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Improving the treatment of near-wall regions for multiple-correction k -exact schemes

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Abstract: A high-order wall treatment is investigated for a family of high-order Godunov-type finite volume schemes based on k -exact polynomial reconstructions of the primitive variables in each cell via a successive-corrections procedure of the derivatives. We focus more specifically on the 1-exact and 2-exact schemes, which offer a good trade-off between accuracy and computational efficiency. The modifications are threefold. First, wall curvature is taken into account by introducing a new model of the wall surfaces based on bicubic Bézier patches. A low-Mach correction is then introduced in the Riemann solver to retrieve accuracy in near-wall regions characterized by low velocity. Finally, the reconstruction stencil is enriched at the boundaries. The benefits of these numerical treatments are studied for two inviscid test cases, namely the flow past a Gaussian bump and the subsonic flow around a circular cylinder.

Keywords: Computational Fluid Dynamics, Finite volumes, High-order methods, k -exact reconstruction, Curvature, Bicubic Bézier patches, Low-Mach-number flows.

1 Introduction

In the context of Computational Fluid Dynamics, high-order finite volume schemes are praised for their ability to provide high-fidelity solutions of challenging configurations including for example flow unsteadiness, boundary layers separation or steep gradients.

A widely studied family of high-order schemes, initially introduced by Barth and Frederickson [1, 2]

and Vandersbilck and Deconinck [3], is represented by the so-called k -exact schemes. These combine high-order piecewise polynomial reconstructions of the solution and its derivatives over mesh cells and high-order formulae for the calculation of flux integrals [4, 5, 6, 7]. In previous work [6, 8, 7], an efficient successive-corrections procedure for computing high-order approximations of the solution and its successive derivatives was introduced and assessed against several steady and unsteady flow configurations [7], showing its capability of ensuring high-order accuracy on fully unstructured grids.

However, for practical reasons, the reconstruction stencil is simplified at solid walls. Moreover, the wall surface is approximated by polygonal faces, introducing a spatial error of order $\mathcal{O}(h^2)$, with h the mesh size [9]. These two features can prevent the scheme to reach its nominal order of accuracy in certain flow configurations with critical wall phenomena or with curved boundaries [10].

Additionally, near-wall flow regions, which are characterized by strong gradients (boundary layers, shock/boundary layer interactions, heat transfer), exhibit low local values of the Reynolds and Mach numbers. A degradation of the accuracy of Reynolds Averaged Navier-Stokes (RANS) method has been observed for low-Mach configurations [11] due to the inability of compressible solvers to reproduce flows in the incompressible limit [11].

In the aim of progressing toward a more accurate representation of wall-bounded flow configurations with curved boundaries, an improved wall treatment is investigated in the present paper. First, a high-order representation of the wall based on bicubic Bézier patches [12, 13, 14] is added to the model and employed to calculate the geometrical moments required by the solver to integrate accurately the fluxes in our methodology. Secondly, a low-Mach correction is introduced to recenter the upwind scheme in low-velocity regions which can appear close to walls. Thirdly, a methodology to enrich the reconstruction stencil in the vicinity of walls consistently with the baseline scheme is investigated.

The present developments are implemented within the industrial in-house code FLUSEPA¹, developed by the European aerospace company ArianeGroup to model external space launcher flows during all the phases of flight [15] (takeoff, stage separation, reentry). The solver can simulate tridimensional, unsteady, compressible, viscous and reactive flows loaded with particles over bodies in relative motion. A CHIMERA-like strategy based on 3D intersections coupled with an Adaptive Mesh Refinement module [16] is used to deal with complex geometries.

This article is organized as follows. In Section 2, the existing spatial solver implemented in FLUSEPA is recalled. In Section 3, the surface model chosen to account for boundary curvature and its incorporation within the solver are described. A low-Mach correction is proposed and detailed in Section 4. Near-wall discretization is investigated in Section 5. Numerical validations on selected inviscid cases are presented and

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discussed in Section 6. Conclusions and perspectives for future work are made in Section 7.

2 Numerical scheme

In this Section, the current-point spatial discretization schemes introduced in [6, 7] are recalled for the inviscid case. Let $\mathbf{w}(\mathbf{x}, t)$ be the vector of conservative variables verifying a set of conservation equations of the form

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{w}) = 0. \quad (1)$$

where $\mathbf{F}(\mathbf{w})$ denotes the convective flux tensor.

Following a cell-centered finite volume approach, the computational domain Ω is divided into N non-overlapping cells $(\Omega_J)_{J \in [1, N]}$ of volume $|\Omega_J|$ and cell center $\mathbf{x}_J = \frac{1}{|\Omega_J|} \iiint_{\Omega_J} \mathbf{x} dV$. For a given function ψ , let us introduce the space average over cell Ω_J :

$$\bar{\psi}_J = \frac{1}{|\Omega_J|} \iiint_{\Omega_J} \psi dV. \quad (2)$$

Integration of Eq. (1) over Ω_J and application of Gauss' theorem yields

$$|\Omega_J| \frac{d\bar{\mathbf{w}}_J}{dt} + \sum_{K \in s(J)} \iint_{S_{JK}} \mathbf{F}(\mathbf{w}) \mathbf{n} dS = 0 \quad (3)$$

with S_{JK} the interface between cell J and its neighbour K , $\mathbf{n}$ the local outward-pointing normal, $(\bar{\mathbf{w}}_J)_{J \in [1, N]}$ the unknowns of the problem and $s(J)$ the direct neighbourhood of cell Ω_J , which comprises Ω_J and the cells Ω_K that share a common face with Ω_J .

For robustness reasons, the primitive variables $\mathbf{q}$ are used in the calculation instead of the conservative ones.

In case of the Euler equations, the conservative variables are $\mathbf{w} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho e \end{pmatrix}$ (with ρ the density, $\mathbf{u}$ the velocity vector and e the specific total energy) and the primitive variables $\mathbf{q}$ can be expressed as functions of the components of $\mathbf{w}$. For a perfect gas with constant thermodynamical properties:

$$\mathbf{q}(\mathbf{w}) = \begin{pmatrix} P \\ \mathbf{u} \\ T \end{pmatrix} = \begin{pmatrix} (\gamma - 1)(\rho e - \frac{1}{2} \frac{|\rho \mathbf{u}|^2}{\rho}) \\ \frac{\rho \mathbf{u}}{\rho} \\ \frac{P}{\rho R} \end{pmatrix} \quad (4)$$

with P and T the pressure and temperature. Denote by q a component of $\mathbf{q}$. In practice, instead of $\bar{q}_J$,

an approximation $\hat{q}_J$ accurate to the order $k + 1$ is used in the k -exact reconstruction process. $\hat{q}_J$ can be obtained as a function of $\tilde{q}_J = q(\bar{w}_J)$ plus a correction term (see [7] for more details). It is shown in [7] that such a relation can be obtained using only geometric moments of the cell Ω_J and derivatives of order $l \leq k$ at the current cell center $\mathbf{x}_j$. For the sake of simplicity, in the following, we do not detail this aspect and directly express the equations with the variables $\bar{q}_J$ instead of $\hat{q}_J$.

Additionally, let us introduce the definition of two geometric quantities that will be useful in the following. The volume moment of order $m \in \mathbb{N}$ associated to cell Ω_J is a tensor of order m defined by:

$$\mathcal{M}_J^{(m)} = \frac{1}{|\Omega_J|} \iiint_{\Omega_J} (\mathbf{x} - \mathbf{x}_j)^{\otimes m} dV = \frac{1}{|\Omega_J|} \iiint_{\Omega_J} \underbrace{(\mathbf{x} - \mathbf{x}_j) \otimes \dots \otimes (\mathbf{x} - \mathbf{x}_j)}_{m \text{ times}} dV \quad (5)$$

with $\otimes$ the tensor product. Note that $\mathcal{M}_J^{(0)} = 1$ and $\mathcal{M}_J^{(1)} = \mathbf{0}$

The surface moment of order $m \in \mathbb{N}$ associated to face S_{JK} is a tensor of order $m + 1$ defined by:

$$\mathcal{S}_{S_{JK}}^{(m)} = \iint_{S_{JK}} (\mathbf{x} - \mathbf{x}_\Gamma)^{\otimes m} \mathbf{n} dS = \iint_{S_{JK}} \underbrace{(\mathbf{x} - \mathbf{x}_\Gamma) \otimes \dots \otimes (\mathbf{x} - \mathbf{x}_\Gamma)}_{m \text{ times}} \mathbf{n} dS \quad (6)$$

with Γ a chosen point associated to S_{JK} called integration point. Note that the surface moment of order 0 is the surface vector $\mathcal{S}_{S_{JK}}$ of face S_{JK} .

The solver consists in three steps, following the pattern of k -exact schemes [1, 2]:

- Reconstruction of the solution in each cell by a polynomial of degree k ;
- Flux integration along the faces accurate to order $k + 1$;
- Advance in time.

Time integration is here performed with either a consistent local time stepping [17] based on the second-order accurate Heun explicit scheme or an implicit first-order scheme based on a Newton-GMRES method [18]. Advance in time is beyond the scope of the present paper. For more information about this subject, the reader is invited to refer to [8].

2.1 Solution reconstruction

To reconstruct the solution in a given cell Ω_J , a Taylor expansion of order k at the cell center $\mathbf{x}_j$ is performed for a k -exact scheme:

$$q(\mathbf{x}) = q_j + \sum_{l=1}^k \frac{1}{l!} D^{(l)} q_j \bullet (\mathbf{x} - \mathbf{x}_j)^{\otimes l} + \mathcal{O}(h^{k+1}) \quad (7)$$

with $\mathbf{D}^{(l)}q$ the l^{th} derivative tensor of q , index j referring to pointwise value at $\mathbf{x}_j$ and $\bullet$ the contracted product.

A classical method to determine a reconstruction is to solve the system of Eq. (7) such that the derivatives verify conservativity relations in a least-squares sense on a sufficient number of neighbouring cells of Ω_J [19, 20]. Such method has the drawback to be sensitive to mesh stretching [21]. After recalling definitions about k -exactness, the reconstruction procedure used in this work is described.

2.1.1 Preliminary definitions about k -exactness

In this section, the notions of k -exact function, k -exact reconstruction operator and k -exact differentiation operator are introduced.

Definition: *k -exact function*

Let k be a positive integer, and ${}^k\phi \in \mathcal{F}(\Omega, \mathbb{R})$ a function. ${}^k\phi$ is said to be k -exact over the stencil $s(J)$ if and only if its restriction to $s(J)$ is a polynomial of degree lower or equal to k :

$$\exists p \in \mathcal{P}_k(\Omega, \mathbb{R}), {}^k\phi|_{s(J)} = p|_{s(J)} \quad (8)$$

Definition: *k -exact reconstruction*

Denote by $[\bar{\phi}]_J \equiv (\bar{\phi}_K)_{K \in s(J)}$ the $(P+1)$ averages of a given function ϕ over the cells of $s(J)$, with P the number of neighbours of cell Ω_J . Let ${}^k\mathcal{R} : \mathbb{R}^{P+1} \rightarrow \mathcal{F}(\Omega, \mathbb{R})$ be a linear operator. ${}^k\mathcal{R}$ is said to be a k -exact reconstruction over $s(J)$ if and only if it satisfies the two criteria originally given by Barth in [1]:

- *conservation of the mean:* the reconstructed function has the same averages on $s(J)$ than the original function ;
- *k -exactness:* for any collection of $(P+1)$ values $\mathbf{a} = (a_1, \dots, a_{P+1})$, ${}^k\mathcal{R}\mathbf{a}$ returns the polynomial p of degree at most k (may it exists) such that $\forall m \in [1, P+1], \bar{p}_m = a_m$.

Two straightforward consequences of this definition is that for any m -exact function ${}^m\phi$, with $m \leq k$:

$$({}^k\mathcal{R} [{}^m\bar{\phi}]_J)|_{s(J)} = {}^m\phi|_{s(J)} \quad (9)$$

and for a general function $\phi \in \mathcal{F}(\Omega, \mathbb{R})$:

$$({}^k\mathcal{R} [\bar{\phi}]_J)|_{s(J)} = \phi|_{s(J)} + \mathcal{O}(h^{k+1}). \quad (10)$$

Definition: *k-exact differentiation operator*

Let d be a strictly positive integer. The linear operator ${}^k\mathcal{D}^{(d)} : \mathbb{R}^{P+1} \rightarrow \mathcal{M}_d$, with $\mathcal{M}_d$ the space of tensors of order d is called a k -exact differentiation operator over $s(J)$ if and only if for any m -exact function ${}^m\phi$, with $m \leq k$:

$${}^k\mathcal{D}^{(d)} [{}^m\bar{\phi}]_J = D^{(d)m} \phi_j \quad (11)$$

where $D^{(d)m} \phi_j$ is the d^{th} spatial tensor derivative of ${}^m\phi$ evaluated at cell center $\mathbf{x}_j$. Let r be the spatial order of accuracy of a k -exact differentiation operator, denoted by $\mathcal{D}_r^{(d)}$. For a general function ϕ , we can show directly by using a Taylor expansion that $r = k + 1 - d$, that is:

$${}^k\mathcal{D}^{(d)} [\bar{\phi}]_J = D^{(d)} \phi_j + \mathcal{O}(h^{k+1-d}) = {}^k\mathcal{D}_{k+1-d}^{(d)} [\bar{\phi}]_J. \quad (12)$$

2.1.2 Linear reconstruction of the solution

For a second-order accurate (1-exact) reconstruction, a first-order Taylor expansion of q at cell center $\mathbf{x}_j$ is performed:

$$q(\mathbf{x}) = q_j + D^{(1)} q_j \cdot (\mathbf{x} - \mathbf{x}_j) + \mathcal{O}(h^2) \quad (13)$$

We need to calculate q_j at the second order of accuracy and $D^{(1)} q_j$ at the first order. For this purpose we construct a 1-exact differentiation operator for the gradient ${}^1\mathcal{D}^{(1)} [\bar{q}]_J = \mathcal{D}_{1+1-1}^{(1)} [\bar{q}]_J = \mathcal{D}_1^{(1)} [\bar{q}]_J$. Recall that in this notation:

- the *italic* font means that the function is a linear approximation operator;
- the index letter refers to the reconstruction stencil. Here it is J , hence the operator is defined on $s(J)$;
- $\bar{q} = (\bar{q}_{K \in s(J)})$ are the averaged variables of the conservative variables on the stencil $s(J)$. The left and right brackets in $[\bar{q}]$ means that the gradient is constructed as a function of these variables;
- the exponent refers to the derivation order. Here the operator approximates the first derivatives (solution gradient);
- the index number refers to the accuracy order. Here we are looking for a first-order accurate operator

on $s(J)$, *i.e.* such that

$$\forall \mathbf{x} \in s(J), \mathcal{D}_1^{(1)} [\bar{q}]_J (\mathbf{x}) = \mathbf{D}^{(1)} q(\mathbf{x}) + \mathcal{O}(h). \quad (14)$$

To compute a 1-exact gradient, a technique based on a modification of the Green-Gauss method is introduced [7, 8]. Gauss' theorem is applied to the exact gradient to bring back the problem to the calculation of surface integrals (Eq. (15)), which are in turn approximated by a linear interpolation between the left and right averaged cells values (Eq. (16)):

$$\frac{1}{|\Omega_J|} \iiint_{\Omega_J} \mathbf{D}^{(1)} q(\mathbf{x}) dV = \frac{1}{|\Omega_J|} \sum_{K=1}^P \iint_{S_{JK}} q \mathbf{n} dS = \mathbf{D}^{(1)} q_J + \mathcal{O}(h) \quad (15)$$

$$\iint_{S_{JK}} q \mathbf{n} dS = (\beta_K \bar{q}_K + (1 - \beta_K) \bar{q}_J) \mathbf{S}_{JK}, \text{ with } \beta_K = \frac{\|\mathbf{x}_j - \mathbf{x}_r\|}{\|\mathbf{x}_j - \mathbf{x}_r\| + \|\mathbf{x}_k - \mathbf{x}_r\|} \in [0, 1] \quad (16)$$

with $\mathbf{S}_{JK} \equiv \mathcal{O}(h^2) \mathbf{n}_{JK}$ the surface vector of interface S_{JK} , $|\Omega_J| \equiv \mathcal{O}(h^3)$ the volume of cell J , and $\mathbf{x}_r$ the integration point related to the interface. The choice of such a point will be discussed in Section 3. Eq. (16) is not used for wall faces: the original solver sets $\beta_K = 0$ for these faces such that the contribution of the wall is totally excluded from the reconstruction stencil. This will be modified in Section 5.

The preceding approximation is not consistent on general grids. To fix the problem, the k -exact methodology is applied. A 1-exact gradient operator should satisfy the conditions [7]:

$$\begin{cases} \sum_{K \in s(J)} \mathbf{W}_{JK} &= \mathbf{0} & \text{(0-exactness)} \\ \forall \boldsymbol{\alpha} \in \mathbb{R}^3, \sum_{K \in s(J)} (\boldsymbol{\alpha} \cdot (\mathbf{x}_k - \mathbf{x}_j)) \mathbf{W}_{JK} &= \boldsymbol{\alpha} & \text{(1-exactness).} \end{cases} \quad (17)$$

where $(\mathbf{W}_{JK})_{K \in s(J)}$ are weight vectors such that $\mathcal{D}_1^{(1)} [\bar{q}]_J = \sum_{K \in s(J)} \mathbf{W}_{JK} \bar{q}_K$.

To construct the 1-exact gradient, we use a correction procedure, starting from a 0-exact approximation obtained by combining Eqs. (15) and (16):

$$\mathcal{D}_0^{(1)} [\bar{q}]_J = \frac{1}{|\Omega_J|} \sum_{K \in s(J)} (\beta_K \bar{q}_K + (1 - \beta_K) \bar{q}_J) \mathbf{S}_{JK} \quad (18)$$

To recover consistency on general unstructured grids, a corrective operator is built, with associated matrix $M_1(J)$ called simple correction matrix [8]. This matrix $M_1(J)$ for the linear operator (18) is obtained by applying the conditions (17) to the canonical basis of the polynomials of degree 1 $(x - x_j, y - y_j, z - z_j)$. This gives

$$M_1(J) \mathcal{D}_1^{(1)} [\bar{q}]_J = \mathcal{D}_0^{(1)} [\bar{q}]_J \quad (19)$$

which leads to the so-called Quasi-Green 1-exact gradient operator

$$\mathcal{D}_1^{(1)} [\bar{q}]_J = M_1^{-1}(J) \frac{1}{|\Omega_J|} \sum_{K \in s(J)} (\beta_K \bar{q}_K + (1 - \beta_K) \bar{q}_J) \mathbf{S}_{JK}, \text{ with } M_1(J) = \frac{1}{|\Omega_J|} \sum_{K \in s(J)} \beta_K \mathbf{S}_{JK} \otimes (\mathbf{x}_K - \mathbf{x}_J) \quad (20)$$

Note that for cartesian meshes, $M_1(J)$ is the identity matrix.

With this procedure, a second-order accurate representation of the variable is achieved.

2.1.3 Parabolic reconstruction of the solution

For a third-order accurate (2-exact) reconstruction, a second-order Taylor expansion of q at cell center $\mathbf{x}_j$ is performed:

$$q(\mathbf{x}) = q_j + \mathbf{D}^{(1)} q_j \cdot (\mathbf{x} - \mathbf{x}_j) + \frac{1}{2} \mathbf{D}^{(2)} q_j \bullet [(\mathbf{x} - \mathbf{x}_j) \otimes (\mathbf{x} - \mathbf{x}_j)] + \mathcal{O}(h^3). \quad (21)$$

We need an approximation of q_j at the third order of accuracy, $\mathbf{D}^{(1)} q_j$ at the second order and $\mathbf{D}^{(2)} q_j$ at the first order.

For this purpose, successive corrections of the derivatives are carried out [6].

First, the 1-exact gradient operator (20) is composed with itself, leading to an inconsistent approximation of the second derivatives in $\mathbf{x}_j$

$$\mathcal{D}_0^{(2)} [\bar{q}]_J = \mathcal{D}_1^{(1)} \left[\mathcal{D}_1^{(1)} [\bar{q}]_K \right]_K \quad (22)$$

Then, the consistency of the second-order derivatives is restored on general meshes thanks to the linearity properties of the operator (22) and also to the 2-exactness constraints (see [22]). Even if the algorithm remains computationally compact (using only the first neighbours), this gives a third-order tensor $M_2(J)$, which actually depends, for each cell Ω_J , on the geometrical characteristics of its so called Von Neumann neighbourhood of range 2:

$$\mathcal{D}_1^{(2)} [\bar{q}]_J = M_2(J)^{-1} \mathcal{D}_0^{(2)} [\bar{q}]_J \quad (23)$$

Finally, the first-order approximation of gradients is improved to reach second order, using the fact that the first-order truncation error is linearly related to the second derivatives. For each component $p \in \{1, 3\}$ of the gradient, a so-called upgrading matrix $H_1(J)_p$ is built, which depends only on the geometry of the

direct neighbourhood $s(J)$:

$$(\mathbf{D}^{(1)}q_j)_p = \overbrace{(\mathcal{D}_1^{(1)}[\bar{q}]_J)_p - \mathbf{D}^{(2)}q_j \bullet H_1(J)_p}^{(\mathcal{D}_2^{(1)}[\bar{q}]_J)_p} + \mathcal{O}(h^2) \quad (24)$$

with $\mathbf{D}^{(2)}q_j$ given by Eq. (23). This procedure achieves a third-order accurate polynomial reconstruction of the variables.

2.2 Flux integration

Flux integration along face S_{JK} is achieved by using a one-point integration formula, based on a Taylor expansion in accordance with the accuracy of the scheme. Given a k -exact reconstruction scheme, we look for an integration of the fluxes accurate to order $k + 1$:

$$\iint_{S_{JK}} \mathbf{F} \cdot \mathbf{n} dS = \mathbf{F}_\Gamma \cdot \mathbf{S}_{JK}^{(0)} + \sum_{l=1}^k \frac{1}{l!} \mathbf{D}^{(l)} \mathbf{F}_\Gamma \bullet \mathbf{S}_{JK}^{(l)} + \mathcal{O}(h^{k+1}) \|\mathbf{S}_{JK}^{(0)}\| \quad (25)$$

where $\mathbf{x}_\Gamma$ is the integration point and $\mathbf{S}_{JK}^{(l)}$ are the surface moments of order l of face S_{JK} .

Calculating these quantities and the location of the integration point in order to reflect accurately the shape of the boundary when curvature is present is the object of Section 3. Note that Eq. (25) is different from the classical choice for which a Gaussian quadrature is involved to integrate fluxes.

Since $\frac{\mathbf{S}_{JK}^{(l)}}{\|S_{JK}\|}$ sizes $\mathcal{O}(h^l)$, a second-order accurate approximation of $\mathbf{F}_\Gamma$ and a first-order accurate approximation of $\mathbf{D}^{(1)}\mathbf{F}_\Gamma$ are needed to achieve a second-order accurate reconstruction process (1-exactness). For a third-order reconstruction process, (2-exactness), we need $\mathbf{F}_\Gamma$ at order 3, $\mathbf{D}^{(1)}\mathbf{F}_\Gamma$ at order 2 and $\mathbf{D}^{(2)}\mathbf{F}_\Gamma$ at order 1.

Approximation of $\mathbf{F}_\Gamma$ is obtained by solving a Riemann problem at the interface S_{JK} . Left and right values of the variable at point Γ are given by the solution reconstruction based on the derivative approximates established previously on both sides of the interface, following a MUSCL-type approach [23].

Derivatives $\mathbf{D}^{(l)}\mathbf{F}_\Gamma$ used in Eq. (25) are evaluated using an approximation of the flux Jacobian matrix and derivatives of the primitive variables involved in the reconstruction process.

3 Wall curvature

Taking into account wall curvature is essential to perform high-order boundary treatment. In the baseline implementation of FLUSEPA, boundary faces are represented by flat faces², introducing a second-order

²More precisely, triangular faces are used. Quadrangular faces are treated by being split into two triangles.

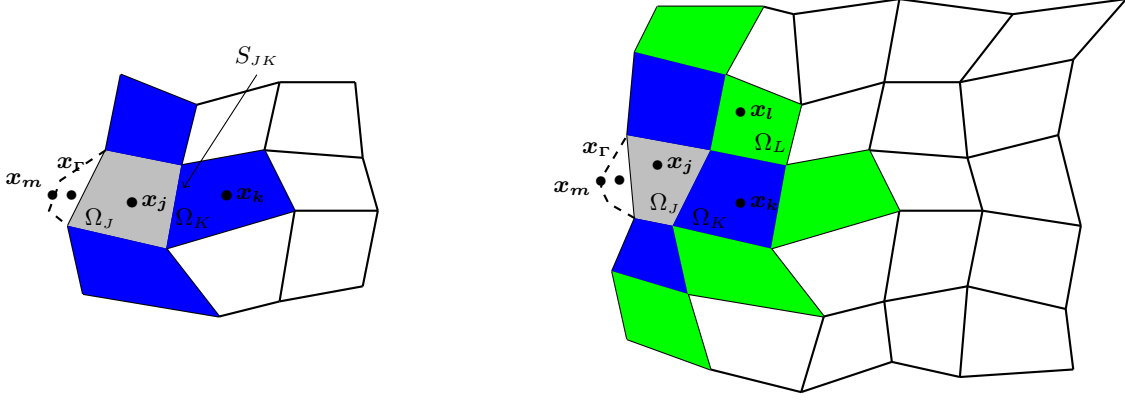


Figure 1: Discretization stencil at the wall and curved wall face for the: (a) 1-exact reconstruction scheme; (b) 2-exact reconstruction scheme.

spatial error compared to the actual surface [9]. However, the approximation of the surface has to be at least of the order of the numerical scheme to ensure the nominal order of accuracy close to the wall [10]. In the case of parametric surfaces using a polynomial basis, and for a third-order accurate scheme, the polynomials have to be at least of degree two.

For this purpose, we use the geometric module BSHAPE developed by INRIA [14], which generates from a surface mesh a composite surface made of [quadrangular and triangular](#) bicubic Bézier patches.

A bicubic Bézier patch is a parametric polynomial surface of degree three originated by De Casteljau and Bézier [12] [and defined by](#) a set of [so-called](#) control points .

[Bicubic Bézier quadrangles](#) are defined by 16 control points $(\mathbf{P}_{ij})_{(i,j) \in \{0,1,2,3\}^2}$, [and](#) the equation of the patch is given by:

$$\forall (u, v) \in [0, 1]^2, \mathbf{f}_{\mathbf{Q}}(u, v) = \sum_{i=0}^3 \sum_{j=0}^3 B_i^{Q,3}(u) B_j^{Q,3}(v) \mathbf{P}_{ij} \quad (26)$$

where $(B_k^{Q,3})_{k \in \{0,3\}}$ are the Bernstein polynomials of degree three [for quadrangles](#) [13]. [A representation of](#) a bicubic Bézier quadrangle is given in Figure 2a.

[Bicubic Bézier triangles](#) are defined by 10 control points $(\mathbf{P}_{ij})_{(i,j,k) \geq 0, i+j+k=3}$, [and](#) the equation of the patch is given by:

$$\forall u \geq 0, v \geq 0, u + v \leq 1, \mathbf{f}_{\mathbf{T}}(u, v) = \sum_{i,j,k \geq 0, i+j+k=3} B_i^{T,3}(u) B_j^{T,3}(v) B_k^{T,3}(1-u-v) \mathbf{P}_{ijk} \quad (27)$$

where $(B_k^{T,3})_{k \in \{0,3\}}$ are the Bernstein polynomials of degree three for triangles [13]. An representation of a bicubic Bézier triangle is given in Figure 2b.

BSHAPE module is based on a point normal interpolation meshing strategy [24]: the patches are generated with the points coordinates and associated normals of the mesh. The model ensures G1 continuity, *i.e.* tangent plane continuity, between the patches.

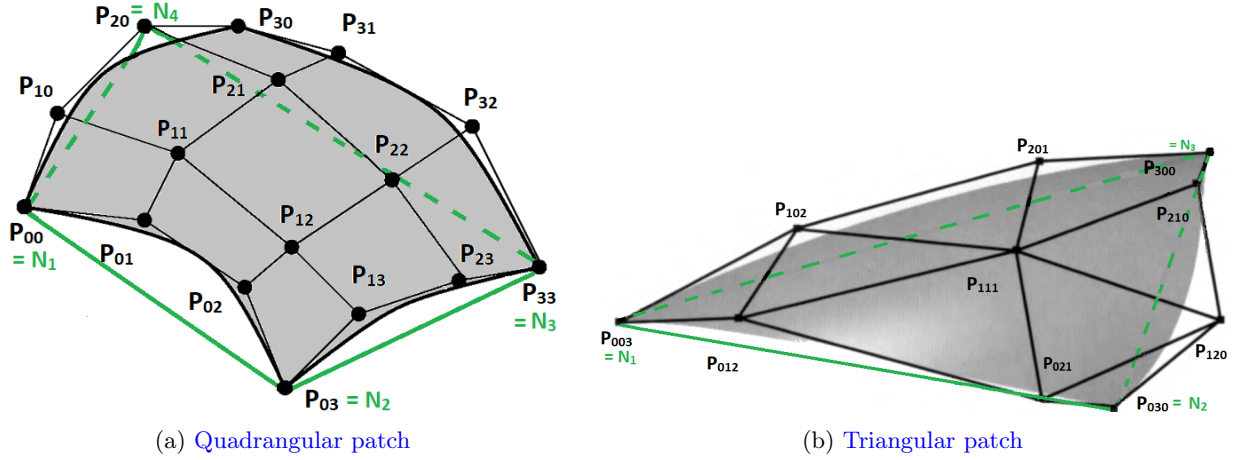
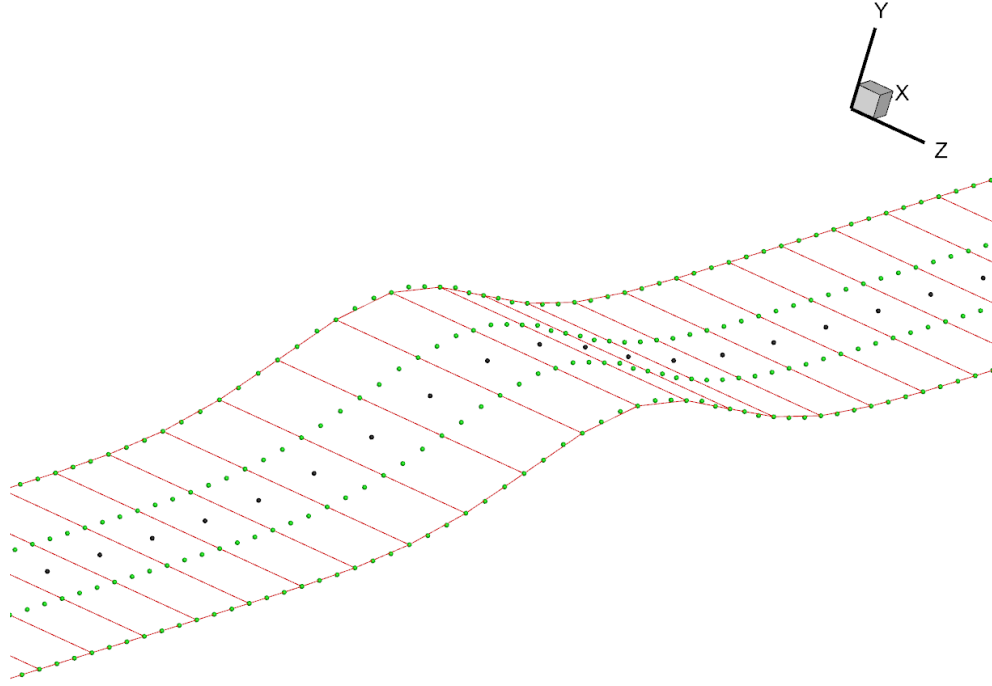


Figure 2: Representation of bicubic Bézier patches. In green: original mesh elements ; in grey: Bézier patch ; in black: associated control net.

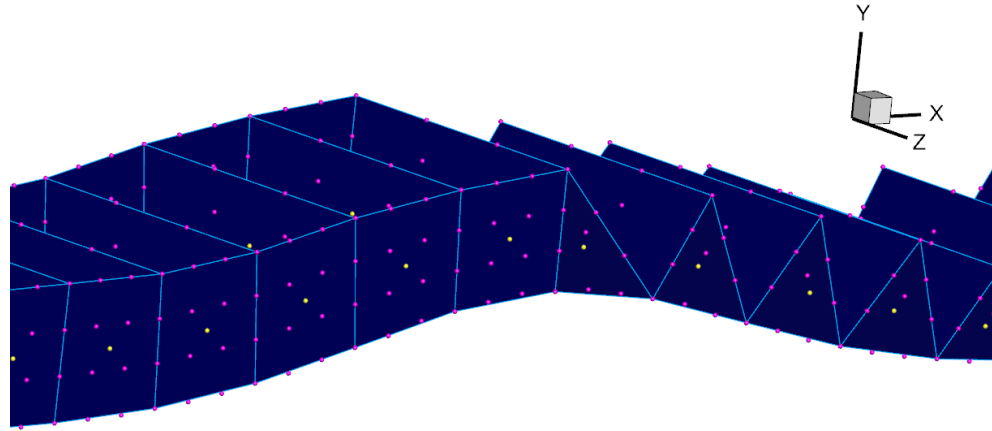
The module has been incorporated in FLUSEPA. The geometry routine selects the wall faces and the side faces sharing an edge with the wall faces, and generates a set of control points corresponding to a patch for each wall face. The composite surface is not directly seen by the solver, that is, a new mesh is not created, but the information about the patches, namely the control points, is used to integrate the boundary geometric quantities required for the scheme introduced in Section 2. The surface moments of the wall faces are derived by integrating analytically the constitutive equation of the patch. Then volume moments and centroids of the boundary cells are deduced by the application of Gauss' theorem.

The generation of the control and integration points is illustrated respectively on Figure 3b for the wall faces and on Figure 3a for the side faces for one of the hybrid meshes which will be used for the Bump test case in Section 6. In the present work, only two-dimensional cases are considered (which are run by the solver as extruded 3D cases with one cell in the third direction), however, Figure 3 provides a glimpse of the potential of the method to deal with three-dimensional configurations.

This strategy was tested on simple configurations for which the analytical equation of the surface is known, such as a parabolic channel and a sphere. A decrease of the geometrical error from 1 up to 2 orders of accuracy was observed in the calculation of boundary cell volumes and geometrical moments of order 1



(a) Wall faces (in white surrounded by red edges), associated control points (in green) and associated integration points (in black)



(b) Side faces (in blue), associated control points (in pink) and associated integration points (in yellow)

Figure 3: Generation of the control and integration points for a hybrid mesh of the Bump test case

and 2.

3.1 Choice of integration point

In the original approach of [7], the integration point $\mathbf{r}$ is constructed to minimize the first-order truncature error in the flux integration:

$$\mathbf{x}_\Gamma = \begin{pmatrix} \frac{1}{\|\mathbf{S}_{JK}\|} \left(\iint_{S_{JK}} x \mathbf{n} dS \right) \cdot \mathbf{n}_{JK} \\ \frac{1}{\|\mathbf{S}_{JK}\|} \left(\iint_{S_{JK}} y \mathbf{n} dS \right) \cdot \mathbf{n}_{JK} \\ \frac{1}{\|\mathbf{S}_{JK}\|} \left(\iint_{S_{JK}} z \mathbf{n} dS \right) \cdot \mathbf{n}_{JK} \end{pmatrix} \quad (28)$$

with $\mathbf{n}_{JK} = \frac{\mathbf{S}_{JK}}{\|\mathbf{S}_{JK}\|}$. Note that the surface moment vectors of order 1 related to this point are orthogonal to $\mathbf{S}_{JK}$.

Moreover, for a flat face we have $\forall \mathbf{x} \in \mathbf{S}_{JK}, \mathbf{n}(\mathbf{x}) = \mathbf{n}_{JK}$, consequently the Γ point defined by (28) coincides with the gravity center and is on the face. However, when the face is curved, such a point is generally not on the face: this feature is likely to deteriorate the accuracy of the fluxes at the wall when a Riemann solver is used, as explained hereafter.

Let n be the order of the reconstruction ($n = k + 1$ for a k -exact reconstruction).

Consider a face between two cells and assume a (sufficiently) smooth solution around this interface. Then the difference between the left and right reconstructed values at the integration point is of order $\mathcal{O}(h^n)$. Consequently, the jump discontinuity of the Riemann problem is also of order $\mathcal{O}(h^n)$, and so is the solution given by the Riemann solver.

Now consider a curved wall face. Then with the definition (28), the order of the discontinuity between left and right values introduced into the Riemann problem changes. The integration point is further from the wall at a distance of order $\mathcal{O}(h^2)$ for a smooth wall, and of order $\mathcal{O}(h)$ for a general wall. Hence, the corresponding pointwise values of the variables carry differences compared to the wall values of the same order.

An explicit example is given by a no-slip condition $\mathbf{u} = \mathbf{0}$ at a curved non-moving wall in a Navier-Stokes configuration. Then the choice (28) will lead to a non-zero numerical speed at $\mathbf{x}_\Gamma$ of order $\mathcal{O}(h)$ or $\mathcal{O}(h^2)$ instead of $\mathcal{O}(h^n)$. For a second-order scheme, the method will remain consistent for smooth walls. However, for any polynomial reconstruction of degree greater or equal to 3, that is for any k -exact reconstruction higher or equal to $k = 2$, the accuracy will be limited to second order in the near-wall region, because of the inconsistency of the flux given by the Riemann solver at point Γ .

Note that this problem at the wall is specific to the use of Riemann solvers, which requires two states values: one at the left side of the face, extrapolated from the cell center of the boundary cell, and one at the right of the face, determined by symmetry relations (mirror cell). This issue can certainly be fixed by extrapolating the required values from the wall. This process is however not straightforward since the symmetry relations are only consistent at the wall.

For these reasons, the definition of the integration point is modified for wall faces with the prerequisite

that it lies on the face. In order to remain consistent, the local normal at point Γ must be used instead of the integrated face normal $\mathbf{n}_{JK}$ in the Riemann solver. For interior faces, formula (28) is however kept to preserve the minimization of the first-order error in the fluxes' integration.

Several ideas have been considered for the location of the integration point on the Bézier patch. These are all summed up in Figure 4, and explained hereafter.

A first approach consists in choosing intuitively the middle point in the patch in a parametrical point of view ($\mathbf{f}_Q(u = \frac{1}{2}, v = \frac{1}{2})$ in case of a quadrangular patch, $\mathbf{f}_T(u = \frac{1}{3}, v = \frac{1}{3})$ in case of a triangular patch), as shown in Figure 4 a). Indeed for faces not very twisted this point will not be far from the gravity center of the face.

A second, more physical approach, presented in Figure 4 b), considers that the integrated normal along the wall face is the most representative information of the face, and thus choose the integration point as the point whose local normal is the closest to $\mathbf{n}_{JK}$.

A third approach, see Figure 4 c), consists in choosing $\mathbf{x}_\Gamma$ such that $(\mathbf{x}_j - \mathbf{x}_\Gamma) \wedge \mathbf{n}_\Gamma = \mathbf{0}$, that is ΓJ is colinear with the local normal at point Γ . It is this method that will be retained after considerations about boundary conditions, see Section 5.

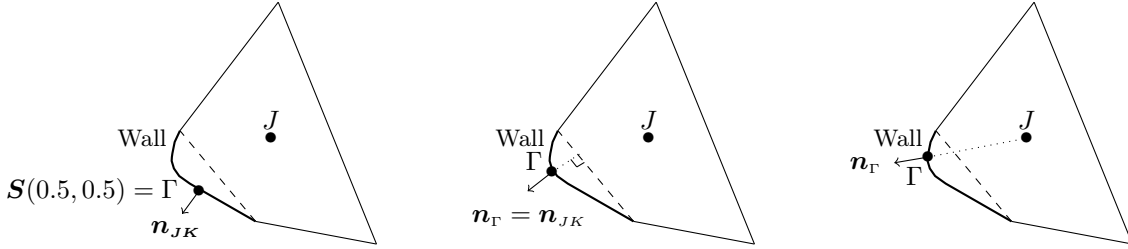


Figure 4: Several possibilities for the location of the integration point Γ : a) middle point on the patch ; b) point whose normal is colinear to the integrated normal ; c) point whose normal is aligned with the cell center.

4 Low-Mach recentering

Boundary layers being characterized by low-speed flow, constructing a compressible scheme accurate at low-Mach numbers appears as an important condition to deal with wall phenomena. In a pioneering article, Volpe [25] investigated the performances of compressible codes at very low-Mach numbers and observed that some of the codes suffered from a lack of accuracy in the low-Mach regime.

Some specific spatial configurations are exempted from this accuracy problem. Non-linear Godunov-type schemes do not exhibit low-Mach inaccuracy on 1D grids, nor on 2D (resp. 3D) grids with triangular (resp.

tetrahedral) cells [26, 27, 28]. Such a result is however only proved for periodic or unbounded spatial domains [26].

By **rendering dimensionless** the Euler equations with two different time scales and expanding the variables in power of the Mach number, two limit solutions are deduced for low-speed flows [29]: an incompressible one and an acoustic one which corresponds to fluctuations of the flow. These limits are both physically relevant and generally coexist.

Guillard and Viozat [30] and Guillard and Murrone [31] showed, by means of **an** asymptotic analysis, that the order of the pressure fluctuations are responsible for the limit to which the solution of the scheme will converge to :

$$\begin{cases} P = P_0 + \mathcal{O}(M^2) \text{ in the incompressible limit} \\ P = P_0 + \mathcal{O}(M) \text{ in the acoustic limit} \end{cases} \quad (29)$$

Rieper [32] pointed out that a too important normal jump of velocity in the Riemann problem at cell interfaces leads to an incorrect pressure scaling.

A review of the main strategies to address the low-Mach problem for compressible flows is given in [29]. The goal of all these curative solutions is to remove the acoustic perturbations. Preconditioning techniques, initially inspired by Turkel's approach [33] are used to derive initial data that permit to recover a correct scaling of the pressure fluctuations, as given in Eq. (29) [32, 31, 30, 34].

In the present work, a low-Mach recentering approach is chosen, which consists in adapting the numerical viscosity of the scheme to suppress the unwanted acoustic part of the solution. This procedure is justified by a physical approach in the next Section.

4.1 Derivation of the recentering functions for the Godunov Riemann solver

In this Section, a linearized version of the governing equations is considered as a prototype for investigating the behaviour of numerical schemes in the low-Mach number limit and developing suitable corrections for our Riemann solver.

In the low-Mach regime, the non-linear part of the Riemann solver is negligible compared to its linear part. We can thus study the linearized Euler equations along with a linear Riemann solver to understand the low-Mach problem.

To simplify the demonstration, we consider the isentropic Euler equations, for which the pressure is a single-valued function of the density

$$\begin{cases} \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{u}) = 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \mathbf{D}^{(1)} P = \mathbf{0} \end{cases} \quad (30)$$

and inject first-order perturbations of a steady flow $\rho = \rho_0 + \rho'$, $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$ and $P = P_0 + c^2 \rho'$ into Eqs. (30), c being the speed of sound. This yields to the linearized Euler equations:

$$\begin{cases} \frac{\partial P'}{\partial t} + \mathbf{u}_0 \cdot \mathbf{D}^{(1)} P' + \rho_0 c^2 \text{div}(\mathbf{u}') = 0 \\ \frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}'(\mathbf{D}^{(1)} \mathbf{u}') \mathbf{u}_0 + \frac{1}{\rho_0} \mathbf{D}^{(1)} P' = \mathbf{0} \end{cases} \quad (31)$$

In the following, the prime symbol is dropped to alleviate the notations. Eqs. (31) are then integrated on Ω_J according to a finite volume approach, and Green-Gauss theorem is applied:

$$\begin{cases} |\Omega_J| \frac{d\overline{P_J}}{dt} + \sum_K (\mathbf{u}_0 \cdot \mathbf{S}_{JK}) P_\Gamma + \rho_0 c^2 \sum_K \mathbf{u}_\Gamma \cdot \mathbf{S}_{JK} = 0 \\ |\Omega_J| \frac{d\overline{\mathbf{u}_J}}{dt} + \sum_K (\mathbf{u}_0 \cdot \mathbf{S}_{JK}) \mathbf{u}_\Gamma + \frac{1}{\rho_0} \sum_K P_\Gamma \mathbf{S}_{JK} = \mathbf{0} \end{cases} \quad (32)$$

where P_Γ and $\mathbf{u}_\Gamma$ are the pressure and velocity values at point Γ . Again, the bar over the variables indicating averaged quantities are dropped in the notations from now on.

For simplicity, we assume that time integration is performed using the first-order forward Euler scheme. Then the system (32) can be written again as

$$\begin{cases} P_J^{n+1} &= P_J^n - \frac{\Delta t}{|\Omega_J|} \sum_K (\mathbf{u}_0 \cdot \mathbf{S}_{JK}) P_\Gamma^n - \frac{\Delta t}{|\Omega_J|} \rho_0 c^2 \sum_K \mathbf{u}_\Gamma^n \cdot \mathbf{S}_{JK} \\ \mathbf{u}_J^{n+1} &= \mathbf{u}_J^n - \frac{\Delta t}{|\Omega_J|} \sum_K (\mathbf{u}_0 \cdot \mathbf{S}_{JK}) \mathbf{u}_\Gamma^n - \frac{\Delta t}{|\Omega_J|} \frac{1}{\rho_0} \sum_K P_\Gamma^n \mathbf{S}_{JK} \end{cases} \quad (33)$$

We denote $M_{JK} = \frac{\mathbf{u}_0 \cdot \mathbf{n}_{JK}}{c}$ the interface Mach number in the direction normal to face S_{JK} . P_Γ^n and $\mathbf{u}_\Gamma^n$ can be calculated with a classical first-order Godunov scheme [35]:

If $|M_{JK}| \geq 1$ then:

if $M_{JK} \geq 1$ then:

$$\begin{cases} P_\Gamma^n = P_J^n \\ \mathbf{u}_\Gamma^n = \mathbf{u}_J^n \end{cases} \quad (34)$$

else if $M_{JK} \leq -1$ then:

$$\begin{cases} P_\Gamma^n = P_K^n \\ \mathbf{u}_\Gamma^n = \mathbf{u}_K^n \end{cases} \quad (35)$$

If $|M_{JK}| \leq 1$ then:

$$\begin{cases} P_\Gamma^n = \frac{P_J^n + P_K^n}{2} + \frac{\rho_0 c}{2} (\mathbf{u}_J^n - \mathbf{u}_K^n) \cdot \mathbf{n}_{JK} \\ \mathbf{u}_\Gamma^n = \frac{\mathbf{u}_J^n + \mathbf{u}_K^n}{2} + \frac{1}{2\rho_0 c} (P_J^n - P_K^n) \mathbf{n}_{JK} \end{cases} \quad (36)$$

In low-Mach configurations, the calculation of the interface values is close to what is done in Eqs. (36), even when a non-linear Godunov solver is used. We see that these two equations are composed of a centered term and of the equivalent of an upwind term whose role is to stabilize the solution through numerical dissipation. Consequently, the loss of accuracy observed in the incompressible limit is due to these "upwind" terms, which become too **large** when the Mach number **tends** to zero. This motivates the idea of recentering these terms with some recentering functions that will be noted ψ_{JK}^P for the pressure equations and ψ_{JK}^u for the velocity equation:

$$\begin{cases} P_\Gamma^n = \frac{P_J^n + P_K^n}{2} + \frac{\rho_0 c}{2} \psi_{JK}^P (\mathbf{u}_J^n - \mathbf{u}_K^n) \cdot \mathbf{n}_{JK} \\ \mathbf{u}_\Gamma^n = \frac{\mathbf{u}_J^n + \mathbf{u}_K^n}{2} + \frac{1}{2\rho_0 c} \psi_{JK}^u (P_J^n - P_K^n) \mathbf{n}_{JK} \end{cases} \quad (37)$$

Two natural conditions on the functions ψ_{JK}^P and ψ_{JK}^u can be derived from physical arguments. First, the two functions must be comprised between zero and one, since their role is to decrease the upwind term without changing its sign in the normal direction $\mathbf{n}_{JK}$. Secondly, ψ_{JK}^P and ψ_{JK}^u must be continuous functions of the normal Mach number M_{JK} , and they must allow to recover the classical Godunov scheme for Mach numbers greater than one. These conditions are **summarized** below

$$\begin{cases} 0 \leq \psi_{JK}^P(M_{JK}) \leq 1 \\ 0 \leq \psi_{JK}^u(M_{JK}) \leq 1 \\ \psi_{JK}^P(M_{JK} \geq 1) = 1 \\ \psi_{JK}^u(M_{JK} \geq 1) = 1 \end{cases} \quad (38)$$

In order to find suitable expressions for ψ_{JK}^P and ψ_{JK}^u , relations (34), (35) and (37) are plugged into (32). After calculations, a linear expression valid for all Mach numbers is finally obtained:

$$\begin{cases} P_J^{n+1} = P_J^n - \frac{\Delta t}{2|\Omega_J|} \sum_K (\alpha_{JK} P_J^n - \alpha_{KJ} P_K^n) \\ \quad - \rho_0 c \frac{\Delta t}{2|\Omega_J|} \sum_K (\lambda_{JK}^P \mathbf{u}_K^n + \lambda_{KJ}^P \mathbf{u}_J^n) \cdot \mathbf{n}_{JK} \\ \mathbf{u}_J^{n+1} = \mathbf{u}_J^n - \frac{\Delta t}{2|\Omega_J|} \sum_K (a_{JK} \mathbf{u}_J^n - a_{KJ} \mathbf{u}_K^n) - \frac{\Delta t}{2|\Omega_J|} \sum_K ((b_{JK}^P \mathbf{u}_J^n - b_{KJ}^P \mathbf{u}_K^n) \cdot \mathbf{n}_{JK}) \mathbf{n}_{JK} \\ \quad - \frac{\Delta t}{2|\Omega_J| \rho_0 c} \sum_K (\lambda_{JK}^u P_J^n + \lambda_{KJ}^u P_K^n) \mathbf{n}_{JK} \end{cases} \quad (39)$$

$$\begin{cases} a_{JK} = c||\mathbf{S}_{JK}||\max(0, 2M_{JK}) \\ b_{JK}^q = c||\mathbf{S}_{JK}||(\Psi_{JK}^q - |M_{JK}|), \text{ with } q = P \text{ or } q = u \\ \alpha_{JK} = a_{JK} + b_{JK}^u \\ \lambda_{JK}^q = c||\mathbf{S}_{JK}||\max(0, \min(2, 1 + \Psi_{JK}^q M_{JK})), \text{ with } q = P \text{ or } q = u \end{cases} \quad (40)$$

Here index JK refers to the left of the interface S_{JK} and index KJ to the right of the same interface. Note that $M_{KJ} = -M_{JK}$.

By noting $\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ and considering the particular direction $\mathbf{e}_x$ (x-axis), we can write Eq. (39):

$$\begin{aligned} u_J^{n+1} = u_J^n & - \frac{\Delta t}{2|\Omega_J|} \sum_K (a_{JK} + b_{JK}^P n_{JK,x}^2) u_J^n + \frac{\Delta t}{2|\Omega_J|} \sum_K (a_{KJ} + b_{KJ}^P n_{JK,x}^2) u_K^n \\ & - \frac{\Delta t}{2|\Omega_J|} \sum_K (b_{JK}^P v_J^n - b_{KJ}^P v_K^n) n_{JK,x} n_{JK,y} + \frac{\Delta t}{2|\Omega_J|} \sum_K (b_{JK}^P w_J^n - b_{KJ}^P w_K^n) n_{JK,x} n_{JK,z} \\ & - \frac{\Delta t}{2|\Omega_J|\rho_0 c} \sum_K (\lambda_{JK}^u P_J^n + \lambda_{KJ}^u P_K^n) n_{JK,x} \end{aligned} \quad (41)$$

The first line of formula (41) represents the evolution of the component $u = \mathbf{u} \cdot \mathbf{e}_x$ which is not coupled with the other variables (pressure field and other velocity components). Since Eq. (39) is invariant by rotation, we can generalize easily this non-coupled part of the formula to the projection of the velocity in any direction $\mathbf{j}$:

$$\mathbf{u}_J^{n+1} \cdot \mathbf{j} = \left(1 - \frac{\Delta t}{2|\Omega_J|} \sum_K (a_{JK} + b_{JK}^P (\mathbf{n}_{JK} \cdot \mathbf{j})^2)\right) \mathbf{u}_J^n \cdot \mathbf{j} + \frac{\Delta t}{2|\Omega_J|} \sum_K (a_{KJ} + b_{KJ}^P (\mathbf{n}_{JK} \cdot \mathbf{j})^2) \mathbf{u}_K^n \cdot \mathbf{j} + \text{coupling terms} \quad (42)$$

4.2 Stability condition

A critical requirement for the recentered scheme is stability. In the following, we look for a strong stability condition given by the positivity of the scheme: the linear scheme (39) is said to be positive if and only if all its constant coefficients are positive (and their sum is equal to unity³). This is a priori not a straightforward task to achieve because of the coupling that exists between pressure and the velocity components in equations (39). To eliminate the coupling issue, the system (31) is written again in order to exhibit its eigenfunctions. Let $\mathbf{i}$ and $\mathbf{j}$ be two given projection directions. Making the combination (31.1) + $\rho_0 c$ (31.2) $\cdot \mathbf{i}$ on the one hand and (31.2) $\cdot \mathbf{j}$ on the other hand leads to:

³This last requirement is achieved because the control volumes are closed (hence the sum of the surface vectors is null, which can be used to demonstrate this property).

$$\left\{ \begin{array}{l} \frac{\partial(P + \rho_0 c \mathbf{u} \cdot \mathbf{i})}{\partial t} + (\mathbf{u}_0 + c\mathbf{i}) \cdot \mathbf{D}^{(1)}(P + \rho_0 c \mathbf{u} \cdot \mathbf{i}) = -\rho_0 c^2 (\text{div}(\mathbf{u}) - \mathbf{i} \cdot \mathbf{D}^{(1)}(\mathbf{u} \cdot \mathbf{i})) \\ \frac{\partial \mathbf{u} \cdot \mathbf{j}}{\partial t} + \mathbf{u}_0 \cdot \mathbf{D}^{(1)}(\mathbf{u} \cdot \mathbf{j}) = -\frac{1}{\rho_0} \mathbf{D}^{(1)} P \cdot \mathbf{j} \end{array} \right. \quad (43)$$

If there exists directions $\mathbf{i}$ and $\mathbf{j}$ that cancel out the coupling terms in system (43) (right-hand side terms), the corresponding functions $\phi(\mathbf{i}) = P + \rho_0 c \mathbf{u} \cdot \mathbf{i}$ and $\mathbf{u} \cdot \mathbf{j}$ are eigenfunctions of the system, convected respectively at the speeds $\mathbf{u}_0 + c\mathbf{i}$ and $\mathbf{u}_0$. Such directions must verify:

$$\left\{ \begin{array}{l} \mathbf{u} = \mathbf{i} \cdot \mathbf{D}^{(1)}(\mathbf{u} \cdot \mathbf{i}) \\ \mathbf{D}^{(1)} P \cdot \mathbf{j} = 0 \end{array} \right. \quad (44)$$

Using Eqs (39), the discretized equation for the function ϕ is established:

$$\forall \mathbf{i} \in \mathbb{R}^3, \phi_J^{n+1}(\mathbf{i}) = \left(1 - \frac{\Delta t}{2|\Omega_J|} \sum_K A_{JK} \right) \phi_J^n + \frac{\Delta t}{2|\Omega_J|} \sum_K A_{KJ} \phi_K^n + \frac{\Delta t}{2|\Omega_J|} \sum_K (\mathbf{C}_{JK} \cdot \mathbf{u}_J - \mathbf{C}_{KJ} \cdot \mathbf{u}_K) \quad (45)$$

with

$$\left\{ \begin{array}{l} A_{JK} = \alpha_{JK} + \lambda_{JK}^u \mathbf{i} \cdot \mathbf{n}_{JK} \\ \mathbf{C}_{JK} = \rho_0 c (b_{JK} \bar{\bar{T}}_{JK} \mathbf{i} + \lambda_{JK} \mathbf{i} \cdot \mathbf{n}_{JK} \mathbf{i} + \|\mathbf{S}_{JK}\| c (\psi_{JK}^u - \psi_{JK}^P)) \end{array} \right. \quad (46)$$

and $\bar{\bar{T}}_{JK} = \bar{\bar{I}} - \mathbf{n}_{JK} \otimes \mathbf{n}_{JK}$ is the projection operator onto the plane orthogonal to $\mathbf{n}_{JK}$.

Hereafter, we prove that our recentered scheme is positive for any existing eigenfunction $\phi(\mathbf{i})$ or $\mathbf{u} \cdot \mathbf{j}$ of the system without coupling. Even more, we impose that the numerical scheme stays positive and thus stable whatever the coupling terms are. Indeed, by saturating the basis of potential eigenfunctions, we are able to guarantee the stability of any linear combination of these.

The non-coupled part of equation (42) yields to the following conditions for the positivity of the function $\mathbf{u} \cdot \mathbf{j}$ ⁴:

$$a_{KJ} + b_{KJ}^P (\mathbf{n}_{JK} \cdot \mathbf{j})^2 \geq 0 \quad (47)$$

and

$$1 - \frac{\Delta t}{2|\Omega_J|} \sum_K (a_{JK} + b_{JK}^P (\mathbf{n}_{JK} \cdot \mathbf{j})^2) \geq 0 \quad (48)$$

⁴For an implicit scheme, ensuring positivity is much more straightforward (for example when replacing $\mathbf{u}_J^{n+1}$ by $\mathbf{u}_J^n$ in Eq. (42)). It will not be further described here.

For (47), since $a_{KJ} \geq 0$, a simple sufficient condition on the pressure recentering function is derived:

$$b_{JK}^P \geq 0 \Rightarrow \Psi_{JK}^P \geq \min(1, |M_{JK}|) \quad (49)$$

From (48) a first CFL-like stability condition is derived:

$$\Delta t \leq \frac{2|\Omega_J|}{\sum_K (a_{JK} + b_{JK}^P (\mathbf{n}_{JK} \cdot \mathbf{j})^2)} = \tau_0 \quad (50)$$

Concerning the function $\phi(\mathbf{i})$, the non-coupled part of equations (45) is

$$\forall \mathbf{i} \in \mathbb{R}^3, \phi_J^{n+1}(\mathbf{i}) = (1 - \frac{\Delta t}{2|\Omega_J|} \sum_K A_{JK}) \phi_J^n + \frac{\Delta t}{2|\Omega_J|} \sum_K A_{KJ} \phi_K^n \quad (51)$$

The positivity conditions for an hypothetical eigenfunctions $\phi(\mathbf{i})$ are given by

$$\alpha_{KJ} + \lambda_{KJ}^u \mathbf{i} \cdot \mathbf{n}_{KJ} \geq 0 \quad (52)$$

and

$$1 - \frac{\Delta t(\mathbf{i})}{2|\Omega_J|} \sum_K A_{JK} \geq 0 \Leftrightarrow \Delta t(\mathbf{i}) \leq \frac{2|\Omega_J|}{\sum_K (\alpha_{JK} + \beta_{JK} \mathbf{i} \cdot \mathbf{n}_{JK})} \quad (53)$$

We would like these relations to be verified for all directions $\mathbf{i}$. Since acoustic waves can not propagate faster than the sound velocity relatively to a frame of reference moving at the fluid velocity, then $\|\mathbf{i}\| \leq 1$. Thus a sufficient condition to fulfill (52) is that $\lambda_{JK}^u \leq \alpha_{JK}$. This leads to

$$\Psi_{JK}^u = 1 \quad (54)$$

For the second condition, we notice that

$$\frac{2|\Omega_J|}{\sum_K (\alpha_{JK} + \beta_{JK} \mathbf{i} \cdot \mathbf{n}_{JK})} \geq \frac{2|\Omega_J|}{\sum_K (|\alpha_{JK}| + |\beta_{JK}|)} = \tau_1 \quad (55)$$

This leads finally to the CFL-like stability condition

$$\Delta t \leq \tau_1 = \frac{2|\Omega_J|}{\sum_K (\max(0, 2M_{JK}, M_{JK} + 1) c \|\mathbf{S}_{JK}\|) + \|\sum_K \max(0, \min(2, M_{JK} + 1)) c \mathbf{S}_{JK}\|}. \quad (56)$$

Note that $\tau_1 \leq \tau_0$. The stability condition (56) is thus stronger than the condition (50) and can be

retained as the stability condition of the resulting recentered scheme.

4.3 Choice of the recentering function

Conditions (49) and (54) obtained for the recentering functions are coherent with the recentering procedures proposed in the litterature. Rieper applies the recentering function $\min(1, |M_{JK}|)$ to the Roe scheme [32]. Dellacherie also recenter the pressure gradients in the Godunov scheme in his all-Mach scheme [36].

In the following, we propose and justify the choice of some recentering functions that will be assessed for our scheme. To construct such a function, one has to pay attention to its recentering effects by studying mathematically the behaviour of the function. In the following, three formulations of pressure recentering are considered. These will be compared in Section 6.

The first studied function is the one chosen by Rieper in his low-Mach version of the Roe scheme:

$$r(M_{JK}) = \min(|M_{JK}|, 1) \quad (57)$$

This function verifies the conditions (49) and (54), and corresponds to a linear recentering on the Mach number $r(|M_{JK}| \leq 1) = |M_{JK}|$ when $|M_{JK}| \leq 1$. Such a pressure recentering is easy to implement but has the drawback that it may recenter the pressure too much. Indeed, additionally to not recentering in supersonic regime, an ideal function is expected to little recenter in close-to-sonic regime where the compressible scheme still behaves correctly, and to have a smooth transition between these two regimes. In practice, having a tangent equal to zero in $M_{JK} = 1$ would permit this, which is not the case of function r .

In the perspective of being able to adjust the Mach number value from which the recentering will be applied, we introduce the family of recentering functions $(f_s)_{s \in [0,1]}$ given by

$$\forall s \in [0, 1], f_s(M_{JK}) = \frac{\min(|M_{JK}|, s)}{1 + (1 - \frac{2}{s})\min(|M_{JK}|, s) + \frac{1}{s^2}\min(|M_{JK}|, s)^2}. \quad (58)$$

By construction, when $|M_{JK}| \geq s$, $f_s(M_{JK}) = 1$. The parameter s can thus be seen as a cutoff value above which there is no recentering of the pressure. Moreover, f_s verifies the conditions (49) and (54), and a change of variable $X = \min(|M_{JK}|, s)$ along with a simple Taylor expansion in 0 for the variable X show that these functions have a dominant term in $|M_{JK}|$ in the zero Mach number limit (since $\min(1, |M_{JK}|) = |M_{JK}|$ in this limit). Thus, the functions f_s will behave like Rieper's function in the incompressible limit but will produce recentering only from absolute Mach values equal to s chosen by the user.

Another idea is to focus on stability issues due to the too strong recentering introduced by the functions r and f_s for very low Mach numbers. For this purpose, a third function is investigated :

$$g(M_{JK}) = 1 - \max(0, 1 - |M_{JK}|)^2 \quad (59)$$

The function g also verifies the conditions (49) and (54). Moreover, by performing a Taylor expansion in 0 for the variable $X = \min(|M_{JK}|, 1)$ as previously, we see that in the incompressible limit, the leading term of the function is $2|M_{JK}|$. We thus have a stronger recentering for small Mach numbers, which means that there is more numerical viscosity and that the scheme is likely to be more robust.

Note that for Mach numbers approaching 1 from smaller values, all the functions introduced in the above have a null derivative, which is a nice feature to have a smooth transition to the classical upwind formulation.

All the described recentering effects of the presented functions are illustrated in Figure 5 where r , f_s (for $s = 0.1$ and $s = 1$) and g are represented for $M_{JK} \in [0, 1.5]$.

These functions will be used in the present work within a Godunov-type solver using a MUSCL approach (see Section 6). Note that condition (49) imposes that the pressure recentering function is above the pink curve in Figure 5 for $M_{JK} \in [0, 1]$.

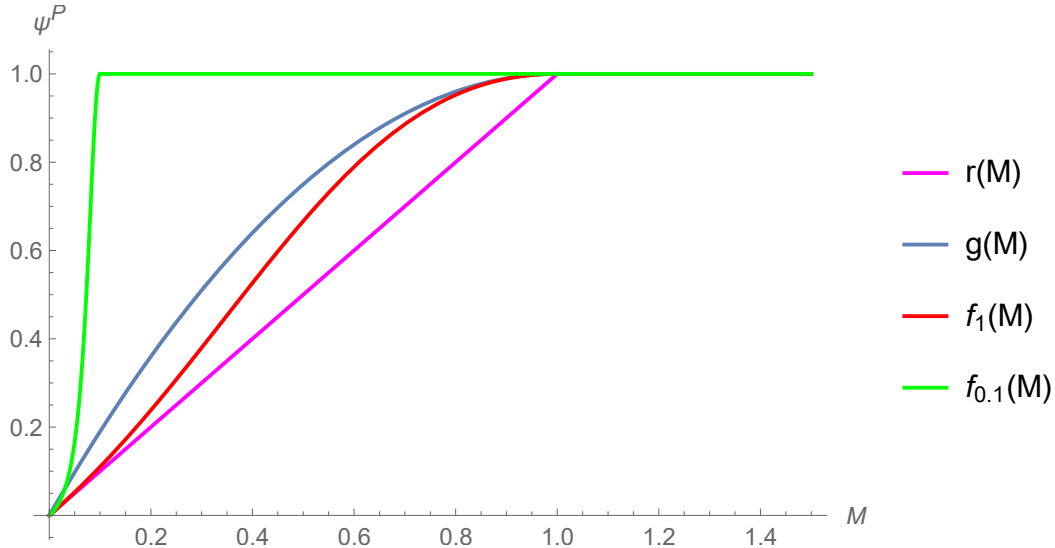


Figure 5: Representation of some of the recentering functions . In the Figure's legend, we note f_1 (resp. $f_{0.1}$) the recentering function of Eq. (58) with $s = 1$ (resp. 0.1).

5 Near-wall discretization

Even if the methodology is only applied to inviscid cases in the the present work, the final goal of the study is the resolution of the Navier-Stokes equations. In practice, for the latter, Dirichlet conditions are imposed on all primitive variables with the exception of pressure (and chemical species if any) where a Neumann-type

boundary condition prevails. For this reason, a Dirichlet-type boundary condition is assumed in the main numerical strategy of Section 5.1. Extension to other types of boundary condition is investigated in Section 5.2.

5.1 Numerical strategy for a Dirichlet-type boundary condition

In this part, the 1-exact and 2-exact reconstruction schemes presented in Section 2 are modified in the vicinity of wall boundaries. The baseline formulation, obtained by simplifying the reconstruction stencil, is first introduced. Then, the reconstruction stencil is enriched and the successive-corrections procedure is adapted accordingly. We will assume here that a Dirichlet boundary condition $q_D(\mathbf{x} \in \partial\Omega)$ is prescribed at the wall. The treatment of Neumann-type boundary conditions is discussed in part 5.2.

The baseline solver of FLUSEPA does not take into account the wall in the polynomial reconstruction of the solution variables (the parameter β introduced in Section 2 is set to 0). The corresponding stencils used respectively for the second and third-order schemes for a boundary cell Ω_J are shown in Figure 1.

To account for the wall contribution in the approximation of the derivatives, we introduce a ghost cell Ω_W which corresponds to the wall boundary condition of the cell Ω_J : this is not an average quantity but a pointwise value. Let $s^*(J) = s(J) \cup \Omega_W$ be the extended neighbourhood of Ω_J . Note that $s^*(J) = s(J)$ when Ω_J is inside the domain.

To build a second-order polynomial representation of the variable q that uses the boundary conditions, the Quasi-Green gradient must be extended: as in Eq. (15), we want

$$\iint_{S_{JW}} q \mathbf{n} dS = (\beta_W q(\mathbf{x}_W) + (1 - \beta_W) \bar{q}_J) \mathbf{S}_{JW} + \mathcal{O}(h^m) \mathbf{S}_{JW} \quad (60)$$

If we perform a Taylor expansion of the Dirichlet function at any point of the wall face $\mathbf{x}_W \in S_{JW}$ in the integral $\iint_{S_{JW}} q \mathbf{n} dS$, we obtain a relation of the form of Eq. (60) with $\beta_W = 1$. Moreover, when the wall face is flat and $\mathbf{x}_W$ is the center of gravity of the face, the approximation is second-order accurate ($m = 2$) and the Green integral is consistent even without the correction matrix $M_1^*(J)$. On general grids, however, this integral is only first-order accurate ($m = 1$) and thus no longer consistent with the Green-Gauss method. However, by nature of the Quasi-Green approach, the consistency of the overall gradient operator will be restored thanks to the simple correction matrix $M_1(J)$, if this matrix is modified accordingly, giving a matrix $M_1^*(J)$ that satisfies the 1-exactness consistency condition of Eq. (17):

$$M_1^*(J) = \frac{1}{|\Omega_J|} \sum_{K \in s^*(J)} \beta_K \mathbf{S}_{JK} \otimes (\mathbf{x}_K - \mathbf{x}_J), \text{ with } \beta_W = 1 \quad (61)$$

In the end, the enriched 1-exact gradient operator $\mathcal{D}_1^{*(1)}$ is

$$\mathcal{D}_1^{*(1)} [\bar{q}]_J = M_1^{*-1}(J) \frac{1}{|\Omega_J|} \sum_{K \in s^*(J)} (\beta_K \bar{q}_K + (1 - \beta_K) \bar{q}_J) \mathbf{S}_{JK}. \quad (62)$$

with the notation $\bar{q}_w = q_w$.

To build a second-order polynomial reconstruction of the solution variables, the strategy is not as straightforward. Indeed, as can be seen in Figure 1 (b), the Quasi-Green operator $\mathcal{D}_1^{*(1)} [\bar{q}]_J$ can not be applied twice since gradients in the neighbourhood of Ω_J are missing.

We apply the enriched gradient operator and its associated correction matrix whenever possible, and keep the baseline approximation otherwise. With this choice, the initial inconsistent approximation of the second derivatives is obtained by applying the original 1-exact operator $\mathcal{D}_1^{(1)}$ to the modified operator $\mathcal{D}_1^{*(1)}$:

$$\mathcal{D}_0^{*(2)} [\bar{q}]_J = \mathcal{D}_1^{(1)} \left[\mathcal{D}_1^{*(1)} [\bar{q}]_J \right]_K \quad (63)$$

The correction matrix $M_2(J)$ is modified in a similar way and a matrix $M_2^*(J)$ is derived.

Then, the **upgrading** matrices $(H_1(J))_{i \in \{1,3\}}$ of Eq. (24), which **correct the truncation error of gradients to reach the second-order of accuracy** are changed into $(H_1^*(J))_{i \in \{1,3\}}$ by adding **the** wall contribution **with** $\beta_w = 1$ and **setting** the volume moments of the fictitious cell Ω_w **equal to zero**.

This strategy to improve the accuracy of the k -exact reconstruction in boundary cells can be theoretically perform for any order of k -exactness by applying the original derivative operator successively ($\mathcal{D}_1^{*(1)}$ is only used for the first step of the composition). However, one critical requirement must be satisfied for the strategy to be consistent : for a k -exact reconstruction, the boundary value $q(\mathbf{x}_r)$ must be known at least to the order $k + 1$.

5.2 Other boundary condition types

5.2.1 Steady case

In the numerical strategy presented in part 5.1, we assumed that a Dirichlet condition was prescribed at the wall. In the following, we focus on the case for which a Neumann boundary condition is imposed at the wall:

$$\frac{\partial q}{\partial \mathbf{n}}(\mathbf{x} \in \partial\Omega) = \mathbf{D}^{(1)} \mathbf{q}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) \quad (64)$$

To avoid multiple definitions of the correction matrices, which would be cumbersome and costly, especially when increasing the reconstruction order, we choose to calculate an equivalent fictitious Dirichlet value at

the wall q_w^{fict} . An essential requirement to ensure the consistency of this method is that q_w^{fict} is known at least to the $(k+1)^{th}$ order of accuracy for a k -exact reconstruction scheme.

For simplicity, assume that the flow is steady. Denote by $\frac{\partial q}{\partial n_w}$ the physical gradient imposed at the wall in the normal direction. At a given time step $n+1$, a predicted gradient can be derived by using the previous wall value q_w^n in the reconstruction scheme, noted $\nabla_w^k(q_w^n, \bar{q}^{n+1})$. The wall value at the current time step q_w^{n+1} is then established by the formula (see Appendix 7)

$$q_w^{n+1} = q_w^n + C_n \left(\frac{\partial q}{\partial n_w} - \nabla_w^k(q_w^n, \bar{q}^{n+1}) \cdot \mathbf{n} \right) + \mathcal{O}(h^{k+1}) \quad (65)$$

where C_n is a constant that depends on the reconstruction scheme in use.

An additional constraint to get a convenient method is to have a single definition of the wall point for any kind of boundary condition. Yet, in the Dirichlet case, any point can be chosen since the consistency of the approximation will be restored with the correction matrix $M_1^*(J)$ (see part 5.1). We thus look for a point suitable to the Neumann boundary condition. The point W whose corresponding normal vector to the local surface is colinear to $\mathbf{x}_j - \mathbf{x}_w$ is chosen. This point was introduced in Section 3.1. It is also chosen as the integration point Γ for the flux integration in our calculations.

In the **non-diffusive case (among which is the inviscid case)**, there is no information directly available at the wall. This information (entropy, **tangential velocity, concentration of the chemical species ...**) generally comes from the initial conditions or the stagnation points by gliding along the wall. This case is recast in a similar way, by using extrapolation from the inside of the domain, assuming a sufficient smoothness of the solution:

$$q_w^{n+1} = q_w^n + C_e (R^k(q_w^n, \bar{q}^{n+1}) - q_w^n) + \mathcal{O}(h^{k+1}) \quad (66)$$

with C_e a constant for the extrapolation case dependent on the k -exact reconstruction operator R^k , see Appendix 7.

5.2.2 Unsteady case

The previous strategy is not suitable for unsteady cases. Using the derivatives at the previous timestep to derive a value of the variable at the wall is inconsistent for unsteady flows and can thus not lead to a correct temporal approximation. A modification of the method to palliate this problem is proposed in the following. It has however not been tested in the present work, in which only steady cases are studied⁵.

Let us consider a wall where an unsteady Neumann boundary condition is known to the required order of

⁵Note that however the methodology proposed here could be useful to accelerate the convergence of steady cases.

accuracy. To take into account the unsteadiness of the problem when evaluating the wall value q , an evolution equation of this value must be derived, deduced from the physical equations of the problem. Indeed,

$$q_W^{n+1} = q_W^n + \int_0^{\Delta t} \frac{\partial q}{\partial t}|_W dt = q_W(t_{n+1}) + \mathcal{O}(h^{k+1}) \quad (67)$$

since $q_W^n = q_W(t_n) + \mathcal{O}(h^{k+1})$.

In a first approach, the crude approximation $\frac{\partial q}{\partial t}|_W = \frac{\partial q}{\partial t}_j + \mathcal{O}(\Delta x)$ can be used in (67). By integrating it relatively to time, one gets:

$$\int_0^{\Delta t} \frac{\partial q}{\partial t}_w dt = \int_0^{\Delta t} \frac{\partial q}{\partial t}_j dt + \mathcal{O}(\Delta x \Delta t) \quad (68)$$

Yet

$$\int_0^{\Delta t} \frac{\partial q}{\partial t}_j dt = q_j^{n+1} - q_j^n + \mathcal{O}(\Delta x^{k+1}) + \mathcal{O}(\Delta t^m) \quad (69)$$

with m the order of accuracy of the integration temporal scheme. By injecting (69) in (67), we finally obtain

$$q_W^{n+1} = q_W^n + q_j^{n+1} - q_j^n = q_W(t_{n+1}) + \mathcal{O}(\Delta x \Delta t + \Delta t^m) + \mathcal{O}(\Delta x^{k+1}) \quad (70)$$

With our numerical scheme we have $k + 1 \geq 2$ and $m \geq 2$. The overall temporal order depends on the temporal scheme used and of its nature: for instance, with our second-order temporal explicit scheme, $\Delta t \equiv \Delta x$, then the formula (70) will be close to the second-order of accuracy in time. In all cases, the approximation will be consistent in time and accurate **accurate to the order $k + 1$** in space which is sufficient to build a k -exact reconstruction.

Due to the temporal error, the extrapolation (70) can not be used alone. It must be corrected so that there is no drift of the wall value when advancing in time. Therefore, we propose to perform at each time step a predictor-corrector process: knowing $q_W^n = q_W(t_n) + \mathcal{O}(h^{k+1})$, the predictor step consists in evaluating $\check{q}_W = q_W^n + \mathcal{O}(\Delta t^m) + \mathcal{O}(\Delta x^{k+1})$ with $m \geq 1$.

$$\check{q}_W = q_W^n + q_W^n + q_j^{n+1} - q_j^n \quad (71)$$

for example by using a formula analogous to Eq. (70).

The second step consists in computing the gradient $\frac{d\check{q}_W}{dn} = \frac{dq}{dn}(\check{q}_W, \bar{q}^{n+1})_W$ and **using** it along with the true

Neumann boundary condition at time t_{n+1} , $\frac{dq_w}{dn}(t_{n+1})$ to correct q_w^n as in the previous part:

$$q_w^{n+1} = q_w^n + C_n \left(\frac{dq_w}{dn}(t_{n+1}) - \frac{d\check{q}_w}{dn} \right) + (q_j^{n+1} - q_j^n) = q(t_{n+1}) + \mathcal{O}(h^{k+1}) \quad (72)$$

6 Numerical results

6.1 Subsonic inviscid flow past a smooth Gaussian bump

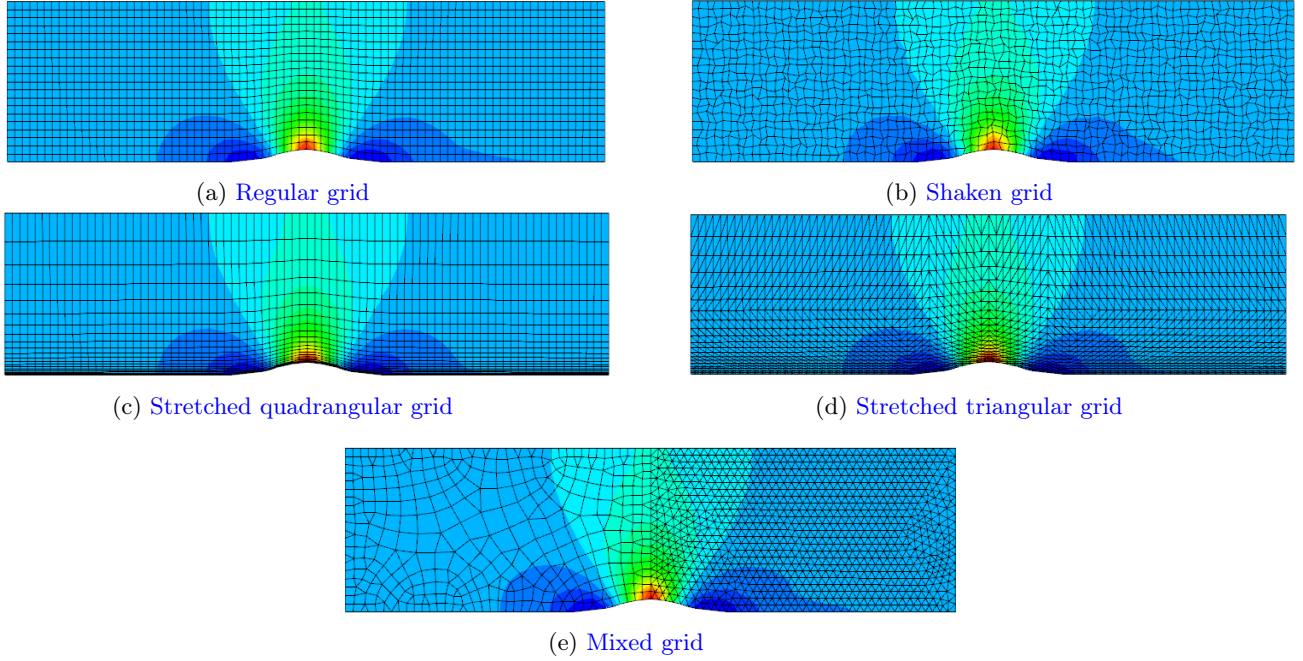


Figure 6: x -component of the velocity for the different grids used for the calculations of the Bump test case - 40 wall faces along the bump

To evaluate the models presented in the previous Sections, we consider a subsonic inviscid flow within a channel with a smooth Gaussian bump of equation:

$$y = 0.0625e^{-25x^2} \quad (73)$$

The height of the canal is $h = 0.8m$ and the characteristic chord length of the bump is equal to $c = 0.8m$. Left and right boundary conditions are respectively a subsonic inflow and a subsonic outflow with imposed pressure, and the upper and lower boundaries are slip walls. The inlet flow angle is equal to 0.

First, a Mach number of $M = 0.5$ is imposed at the inflow, and a mesh convergence study for several

types of grid is performed to compare the models. Then the inflow Mach number is decreased to $M = 0.01$ to assess qualitatively the low Mach recentering scheme of Section 4.

6.1.1 Inflow Mach number of 0.5

In the following, since the analytical solution for the conservative variables is unknown, entropy is used as an measure of accuracy. This should be constant and equal to the inflow value throughout the flow for the present steady inviscid flow (homotropic flow).

In Figure 7, the entropy error for the 2-exact reconstruction scheme on a mesh having 80 wall faces along the bump is compared with and without the curvature model. Spurious entropy generation is observed downstream of the bump, which is significantly reduced by taking into account the curvature. In the rest of the domain where curvature is smaller, entropy is already correctly computed by the baseline method.

An initial mesh convergence study is carried out for the entropy error, on a set of five regular grids composed of 400, 1600, 6400, 25600 and 102400 quadrangular cells (see Figure 6a), corresponding respectively to 40, 80, 160, 320, and 640 wall faces along the bump. Then the ability of the method to preserve its accuracy on strongly deformed meshes is assessed on four additional types of grid for which a similar mesh convergence study is performed: shaken grid obtained by random perturbation of the nodes (Figure 6b), mixed grid made of unstructured triangles and quadrangles (Figure 6e), and highly-stretched quadrangular (Figure 6c) and triangular grids (Figure 6d) for which the height of the first cell close to the bump wall is equal to $0.001h$. Results are provided for the 1-exact and 2-exact reconstruction schemes, by comparing for all these types of grid the baseline solver, the solver with curved boundaries, and the solver with curved boundaries and an enriched stencil at the wall.

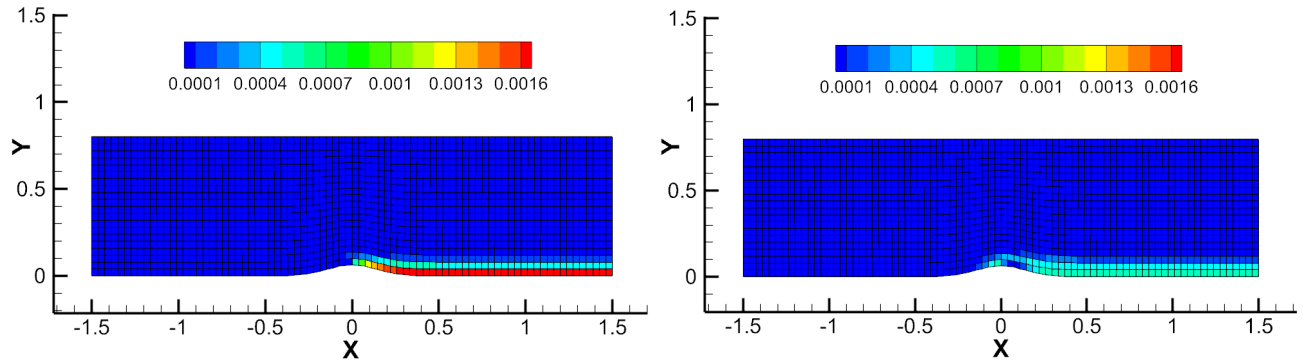


Figure 7: Inviscid Gaussian bump: entropy error generated downstream of the bump with the 2-exact scheme (a) with no modification; (b) with the wall curvature model

Note that grid refinements are based on the number of wall faces along the bump: hence, the number of cells for a given refinement level can vary from one type of grid to the other. The L_2 and L_∞ norms of the

error are thus plotted versus a characteristic mesh size given by $\frac{1}{\sqrt{N_{cells}}}$. Grid convergence plots are shown in Figure 8 for the 1-exact reconstruction scheme and in Figure 9 for the 2-exact reconstruction scheme. The corresponding orders of accuracy given by the asymptotic slopes of the curves are respectively listed in Table 1 and 2.

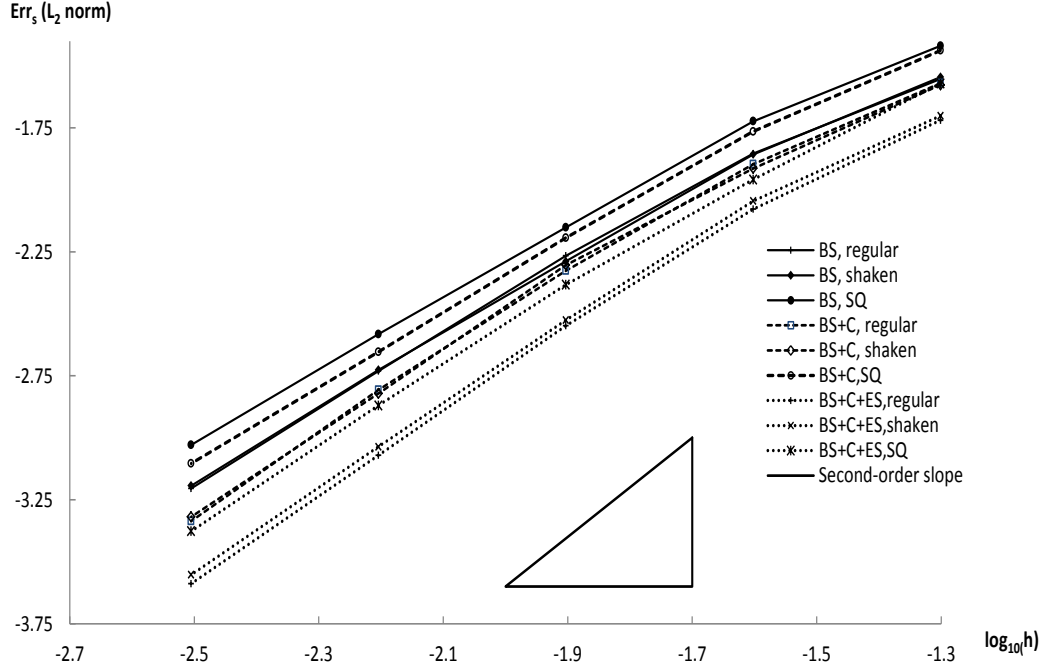
Satisfying orders of accuracy are obtained for the reconstruction schemes, and a decrease of the entropy error is observed on all types of grid with the new models. A decrease of the accuracy order when using meshes composed of triangular faces (hybrid and stretched triangular meshes) can be noticed: this is partly due to the reduction of the size of the stencil.

For the scheme with 1-exact reconstruction, we see that the convergence slope when adding the curvature model to the scheme increases compared to using the unmodified scheme. However, the entropy error reduction is low. This is not surprising: as it was stressed before, the spatial error introduced when approximating curved surfaces with flat faces is of order 2. Yet, the approximation of the boundary should at least represent the real boundary with the same order of accuracy as the discretization scheme [10]. Here the original scheme with 1-exact reconstruction exhibits a convergence of order 2 at most, as can be seen in Figure 8. Even if the Bézier patches approximate the wall with third-order accuracy, the gain is expected to be small when only this improvement is used in the solver.

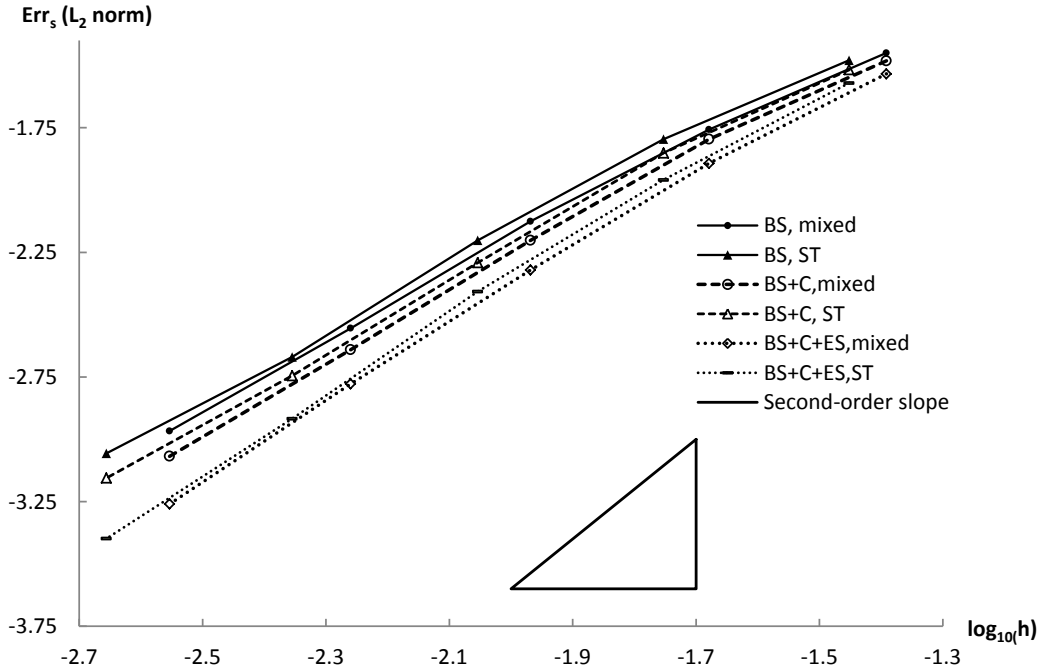
When using the model with extended stencil in boundary cells along with curved walls, much larger improvements are obtained: the entropy error drops and the convergence slope is again increased. The information at the wall is accurate enough (thanks to the modelling of the curvature and to a proper handling of boundary conditions even if the available information at the wall for the Euler equations is **poor**) to calculate a 1-exact approximation of the gradient.

For the scheme with 2-exact reconstruction, satisfying results are obtained with the curvature model, which approximates the physical surface with a spatial error of order 3. The entropy error is reduced and more importantly, the convergence slope is significantly increased with this model, getting closer to third order.

When considering the curvature model coupled with the parabolic reconstruction with a more symmetrical stencil in boundary cells, no significant gain is observed **for the meshes with quadrangular cells**. This is due to the fact that the boundary information **for the inviscid flow equations** is quite limited (a lot of quantities must be extrapolated **from the interior of the domain**) and is thus not very efficient to improve the overall parabolic reconstruction of the solution, contrary to the 1-exact scheme for which the first-order approximate gradients are significantly improved. **However, a much more significant improvement is achieved on hybrid and stretched triangular grids for this coupled model. Indeed, enriching the stencil with the wall point has a stronger impact on triangular cells compared to quadrangular cells.**



(a) Regular, shaken and Stretched Quadrangle grids



(b) Mixed and Stretched Triangle grids

Figure 8: Inviscid flow past a bump: grid convergence on entropy error. Comparison of the different models using the 1-exact reconstruction scheme for several types of grid. BS: Baseline solver, C: Curvature model, ES: Enriched Stencil.

6.1.2 Inflow Mach number of $M = 10^{-2}$

Subsequently, similarly as in [37], a preliminary qualitative test of the low-Mach correction presented in Section 4 is performed, by reducing the inflow Mach number to $M = 10^{-2}$. Two of the recentring functions

Slopes	Regular	Shaken	Mixed	Stretched Quadrangles	Stretched Triangles
BS	1.5593	1.5036	1.4368	1.4565	1.4196
BS + C	1.678	1.6869	1.4806	1.5109	1.4354
BS + C + ES	1.7077	1.6938	1.6043	1.6509	1.6464

Table 1: Asymptotic slopes of the grid convergences using the 1-exact reconstruction

Slopes	Regular	Shaken	Mixed	Stretched Quadrangles	Stretched Triangles
BS	2.6342	2.6397	2.3714	2.6243	2.5084
BS + C	2.7803	2.7556	2.4549	2.7245	2.5497
BS + C + ES	2.7884	2.7602	2.4663	2.7292	2.6221

Table 2: Asymptotic slopes of the grid convergences using the 2-exact reconstruction

of part 4.3 are tested : f_s with $s = 0.02$ and g , see Eqs. (58) and (59) respectively. These have been chosen because they are representative of all the recentering functions presented in the present paper. Indeed, the Mach number used here is very low and the functions r and f_s using any other value for s present the same asymptotic behaviour.

The iso-Mach and iso-pressure contours are presented on a fixed quadrangular mesh of 400 cells, by using successively the baseline solver and the recentered solvers for $M = 10^{-2}$ in Figure 10. For the pressure, 12 levels of iso-lines between $P_{min} = 101316$ and $P_{max} = 101336$ are displayed. For the Mach number, 16 levels of isolines are shown, between $M_{min} = 0.0094$ and $M_{min} = 0.0122$. The expected behaviour of the solution is shown in [38].

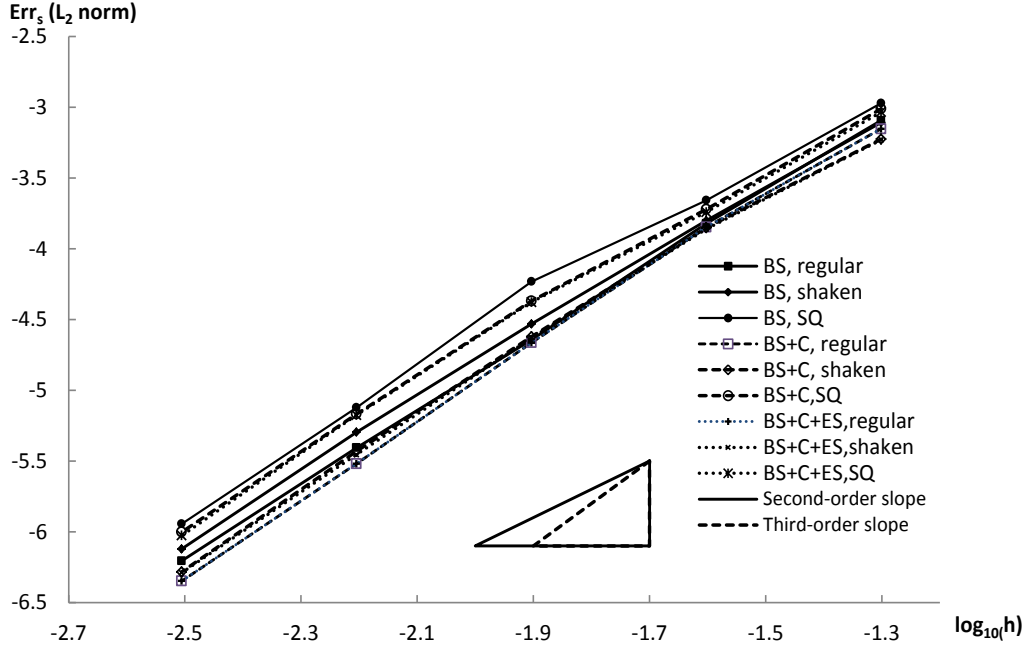
It appears that the baseline solver does not reproduce properly the solution. The expected shape of the Mach number iso-lines is retrieved to some extent when recentering with the function $f_{0.02}$, even if there still exists a lack of symmetry in the results. Recentering with the function g symmetrizes a little bit more the Mach number and pressure contours.

6.2 Subsonic flow around a circular cylinder

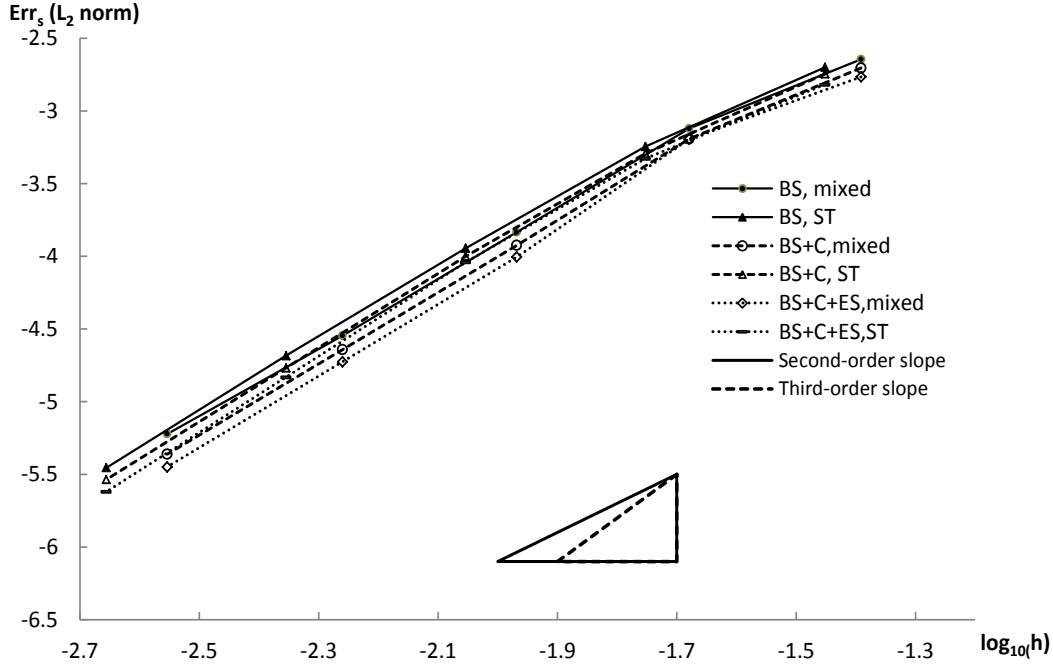
We assess more rigorously the low-Mach correction for the inviscid flow around a circular cylinder. The advantage of this case is that an analytical solution is available in the incompressible regime, given by the well-known potential flow around a circular cylinder:

$$\begin{cases} u_{r,ref}(r, \theta) = U_\infty(1 - \frac{r_0^2}{r^2})\cos(\theta) \\ u_{\theta,ref}(r, \theta) = -U_\infty(1 + \frac{r_0^2}{r^2})\sin(\theta) \end{cases} \quad (74)$$

$$P_{ref}(r, \theta) = P_\infty + \frac{1}{2}\rho U_\infty^2(2\frac{r_0^2}{r^2}\cos(2\theta) - \frac{r_0^4}{r^4}) \quad (75)$$



(a) Regular, shaken and Stretched Quadrangle grids



(b) Mixed and Stretched Triangle grids

Figure 9: Inviscid flow past a bump: grid convergence on entropy error. Comparison of the different models using the 2-exact reconstruction scheme for several types of grid. BS: Baseline solver, C: Curvature model, ES: Enriched Stencil.

where $\mathbf{u}_\infty = U_\infty \mathbf{e}_x$ and P_∞ are the freestream velocity and pressure, and r_0 is the radius of the cylinder.

The computational domain is $\Omega = [r_0, r_1] \times [0, 2\pi]$, with $r_0 = 0.5$ and $r_1 = 5$. Simulations are first

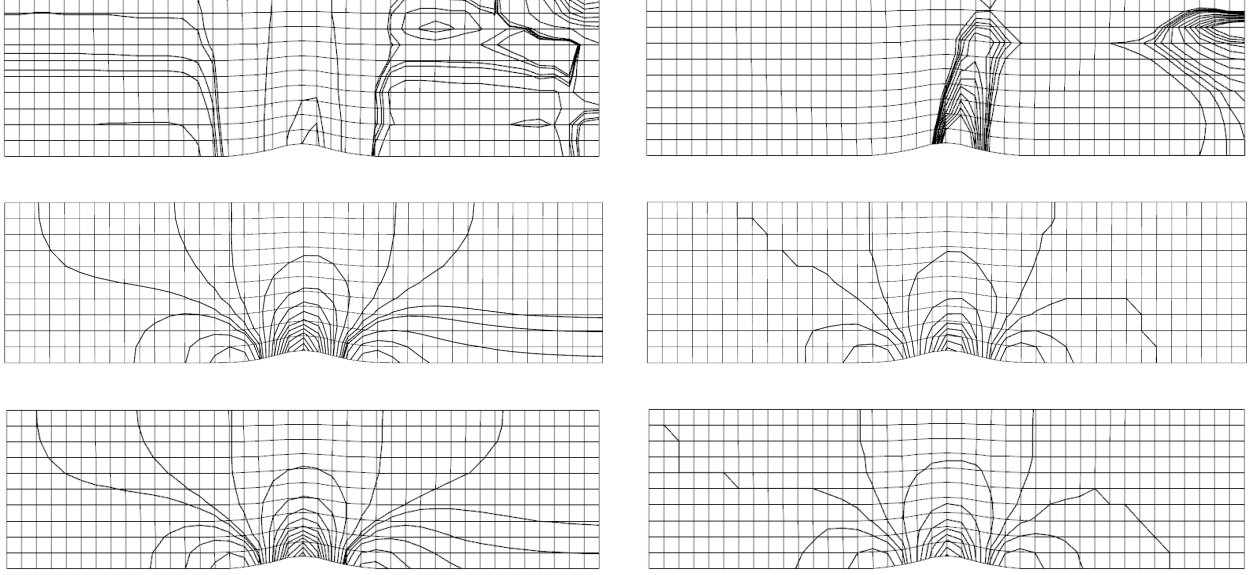


Figure 10: Iso-lines of the Mach number (left) and pressure (right), 1-exact reconstruction scheme, 400cells, inflow Mach number of $M = 10^{-2}$: Baseline solver (top) ; Recentered scheme using $f_{0.02}$ (middle), Recentering scheme using g (bottom)

performed on a fixed mesh of 7500 cells (50 cells in the radial direction, 150 in the azimuthal direction), composed of a far field and of a slip wall boundary condition for the cylinder. Four different Mach numbers approaching the incompressible limit are studied: $M = 10^{-1}$, 10^{-2} , 10^{-3} and 10^{-4} . The incoming velocity U_∞ is set to $U_\infty = Mc$, with c the speed of sound.

The L_2 norm of the pressure error $\delta P = P - P_{ref}$ is investigated for several wall treatments: baseline solver, solver with low-Mach recentering, solver with curvature model for the wall faces, and solver with both curvature and low-Mach recentering. From the results of part 6.1.2, we decide to use $f_{0.02}$ as our recentering function. Indeed, both functions $f_{0.02}$ and g gave good results on the bump test case, and the family of functions $(f_s)_{s \in [0,1]}$ is less diffusive in the low Mach limit.

The results are reported for the 1-exact scheme in Table 3 and for the 2-exact scheme in Table 4. In order to take into account the proximity to the incompressible regime when comparing each simulation to the analytical solution, δP is scaled by ρU_∞^2 , instead of being normalized by P_{ref} as it is generally done (see [32] for instance).

When using the 1-exact reconstruction scheme (Table 3), we see that the baseline solver gives acceptable results at $M = 10^{-1}$ for the pressure error, similar to the one obtained by using the low-Mach correction.

However, for $M = 10^{-2}$ or lower, the baseline scheme error increases as shown in Figures 12 and 13. Moreover, the pressure error do not converge well at $M = 10^{-4}$.

Introducing the low-Mach correction improves the results significantly, both in terms of accuracy and of

M	BS	BS+C	BS+LM ($f_{0.02}$)	BS+C+LM ($f_{0.02}$)
10^{-1}	1.12×10^{-4}	7.53×10^{-5}	1.86×10^{-4}	7.96×10^{-5}
10^{-2}	2.33×10^{-3}	2.3×10^{-3}	1.52×10^{-4}	4.82×10^{-5}
10^{-3}	1.08×10^{-1}	9.32×10^{-2}	2.31×10^{-4}	6.53×10^{-5}
10^{-4}			2.26×10^{-4}	5.74×10^{-5}

Table 3: Pressure error δP scaled with ρU_∞^2 , using the 1-exact scheme. BS = Baseline Solver, LM = Low-Mach recentered scheme, C = Curvature model, gray cell : did not converge.

M	BS	BS+C	BS+LM ($f_{0.02}$)	BS+C+LM ($f_{0.02}$)
10^{-1}	3.44×10^{-5}	3.33×10^{-5}	3.46×10^{-5}	3.38×10^{-5}
10^{-2}	1.65×10^{-4}	1.62×10^{-4}	1.19×10^{-5}	1.12×10^{-5}
10^{-3}	2.72×10^{-3}	2.67×10^{-3}	2.51×10^{-5}	2.45×10^{-5}
10^{-4}	2×10^{-2}	1.99×10^{-2}	3.37×10^{-5}	3.34×10^{-5}

Table 4: Pressure error δP scaled with ρU_∞^2 , using the 2-exact scheme. BS = Baseline Solver, LM = Low-Mach recentered scheme, C = Curvature model.

convergence. The error of the corrected scheme is several orders of magnitude lower than the baseline one for $M = 10^{-3}$ and $M = 10^{-4}$.

Levels of the rescaled pressure error with the recentered scheme remains the same for all considered Mach (see Table 3), which is an expected feature showing the ability of the recentering strategy to solve the low-Mach accuracy problem of our compressible solver. Indeed, with the choice of scaling δP with ρU_∞^2 , an acceptable error is supposed to remain of the same order for all Mach numbers: when the Mach number decreases, $(P - P_{ref})$ decreases (because we are closer to the incompressible regime) and U_∞ decreases too such that $(P - P_{ref})$ is scaled by a smaller quantity (linked to the incompressible Bernouilli formulation): consequently $\frac{P - P_{ref}}{\rho U_\infty^2}$ remains of the same order.

Similar behaviours are obtained for the 2-exact scheme (Table 4) with the low-Mach recentering scheme. In this case however, the baseline scheme is more accurate and introduces smaller errors even without low-Mach correction. As a consequence, the benefit of low-Mach recentering is less significant. **This is an expected result, since higher-order schemes introduce a lower intrinsic dissipation and thus provide more accurate results already without low-Mach correction.** Once again, we stress the importance of introducing the curvature correction in conjunction with high-order schemes. Curvature corrections not only reduce the overall error levels but also alleviate the checkerboard oscillations that can appear in the error [39].

Lastly, a mesh convergence study is made in Figure 11, for the 1-exact and 2-exact reconstruction schemes for a fixed Mach number of $M = 10^{-3}$, on three grids of 600, 2400 and 9600 cells (corresponding respectively to azimuthal and radial discretizations of 20×30 , 40×60 and 80×120) with the low-Mach scheme on the one hand, and with the low-Mach scheme coupled with the curvature model on the other hand. For the 1-exact

reconstruction scheme, second order is retrieved, and the coupling of the low-Mach and curvature models yields to a slightly increase of the order of accuracy. For the 2-exact reconstruction scheme, third-order is retrieved. Here again, coupling the low-Mach with the curvature model have a slight effect on the overall accuracy of the simulation. This is due to the fact that the curvature model, through the calculation of more accurate geometrical moments, has an impact on the boundary conditions (those which are extrapolated because poorly prescribed in the inviscid case) which is more seeable for the 1-exact reconstruction than for the 2-exact reconstruction. The norm of the L_∞ error is displayed in Figure 11b to emphasize the effect of the correction in the near-wall region.

7 Conclusions

An improved treatment of near-wall regions is investigated for a family of high-order k -exact Godunov-type schemes, with an emphasis on the 1-exact and 2-exact reconstructions.

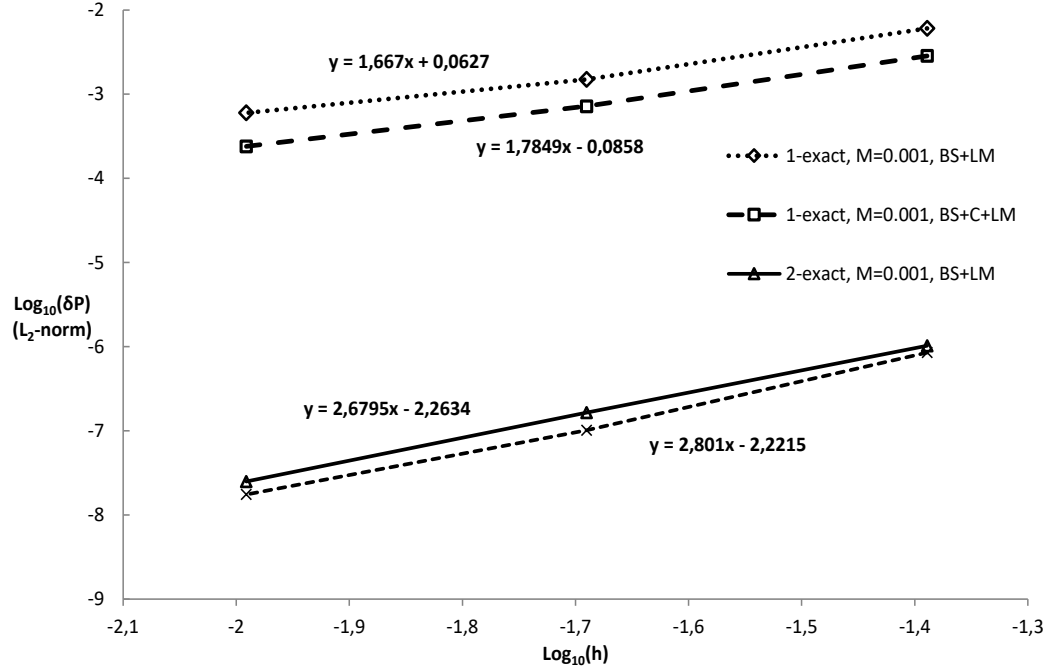
First, a high-order composite representation of the wall based on bicubic Bézier patches is incorporated into the solver, and the computation of geometrical quantities is modified close to the wall to conform with the patches. In particular, the choice of the integration point $\mathbf{x}_\mathbf{r}$ is essential to an accurate flux integration.

Secondly, a low-Mach correction based on the normal Mach number at the face is introduced in the Riemann solver used to calculate the convective fluxes. The scheme remains unchanged for Mach numbers whose absolute values are greater than one and is recentered otherwise, which makes the changes consistent with the baseline method for high-velocity flows.

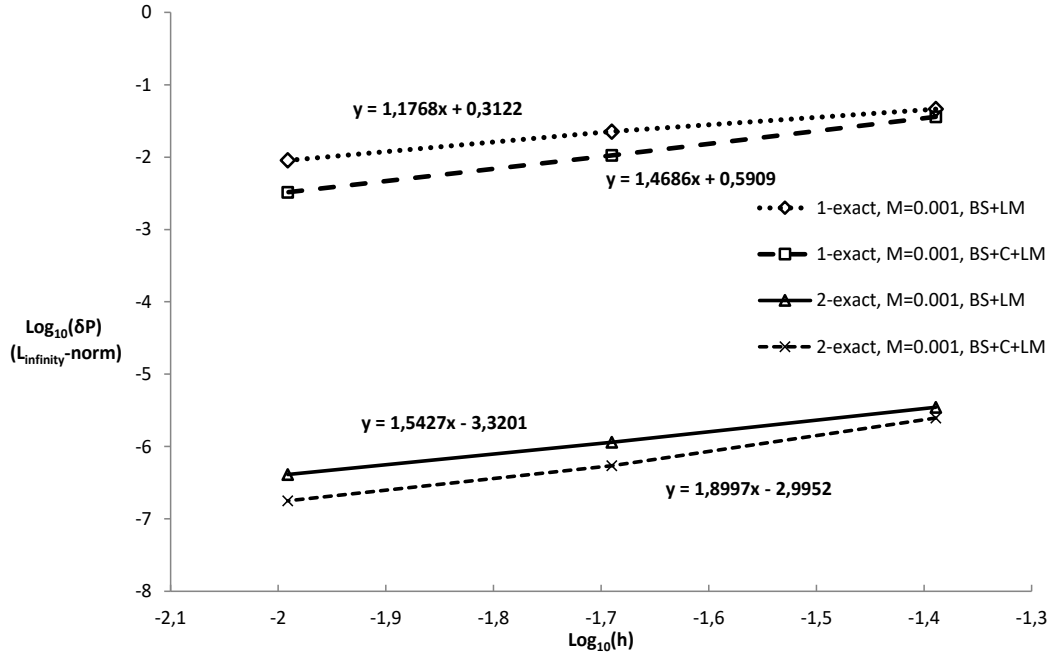
Thirdly, a numerical strategy is presented to enrich the stencil, based on an assumption of Dirichlet-type boundary condition, by using the boundary information in the reconstruction procedure. The modification is quite straightforward for the 1-exact scheme. For the 2-exact scheme, the approximate gradient operator with enriched stencil is applied only one time in the successive-corrections procedure, and correction matrices are modified in a similar way. To unify the method for Neumann and other boundary condition types, a surrogate value at the wall is necessary, at least accurate to the same order than the reconstruction. When the boundary condition is not of Dirichlet type, the fictitious wall value is derived from the boundary condition by a corrective procedure.

These modifications are then assessed against two inviscid flow problems, namely, the flow within a channel with a smooth Gaussian bump and the flow past a circular cylinder.

The results show that, in all cases, accounting for wall curvature improves the solution accuracy significantly. The benefit is larger for the 2-exact scheme, which includes more geometrical information. Conversely, low-Mach corrections are more influential for the more dissipative 1-exact scheme, although important im-



(a) With respect to the L_2 -norm



(b) With respect to the L_∞ -norm

Figure 11: Mesh convergence study for the subsonic cylinder, at $M = 10^{-3}$, with δP scaled by P_{ref} . BS: Baseline solver, LM: Low-Mach recentered scheme, C: Curvature model

provements are observed at all orders. Enriching the reconstruction stencil close to the wall also leads to solution improvements, but less significant than the two former corrections, especially for the 2-exact scheme.

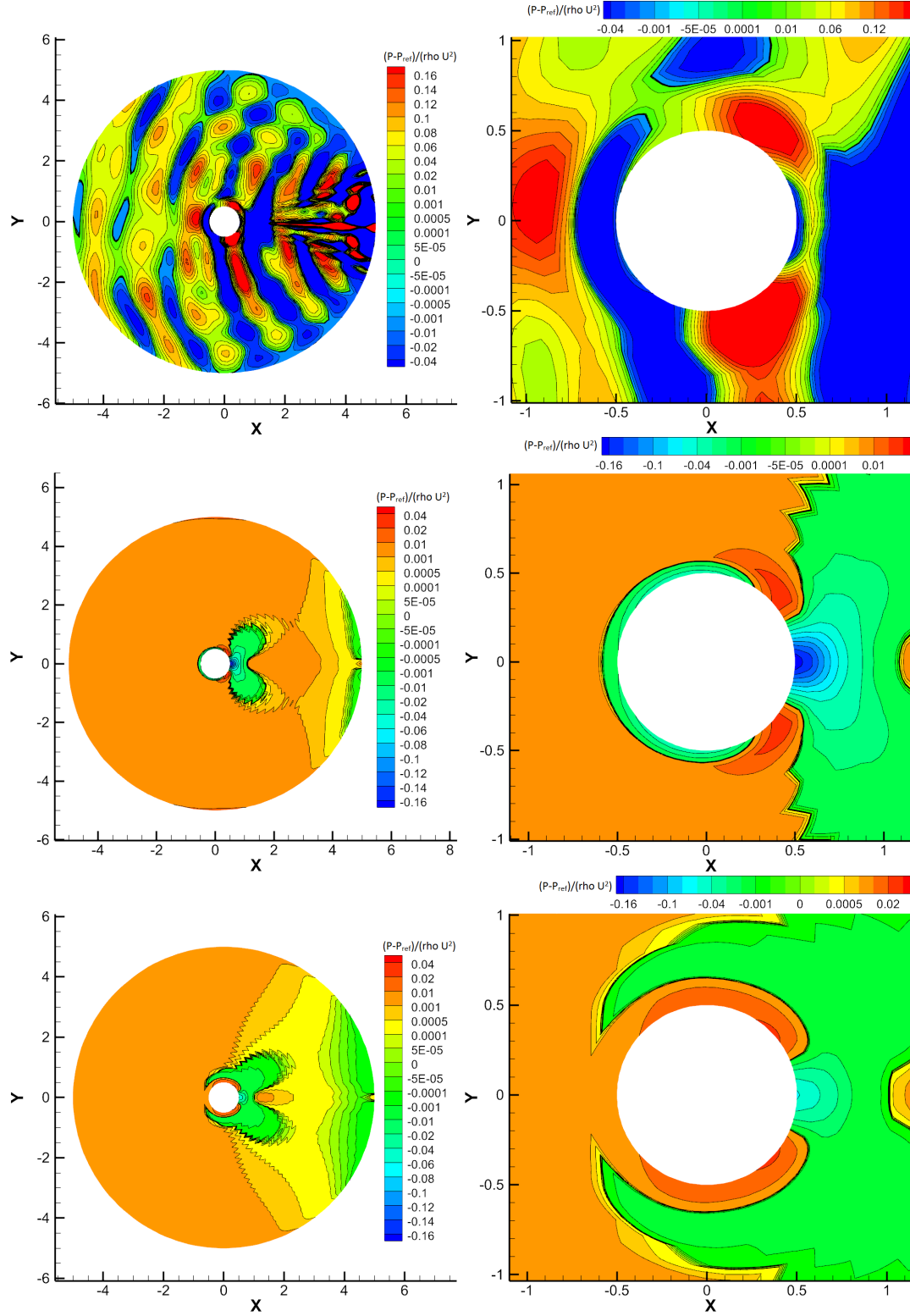


Figure 12: Pressure error levels and contours, $M = 10^{-2}$. Entire domain at the left, focus (zoom) on the cylinder at the right. Top : Baseline solver , middle : solver with low-Mach recentering , bottom : solver with low-Mach recentering and model of curvature.

This was predictable given the characteristics of the wall boundary conditions for non-viscous flows: for this type of flow, many information is extrapolated from the interior of the domain, so the information actually added by the enriched discretization is limited.

Future work comprises application of the schemes to unsteady cases and near-wall treatment of viscous fluxes to deal with Navier-Stokes configurations.

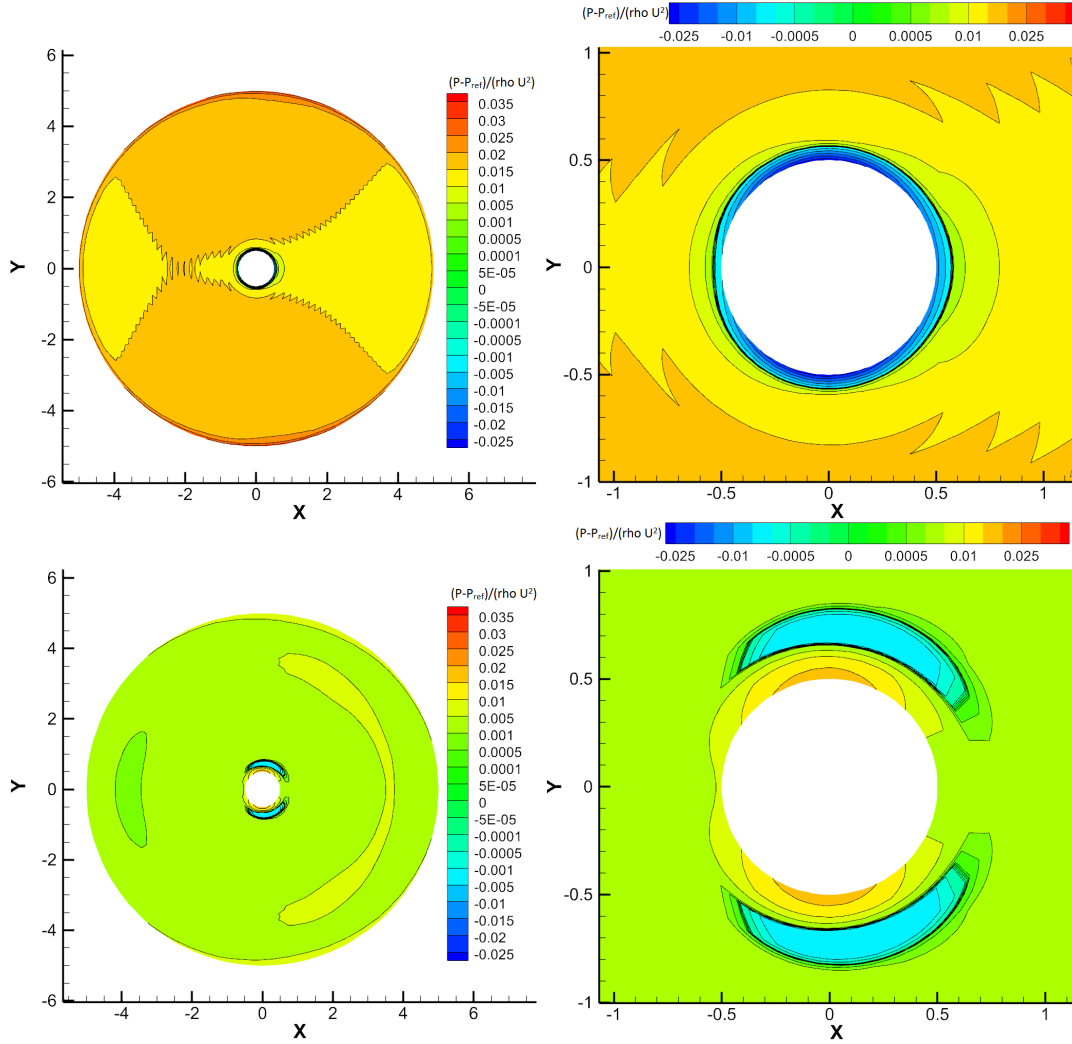


Figure 13: Pressure error levels and contours, $M = 10^{-4}$. Entire domain at the left, focus (zoom) on the cylinder at the right. Top: solver with low-Mach recentering , bottom: solver with low-Mach recentering and model of curvature.

Appendix : Formulae for a fictitious boundary value for other boundary conditions than Dirichlet's

Neumann boundary condition

In the following, we give a formal demonstration of the formula (65) to calculate a fictitious boundary value when a Neumann boundary condition is prescribed at the wall. Denote by ∇_W^k the approximate gradient operator which is k^{th} -order accurate, by $\frac{\partial q}{\partial n_W}$ the gradient imposed at the wall in the normal direction, and by $\bar{q}$ the averaged variables $(\bar{q}_K)_{K \in s(J)}$.

We look for a pointwise value q_W^{n+1} at time $n + 1$ such that

$$\frac{\partial q}{\partial n_W} = \nabla_W^k(q_W^{n+1}, \bar{q}^{n+1}) \cdot \mathbf{n} \quad (76)$$

However we can only evaluate $\nabla_W^k(q_W^n, \bar{q}^{n+1}) \cdot \mathbf{n}$. The main idea is that by construction, the approximate gradient can be expressed as

$$\nabla_W^k(q_W, \bar{q}) = \nabla_W^k(q_W, 0) + \nabla_W^k(0, \bar{q}) \quad (77)$$

Then rewriting the previous equations, we get:

$$\begin{cases} \frac{\partial q}{\partial n_W} = \nabla_W^k(q_W^{n+1}, 0) \cdot \mathbf{n} + \nabla_W^k(0, \bar{q}^{n+1}) \cdot \mathbf{n} \\ \nabla_W^k(q_W^n, \bar{q}^{n+1}) \cdot \mathbf{n} = \nabla_W^k(q_W^n, 0) \cdot \mathbf{n} + \nabla_W^k(0, \bar{q}^{n+1}) \cdot \mathbf{n} \end{cases} \quad (78)$$

Introduce $\gamma = \nabla_W^k(1, 0) \cdot \mathbf{n}$. By linearity of the gradient operator relatively to its first argument (the wall value), the system (79) becomes

$$\begin{cases} \frac{\partial q}{\partial n_W} = \gamma q_W^{n+1} + \nabla_W^k(0, \bar{q}^{n+1}) \cdot \mathbf{n} \\ \nabla_W^k(q_W^n, \bar{q}^{n+1}) \cdot \mathbf{n} = \gamma q_W^n + \nabla_W^k(0, \bar{q}^{n+1}) \cdot \mathbf{n} \end{cases} \quad (79)$$

With the notation $C_n = \frac{1}{\gamma}$ (under the assumption that $\gamma \neq 0$), we finally obtain:

$$q_W^{n+1} = q_W^n + C_n \left(\frac{\partial q}{\partial n_W} - \nabla_W^k(q_W^n, \bar{q}^{n+1}) \cdot \mathbf{n} \right) = q_W + \mathcal{O}(h^{k+1}) \quad (80)$$

with C_n a constant dependent on the reconstruction, scaling in $\mathcal{O}(h)$. The value of C_n for a 1-exact reconstruction on a one-dimensional cartesian mesh, under the assumption that $\beta_W = 1$ is $C_n = h$. The calculation of C_n in the three-dimensional, unstructured mesh case is cumbersome and is not detailed in the

present paper.

Extrapolation

In the following, we give a formal demonstration of the extrapolation formula (66) to calculate a boundary value in case no information is available at the wall. Denote by R^k the k -exact reconstruction operator and by $\bar{q}$ the averaged variables $(\bar{q}_K)_{K \in s(J)}$. We look for a pointwise value q_w^{n+1} at time $n+1$ which will be invariable when reconstructed, that is to say such that

$$q_w^{n+1} = R^k(q_w^{n+1}, \bar{q}^{n+1}) \quad (81)$$

However, at this point we can only evaluate

$$\tilde{q}_w^{n+1} = R^k(q_w^n, \bar{q}^{n+1}) \quad (82)$$

By construction, the reconstruction R^k can be expressed as:

$$R^k(q_w, \bar{q}) = R^k(q_w, 0) + R^k(0, \bar{q}) \quad (83)$$

Let us introduce $\lambda = R^k(1, 0)$. Then, by linearity of the k -exact reconstruction operator relatively to its first argument (the wall value), we can rewrite (81) and (82):

$$\begin{cases} q_w^{n+1} = \lambda q_w^{n+1} + R^k(0, \bar{q}^{n+1}) \\ \tilde{q}_w^{n+1} = \lambda q_w^n + R^k(0, \bar{q}^{n+1}) \end{cases} \quad (84)$$

This leads to:

$$q_w^{n+1} = \frac{R^k(q_w^n, \bar{q}) - \lambda q_w^n}{1 - \lambda} \quad (85)$$

With the notation $C_e = \frac{1}{1-\lambda}$ and by noticing that $1 - C_e = -\frac{\lambda}{1-\lambda}$, we finally obtain:

$$q_w^{n+1} = q_w^n + C_e (R^k(q_w^n, \bar{q}) - q_w^n) = q_w + \mathcal{O}(h^{k+1}) \quad (86)$$

with C_e a constant dependent on the reconstruction. The value of C_e for a 1-exact reconstruction on a one-dimensional cartesian mesh is given hereafter as an example:

$$C_e = \frac{\beta_w}{1 + \beta_w}. \quad (87)$$

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