian chain. Eqs.5 and 6 represent the analytical solution in the trivial case of an aligned constrained pedestrian chain as illustrated in Fig.3a, they show the dependence on the mass and the Lagrange multipliers (and hence the contact forces).

$$\begin{cases} \lambda^{1,2} = -m^1(u - u^{1,-}), \\ \lambda^{i,i+1} = \lambda^{i-1,i} - m^i(u - u^{i,-}), \quad 2 \leq i < n_p. \end{cases} \quad (5)$$

and

$$u^{i,+} = u = \frac{\sum_{i=1}^{n_p} m^i u^{i,-}}{\sum_{i=1}^{n_p} m^i}, \quad 1 \leq i \leq n_p, \quad (6)$$

In addition, Fig.3b shows the difference in the percussion suffered by each pedestrian, in the cases where pedestrians have identical and different masses.



**Figure 3:** (a) Constrained pedestrian chain, (b) the generated percussion, red line: pedestrians have the same masses, blue line: pedestrians have different masses.

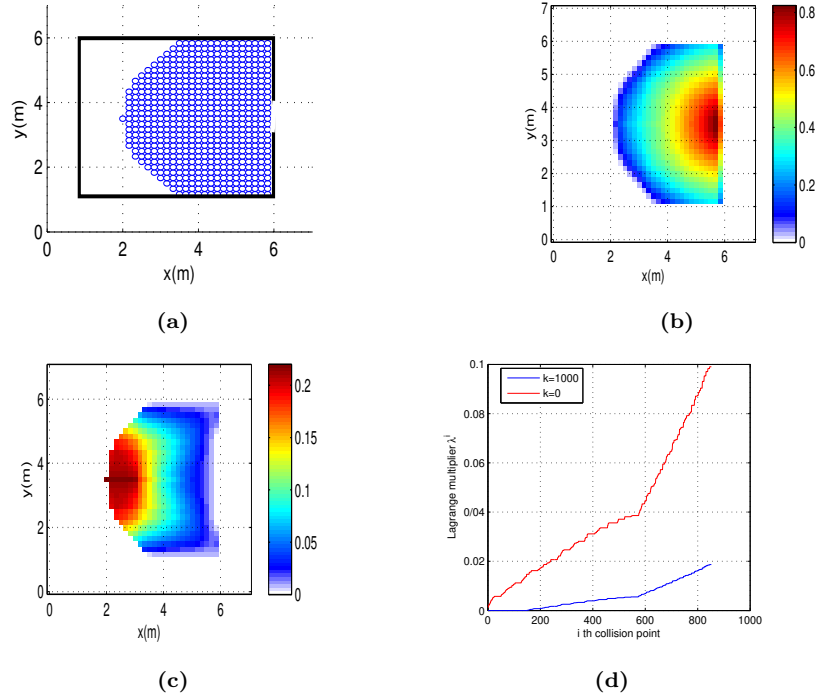
In the 2D simulation case, we are interested in estimating the contact forces generated during the evacuation of a room, we choose the instant correspond to the formation of the ark around the exit, see Fig.4a. The number of pedestrians considered is 594 with the same mass  $m = 75Kg$  and a maximum density of 9 pedestrians per square meter. The contacts are assumed purely inelastic (i.e.  $k = 0$ ), and pedestrians have a desired velocity towards the exit of a norm equals to  $2m.s^{-1}$ . Right-velocity and the contact forces are given by:

$$\mathbf{u}^+ = \mathbf{u}^- + \mathbf{C}^T \lambda, \quad (7)$$

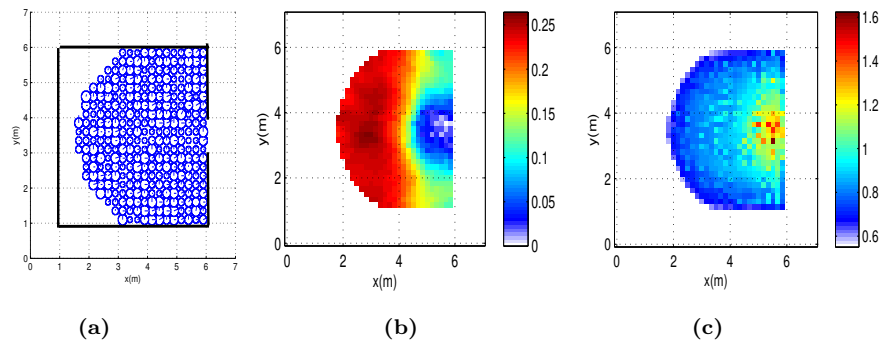
$$\text{where } \mathbf{W} \lambda = \mathbf{C} \mathbf{u}^-. \quad (8)$$

Obtained results are given in Fig.4: We notice that the force is maximum in the neighborhood of the exit Fig. 4b, where the movement of the crowd is rallied Fig.4c. As we mentioned, the proposed approach is general, in a way that we can take different masses, to produce a realistic and different behavior according to the calibration of the coefficient of dissipation  $k$ .

Fig. 5 shows the obtained results in the case where pedestrians have different masses, the left-velocity is chosen uniformly, and yet, we notice that neither the speed on the right nor the pressure are uniform with respect to the distance to the exit.



**Figure 4:** Estimation of contact forces in the situation of an arch formation with pedestrians of the same masses. (a) Initial configuration as an arch, (b) Right-velocity, (c) Norm of the corresponding contact force applied on each pedestrian, (d) Influence of the dissipation coefficient  $k$  on the contact force.



**Figure 5:** Estimation of contact forces in the situation of an arch formation with pedestrians of different masses. (a) Initial configuration as an arch, (b) Right-velocity, (c) Norm of the corresponding contact force applied on each pedestrian.

### 2.3.2. The proposed approach versus social force model

In [9], **the authors** have defined contact force  $f_{ij}$  between two pedestrians  $i$  and  $j$  by:

$$\mathbf{f}_{ij} = \{A_i \exp[(r_{ij} - d_{ij})/B_i] + \varrho g(r_{ij} - d_{ij})\} \mathbf{n}_{ij} + \varsigma, g(r_{ij} - d_{ij}) \Delta v_{ij}^t \mathbf{t}_{ij} \quad (9)$$

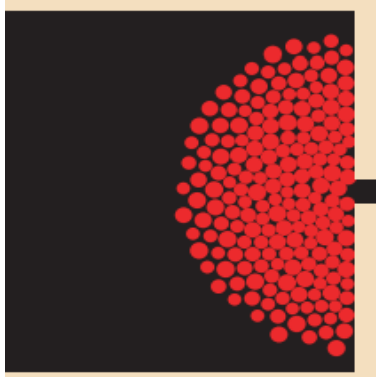
where:

- $A_i$  and  $B_i$  are constants,  $A_i = 2.10^3 N$  and  $B_i = 0.08m$ . [9].
- $r_{ij} := r_i + r_j$  the sum of radius  $r_i$  and  $r_j$  of pedestrians  $i$  and  $j$ .
- $d_{ij}$  denotes the distance between the pedestrians' mass center.
- $\mathbf{n}_{ij}$  is the normalized vector pointing from pedestrian  $j$  to pedestrian  $i$ , and  $\mathbf{t}_{ij}$  the tangential direction.
- The function  $g(x)$  is zero if the pedestrians are not in contact ( $d_{ij} \geq r_{ij}$ ), and is otherwise equal to the argument  $x$ .
- The parameters  $\varrho = 1.2 \times 10^5 kg.s^{-2}$  and  $\varsigma = 2.4 \times 10^5 kg.m^{-1}.s^{-1}$  determine the obstruction effects in case of physical interactions.

Let us take again the one-dimensional example of the constrained pedestrian chain Fig.3a, according to the formula (9), if pedestrians are considered as rigid disks / spheres, than the contact force between each pair of pedestrians  $i$  and  $j$  is constant throughout the chain of equal intensity  $\|\mathbf{f}_{i,j}\| = A_i$  (because  $d_{i,j} = r_{i,j}$ ), which of course is not realistic. Else if the pedestrians are modeled as a deformable disk, the contact forces should be calculated over the entire contact surface and not at a single point of a contact, as in formula (9).

In the two-dimensional case, we consider the same evacuation simulation of a room made by the authors in [9]. Fig.6 clearly shows that there is no overlap between the pedestrians, which is, to say, still according to the formula (9), that the contact forces are constant independently of the masses, positions, desired velocities and also the density of the crowd. Therefore, the main difference between the proposed approach and the social force model, at the level of the

estimate of the contact force, **is due to the physical basis of each approach.** In the social force model, which is inspired by the smooth dynamics theory, contact forces are introduced as a penalty term in order to avoid the overlapping of individuals, Conversely to the proposed approach based on the non-smooth theory of contact dynamics, and rely on the notion of percussion which is the integral of the contact force during the collision. When the duration of the contact becomes important, the percussion converges towards the contact force [19]. An alternative contact force is proposed by the authors in [10], where the



**Figure 6:** Evacuation of a room by authors in [9], where there is no overlap between pedestrians, which gives a constant contact force between all pedestrians in contact

contact force  $\mathbf{f}_i$  is applied to a pedestrian  $i$  by another pedestrian  $j$ , their model is given by:

$$\mathbf{f}_i = \begin{cases} \frac{-f_{max}}{1+d_i^2} & \text{if } d_{i,j} \leq r_{i,j}, \\ \frac{-2f_{max}}{1+d_i^2} & \text{if } d_{i,j} > r_{i,j}, \end{cases} \quad (10)$$

where:  $d_{i,j}$  the distance between pedestrian  $i$  and pedestrian  $j$ ,  $d_i = \frac{d_{i,j}}{r_i}$ ,  $f_{max}$  is a maximum force such that  $\frac{f_{max}}{m_i} = O(max(\alpha_i, \alpha_j))$ , with  $m_i$  the mass of the pedestrian  $i$ , and  $\alpha_i$  its acceleration. The formula shows that the force depends on the distance between the pedestrians, and that the contact force can be non-zero even if no contact takes place, contrary to the proposed approach where the contact force is active only at the instant of contact.

To conclude, the proposed approach based on the theory of rigid solid dynamics allows to manage the simultaneous contact in a crowd of pedestrians with different masses, and to estimate the corresponding contact forces which is rigorously defined from both a mathematical and physical point of view. It is also possible to show the effect of the shock wave responsible for non-local interactions. Managing and treating multiple and simultaneous collisions (including the non interpenetration condition) in a dense crowd, where panic situations are very common, is the key point of the approach developed above. However, to model a pedestrian as a rigid disk stays a restrictive hypothesis. It is because of this problem, that we are focusing to treat in the following section by identifying the entire crowd as a deformable solid.

### 3. A macroscopic approach: Crowd is modeled as a deformable solid

The proposed continuous approach is also inspired by the **non-smooth** view of contact dynamics for deformable solids developed by Frémond [6]. As previously pointed out, we are interested in the problem of contact in a dense crowd, i.e. calculating the right-velocity and the generated pressure. Generally, macroscopic models consist on transporting the density, which represents the evolution of the crowd, by an admissible velocity, this latter can depend on several quantities such as the desired direction, overpopulated zones, flow direction, etc. We assume that all these considerations are taken into account by the left-velocity (velocity before the impact). We take inspiration from the basic approach established by Frémond to model the collision of a deformable solid with a rigid plane, the contact forces between pedestrians are given by the stress tensor, and pedestrians-obstacles contact forces are deduced from the constraints of non interpenetration.

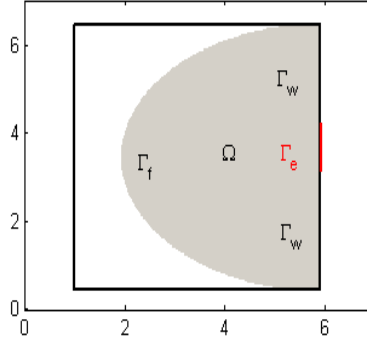
#### 3.1. The model: Mathematical and physical framework

We consider a dense moving crowd, whose density is denoted at instant  $t$  and point  $\mathbf{x}$  by  $\rho(t, \mathbf{x})$ , the left-velocity is  $\mathbf{u}^-(t_c, \mathbf{x})$ , where  $t_c$  is a collision instant.

Applying the principle of virtual work on the crowd, we obtain at collision instant the motion equations given by:

$$\begin{aligned}\rho(t_c, \mathbf{x}) (\mathbf{u}^+(t_c, \mathbf{x}) - \mathbf{u}^-(t_c, \mathbf{x})) &= \operatorname{div} \sigma(t_c, \mathbf{x}), \mathbf{x} \in \Omega, \\ \sigma(t_c, \mathbf{x}) \mathbf{n}(t_c, \mathbf{x}) &= -\mathbf{r}_{int}(t_c, \mathbf{x}), \mathbf{x} \in \Gamma_w, \\ \sigma(t_c, \mathbf{x}) \mathbf{n}(t_c, \mathbf{x}) &= \mathbf{r}_{ext}(t_c, \mathbf{x}), \mathbf{x} \in \Gamma_f, \\ \sigma(t_c, \mathbf{x}) \mathbf{n}(t_c, \mathbf{x}) &= 0, \mathbf{x} \in \Gamma_e,\end{aligned}$$

where  $\Omega$  is the area occupied by the crowd, whose border  $\partial\Omega$  is defined by:  $\partial\Omega = \Gamma_w \cup \Gamma_f \cup \Gamma_e$ , where  $\Gamma_w$  are the contact points with obstacles (walls, ...),  $\Gamma_e$  the exits, and  $\Gamma_f$  the free border that has no contact with any obstacle, see Fig.7. **Constitutive laws are needed to define the contact percussions  $\mathbf{r}_{int}$**



**Figure 7:** 2D domain of evolution.

, and the percussion stresses  $\sigma$ . **They are defined respectively** by a surface pseudo-potential of dissipation  $\phi_s$ ? that takes into account the impenetrability condition, and a volume pseudo-potential of dissipation  $\phi_v$  that defines the volumetric constraints:

$$\begin{aligned}\phi_s \left( D \left( \frac{\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x})}{2} \right) \right) &= k_s \left( \frac{\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x})}{2} \right)^2 \dots \\ &\quad + I_- (\mathbf{u}^+(t_c, \mathbf{x}) \cdot \mathbf{n}(t_c, \mathbf{x})), \\ \phi_v \left( D \left( \frac{\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x})}{2} \right) \right) &= k_v \left\{ D \left( \frac{\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x})}{2} \right) \right\}^2,\end{aligned}$$

with  $D(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$ .

$\mathbf{r}_{int}$  (respectively  $\sigma$ ) is the sub-differential of  $\phi_s$  (respectively of  $\phi_v$ ) at point  $\frac{\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x})}{2}$ .

$$\begin{aligned} \mathbf{r}_{int}(t_c, \mathbf{x}) &= k_s (\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x})) \dots \\ &\quad + \partial I_- (\mathbf{u}^+(t_c, \mathbf{x}), \mathbf{n}(t_c, \mathbf{x})) \cdot \mathbf{n}(t_c, \mathbf{x}), \\ \sigma(t_c, \mathbf{x}) &= k_v (D (\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x}))). \end{aligned}$$

Finally, the problem describing the crowd at the instant of collision is:

$$\left\{ \begin{array}{l} \rho(t_c, \mathbf{x}) [\mathbf{u}(t_c, \mathbf{x})] = k_v \operatorname{div} (D (\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x}))), \mathbf{x} \in \Omega, \\ \sigma(t_c, \mathbf{x}) \mathbf{n}(t_c, \mathbf{x}) + k_s (\mathbf{u}^+(t_c, \mathbf{x}) + \mathbf{u}^-(t_c, \mathbf{x})) \dots \\ \quad + \partial I_- (\mathbf{u}^+(t_c, \mathbf{x}), \mathbf{n}(t_c, \mathbf{x})) \cdot \mathbf{n}(t_c, \mathbf{x}) \ni 0, \mathbf{x} \in \Gamma_w, \\ \sigma(t_c, \mathbf{x}) \mathbf{n}(t_c, \mathbf{x}) = \mathbf{r}_{ext}(t_c, \mathbf{x}), \mathbf{x} \in \Gamma_f, \\ \sigma(t_c, \mathbf{x}) \mathbf{n}(t_c, \mathbf{x}) = 0, \mathbf{x} \in \Gamma_e. \end{array} \right. \quad (11)$$

$[\mathbf{u}(t_c, \mathbf{x})] := (\mathbf{u}^+(t_c, \mathbf{x}) - \mathbf{u}^-(t_c, \mathbf{x}))$  the jump of velocity discontinuity. For the existence and uniqueness of the solution of the problem described by Eqs.(11), it is shown in [5] that at the collision instant, if the domain  $\Omega$  is regular, with  $\operatorname{mes}(\Gamma_w) > 0$ , and  $\rho(t_c, \mathbf{x}) \geq \rho_0 > 0$ , if  $\mathbf{u}^-(t_c, \cdot) \in H^1(\Omega)$ ,  $\mathbf{r}_{ext}(t_c, \cdot) \in L^2(\Omega)$  then there is an unique right-velocity  $\mathbf{u}^+(t_c, \cdot) \in H^1(\Omega)$  solution of the problem (11).

### 3.2. Numerical simulations and comparisons

#### 3.2.1. Validation tests: Jostling in a pedestrians chain.

We take the one-dimensional example of the pedestrians chain introduced previously. For a constrained pedestrian chain, i.e. a fixed obstacle is placed in front of pedestrians at the point  $x = l$ , see Fig.3a. The problem (11) becomes :

$$\left\{ \begin{array}{l} \rho(t_c, x) (\mathbf{u}^+(t_c, x) - \mathbf{u}^-(t_c, x)) = k_v \frac{\partial^2 (\mathbf{u}^+(t_c, x) + \mathbf{u}^-(t_c, x))}{\partial x^2}, x \in ]0, l[, \\ -k_v \frac{\partial (\mathbf{u}^+(t_c, l) + \mathbf{u}^-(t_c, l))}{\partial x} + k_s (\mathbf{u}^+(t_c, l) + \mathbf{u}^-(t_c, l)) + \partial I_- (\mathbf{u}^+(l)) \ni 0. \\ \frac{\partial (\mathbf{u}^+(t_c, 0) + \mathbf{u}^-(t_c, 0))}{\partial x} = 0. \end{array} \right. \quad (12)$$



For a constant left-velocity and an uniform density, under the following assumption:

$$\left\{ \tilde{k} \geq \frac{e^{-2l/h} - 1}{(e^{-2l/h} + 1)}, \tilde{k} = \frac{k_s}{k_v}, \frac{1}{h^2} = \frac{\rho}{k_v} \right\}, \quad (13)$$

the solution of the unidimensional problem (12) is:

$$u^+(t_c, x) = u^-(t_c, x) \left( 1 - \frac{e^{x/h} + e^{-x/h}}{e^{l/h} + e^{-l/h}} \right), \text{ with } h = \sqrt{\frac{\tilde{k}}{\rho}},$$

and internal percussion is given by (14), it becomes maximal as  $x$  goes to  $l$ .

$$r_{int}(t_c, x) = u^-(t_c, x) \frac{e^{x/h} - e^{-x/h}}{h(e^{l/h} + e^{-l/h})} \quad (14)$$

The condition 13 implies that  $u^+(l) = 0$ , **i.e.**, the right pedestrian remains crushed against the obstacle. However, the other pedestrians stay glued to each other **after the impact**. If the condition (13) is not verified, rebounds will appear and the solution must be changed. Note that for an unconstrained pedestrians chain with a constant left-velocity, the right-velocity after collision is constant and the pressure is equal to zero. However, for an unconstrained pedestrian chain with a decreasing left-velocity, we get the same result as in the rigid case.

By analogy to the constrained pedestrian chain in the second-order microscopic model, we evaluate the pressure for a uniform density and a variable density. For this, it is quite enough to mark that the percussion given by equation 14 is an increasing function, with respect to the density  $\rho$ . This is exactly the same result as in microscopic case.

### 3.2.2. The proposed macroscopic approach versus continuous approach for modeling crowd turbulence

In [10], inter-pedestrians stress is given by a symmetric  $2 \times 2$  tensor field representing the internal forces in the crowd:

$$\sigma = p\mathbb{I} + \mathbf{S} \quad (15)$$

where the normal component  $p$  represents the normal or repulsive forces, while the deviatoric stress  $\mathbf{S}$  represents the frictional forces. The pressure  $p$  is defined by an increasing density function  $g$  :

$$g(\rho) = \begin{cases} 0 & \text{if } \rho \leq \rho_c \\ c_1 \frac{\rho}{\sqrt{\rho_y - \rho + \epsilon}} - c_2 & \text{if } \rho_c < \rho < \rho_y \\ c_3 \rho - c_2 & \text{if } \rho \geq \rho_y. \end{cases} \quad (16)$$

where the values of the parameters  $\rho_c, \rho_y, c_1, c_2, c_3$  and  $\epsilon$  are given in the table 1. The value of frictional stress  $\mathbf{S}$  is determined by minimizing the kinetic energy  $\mathcal{E}$ :

$$\mathcal{E} = \frac{1}{\rho_y} \int \|\rho(\mathbf{v} - \nabla p) + \delta t \nabla \cdot \mathbf{S}\|^2 dV \quad (17)$$

subjected to the following constraint:

$$\|\mathbf{S}\|_F \leq \sqrt{3}\beta g(\rho) \quad (18)$$

with  $\|\cdot\|_F$  is the Frobenius norm, and  $\alpha$  the coefficient of friction.

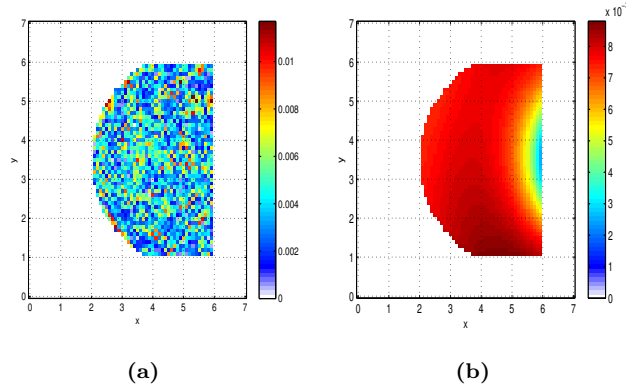
The contact forces are given by:

$$\nabla \cdot \sigma = \nabla p + \nabla \cdot \mathbf{S} \quad (19)$$

To compare this approach with our proposed macroscopic model, we consider the example of an arch formation during an emergency evacuation of a room, previously introduced in the discrete case. The room is a square with exit side of  $1m$ . Desired velocity  $\mathbf{u}_d$  is defined by:  $\mathbf{u}_d = u_d \mathbf{e}_d$ , where  $u_d$  is the free speed and  $\mathbf{e}_d$  the desired direction,  $\mathbf{e}_d(\mathbf{x}) = \frac{\mathbf{x} - P_{\Gamma_e}(\mathbf{x})}{\|\mathbf{x} - P_{\Gamma_e}(\mathbf{x})\|}$ , where  $P_{\Gamma_e}(\mathbf{x})$  is the projection of point  $\mathbf{x}$  on the exit.

Fig.8 shows the error corresponding to the difference between the contact force intensity, found by the authors in [10], and our proposed approach. We note the following remarks:

- We obtained similar results for  $k_v \geq 4$  and a time step  $\delta t = 0.01s$  (error of the order of  $10^{-2}$ ). As in the discrete case, the dissipation coefficient reflects the aggressive nature of pedestrians [2].



**Figure 8:** The norm of the difference between the contact force modulus calculated by the authors [10] and that by the proposed approach.(a) Error corresponding to a random desired velocity of an average  $1.7m/s$ , standard deviation  $0.5m/s$  and an uniform density equals to 7 pedestrians per square meter.(b) Error for an uniform desired velocity equals to  $1.23 m : s$  and density equals to 7 pedestrians per square meter

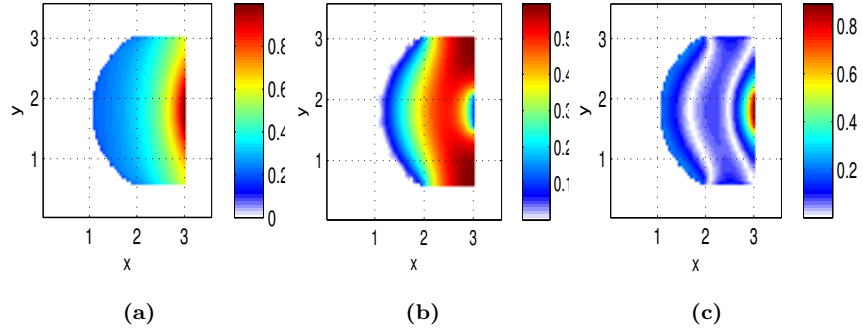
- In both approaches, the estimation of contact forces is inspired by the modeling of granular media. Authors in [10] proposed a constitutive law allowing to define the density-pressure dependence, and frictional forces minimizing kinetic energy over a short period. In the proposed approach, all these considerations have been taken into account by the pseudo-potential of dissipation that defines the percussions tensor.
- The contact forces must be calculated independently to the time discretization step, this is not possible within the framework of the smooth dynamics that is the basis of the approach in [10].
- The dependence of the kinetic energy by the step of discretization in time gives different values of frictional force according to the time step.

### 3.2.3. The proposed approach versus a first order macroscopic model

In [3], the authors proposed a first order continuous model based on the mass conservation, in which the pressure is characterized by the following place equation:

$$\begin{cases} -\Delta p = -\nabla \cdot \mathbf{u}_d & \text{in } \Omega \\ \frac{\partial p}{\partial n} = 0 & \text{on } \Gamma_w \\ p = 0 & \text{on } \Gamma_e \cup \Gamma_f \end{cases} \quad (20)$$

Fig.9 shows the results obtained by the proposed approach and the first order model for a uniform desired velocity  $u_d = 1.7m/s$  and a uniform density  $\rho = 7$  pedestrians per square meter. The proposed approach gives a Gaussian form of pressure concentrated around the exit, this result reflects the accumulation of forces near the exit. However, in the case of the first order macroscopic model, the pressure is minimal around the exit. Consequently, as the authors in [3] confirm, the velocity is maximum at these zones, which is contradictory with reality.



**Figure 9:** pressure estimation in the case of arch formation, (a) the proposed approach, (b) the first order macroscopic model, (c) the relative error.

### 3.2.4. The proposed approach versus constitutive laws

In this part, we will compare the variation of pressure and the right-velocity with respect to the density in the case of our approach, with constitutive laws

describing the density-pressure and the density-speed dependence, Fig.10.

Fig. 10a shows pressure as a function of the right-velocity.

We are interested in tracing the behavior of the density-pressure and density-right-velocity, then compare it with others models. For the dependence density-pressure, we compare our result with two models. The first one is proposed in [10] and it is also restrained by the authors in [12]. Pressure-density relation is given by:

$$(M_1) \quad p(\mathbf{x}, t) = \rho(\mathbf{x}, t)Var(V)$$

where  $\rho(\mathbf{x}, t)$  indicates the density at location  $\mathbf{x}$ ,  $Var(V)$  is the current velocity variance, and  $V(\mathbf{x}, t) = \sum_i ||u_i^+||f(d_{i,\mathbf{x}})$ ,  $d_{i,\mathbf{x}}$  is the distance of location  $\mathbf{x}$  and pedestrian  $i$ , and  $f(d_{i,\mathbf{x}}) = \frac{1}{\pi R^2}exp(-\frac{d^2}{R^2})$ , where  $R$  is a measurement parameter. In the second model [13], the density-pressure relation is defined by the following constitutive law:

$$(M_2) \quad p(\rho) = p_0\rho^\gamma, p_0 > 0, \gamma > 1.$$

where  $p_0$  is the pressure coefficient and  $\gamma$  the adiabatic exponent.

Concerning the variation of the speed as a function of the density, we compare our result with two constitutive laws, the first is adapted by authors in [13], which is given by:

$$(L_1) \quad \rho(u) = u_{max}e^{-\alpha(\frac{\rho}{\rho_{max}})^2},$$

the second is the Kladek formula introduced by Weidmann [22], it is adopted in the revisited version proposed in [14]:

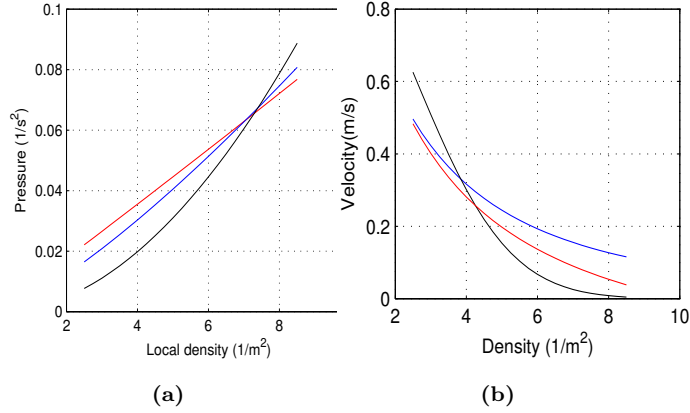
$$(L_2) \quad \rho(u) = u_{max}\left(1 - e^{-\gamma(\frac{1}{\rho} - \frac{1}{\rho_{max}})}\right),$$

where  $\rho_{max}$  is the congestion density,  $u_{max}$  is the maximum walking speed in unconstrained conditions (also called free speed), and  $\gamma$  is an exponent.

The simulation data are given in the following table 1, note that we kept the same values of the parameters appearing in the formulas  $(M_1)$ ,  $(M_2)$ ,  $(L_1)$  and  $(L_2)$ , that are given in [12], [13] and [14].

Parameter name	Symbol	Value	Units
volume dissipation coefficient	$k_v$	9	$N.s^2.m^{-2}$
surface dissipation coefficient	$k_s$	1	$N.s^2.m^{-3}$
desired speed	$u_{max}$	2	$m.s^{-1}$
maximal density	$\rho_{max}$	8.5	$ped.m^{-2}$
density-speed coefficient	$\alpha$	7.5	-
adiabatic exponent	$\gamma$	2	-
pressure coefficient	$p_0$	0.005	$ped^{1-\gamma}.m^{2+\gamma}.s^{-2}$
measurement parameter	R	0.7	m
yield density	$\rho_y$	5.5	$ped/m^2$
critical density	$\rho_c$	3.5	$ped/m^2$
epsilon	$\epsilon$	0.01	-
coefficient 1	$c_1$	13.33	-
coefficient 2	$c_2$	1.5	-
coefficient 3	$c_3$	$\frac{c_1}{\sqrt{\epsilon}}$	-
frictional coefficient	$\beta$	0.75	-
step time	$\delta t$	0.01	s

**Table 1:** Parameter values used in the numerical simulations

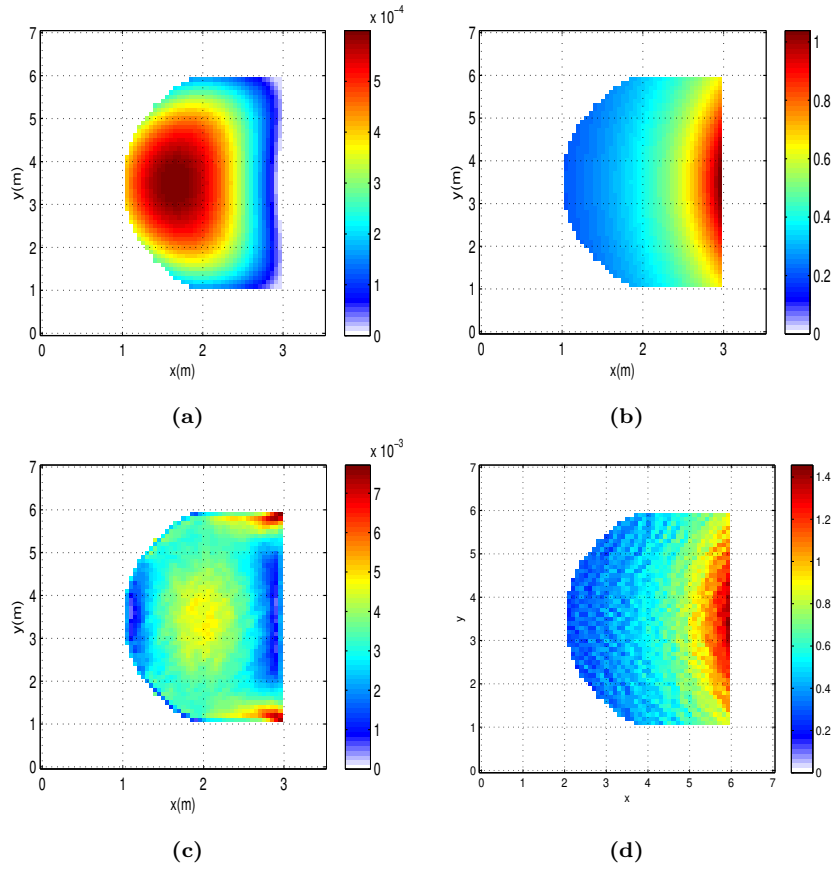


**Figure 10:** (a) Pressure versus density, Red line ( $M_1$ ), black line ( $M_2$ ) blue line (proposed approach), (b) Density versus velocity, red line ( $L_1$ ), black line ( $L_2$ ) blue line (proposed approach).

The complete congestion is undesirable, as it can be seen in actual emergency evacuation situations, this is consistent with the result obtained by the proposed approach Fig.10b (blue line), unlike the two constitutive laws Fig.10b (black and red line).

Figs.11a and 11b represent respectively the right-velocity and the corresponding pressure for a given uniform left-velocity and initial density. Figs. 11c and 11d correspond respectively, to the right-velocity and the generated pressure for a given random left-velocity.

According to Fig11, the right-velocity is canceled at the boundary of the domain which reflects the condition of non-interpenetration with the wall, and the pressure is maximal around the exit, as for the microscopic case. Particularly, the general form of the right-velocity depends on the left-velocity, for (a) and (b) where the left-velocity is chosen uniformly with respect to the distance to the exit, the right-velocity keeps the same form. For a random left-velocity, we find a random right-velocity. We recall here for this example, the density is chosen uniformly, the speed also depends on the density.



**Figure 11:** Right-velocity (a) and the correspond pressure (b) for an uniform left-velocity. (c) right-velocity and (d) pressure for a random left-velocity of average  $1.34m/s$  and standard deviation of  $0.26m/s$ .



#### 4. Conclusion

In this paper, we are interested in estimating contact forces generated in a dense crowd from both discrete and continuous point of view. The mathematical and physical framework is inspired by the non-smooth contact dynamics theory developed by Frémond and his collaborators for rigid and deformable bodies assemblies. Starting with the microscopic approach, it has been shown the ability of this approach to manage multiple and simultaneous contacts, and to estimate the corresponding contact force via the notion of percussion. The case of arch formation is studied, the obtained results and the comparative study with other contact force approximation techniques show the superiority of the proposed approach. The pedestrian choice form remains the main criticism of all discrete models, different values can be obtained depending on the pedestrian representation and the type of contact. To overcome this challenge, a continuous version of the discrete approach is introduced. It make it possible to keep highlights of the discrete case and to estimate rigorously the contact forces by the stress tensor. It has been shown also that, for a good choice of the values of modeling parameters, we find results more significant and realistic than those found in the literature. As perspectives, the performance of the continuous approach is considered. In particular, taking into account the condition of non-interpenetration between pedestrians as in the discrete case and the study of the convergence of the microscopic approach towards the macroscopic approach. Exploring the case where each pedestrian is modeled as a deformable solid also holds our attention in future work.

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