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Scale effect and higher-order boundary conditions for generalized lattices, with direct and indirect interactions

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Abstract – The effects of higher-order boundary conditions in the dynamics behavior of some higher-order lattices are studied from exact and asymptotic solutions. Higher-order lattices considered herein are generalized axial lattices with direct and indirect symmetrical elastic interactions. More specifically, this two-neighbour interaction lattice is composed of different springs connected to adjacent nodes and to next to adjacent nodes, with possible different stiffness values for each interaction. The boundary nodes at each extremity of this generalized lattice are assumed to be fixed. The natural frequencies of such a fixed-fixed generalized lattice with both symmetrical and truncated higher-order boundary conditions are analytically calculated, from the resolution of a fourth-order boundary difference value problem. The physical meaning of both higher-order boundary conditions is discussed. Whereas the so-called symmetrical higher-order boundary condition is associated with a boundary spring twice the internal one, the truncated higher-order boundary condition preserves the stiffness value of the boundary spring to the internal one. For both higher-order boundary conditions, the vibration modes and dimensionless frequencies are exactly calculated. In both cases, the dimensionless frequency of the general lattice is shown to be lower than the asymptotic continuous one. However, an asymptotic analysis shows that the scaling law for such generalized lattice is strongly sensitive to each higher-order boundary condition. A power law of order 1 or order 2 is obtained for the scaling laws associated with each higher-order boundary condition. As generalized lattices can be also understood as the physical discrete support of some distributed nonlocal elastic models with continuous kernels, it is expected that the strong scale dependence observed in this paper also concerns nonlocal elastic problems.

Keywords: One-dimensional lattice; Long range interaction; Direct and indirect neighbouring; Vibrations; Nonlocal elasticity; Exact solutions; Scale effects; Higher-order boundary conditions

1. Introduction

This article is devoted to the vibration behaviour of higher-order finite lattices, where interactions between concentrated masses are composed of direct and indirect elastic interactions. More specifically, the natural frequencies of this co-called generalized lattice will be calculated for fixed-fixed boundary conditions. Higher-order finite lattices need higher-order boundary conditions for a well-posed eigenvalue problem. Therefore, the paper will be focused on the key role of these higher-order boundary conditions, and their consequences on the scale effect at a macroscopic scale. Although the eigenfrequency solution of simple finite lattice composed of direct interactions goes back to the XVIIIth century with the works of Lagrange (1759; 1788), the consideration of finite lattice with more generalized interactions (short range and long range interactions) is more recent and dates from the XXth century. The natural frequencies of a fixed-fixed finite generalized lattice (with direct and indirect neighbouring interactions) were calculated by Pipes (1966), Chen (1970), Chen (1971), Eaton and Peddieson (1973). Pipes (1966), Chen (1970) and Chen (1971) obtained the eigenfrequencies solution

of a particular generalized finite lattice with N-neighbour interaction, characterized by equal stiffness for each interaction. The fixed-fixed generalized lattice that they considered is also free of higher-order interactions for the fixed boundary nodes. Eaton and Peddieson (1973) studied the more generalized finite lattice with N-neighbour interaction, using symmetrical higher-order boundary conditions. Eaton and Peddieson (1973) or Rosenau (1987) studied the possible approximation of such generalized lattice systems with equivalent quasi-continuous operators. Charlotte and Truskinovsky (2002), Charlotte and Truskinovsky (2008) investigated the static behaviour of a lattice composed of direct and next to direct elastic interaction and shown the strong influence of higher-order boundary conditions, through a soft and a hard loading device (displacement-based or force-based boundary conditions). Kaplunov and Pichugin (2009) developed some higher-order boundary conditions for a gradient elasticity model built from continualization of a lattice model with direct neighbouring interactions and with displacement-based boundary conditions (Dirichlet boundary conditions with non-vanishing constraints for the end displacements). Recently, Challamel et al (2018) revisited this generalized finite lattice with N-neighbour interaction, using symmetrical higher-order boundary

conditions (as considered by Eaton and Peddieson, 1973). They also found that the generalized lattice eigenfrequencies of the generalized lattices are lower than the asymptotic continuous one. An asymptotic analysis shows that the scaling law for such generalized lattice asymptotically behaves as a power law of the lattice spacing or order 2. We shall show herein that the scaling law may be very sensitive to the choice of higher-order boundary conditions.

This paper is also concerned with the role of boundary conditions in the modeling of finite integral-based nonlocal systems. Eringen and Kim (1977) already showed the link between lattice and integral-based nonlocal media, where the kernel of nonlocal systems may be related to the generalized discrete interactions of the generalized lattice system (see also Challamel, 2018 for the possible link between lattice, nonlocal and peridynamic models). The discussion on the role of boundary conditions for nonlocal media is an old and difficult topic which is still under debate. For instance, Ciesielski. and Leszczynski (2006) considered a fractional nonlocal diffusion equation, with generalized Dirichlet boundary conditions. They decided to fix the values of nodes outside the finite domain to the ones at the boundaries. This method will be investigated in the present paper for application to generalized lattices, under the so-called truncated boundary conditions. Recently, Sumelka (2017-a) and Challamel (2018) discussed the role of boundary conditions for nonlocal elastic bars with finite kernels, including periodic and anti-periodic boundary conditions. Sumelka (2017-b) studied nonlocal bars with variable length scale, which preserves the locality of the constitutive law close to the boundaries; thereby avoiding the possible introduction of external virtual boundary conditions.

2. Lattice model with two-neighbour interaction

The fixed-fixed axial lattice with two-neighbour interactions is considered, as shown in Figure 1 and Figure 2, respectively. This lattice is composed of $(n+1)$ concentrated masses. The axial displacement of a mass at node i is denoted by u_i . The spacing between each mass is assumed to be uniform and is denoted by a . The length of this finite lattice is equal to L so that the lattice spacing a is related to the total length L of the lattice by the scaling law $a=L/n$. The axial spring stiffness of the direct neighbouring interaction is denoted by k_1 , whereas the axial spring stiffness of the second neighbouring interaction is denoted by k_2 .

The stiffness parameters are calibrated from the following scaling laws

$$k_1 = \beta_1 \frac{EA}{a} \text{ and } k_2 = \beta_2 \frac{EA}{4a} \quad (1)$$

with the following parameter constraints $\beta_1 \geq 0, \beta_2 \geq 0$ in order to ensure the definite positiveness of the potential energy. EA is the axial rigidity of the continuous axial bar asymptotically obtained for a number n of nodes tending to infinite. We also have $\beta_1 + \beta_2 = 1$ so that the parameters β_1 and β_2 can be considered as weighting coefficients for the short and long range interactions. The lattice is composed of equal concentrated masses m attached at each node, except at the end nodes of the lattice. The mass parameter is calibrated from

$$m = \rho Aa \quad (2)$$

where ρA is the linear mass density of the continuous axial bar asymptotically obtained for an infinite number n of nodes.

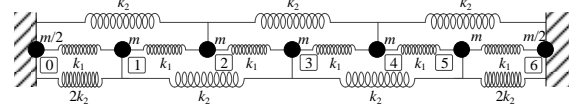


Figure 1 – Generalized lattice with direct and indirect neighbouring interactions – Mechanical system with symmetrical higher-order boundary conditions – case $n = 6$

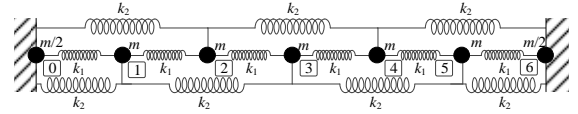


Figure 2 – Generalized lattice with direct and indirect neighbouring interactions – Mechanical system with truncated higher-order boundary conditions – case $n = 6$

The difference equations for such a generalized lattice are given by Challamel et al. (2018), see also Eaton and Peddieson (1973) or Rosenau (1987)

$$EA \left[\beta_1 \frac{u_{i+1} - 2u_i + u_{i-1}}{a^2} + \beta_2 \frac{u_{i+2} - 2u_i + u_{i-2}}{4a^2} \right] - \rho A \ddot{u}_i = 0 \quad (3)$$

This difference equation can be obtained from the coupled system of difference equations, in terms of node displacements and element normal forces

$$\frac{N_{i+1/2} - N_{i-1/2}}{a} - \rho A \ddot{u}_i = 0 \quad \text{and} \quad N_{i+1/2} = EA \left[\beta_1 \frac{u_{i+1} - u_i}{a} + \beta_2 \frac{u_{i+2} - u_i}{4a} + \beta_2 \frac{u_{i+1} - u_{i-1}}{4a} \right] \quad (4)$$

In Eq. (4), $N_{i+1/2}$ is the normal force obtained as a resultant of the springs between nodes i and $i+1$. This notation is implicitly used by Jaberolanssar and Peddieson (1981) for lattices with direct interaction, where the normal force is defined at the spring midpoints. Considering a harmonic motion $u_i(t) = u_i e^{j\omega t}$ where $j^2 = -1$ and ω is the natural frequency of vibration, the difference equation Eq. (3) may be written as

$$EA \left[\beta_1 \frac{u_{i+1} - 2u_i + u_{i-1}}{a^2} + \beta_2 \frac{u_{i+2} - 2u_i + u_{i-2}}{4a^2} \right] + \rho A \omega^2 u_i = 0 \quad (5)$$

In this paper, fixed-fixed boundary conditions will be studied, which is equivalent to the constraint of the axial displacement at the boundaries :

$$u_0 = u_n = 0 \quad (6)$$

Two families of higher-order boundary conditions are investigated, namely symmetric and truncated higher-order boundary conditions. For symmetric higher-order boundary conditions, one has to consider fictitious nodes outside the domain

$$u_{-1} = -u_1 \text{ and } u_{n+1} = -u_{n-1} \quad (7)$$

In view of Eq. (4), Eq. (6) and Eq. (7), the normal forces in the adjacent element in contact with the fixed nodes can be expressed by

$$N_{1/2} = EA \left[\beta_1 \frac{u_1}{a} + \beta_2 \frac{u_2}{4a} + \beta_2 \frac{u_1}{2a} \right] \text{ and } N_{n-1/2} = -EA \left[\beta_1 \frac{u_{n-1}}{a} + \beta_2 \frac{u_{n-2}}{4a} + \beta_2 \frac{u_{n-1}}{2a} \right] \quad (8)$$

It is seen from Eq. (8) that this fixed boundary condition with symmetrical higher-order boundary condition is equivalent to fixing the first node of the lattice and by assuming that the second node is attached to the fixed node with a second neighbouring spring stiffness $2k_2$ that is twice the one inside the lattice (the same conclusion was addressed in Challamel et al, 2018 – see Figure 1). Such a stiffness calibration is similar to the one considered for calibrating the rotational stiffness at the lattice clamped boundary condition of the Hencky system (see Hencky, 1920 who already mentioned that the calibrated stiffness of the clamped section is twice the one inside the beam domain, or Challamel et al, 2014, or Wang et al, 2017 for a recent discussion of this problem in conjunction with a nonlocal beam mechanics insight).

For truncated higher-order boundary conditions, the displacement outside the domain is assumed to vanish, i.e.

$$u_{-1} = 0 \text{ and } u_{n+1} = 0 \quad (9)$$

In view of Eq. (4), Eq. (6) and Eq. (9), the normal forces in the adjacent element in contact with the fixed nodes can be expressed, for this slightly different system, by

$$N_{1/2} = EA \left[\beta_1 \frac{u_1}{a} + \beta_2 \frac{u_2}{4a} + \beta_2 \frac{u_1}{4a} \right] \text{ and } N_{n-1/2} = -EA \left[\beta_1 \frac{u_{n-1}}{a} + \beta_2 \frac{u_{n-2}}{4a} + \beta_2 \frac{u_{n-1}}{4a} \right] \quad (10)$$

With such a fixed boundary condition with a truncated higher-order boundary condition, the second node is attached to the fixed node with a second neighbouring spring stiffness k_2 equal to the one inside the domain (uniform spring stiffness distribution including the ones at the boundaries – see Figure 2). Both higher-order boundary conditions are related to different choices of the higher-order stiffness at the boundaries. By letting this higher-order stiffness at the boundary to be equal to the value k_2 inside the lattice, one preserves the periodicity of the chain up to the boundaries, whereas considering a

value equal to $2k_2$ preserves some symmetrical properties of the displacement field inside and outside the lattice.

3. Exact solution

The fourth-order linear difference equation to be solved may also be written with the dimensionless frequency variable as

$$\frac{\beta_2}{4} u_{i+2} + \beta_1 u_{i+1} - 2 \left(\beta_1 + \frac{\beta_2}{4} - \frac{\Omega^2}{2n^2} \right) u_i + \beta_1 u_{i-1} + \frac{\beta_2}{4} u_{i-2} = 0 \quad (11)$$

where $\Omega^2 = \rho L^2 \omega^2 / E$ is a dimensionless squared frequency.

The solution is sought in the form

$$u_i = \sum_{p \in \{1,2,3,4\}} C_p f_p^i \quad (12)$$

where the p solutions of the characteristic equation f_p are detailed below

$$\frac{\beta_2}{4} f^2 + \beta_1 f - 2 \left(\beta_1 + \frac{\beta_2}{4} - \frac{\Omega^2}{2n^2} \right) + \beta_1 \frac{1}{f} + \frac{\beta_2}{4} \frac{1}{f^2} = 0 \quad (13)$$

This quartic equation can be transformed as a second-order polynomial equation by using a change of variable

$$\tau = f + \frac{1}{f} \quad (14)$$

leading to

$$\frac{\beta_2}{4} \tau^2 + \beta_1 \tau - 2 \left(\beta_1 + \frac{\beta_2}{2} - \frac{\Omega^2}{2n^2} \right) = 0 \quad (15)$$

which admits two solutions

$$\begin{aligned} \frac{\tau_1}{2} &= -\frac{\beta_1}{\beta_2} + \sqrt{\left(\frac{\beta_1}{\beta_2} + 1 \right)^2 - \frac{\Omega^2}{\beta_2 n^2}} < 1 \quad \text{and} \\ \frac{\tau_2}{2} &= -\frac{\beta_1}{\beta_2} - \sqrt{\left(\frac{\beta_1}{\beta_2} + 1 \right)^2 - \frac{\Omega^2}{\beta_2 n^2}} < -1 \end{aligned} \quad (16)$$

Let us introduce the new variables for a simple expression of the solution

$$\begin{aligned} \cos \varphi &= \frac{\tau_1}{2} = -\frac{\beta_1}{\beta_2} + \sqrt{\left(\frac{\beta_1}{\beta_2} + 1 \right)^2 - \frac{\Omega^2}{\beta_2 n^2}} \quad \text{and} \\ \cosh \theta &= -\frac{\tau_2}{2} = \frac{\beta_1}{\beta_2} + \sqrt{\left(\frac{\beta_1}{\beta_2} + 1 \right)^2 - \frac{\Omega^2}{\beta_2 n^2}} \end{aligned} \quad (17)$$

The solutions of Eq. (13) can finally be expressed by

$$\begin{aligned} f_{1,2} &= \cos \varphi \pm j \sin \varphi \quad \text{and} \\ f_{3,4} &= -\cosh \theta \pm \sinh \theta \end{aligned} \quad (18)$$

The solution of the fourth-order difference equation is finally obtained as a combination of four functions

$$u_i = A_1 \cos(\varphi i) + A_2 \sin(\varphi i) + A_3 (-1)^i \cosh(\theta i) + A_4 (-1)^i \sinh(\theta i) \quad (19)$$

where the parameters φ and θ are defined by

$$\begin{aligned}\varphi &= \arccos \left[-\frac{\beta_1}{\beta_2} + \sqrt{\left(1 + \frac{\beta_1}{\beta_2}\right)^2 - \frac{\Omega^2}{n^2\beta_2}} \right] \text{ and} \\ \theta &= \operatorname{acosh} \left[\frac{\beta_1}{\beta_2} + \sqrt{\left(1 + \frac{\beta_1}{\beta_2}\right)^2 - \frac{\Omega^2}{n^2\beta_2}} \right]\end{aligned}\quad (20)$$

4. Symmetric boundary conditions

By using the four boundary conditions Eq. (6) and Eq. (7), the eigenfrequencies are computed from the simple trigonometric equation

$$\sin(n\varphi) = 0 \quad \Rightarrow \quad \varphi = \frac{k\pi}{n} \quad (21)$$

where k is an integer. This equation admits the solution for the eigenfrequencies

$$-\frac{\beta_1}{\beta_2} + \sqrt{\left(\frac{\beta_1}{\beta_2} + 1\right)^2 - \frac{\Omega^2}{\beta_2 n^2}} = \cos \frac{k\pi}{n} \quad (22)$$

The fundamental frequency ($k=1$) is finally obtained from (see also Eaton and Peddieson, 1973 or Challamel et al, 2018)

$$\Omega = \sqrt{4\beta_1 n^2 \sin^2\left(\frac{\pi}{2n}\right) + \beta_2 n^2 \sin^2\left(\frac{\pi}{n}\right)} \quad (23)$$

An asymptotic expansion shows that the scale dependence with such higher-order boundary conditions is controlled by a power -2 with respect to n .

$$\Omega = \pi \left[1 - \frac{\pi^2}{24} (\beta_1 + 4\beta_2) \frac{1}{n^2} \right] + O\left(\frac{1}{n^4}\right) \quad (24)$$

Figure 3 shows that the asymptotic solution Eq. (24) correctly fits the exact solution of the lattice problem based on the resolution of the fourth-order difference equation. In this case, the scale dependence with such

higher-order boundary conditions is controlled by $O(1/n^2)$ term in bracket.

The eigenvibration mode is obtained with such higher-order boundary condition from $A_1 = A_3 = A_4 = 0$:

$$u_i = A_2 \sin(\varphi i) \quad (25)$$

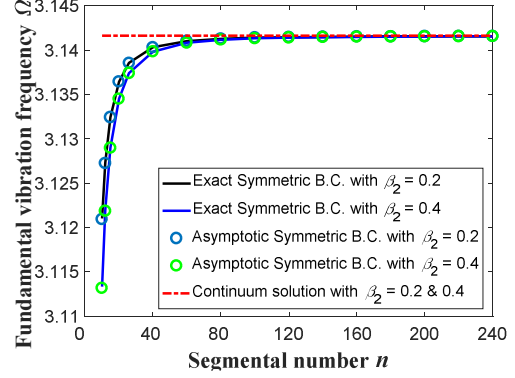


Figure 3 – Scale effect with symmetrical higher-order boundary conditions (short for B.C.) with -2 power law for the scale effect – Comparison between the exact and the asymptotic solution for the fundamental frequency

5. Truncated boundary conditions

The four boundary conditions of the so-called truncated model are

$$u_0 = 0 ; u_{-1} = 0 ; u_n = 0 \text{ and } u_{n+1} = 0 \quad (26)$$

Now, considering the truncated higher-order boundary conditions, the eigenfrequency equation is obtained from the following determinant built from Eq. (6) and Eq. (9):

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ \cos \varphi & -\sin \varphi & -\cosh \theta & \sinh \theta \\ \cos(\varphi n) & \sin(\varphi n) & (-1)^n \cosh(\theta n) & (-1)^n \sinh(\theta n) \\ -\cos(\varphi n + \varphi) & -\sin(\varphi n + \varphi) & (-1)^n \cosh(\theta n + \theta) & (-1)^n \sinh(\theta n + \theta) \end{vmatrix} = 0 \quad (27)$$

which leads to the transcendental equation

$$\begin{aligned} & (-1)^{n+1} 2 \sin(\varphi) \sinh(\theta) + 2 \sin(\varphi n + \varphi) \sinh(\theta n + \theta) + \sin(\varphi n) \sinh(\theta n + 2\theta) + \sin(\varphi n + 2\varphi) \sinh(\theta n) = 0 \end{aligned} \quad (28)$$

We are looking for an asymptotic expansion of Ω in term of small parameter ε which is defined as the inverse of n

$$\Omega = \pi + \Omega_1 \varepsilon + O(\varepsilon^2) \quad \text{with} \quad \varepsilon = \frac{1}{n} \quad (29)$$

The value of Ω_1 is calculated by replacing the frequency defined by Eq. (29) in Eq. (28).

It can be first noted that

$$\cos \varphi = 1 - \frac{\Omega^2}{2} \varepsilon^2 - \beta_2 \frac{\Omega^4}{8} \varepsilon^4 + O(\varepsilon^6) \quad (30)$$

so that

$$\varphi = \Omega \varepsilon \left[1 + \frac{\Omega^2}{24} \varepsilon^2 (1 + 3\beta_2) \right] + O(\varepsilon^5) \quad (31)$$

We have for the same reasons

$$\cosh \theta = \frac{2-\beta_2}{\beta_2} - \frac{\Omega^2}{2} \varepsilon^2 - \beta_2 \frac{\Omega^4}{8} \varepsilon^4 + O(\varepsilon^6) \quad (32)$$

which means that θ is not necessarily small

$$\theta = \theta_0 + O(\varepsilon^2)$$

$$\text{with} \quad \theta_0 = \text{acosh}\left(\frac{2-\beta_2}{\beta_2}\right) \quad (33)$$

For large values of n in Eq. (28), the transcendental equation converges towards the simplified one

$$2e^{\theta_0} \sin(\varphi n + \varphi) + e^{2\theta_0} \sin(\varphi n) + \sin(\varphi n + 2\varphi) = 0 \quad (34)$$

We finally calculate from Eq. (34)

$$\Omega_1 = -2\pi \frac{\cosh \frac{\theta_0}{2}}{(1 + \cosh \theta_0) e^{\frac{\theta_0}{2}}} = -\frac{\pi \beta_2}{2} (1 + e^{-\theta_0}) \quad (35)$$

Furthermore, we have

$$e^{-\theta_0} = \cosh \theta_0 - \sinh \theta_0 = \frac{2-\beta_2}{\beta_2} - \frac{2\sqrt{1-\beta_2}}{\beta_2} \quad (36)$$

As a conclusion, the first-order term can be simplified

$$\Omega_1 = -\pi(1 - \sqrt{\beta_1}) \quad (37)$$

For the truncated boundary conditions, we then have

$$\Omega = \pi \left[1 - (1 - \sqrt{\beta_1}) \frac{1}{n} \right] + O\left(\frac{1}{n^2}\right) \quad (38)$$

In this case, the scale dependence with such higher-order boundary conditions is controlled by $O(1/n)$ term in bracket. It is seen in Figure 4 that this asymptotic solution also correctly fits the exact lattice solution ruled by a linear fourth-order difference equation.

Once the fundamental frequency has been calculated, the vibration mode can be computed from Eq. (19) with:

$$\begin{aligned} A_3 &= -A_1 ; \\ A_4 &= -A_1 \frac{\sin n\varphi(\cos \varphi + \cosh \theta) + \sin \varphi(\cos n\varphi - (-1)^n \cosh n\theta)}{\sinh \theta \sin n\varphi + (-1)^n \sinh n\theta \sin \varphi} \\ \text{and} \\ A_2 &= A_1 \frac{(-1)^n \sinh n\theta(\cos \varphi + \cosh \theta) - \sinh \theta(\cos n\varphi - (-1)^n \cosh n\theta)}{\sinh \theta \sin n\varphi + (-1)^n \sinh n\theta \sin \varphi} \end{aligned} \quad (39)$$

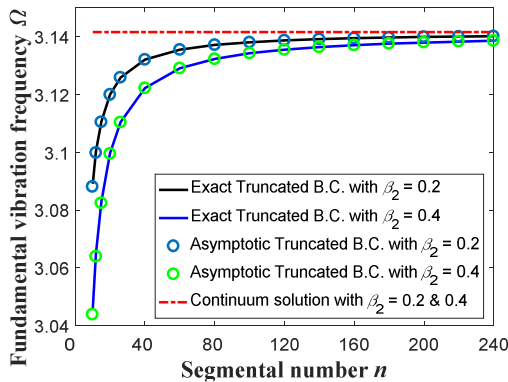


Figure 4 – Scale effect with truncated higher-order boundary conditions with $-I$ power law for the scale effect – Comparison between the exact and the asymptotic solution for the fundamental frequency

The fundamental eigenmodes are shown in Figure 5 for truncated and symmetric boundary conditions with $n=6$ and $\beta_2 = 0.1, 0.3, 0.5$. It is shown that the choice of higher-order boundary conditions affects the shape of the fundamental eigenmode. It is found that the eigenfrequencies for both boundary conditions are determined by n and β_2 whereas the eigenmodes for symmetric boundary conditions are only dependent on n .

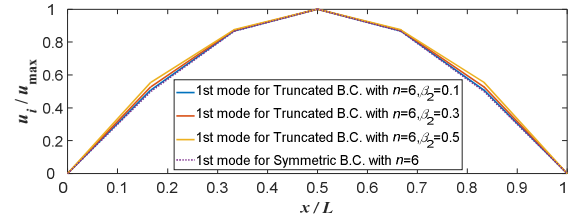


Figure 5 – Fundamental eigenmodes for truncated boundary condition and symmetric boundary condition with $n = 6$; $\beta_2 = 0.1, 0.3, 0.5$

Higher modes are also affected by the choice of higher-order boundary conditions, as shown by Figure 6 for the same value of $n=6$.

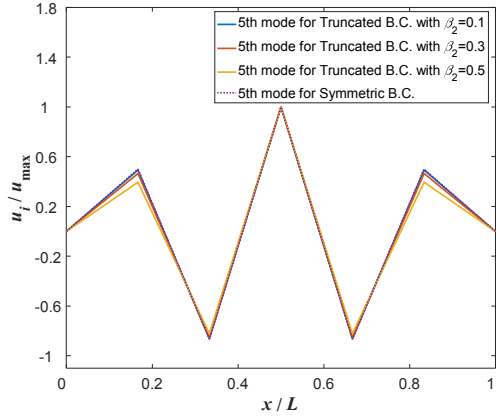


Figure 6 – 5th Eigenmodes for truncated boundary condition and symmetric boundary condition with $n = 6$; $\beta_2 = 0.1, 0.3, 0.5$

Figure 7 and Figure 8 show the first five eigenmodes for generalized lattice system with truncated and symmetric higher-order boundary conditions, respectively (still for $n=6$). By comparing the two figures, it could be seen that higher modes 2nd~5th are more affected by the higher-order boundary conditions. The 2nd mode shape of symmetric boundary conditions (short for SBC) has flat crest and trough while truncated boundary conditions (short for TBC) has sharp ones. The valley of the 3rd SBC is straight while the 3rd TBC is more curved. For SBC, the 4th mode have first and second peak with the same values, whereas the first peak is lower than the second peak for TBC. The first two joints and the last two joints of the 5th mode SBC almost have the same displacements

as the 1st mode SBC mode while the 5th mode TBC does not.

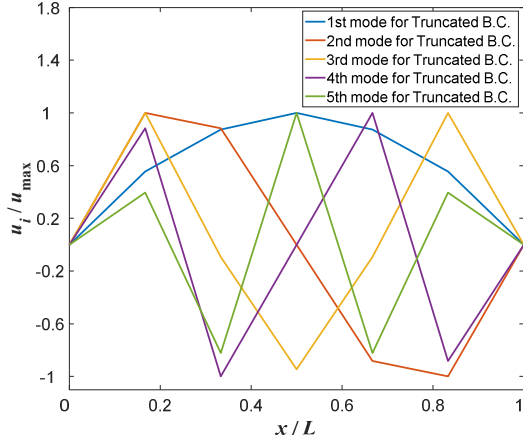


Figure 7 – 1st~5th Eigenmodes for truncated boundary condition with $n = 6$; $\beta_2 = 0.5$

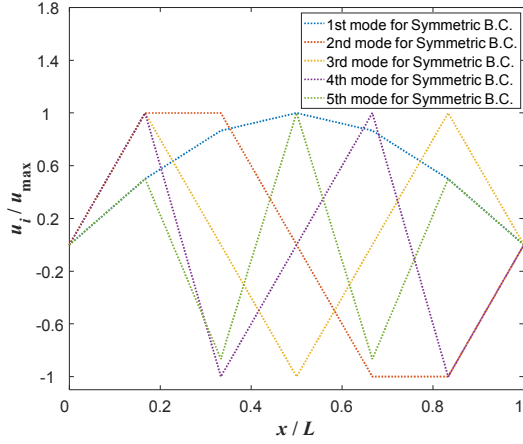


Figure 8 – 1st~5th Eigenmodes for symmetric boundary condition with $n = 6$; $\beta_2 = 0.5$ (symmetric B.C. mode shape is irrelevant to β_2)

Figure 9 and Figure 10 show the first five eigenmodes for larger values of n ($n=10$), and with both symmetric and truncated higher-order boundary conditions.

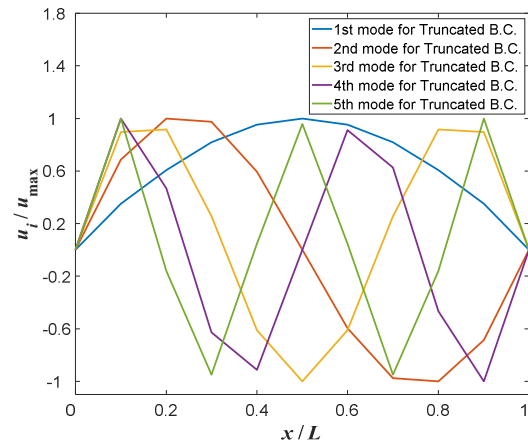


Figure 9 – 1st~5th Eigenmodes for truncated boundary condition with $n = 10$; $\beta_2 = 0.5$

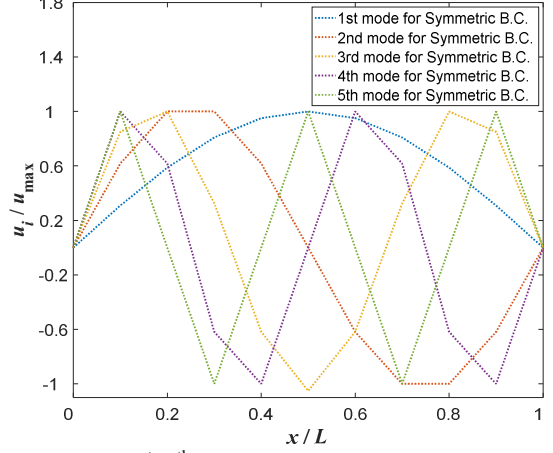


Figure 10 – 1st~5th Eigenmodes for symmetric boundary condition with $n = 10$; $\beta_2 = 0.5$

For larger values of n , i.e. for instance for $n=100$, the lowest eigenmodes apparently converge towards the ones of the continuous bar system, asymptotically obtained when n tends towards infinite:

$$EAu'' - \rho A\ddot{u} = 0 \quad (40)$$

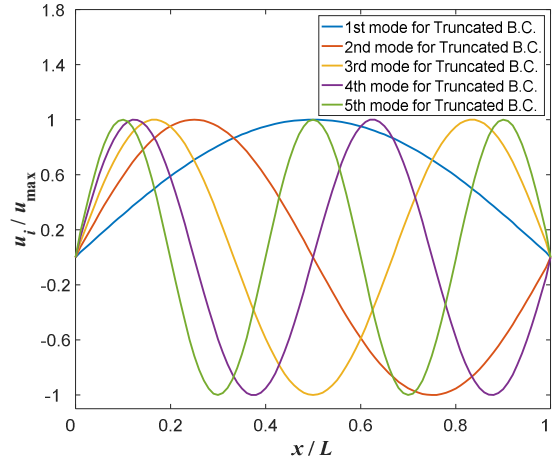


Figure 11 – 1st~5th Eigenmodes for truncated boundary condition with $n = 100$; $\beta_2 = 0.5$

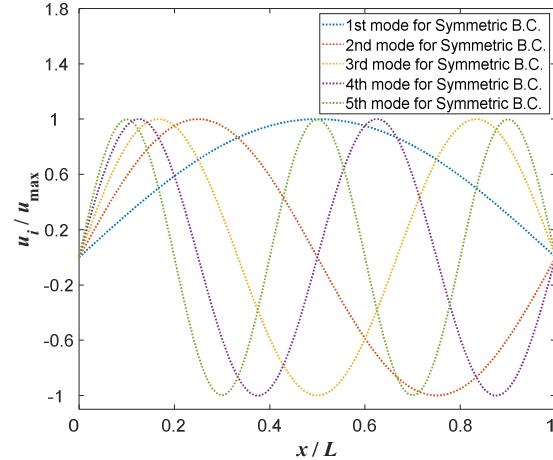


Figure 12 – 1st~5th Eigenmodes for symmetric boundary condition with $n = 100$; $\beta_2 = 0.5$

However, the calculation of the first upward difference highlights a strong boundary layer in Figure 13 for truncated higher-order boundary condition, as opposed to symmetric higher-order boundary conditions.

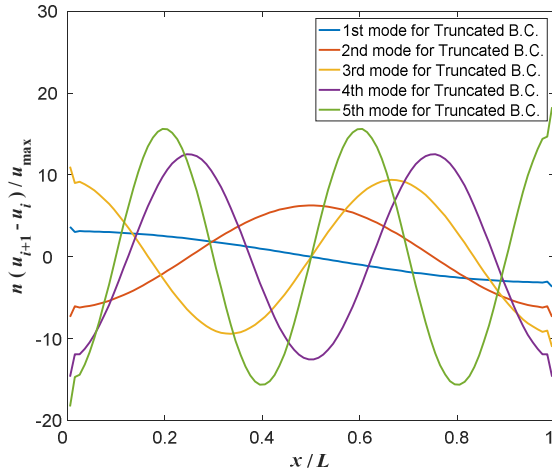


Figure 13 – $1^{\text{st}} \sim 5^{\text{th}} n(u_{i+1} - u_i) / u_{\max}$ for truncated boundary condition with $n = 100$; $\beta_2 = 0.5$

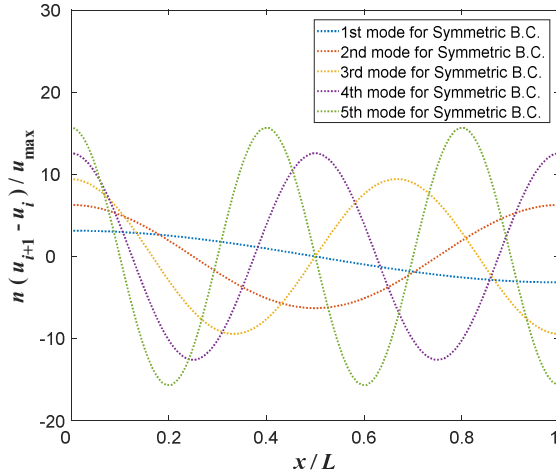


Figure 14 – $1^{\text{st}} \sim 5^{\text{th}} n(u_{i+1} - u_i) / u_{\max}$ for symmetric boundary condition with $n = 100$; $\beta_2 = 0.5$

6. Conclusions

In the present paper, the eigenfrequencies of a fixed-fixed generalized lattice governed by two neighbour elastic interaction have been determined for various higher-order boundary conditions. We first discuss the meaning of each higher-order boundary conditions, namely the symmetrical and truncated higher-order boundary conditions from a mechanical concept. Exact eigenfrequency solutions are obtained for each higher-order boundary conditions. The order of the power dependence on scale effect is controlled by the choice of higher-order boundary conditions. It means that the higher-order boundary conditions not only affect quantitatively the scale dependence, but they also from a qualitative point of view. As generalized lattices can be also understood as the physical discrete support of some distributed nonlocal elastic models with continuous kernels, it is expected that the strong scale dependence observed herein also concerns nonlocal elastic problems.

7. References

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