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► **To cite this version:**

Issam Bartouli, Ammar Derbali, Mohamed Elkhames Chraygui, Jean Lerbet. Cohomology of $Vect(\mathbb{R})$ acting on the space of bilinear differential operators. *Journal of Geometry and Physics*, 2019, 138, pp.317 - 330. 10.1016/j.geomphys.2018.09.015 . hal-03486852

HAL Id: hal-03486852

<https://hal.science/hal-03486852v1>

Submitted on 20 Dec 2021

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COHOMOLOGY OF $\text{Vect}(\mathbb{R})$ ACTING ON THE SPACE OF BILINEAR DIFFERENTIAL OPERATORS

ISSAM BARTOULI*, AMMAR DERBALI, MOHAMED ELKHAMES CHRAYGUI,
AND JEAN LERBET

ABSTRACT. We consider the $\text{Vect}(\mathbb{R})$ -module structure on the spaces of bilinear differential operators acting on the spaces of weighted densities. We compute the first differential cohomology of the vector fields Lie algebra $\text{Vect}(\mathbb{R})$ with coefficients in space $\mathcal{D}_{\lambda,\nu;\mu}$ of bilinear differential operators acting on weighted densities.

1. Introduction

Let \mathbb{M} be a smooth manifold and $\text{Vect}(\mathbb{M})$ the Lie algebra of vector fields on \mathbb{M} . The main purpose of this article is to study the cohomology of $\text{Vect}(\mathbb{M})$ with coefficients in the space of linear differential operators acting on tensor fields. This cohomology is, actually, a natural generalization of the Gelfand-Fuchs cohomology (i.e., of $\text{Vect}(\mathbb{M})$ -cohomology with coefficients in the modules of tensor fields on \mathbb{M}).

Let \mathfrak{g} be a Lie algebra and let \mathcal{M} and \mathcal{N} be two \mathfrak{g} -modules. It is well-known that nontrivial extensions of \mathfrak{g} -modules:

$$0 \rightarrow \mathcal{M} \rightarrow \cdot \rightarrow \mathcal{N} \rightarrow 0$$

are classified by the first cohomology group $H^1(\mathfrak{g}, \text{Hom}(N, M))$ (see, e.g., [9]). Any 1-cocycle Λ generates a new action on $\mathcal{M} \oplus \mathcal{N}$ as follows: for all $g \in \mathfrak{g}$ and for all $(a, b) \in \mathcal{M} \oplus \mathcal{N}$, we define $g * (a, b) := (g * a + \Lambda(b), g * b)$.

Although our results and their applications are essentially geometric, the proofs are algebraic. The considerations are local, so we work in the Euclidean case, replacing \mathbb{M} by \mathbb{R} .

The space of weighted densities of weight λ on \mathbb{R} (or λ -densities for short), denoted by:

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}, \quad \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(T^*\mathbb{R})^{\otimes \lambda}$ for positive integer λ . The Lie algebra $\text{Vect}(\mathbb{R})$ of vector fields $X_h = h \frac{d}{dx}$, where $h \in C^\infty(\mathbb{R})$, acts by the Lie derivative. Alternatively, this action can be written as follows:

$$(1.1) \quad X_h.(f dx^\lambda) = L_{X_h}^\lambda(f) dx^\lambda \text{ with } L_{X_h}^\lambda(f) = hf' + \lambda h' f,$$

*Issam Bartouli.

where f', h' are $\frac{df}{dx}, \frac{dh}{dx}$. Each bilinear differential operator A from $C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ thus gives rise to a morphism from $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ to \mathcal{F}_μ , for any $(\lambda, \nu, \mu) \in \mathbb{R}^3$, by $fdx^\lambda \otimes gdx^\nu \mapsto A(f \otimes g)dx^\mu$. The Lie algebra $\text{Vect}(\mathbb{R})$ acts on the space $\text{Hom}(\mathcal{F}_\lambda \otimes \mathcal{F}_\nu, \mathcal{F}_\mu) = \mathcal{D}_{\lambda, \nu; \mu}$ of these differential operators by:

$$(1.2) \quad X_h \cdot A = L_{X_h}^\mu \circ A - A \circ L_{X_h}^{(\lambda, \nu)},$$

where, $L_{X_h}^{(\lambda, \nu)}$ is the Lie derivative on $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ defined by the Leibniz rule:

$$L_{X_h}^{(\lambda, \nu)}(f \otimes g) = L_{X_h}^\lambda(f) \otimes g + f \otimes L_{X_h}^\nu(g).$$

According to Nijenhuis-Richardson [12], the space $H^1(\mathfrak{g}; \text{End}(V))$ classifies the infinitesimal deformations of a \mathfrak{g} -module V and the obstructions to integrability of a given infinitesimal deformation of V are elements of $H^2(\mathfrak{g}; \text{End}(V))$ while the spaces $H^1(\mathfrak{g}; L(\otimes_s^n V, V))$ appear naturally in the problem of normalization of nonlinear representations of \mathfrak{g} in V . To be more precise, let

$$\mathbb{T} : \mathfrak{g} \rightarrow \sum_{n \geq 0} L(\otimes_s^n V, V), X \mapsto \mathbb{T}_X = \sum \mathbb{T}_X^n,$$

be a nonlinear representation of \mathfrak{g} in V , that is, $\mathbb{T}_{[X, Y]} = [\mathbb{T}_X, \mathbb{T}_Y]$. In [1], it is proved that the representation \mathbb{T} is normalized if \mathbb{T}_X^n is in a supplementary of $B^1(\mathfrak{g}; L(\otimes_s^n V, V))$ in $Z^1(\mathfrak{g}; L(\otimes_s^n V, V))$.

In fact if A is a differential operator on the line, A can be viewed as an homomorphism from \mathcal{F}_λ to \mathcal{F}_μ . If A is with order n , we can define its symbol as an element in $\mathcal{S}_\delta = \bigoplus_{k=0}^n \mathcal{F}_{\delta-k}$, for $\delta = \mu - \lambda$. If n goes to $+\infty$, the space $\mathcal{S}_\delta = \bigoplus_{k \geq 0} \mathcal{F}_{\delta-k}$, appears as the space symbols for all differential operators. The space $H^1(\text{Vect}(\mathbb{R}), L(\otimes_s^2 \mathcal{S}_\delta, \mathcal{S}_\delta))$ can be decomposed as a sum of spaces $H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu})$. Thus, the computation of the spaces $H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu})$ is the first step to normalize any nonlinear representation of $\text{Vect}(\mathbb{R})$ in \mathcal{S}_δ .

For the space of tensor densities of weight $\lambda \in \mathbb{R}$, \mathcal{F}_λ , viewed as a module over the Lie algebra of smooth vector fields $\text{Vect}(\mathbb{R})$, the classification of nontrivial extensions

$$0 \rightarrow \mathcal{F}_\mu \rightarrow \cdot \rightarrow \mathcal{F}_\lambda \rightarrow 0.$$

leads Feigin and Fuks in [10] to compute the cohomology group $H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \text{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu))$. Later, Ovsienko and Bouarroudj in [3] have computed the corresponding relative cohomology group with respect to $\mathfrak{sl}(2)$, namely

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathfrak{sl}(2); \text{Hom}(\mathcal{F}_\lambda, \mathcal{F}_\mu)).$$

Later, Bouarroudj in [2] has computed the corresponding relative cohomology group with respect to $\mathfrak{sl}(2)$, namely

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathfrak{sl}(2); \text{Hom}(\mathcal{F}_\lambda \otimes \mathcal{F}_\nu, \mathcal{F}_\mu)).$$

In this paper, we will compute the first differential cohomology group $H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu})$, where, $\mathcal{D}_{\lambda, \nu; \mu}$ is the space of bilinear differential operators from $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ to \mathcal{F}_μ .

This work allows us to study the first cohomology of the Lie superalgebra $\mathcal{K}(1)$ with coefficients in the superspace of bilinear differential operators.

2. DEFINITIONS AND NOTATIONS

In this section, we recall the main definitions and facts related to the geometry of the space \mathbb{R} . We also recall some fundamental concepts from cohomology theory (see, e.g., [8, 9]).

2.1. Cohomology theory. Let us first recall some fundamental concepts from cohomology theory (see, e.g., [6, 7, 8, 9]). Let \mathfrak{g} be a Lie algebra acting on a vector space V . The space of n -cochains of \mathfrak{g} with values in V is the \mathfrak{g} -module

$$C^n(\mathfrak{g}, V) := \text{Hom}(\Lambda^n \mathfrak{g}; V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g}, V) \rightarrow C^{n+1}(\mathfrak{g}, V)$ is a \mathfrak{g} -map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n , denoted $Z^n(\mathfrak{g}, V)$, is the space of *n -cocycles*, among them, the elements in the range of δ_{n-1} are called *n -coboundaries*. We denote $B^n(\mathfrak{g}, V)$ the space of n -coboundaries.

By definition, the n^{th} cohomology space is the quotient space

$$H^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V) / B^n(\mathfrak{g}, V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0, 1 and 2: for $\Xi \in C^0(\mathfrak{g}, V) = V$, $\delta\Xi(x) := x \cdot \Xi$, and for $\Lambda \in C^1(\mathfrak{g}, V)$,

$$(2.1) \quad \delta(\Lambda)(x, y) := g \cdot \Lambda(y) - y \cdot \Lambda(x) - \Lambda([x, y]) \quad \text{for any } x, y \in \mathfrak{g},$$

2.2. The space of tensor densities on \mathbb{R} . The Lie algebra, $\text{Vect}(\mathbb{R})$, of vector fields on \mathbb{R} naturally acts, by the Lie derivative, on the space

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda : f \in C^\infty(\mathbb{R}) \right\},$$

of weighted densities of degree λ . The Lie derivative L_X^λ of the space \mathcal{F}_λ along the vector field $X \frac{d}{dx}$ is defined by

$$(2.2) \quad L_X^\lambda = X \partial_x + \lambda X',$$

where $X, f \in C^\infty(\mathbb{R})$ and $X' := \frac{dX}{dx}$. More precisely, for all $f dx^\lambda \in \mathcal{F}_\lambda$, we have

$$L_X^\lambda(f dx^\lambda) = (X f' + \lambda f X') dx^\lambda.$$

2.3. The space of bilinear differential operators as a $\text{Vect}(\mathbb{R})$ -module.
The space of bilinear differential operators is a $\text{Vect}(\mathbb{R})$ -module, denoted

$$\mathcal{D}_{\underline{\lambda}, \mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}, \mathcal{F}_{\mu}).$$

The $\text{Vect}(\mathbb{R})$ action is:

$$(2.3) \quad L_X^{\underline{\lambda}, \mu}(A) = L_X^{\mu} \circ A - A \circ L_X^{\underline{\lambda}},$$

where $\underline{\lambda} = (\lambda_1, \lambda_2)$ and $L_X^{(\lambda_1, \lambda_2)}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}$ defined by the Leibnitz rule:

$$L_X^{\underline{\lambda}}(f_1 dx^{\lambda_1} \otimes f_2 dx^{\lambda_2}) = L_X^{\lambda_1}(f_1) \otimes f_2 dx^{\lambda_2} + f_1 dx^{\lambda_1} \otimes L_X^{\lambda_2}(f_2 dx^{\lambda_2}).$$

3. Cohomology of $\text{Vect}(\mathbb{R})$ acting on $\mathcal{D}_{\lambda, \nu; \mu}$

In this section, we compute the "differentiable" cohomology of the Lie algebra $\text{Vect}(\mathbb{R})$ with coefficients in the space of bilinear differential operators $\mathcal{D}_{\lambda, \nu; \mu}$. Namely, we consider only cochains that are given by differentiable maps.

Theorem 3.1. (i) *If $\mu - \lambda - \nu = 0$, then*

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = \mathbb{R} \quad \text{for all } \lambda \text{ and } \nu.$$

(ii) *If $\mu - \lambda - \nu = 1$, then*

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = \begin{cases} \mathbb{R}^3 & \text{if } (\lambda, \nu) = (0, 0), \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

(iii) *If $\mu - \lambda - \nu = 2$, then*

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = \begin{cases} \mathbb{R}^2 & \text{if } (\lambda, \nu) = (0, 0), \\ \mathbb{R} & \text{if } (\lambda, \nu) = (0, -1), (0, \nu), (\lambda, 0), (\lambda, -1 - \lambda), \\ 0 & \text{otherwise.} \end{cases}$$

(iv) *If $\mu - \lambda - \nu = 3$, then*

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = \begin{cases} \mathbb{R}^3 & \text{if } (\lambda, \nu) = (0, 0), \\ \mathbb{R}^2 & \text{if } (\lambda, \nu) = (-2, 0), (0, -2), (-\frac{2}{3}, -\frac{2}{3}), \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

(v) *If $\mu - \lambda - \nu = 4$, then*

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = \mathbb{R}^2 \quad \text{for all } \lambda \text{ and } \nu.$$

(vi) If $\mu - \lambda - \nu = 5$, then

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = \begin{cases} \mathbb{R}^3 & \text{if } (\lambda, \nu) = (0, 0), (0, -4), (-4, 0), \\ \mathbb{R}^2 & \text{if } (\lambda, \nu) \neq (-\frac{3}{2}, -\frac{1}{2}), (-\frac{1}{2}, -2), (-1, -\frac{3}{2}), (-1, -2), \\ & (-\frac{1}{2}, -\frac{3}{2}), (-\frac{3}{2}, -1), (-\frac{3}{2}, -\frac{3}{2}), (-\frac{3}{2}, -2), \\ & (-2, -\frac{1}{2}), (-1, -1), (-2, -1), (-2, -\frac{3}{2}) \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

(vii) If $\mu - \lambda - \nu = 6$, then

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = \begin{cases} \mathbb{R}^3 & \text{if } (\lambda, \nu) = (\frac{-5 \pm \sqrt{19}}{2}, 0), (0, \frac{-5 \pm \sqrt{19}}{2}), \\ & (\frac{\sqrt{19}-5}{2}, \frac{-\sqrt{19}-5}{2}), (-\frac{\sqrt{19}-5}{2}, \frac{\sqrt{19}-5}{2}), \\ \mathbb{R}^2 & \text{if } (\lambda, \nu) \neq (-\frac{1}{2}, -2), (-2, -\frac{1}{2}), (-\frac{1}{2}, -\frac{5}{2}), \\ & (-\frac{5}{2}, -\frac{1}{2}), (-1, -\frac{3}{2}), (-\frac{3}{2}, -1), (-1, -2), \\ & (-\frac{5}{2}, -1), (-1, -\frac{5}{2}), (-\frac{3}{2}, -\frac{3}{2}), (-2, -\frac{3}{2}), \\ & (-\frac{3}{2}, -2), (-\frac{3}{2}, -\frac{5}{2}), (-\frac{5}{2}, -\frac{3}{2}), (-2, -1), \\ & (-2, -2), (-2, -\frac{5}{2}), (-\frac{5}{2}, -2), \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

(viii) If $\mu - \lambda - \nu = 7$, then

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = \begin{cases} \mathbb{R}^2 & \text{if } (\lambda, \nu) = (\frac{-5-\sqrt{19}}{2}, \frac{-5-\sqrt{19}}{2}), (\frac{-5+\sqrt{19}}{2}, \frac{-5+\sqrt{19}}{2}), \\ & (0, \nu), (\lambda, 0), (\lambda, -6 - \lambda), (\sqrt{19} - 1, \frac{-\sqrt{19}-5}{2}), \\ & (\frac{-\sqrt{19}-5}{2}, \sqrt{19} - 1), (-\sqrt{19} - 1, \frac{\sqrt{19}-5}{2}), \\ & (\frac{\sqrt{19}-5}{2}, -\sqrt{19} - 1), \\ \mathbb{R} & \text{if } (\lambda, \nu) \neq (-\frac{1}{2}, -3), (-3, -\frac{1}{2}), (-3, -\frac{5}{2}), \\ & (-\frac{5}{2}, -3), (-1, -3), (-3, -1), (-1, -2), \\ & (-2, -1), (-2, -3), (-3, -2), (-2, -\frac{3}{2}), \\ & (-\frac{1}{2}, -\frac{5}{2}), (-\frac{5}{2}, -\frac{1}{2}), (-\frac{3}{2}, -2), (-\frac{3}{2}, -\frac{5}{2}), \\ & (-\frac{5}{2}, -\frac{3}{2}), (-2, -2), (-2, -\frac{5}{2}), (-\frac{5}{2}, -2), \\ 0 & \text{otherwise.} \end{cases}$$

(ix) If $\mu - \lambda - \nu \geq 8$ but with λ and ν generic then

$$H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 0.$$

Remark 3.2. The values $-\frac{2}{3}$ can be deduced from the work of Grozman [13] and the values $\sqrt{19}$ from the work of Feigin-Fuchs [10].

4. Proof of Theorem 3.1

To prove Theorem 3.1 we proceed by following the three steps:

- We first study the dimension of the space of operators that satisfy the 1-cocycle condition.
- We then study all trivial 1-cocycles, namely, operators of the form $L_Y A_k^{\lambda, \nu}$ where $A_k^{\lambda, \nu}$ is a bilinear operator given by (4.2)
- Lastly, by taking into account Part 1 and Part 2 and depending on λ and ν -the dimension of the cohomology group $H_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu})$ will be equal to

$$\dim(\text{operators that are 1-cocycles}) - \dim(\text{operators of the form } L_Y A_k^{\lambda, \nu}).$$

We need also the following two Lemmas:

Lemma 4.1. *Let Y be a vector fields in $\text{Vect}(\mathbb{R})$ and let $c : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \rightarrow \mathcal{F}_\mu$ be a bilinear differential operator defined as follows. For all $f \in \mathcal{F}_\lambda$ and for all $g \in \mathcal{F}_\nu$:*

$$(4.1) \quad c(f, g) = \sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} f^{(j)} g^{(l)}$$

where $c_{i,j,l}$ are constants. Then, for all $X \in \text{Vect}(\mathbb{R})$, we have

$$\begin{aligned} L_X^{\lambda, \nu; \mu} c(f, g) &= \sum_{i+j+l=k+2} c_{i,j,l} X Y^{(i)} f^{(j)} g^{(l)} + \sum_{i+j+l=k+1} (\mu - \lambda - \nu - j - l) c_{i,j,l} X' Y^{(i)} f^{(j)} g^{(l)} - \\ &\sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} \left(\sum_{s=2}^j C_j^s X^{(s)} f^{(j-s+1)} \right) g^{(l)} - \lambda \sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} \left(\sum_{s=1}^j C_j^s X^{(s+1)} f^{(j-s)} \right) g^{(l)} - \\ &\sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} f^{(j)} \left(\sum_{t=2}^l C_l^t X^{(t)} g^{(l-t+1)} \right) - \nu \sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} f^{(j)} \left(\sum_{t=1}^l C_l^t X^{(t+1)} g^{(l-t)} \right) \end{aligned}$$

Proof. Straightforward computation. By using (2.3). \square

Lemma 4.2. *Let $A_k^{\lambda, \nu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \rightarrow \mathcal{F}_\mu$ be a bilinear differential operator defined as follows. For all $f \in \mathcal{F}_\lambda$ and for all $g \in \mathcal{F}_\nu$:*

$$(4.2) \quad A_k^{\lambda, \nu}(f, g) = \sum_{i+j=k} a_{i,j} f^{(i)} g^{(j)}$$

where $a_{i,j}$ are constants. Then, for all $Y \in \text{Vect}(\mathbb{R})$, we have

$$\begin{aligned} L_Y^{\lambda, \nu; \mu} A_k^{\lambda, \nu}(f, g) &= \sum_{i+j=k} (\mu - \lambda - \nu - i - j) a_{i,j} Y' f^{(i)} g^{(j)} - \sum_{i+j=k} a_{i,j} \left(\sum_{s=2}^i (C_i^s + \right. \\ &\left. \lambda C_i^{s-1}) Y^{(s)} f^{(i-s+1)} \right) g^{(j)} - \sum_{i+j=k} a_{i,j} f^{(i)} \left(\sum_{t=2}^j (C_j^t + \nu C_j^{t-1}) Y^{(t)} g^{(j-t+1)} \right) \end{aligned}$$

Proof. Straightforward computation. By using (2.3). \square

Any 1-cocycle on $\text{Vect}(\mathbb{R})$ should retain the following general form:

$$(4.3) \quad c(f, g) = \sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} f^{(j)} g^{(l)},$$

where $c_{i,j,l}$ are constants. The fact that this 1-cocycle implies that

$$c_{0,j,l} = 0$$

the 1-cocycle condition reads as follows: for all $f \in \mathcal{F}_\lambda$, for all $g \in \mathcal{F}_\nu$ and for all $X, Y \in \text{Vect}(\mathbb{R})$, we have

$$c([X, Y], f, g) - L_X^{\lambda, \nu; \mu} c(Y, f, g) + L_Y^{\lambda, \nu; \mu} c(X, f, g) = 0.$$

(i) The case when $\mu - \lambda - \nu = 0$

$$c(f, g) = c_{1,0,0} Y' f g$$

By a straightforward computation, the 1-cocycle condition above reads

$$\mu - \lambda - \nu = 0$$

(ii) The case when $\mu - \lambda - \nu = 1$

$$c(f, g) = c_{1,1,0} Y' f' g + c_{1,0,1} Y' f g' + c_{2,0,0} Y'' f g$$

Again, by a straightforward computation, the above 1-cocycle condition reads

$$\mu - \lambda - \nu = 1 \text{ and } c_{1,1,0} \lambda + c_{1,0,1} \nu = 0$$

Let us study now the triviality of this 1-cocycle. A straightforward computation shows that

$$L_Y A_1^{\lambda, \nu} = -(a_{1,0} \lambda + a_{0,1} \nu) Y'' f g$$

(1) If $(\lambda, \nu) = (0, 0)$, then

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 3$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 0$$

(2) If $\lambda = 0$ and $\nu \neq 0$, then $c_{1,1,0} = 0$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 1, \text{ generated by}$$

$$\Lambda = -a_{0,1} \nu Y'' f g.$$

The results hold true for $(\lambda \neq 0 \text{ together with } \nu = 0)$ and $(\lambda, \nu) \neq (0, 0)$.

(iii) **The case when $\mu - \lambda - \nu = 2$**

$$\begin{aligned} c(f, g) &= c_{1,2,0}Y'f''g + c_{1,1,1}Y'f'g' + c_{1,0,2}Y'fg'' \\ &+ c_{2,1,0}Y''f'g + c_{2,0,1}Y''fg' + c_{3,0,0}Y'''fg \end{aligned}$$

Thanks to a straightforward computation, the above 1-cocycle condition reads

$$\mu - \lambda - \nu = 2 \text{ and } \begin{cases} c_{1,2,0}(1 + 2\lambda) + c_{1,1,1}\nu = 0 \\ c_{1,2,0}\lambda + c_{1,0,2}\nu = 0 \\ c_{1,1,1}\lambda + c_{1,0,2}(1 + 2\nu) = 0 \end{cases}$$

Let us study the triviality of this 1-cocycle. a straightforward computation shows that

$$L_X A_2^{\lambda, \nu} = -(a_{2,0}(1+2\lambda) + a_{1,1}\nu)Y''f'g - (a_{1,1}\lambda + a_{0,2}(1+2\nu))Y''fg' - (a_{2,0}\lambda + a_{0,2}\nu)Y'''fg$$

(1) If $(\lambda, \nu) = (0, 0)$, then $c_{1,2,0} = c_{1,0,2} = 0$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 4$$

$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2$, generated by

$$\Lambda_1 = -a_{2,0}Y''f'g,$$

$$\Lambda_2 = -a_{0,2}Y''fg'.$$

(2) If $(\lambda, \nu) = (0, -1)$, then $c_{1,2,0} = c_{1,0,2}$ and $c_{1,0,2} = 0$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 4$$

$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 3$, generated by

$$\Lambda_1 = -a_{2,0}Y''f'g,$$

$$\Lambda_2 = a_{1,1}Y''f'g,$$

$$\Lambda_3 = -a_{0,2}(Y''fg' + Y'''fg).$$

The results hold true for $(\lambda \neq 0 \text{ together with } \nu = 0)$, $(\lambda = 0 \text{ together with } \nu \neq 0)$ and $(\lambda, \nu) = (\lambda, -1 - \lambda)$.

(3) for other cases for λ and ν , then $c_{1,2,0} = c_{1,1,1} = c_{1,0,2} = 0$.

Thus, the 1-cocycle is trivial.

(iv) **The case when $\mu - \lambda - \nu = 3$**

$$\begin{aligned} c(f, g) &= c_{1,0,3}Y'fg''' + c_{1,1,2}Y'f'g'' + c_{1,2,1}Y'f''g' + c_{1,3,0}Y'f'''g + c_{2,0,2}Y''fg'' \\ &+ c_{2,1,1}Y''f'g' + c_{2,2,0}Y''f''g + c_{3,1,0}Y'''f'g + c_{3,0,1}Y'''fg' + c_{4,0,0}Y^{(4)}fg \end{aligned}$$

By a straightforward computation, the above 1-cocycle condition reads

$\mu - \lambda - \nu = 3$ and

$$\begin{aligned} c_{1,0,3}(3 + 3\nu) + c_{1,1,2}\lambda &= 0, & c_{1,0,3}(1 + 3\nu) + c_{1,2,1}\lambda &= 0, \\ c_{1,0,3}\nu + c_{1,3,0}\lambda &= 0, & c_{1,1,2}(1 + 2\nu) + c_{1,2,1}(1 + 2\lambda) &= 0, \\ c_{1,1,2}\nu + c_{1,3,0}(1 + 3\lambda) &= 0, & c_{1,2,1}\nu + c_{1,3,0}(3 + 3\lambda) &= 0, \\ c_{2,0,2}\nu + c_{2,2,0}\lambda &= 0, & c_{3,0,1}\nu + c_{3,1,0}\lambda + 2c_{4,0,0} &= 0. \end{aligned}$$

Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{aligned} L_X A_3^{\lambda, \nu} = & - (a_{0,3}(3 + 3\nu) + a_{1,2}\lambda)Y''fg'' - (a_{1,2}(1 + 2\nu) + a_{2,1}(1 + 2\lambda))Y''f'g' \\ & - (a_{2,1}\nu + a_{3,0}(3 + 3\lambda))Y''f''g - (a_{1,2}\nu + a_{3,0}(1 + 3\lambda))Y'''f'g \\ & - (a_{0,3}(1 + 3\nu) + a_{2,1}\lambda)Y'''fg' - (a_{0,3}\nu + a_{3,0}\lambda)Y^{(4)}fg \end{aligned}$$

(1) If $(\lambda, \nu) = (0, 0)$, then $c_{1,0,3} = c_{1,3,0} = c_{4,0,0} = 0$ and $c_{1,1,2} = -c_{1,2,1}$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 6$$

$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 3$, generated by

$$\begin{aligned} \Lambda_1 &= -a_{0,3}(3Y''fg'' + Y'''fg'), \\ \Lambda_2 &= -a_{1,2}Y''f'g', \\ \Lambda_3 &= -a_{2,1}Y''f'g'. \end{aligned}$$

(2) If $(\lambda, \nu) = (-2, 0)$, then $c_{1,0,3} = c_{2,0,2} = 0$, $c_{1,3,0} = 2c_{1,1,2}$ and $c_{4,0,0} = c_{3,1,0}$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 5$$

$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 3$, generated by

$$\begin{aligned} \Lambda_1 &= -a_{0,3}(3Y''fg'' + Y'''fg'), \\ \Lambda_2 &= -a_{1,2}(-2Y''fg'' + Y''f'g'), \\ \Lambda_3 &= a_{2,1}(3Y''f'g' + 2Y'''fg'). \end{aligned}$$

The results hold true for $(0, -2)$ and $(-\frac{2}{3}, -\frac{2}{3})$.

(3) If λ and ν are not like above, then $c_{1,0,3} = c_{1,1,2} = c_{1,2,1} = c_{1,3,0} = 0$, $c_{2,0,2} = -\frac{\lambda}{\nu}c_{2,2,0}$ and $c_{4,0,0} = \frac{1}{2} - \nu c_{3,0,1} - \lambda c_{3,1,0}$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 4$$

$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 3$, generated by

$$\Lambda_1 = a_{0,3} \left(-\frac{(1+2\nu)(3+3\nu)}{\lambda} Y''f'g' - \frac{\nu(3+3\nu)}{\lambda} Y'''f'g + (1+3\nu)Y'''fg' + \nu Y^{(4)}fg \right),$$

$$\Lambda_2 = a_{2,1} \left(-\lambda Y''fg'' + 2(\lambda - \nu)Y''f'g' + \nu Y''f''g - \nu Y'''f'g + \lambda Y'''fg' \right),$$

$$\Lambda_3 = -a_{3,0} \left(-\frac{\lambda(3+3\lambda)}{\nu} Y''fg'' - \frac{(1+2\nu)(3+3\lambda)}{\nu} Y''f'g' + (3+3\lambda)Y''f''g + (1+3\lambda)Y'''f'g + \lambda Y^{(4)}fg \right).$$

(v) **The case when $\mu - \lambda - \nu = 4$**

$$\begin{aligned} c(f, g) &= c_{1,4,0}Y'f^{(4)}g + c_{1,3,1}Y'f'''g' + c_{1,2,2}Y'f''g'' + c_{1,1,3}Y'f'g''' + c_{1,0,4}Y'fg^{(4)} \\ &+ c_{2,3,0}Y''f'''g + c_{2,2,1}Y''f''g' + c_{2,1,2}Y''f'g'' + c_{2,0,3}Y''fg''' + c_{3,2,0}Y'''f''g \\ &+ c_{3,1,1}Y'''f'g' + c_{3,0,2}Y'''fg'' + c_{4,1,0}Y^{(4)}f'g + c_{4,0,1}Y^{(4)}fg' + c_{5,0,0}Y^{(5)}fg \end{aligned}$$

By a straightforward computation, the above 1-cocycle condition reads $\mu - \lambda - \nu = 4$ and

$$\begin{aligned}
c_{1,4,0}(6 + 4\lambda) + c_{1,3,1}\nu &= 0, & c_{1,4,0}(4 + 6\lambda) + c_{1,2,2}\nu &= 0, \\
c_{1,4,0}(1 + 4\lambda) + c_{1,1,3}\nu &= 0, & c_{1,4,0}\lambda + c_{1,0,4}\nu &= 0, \\
c_{1,3,1}(3 + 3\lambda) + c_{1,2,2}(1 + 2\nu) &= 0, & c_{1,3,1}(1 + 3\lambda) + c_{1,1,3}(1 + 3\nu) &= 0, \\
c_{1,3,1}\lambda + c_{1,0,4}(1 + 4\nu) &= 0, & c_{1,2,2}(1 + 2\lambda) + c_{1,1,3}(3 + 3\nu) &= 0, \\
c_{1,2,2}\lambda + c_{1,0,4}(4 + 6\nu) &= 0, & c_{1,1,3}\lambda + c_{1,0,4}(6 + 4\nu) &= 0, \\
c_{2,3,0}(1 + 3\lambda) + c_{2,1,2}\nu &= 0, & c_{2,3,0}\lambda + c_{2,0,3}\nu &= 0, \\
c_{2,2,1}\lambda + c_{2,0,3}(1 + 3\nu) &= 0, & c_{3,2,0}(1 + 2\lambda) + c_{3,1,1}\nu + 2c_{4,1,0} &= 0, \\
c_{3,1,1}\lambda + c_{3,0,2}(1 + 2\nu) + 2c_{4,0,1} &= 0, & c_{4,1,0}\lambda + c_{4,0,1}\nu + 5c_{5,0,0} &= 0.
\end{aligned}$$

The space of solutions of the above system is five-dimensional for $(\lambda, \nu) = (0, 0), (\lambda, 0), (0, \nu)$, and four-dimensional otherwise.

Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{aligned}
L_X A_4^{\lambda, \nu} = & - (a_{4,0}(6 + 4\lambda) + a_{3,1}\nu)Y''f'''g - (a_{3,1}(3 + 3\lambda) + a_{2,2}(1 + 2\nu))Y''f''g' \\
& - (a_{2,2}(1 + 2\lambda) + a_{1,3}(3 + 3\nu))Y''f'g'' - (a_{1,3}\lambda + a_{0,4}(6 + 4\nu))Y''fg''' \\
& - (a_{4,0}(4 + 6\lambda) + a_{2,2}\nu)Y'''f''g - (a_{3,1}(1 + 3\lambda) + a_{1,3}(1 + 3\nu))Y'''f'g' \\
& - (a_{2,2}\lambda + a_{0,4}(4 + 6\nu))Y'''fg'' - (a_{4,0}(1 + 4\lambda) + a_{1,3}\nu)Y^{(4)}f'g \\
& - (a_{3,1}\lambda + a_{0,4}(1 + 4\nu))Y^{(4)}fg' - (a_{4,0}\lambda + a_{0,4}\nu)Y^{(4)}fg
\end{aligned}$$

- (1) If $(\lambda, \nu) = (0, 0)$, then $c_{1,4,0} = c_{1,3,1} = c_{1,2,2} = c_{1,1,3} = c_{1,0,4} = c_{2,3,0} = c_{2,0,3} = c_{5,0,0} = 0$, $c_{3,2,0} = -2c_{4,1,0}$ and $c_{3,0,2} = -2c_{4,0,1}$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 5$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 3, \text{ generated by}$$

$$\begin{aligned}
\Lambda_1 &= -a_{3,1}(3Y''f''g' + Y'''f'g'), \\
\Lambda_2 &= -a_{2,2}(Y''f''g' + Y''f'g''), \\
\Lambda_3 &= -a_{1,3}(3Y''f'g'' + Y'''f'g').
\end{aligned}$$

The results hold true for $(\lambda, \nu) = (\lambda, 0), (0, \nu)$.

- (2) for the other values of λ and ν , then the space $B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu})$ is two-dimensional.

For example, if $(\lambda, \nu) = (-\frac{1}{2}, -\frac{1}{2})$, then $c_{1,4,0} = c_{1,3,1} = c_{1,2,2} = c_{1,1,3} = c_{1,0,4} = 0$, $c_{2,2,1} = -c_{2,0,3}$, $c_{2,1,2} = c_{2,0,3}$, $c_{4,1,0} = \frac{1}{4}c_{3,1,1}$, $c_{4,0,1} = \frac{1}{4}c_{3,1,1}$ and $c_{5,0,0} = \frac{1}{2,0}c_{3,1,1}$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 4$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2, \text{ generated by}$$

$$\begin{aligned}
\Lambda_1 &= a_{4,0}(-3Y''f'''g - 3Y''f''g' + 3Y''f'g'' + 3Y''fg''' - Y'''f''g + Y'''fg'' - Y^{(4)}f'g), \\
\Lambda_2 &= \frac{1}{2}a_{3,1}(Y'''f''g + Y'''fg'')
\end{aligned}$$

(vi) The case when $\nu - \lambda - \nu = 5$

$$\begin{aligned} c(f, g) = & c_{1,5,0}Y'f^{(5)}g + c_{1,4,1}Y'f^{(4)}g' + c_{1,3,2}Y'f'''g'' + c_{1,2,3}Y'f''g''' + c_{1,1,4}Y'f'g^{(4)} \\ & + c_{1,0,5}Y'fg^{(5)} + c_{2,4,0}Y''f^{(4)}g + c_{2,3,1}Y''f'''g' + c_{2,2,2}Y''f''g'' + c_{2,1,3}Y''f'g''' \\ & + c_{2,0,4}Y''fg^{(4)} + c_{3,3,0}Y'''f'''g + c_{3,2,1}Y'''f''g' + c_{3,1,2}Y'''f'g'' + c_{3,0,3}Y'''fg''' \\ & + c_{4,2,0}Y^{(4)}f''g + c_{4,1,1}Y^{(4)}f'g' + c_{4,0,2}Y^{(4)}fg'' + c_{5,1,0}Y^{(5)}f'g \\ & + c_{5,0,1}Y^{(5)}fg' + c_{6,0,0}Y^{(6)}fg \end{aligned}$$

A straightforward computation shows that the above 1-cocycle condition reads

$\mu - \lambda - \nu = 5$ and

$$\begin{aligned} c_{1,5,0}(10 + 5\lambda) + c_{1,4,1}\nu &= 0, & c_{1,5,0}(10 + 10\lambda) + c_{1,3,2}\nu &= 0, \\ c_{1,5,0}(5 + 10\lambda) + c_{1,2,3}\nu &= 0, & c_{1,5,0}(1 + 5\lambda) + c_{1,1,4}\nu &= 0, \\ c_{1,5,0}\lambda + c_{1,0,5}\nu &= 0, & c_{1,4,1}(6 + 4\lambda) + c_{1,3,2}(1 + 2\nu) &= 0, \\ c_{1,4,1}(4 + 6\lambda) + c_{1,2,3}(1 + 3\nu) &= 0, & c_{1,4,1}(1 + 4\lambda) + c_{1,1,4}(1 + 4\nu) &= 0, \\ c_{1,4,1}\lambda + c_{1,0,5}(1 + 5\nu) &= 0, & c_{1,3,2}(3 + 3\lambda) + c_{1,2,3}(3 + 3\nu) &= 0, \\ c_{1,3,2}(1 + 3\lambda) + c_{1,1,4}(4 + 6\nu) &= 0, & c_{1,3,2}\lambda + c_{1,0,5}(5 + 10\nu) &= 0, \\ c_{1,2,3}(1 + 2\lambda) + c_{1,1,4}(6 + 4\nu) &= 0, & c_{1,2,3}\lambda + c_{1,0,5}(10 + 10\nu) &= 0, \\ c_{1,1,4}\lambda + c_{1,0,5}(10 + 5\nu) &= 0, & & \\ c_{2,4,0}(4 + 6\lambda) + c_{2,2,2}\nu &= 0, & c_{2,4,0}(1 + 4\lambda) + c_{2,1,3}\nu &= 0, \\ c_{2,4,0}\lambda + c_{2,0,4}\nu &= 0, & c_{2,3,1}(1 + 3\lambda) + c_{2,1,3}(1 + 3\nu) &= 0, \\ c_{2,3,1}\lambda + c_{2,0,4}(1 + 4\nu) &= 0, & c_{2,3,1}\lambda + c_{2,0,4}(4 + 6\nu) &= 0, \\ (c_{4,2,0} - c_{3,3,0})\lambda + (c_{4,0,2} - c_{3,0,3})\nu + 5c_{6,0,0} &= 0, & c_{3,3,0}(3 + 3\lambda) + c_{3,2,1}\nu + 2c_{4,2,0} &= 0, \\ c_{3,2,1}(1 + 2\lambda) + c_{3,1,2}(1 + 2\nu) + 2c_{4,1,1} &= 0, & c_{3,1,2}\lambda + c_{3,0,3}(3 + 3\nu) + 2c_{4,0,2} &= 0, \\ c_{4,2,0}(1 + 2\lambda) + c_{4,1,1}\nu + 5c_{5,1,0} &= 0, & & \\ c_{4,1,1}\lambda + c_{4,0,2}(1 + 2\nu) + 5c_{5,0,1} &= 0, & c_{5,1,0}\lambda + c_{5,0,1}\nu + 9c_{6,0,0} &= 0. \end{aligned}$$

The space of solutions of the above system is six-dimensional for $(\lambda, \nu) = (0, 0)$, five-dimensional for $(\lambda, \nu) = (0, -2), (-2, 0), (0, -4), (-4, 0)$, four-dimensional for $(\lambda, \nu) = (-\frac{2}{3}, -\frac{2}{3}), (-2, -2), (0, \nu), (\lambda, 0)$, and three-dimensional otherwise. Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{aligned} L_X A_5^{\lambda, \nu} = & - (a_{5,0}(10 + 5\lambda) + a_{4,1}\nu)Y''f^{(4)}g - (a_{4,1}(6 + 4\lambda) + a_{3,2}(1 + 2\nu))Y''f'''g' \\ & - (a_{3,2}(3 + 3\lambda) + a_{2,3}(3 + 3\nu))Y''f''g'' - (a_{2,3}(1 + 2\lambda) + a_{1,4}(6 + 4\nu))Y''f'g''' \\ & - (a_{1,4}\lambda + a_{0,5}(10 + 5\nu))Y''fg^{(4)} - (a_{5,0}(10 + 10\lambda) + a_{3,2}\nu)Y'''f'''g \\ & - (a_{4,1}(4 + 6\lambda) + a_{2,3}(1 + 3\nu))Y'''f''g' - (a_{3,2}(1 + 3\lambda) + a_{1,4}(4 + 6\nu))Y'''f'g'' \\ & - (a_{2,3}\lambda + a_{0,5}(10 + 10\nu))Y'''fg''' - (a_{5,0}(5 + 10\lambda) + a_{2,3}\nu)Y^{(4)}f''g \\ & - (a_{4,1}(1 + 4\lambda) + a_{1,4}(1 + 4\nu))Y^{(4)}f'g' - (a_{3,2}\lambda + a_{0,5}(5 + 10\nu))Y^{(4)}fg'' \\ & - (a_{5,0}(1 + 5\lambda) + a_{1,4}\nu)Y^{(5)}f'g - (a_{4,1}\lambda + a_{0,5}(1 + 5\nu))Y^{(5)}fg' \\ & - (a_{5,0}\lambda + a_{0,5}\nu)Y^{(6)}fg \end{aligned}$$

- (1) If $(\lambda, \nu) = (0, 0)$, then $c_{1,5,0} = c_{1,4,1} = c_{1,3,2} = c_{1,2,3} = c_{1,1,4} = c_{1,0,5} = c_{2,4,0} = c_{2,0,4} = c_{6,0,0} = 0$, $c_{2,3,1} = -c_{2,1,3}$, $c_{3,3,0} = -\frac{1,0}{3}c_{5,1,0}$, $c_{3,2,1} = -c_{3,1,2} - 2c_{4,1,1}$, $c_{3,0,3} = -\frac{1,0}{3}c_{5,0,1}$, $c_{4,2,0} = -5c_{5,1,0}$, $c_{4,0,2} = -5c_{5,0,1}$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 6$$

$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 3$, generated by

$$\begin{aligned}\Lambda_1 &= -a_{4,1}(-6Y''f'''g' + 6Y''f'g''' - 4Y'''f''g' + 4Y'''f'g''), \\ \Lambda_2 &= -a_{3,2}(-Y''f'''g' - 3Y''f''g'' + Y''f'g''' - \frac{1}{3}Y'''f'g'' + \frac{1}{6}Y^{(4)}f'g'), \\ \Lambda_3 &= -a_{2,3}(-3Y''f''g'' - Y'''f''g' + \frac{2}{3}Y'''f'g'' + \frac{1}{6}Y^{(4)}f'g').\end{aligned}$$

- (2) If $(\lambda, \nu) = (0, -4)$, then the constant $a_{2,3}$ and $a_{1,4}$ are arbitrary. On the Other hand,

$$\begin{aligned}a_{0,5} &= 0 \\ a_{5,0} &= \frac{1,4,2}{45}a_{2,3} - \frac{158}{9}a_{1,4} \\ a_{4,1} &= \frac{62}{9}a_{2,3} - \frac{3,0,3}{9}a_{1,4} \\ a_{3,2} &= \frac{1,3}{3}a_{2,3} - \frac{4,0}{3}a_{1,4}\end{aligned}$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2$$

The results hold true for $(\lambda, \nu) = (-4, 0)$.

- (3) If $(\lambda, \nu) = (0, -2)$, then the constant $a_{2,3}$, $a_{1,4}$ and $a_{0,5}$ are arbitrary. On the other hand,

$$\begin{aligned}a_{5,0} &= \frac{3}{5}a_{2,3} - a_{1,4} \\ a_{4,1} &= -2a_{2,3} - 3a_{1,4} \\ a_{3,2} &= \frac{7}{3}a_{2,3} - \frac{8}{3}a_{1,4} \\ a_{0,5} &= 0\end{aligned}$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2$$

The results hold true for $(\lambda, \nu) = (-2, 0)$.

- (4) If $(\lambda, \nu) = (0, \nu)$, then the constant $a_{2,3}$ and $a_{1,4}$ are arbitrary. On the other hand,

$$\begin{aligned}a_{0,5} &= 0 \\ a_{3,2} &= \frac{1-3\nu}{3}a_{2,3} + \frac{24+16\nu}{3}a_{1,4} \\ a_{4,1} &= \frac{3\nu^2-4\nu-2}{9}a_{2,3} + \frac{(17\nu+7)(2\nu+3)}{9}a_{1,4} \\ a_{5,0} &= \frac{-3\nu^2-5\nu+2}{90}a_{2,3} + \frac{(-\nu+7)(2\nu+3)}{90}a_{1,4}\end{aligned}$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2$$

The results hold true for $(\lambda, \nu) = (\lambda, 0)$.

- (5) If $(\lambda, \nu) = (-1, -1)$, then the constant $a_{3,2}$ and $a_{0,5}$ are arbitrary. On the other hand,

$$\begin{aligned}a_{5,0} &= \frac{1}{1,0}a_{3,2} \\ a_{4,1} &= \frac{1}{2}a_{3,2} \\ a_{2,3} &= 10a_{0,5} \\ a_{1,4} &= 5a_{0,5}\end{aligned}$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2$$

The results hold true for $(\lambda, \nu) = (-\frac{3}{2}, -\frac{1}{2}), (-\frac{1}{2}, -2), (-1, -\frac{3}{2}), (-1, -2), (-\frac{1}{2}, -\frac{3}{2}), (-\frac{3}{2}, -1), (-\frac{3}{2}, -\frac{3}{2}), (-\frac{3}{2}, -2), (-2, -\frac{1}{2}), (-2, -1), (-2, -\frac{3}{2})$.

(6) If λ and ν are not like above, then the space $B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu})$ is one-dimensional.

For example, if $(\lambda, \nu) = (-\frac{1}{2}, -\frac{1}{2})$, then the constant $a_{2,3}$ is arbitrary.

On the other hand,

$$\begin{aligned} a_{5,0} &= a_{4,1} = a_{1,4} = a_{0,5} = 0 \\ a_{3,2} &= -a_{2,3} \end{aligned}$$

(vii) **The case when $\mu - \lambda - \nu = 6$**

$$\begin{aligned} c(f, g) &= c_{1,6,0}Y'f^{(6)}g + c_{1,5,1}Y'f^{(5)}g' + c_{1,4,2}Y'f^{(4)}g'' + c_{1,3,3}Y'f'''g''' + c_{1,2,4}Y'f''g^{(4)} \\ &+ c_{1,1,5}Y'f'g^{(5)} + c_{1,0,6}Y'fg^{(6)} + c_{2,5,0}Y''f^{(5)}g + c_{2,4,1}Y''f^{(4)}g' + c_{2,3,2}Y''f'''g'' \\ &+ c_{2,2,3}Y''f''g''' + c_{2,1,4}Y''f'g^{(4)} + c_{2,0,5}Y''fg^{(5)} + c_{3,4,0}Y'''f^{(4)}g + c_{3,3,1}Y'''f'''g' \\ &+ c_{3,2,2}Y'''f''g'' + c_{3,1,3}Y'''f'g''' + c_{3,0,4}Y'''fg^{(4)} + c_{4,3,0}Y^{(4)}f'''g + c_{4,2,1}Y^{(4)}f''g' \\ &+ c_{4,1,2}Y^{(4)}fg'' + c_{4,0,3}Y^{(4)}fg''' + c_{5,2,0}Y^{(5)}f''g + c_{5,1,1}Y^{(5)}f'g' + c_{5,0,2}Y^{(5)}fg'' \\ &+ c_{6,1,0}Y^{(6)}f'g + c_{6,0,1}Y^{(6)}fg' + c_{7,0,0}Y^{(7)}fg \end{aligned}$$

A straightforward computation shows that the above 1-cocycle condition reads

$\mu - \lambda - \nu = 6$ and

$$\begin{aligned} c_{1,6,0}(15 + 6\lambda) + c_{1,5,1}\nu &= 0, & c_{1,6,0}(20 + 15\lambda) + c_{1,4,2}\nu &= 0, \\ c_{1,6,0}(15 + 20\lambda) + c_{1,3,3}\nu &= 0, & c_{1,6,0}(6 + 15\lambda) + c_{1,2,4}\nu &= 0, \\ c_{1,6,0}(1 + 6\lambda) + c_{1,1,5}\nu &= 0, & c_{1,6,0}\lambda + c_{1,0,6}\nu &= 0, \\ c_{1,5,1}(10 + 5\lambda) + c_{1,4,2}(1 + 2\nu) &= 0, & c_{1,5,1}(10 + 10\lambda) + c_{1,3,3}(1 + 3\nu) &= 0, \\ c_{1,5,1}(5 + 10\lambda) + c_{1,2,4}(1 + 4\nu) &= 0, & c_{1,5,1}(1 + 5\lambda) + c_{1,1,5}(1 + 5\nu) &= 0, \\ c_{1,5,1}\lambda + c_{1,0,6}(1 + 6\nu) &= 0, & c_{1,4,2}(6 + 4\lambda) + c_{1,3,3}(3 + 3\nu) &= 0, \\ c_{1,4,2}(4 + 6\lambda) + c_{1,2,4}(4 + 6\nu) &= 0, & c_{1,4,2}(1 + 4\lambda) + c_{1,1,5}(5 + 10\nu) &= 0, \\ c_{1,4,2}\lambda + c_{1,0,6}(6 + 15\nu) &= 0, & c_{1,3,3}(3 + 3\lambda) + c_{1,2,4}(6 + 4\nu) &= 0, \\ c_{1,3,3}(1 + 3\lambda) + c_{1,1,5}(10 + 10\nu) &= 0, & c_{1,3,3}\lambda + c_{1,0,6}(15 + 20\nu) &= 0, \\ c_{1,2,4}(1 + 2\lambda) + c_{1,1,5}(10 + 5\nu) &= 0, & c_{1,2,4}\lambda + c_{1,0,6}(20 + 15\nu) &= 0, \\ c_{1,1,5}\lambda + c_{1,0,6}(15 + 6\nu) &= 0, & c_{2,5,0}(10 + 10\lambda) + c_{2,3,2}\nu &= 0, \\ c_{2,5,0}(5 + 10\lambda) + c_{2,2,3}\nu &= 0, & c_{2,5,0}(1 + 5\lambda) + c_{2,1,4}\nu &= 0, \\ c_{2,5,0}\lambda + c_{2,0,5}\nu &= 0, & c_{2,4,1}(4 + 6\lambda) + c_{2,2,3}(1 + 3\nu) &= 0, \\ c_{2,4,1}(1 + 4\lambda) + c_{2,1,4}(1 + 4\nu) &= 0, & c_{2,4,1}\lambda + c_{2,0,5}(1 + 5\nu) &= 0, \\ c_{2,3,2}(1 + 3\lambda) + c_{2,1,4}(4 + 6\nu) &= 0, & c_{2,3,2}\lambda + c_{2,0,5}(5 + 10\nu) &= 0, \\ c_{2,2,3}\lambda + c_{2,0,5}(10 + 10\nu) &= 0, & c_{3,4,0}(6 + 4\lambda) + c_{3,3,1}\nu + 2c_{4,3,0} &= 0, \\ c_{3,3,1}(3 + 3\lambda) + c_{3,2,2}(1 + 2\nu) + 2c_{4,2,1} &= 0, & c_{6,1,0}\lambda + c_{6,0,1}\nu + 14c_{7,0,0} &= 0, \\ c_{3,2,2}(1 + 2\lambda) + c_{3,1,3}(3 + 3\nu) + 2c_{4,1,2} &= 0, & c_{3,1,3}\lambda + c_{3,0,4}(6 + 4\nu) + 2c_{4,0,3} &= 0, \\ c_{4,3,0}(1 + 3\lambda) - c_{3,4,0}(1 + 4\lambda) + (c_{4,1,2} - c_{3,1,3})\nu & & & \\ + 5c_{6,1,0} &= 0, & c_{4,3,0}(3 + 3\lambda) + c_{4,2,1}\nu + 5c_{4,0,3} &= 0, \\ (c_{3,3,1} - c_{4,2,1})\lambda - c_{3,0,4}(1 + 4\nu) + c_{4,0,3}(1 + 3\nu) & & & \\ + 5c_{6,0,1} &= 0, & c_{4,1,2}\lambda + c_{4,0,3}(3 + 3\nu) + 5c_{5,0,2} &= 0, \\ c_{4,2,1}(1 + 2\lambda) + c_{4,1,2}(1 + 2\nu) + 5c_{5,1,1} &= 0, & c_{5,2,0}(1 + 2\lambda) + c_{5,1,1}\nu + 9c_{6,1,0} &= 0, \\ c_{5,2,0}\lambda + c_{5,0,2}\nu - c_{3,4,0}\lambda - c_{3,0,4}\nu + 14c_{7,0,0} &= 0, & c_{5,1,1}\lambda + c_{5,0,2}(1 + 2\nu) + 9c_{6,0,1} &= 0. \end{aligned}$$

The space of solutions of the above system is four-dimensional for $(\lambda, \nu) =$

$(0, 0), (-\frac{5}{2}, -\frac{5}{2}), (-\frac{5}{2}, 0), (0, -\frac{5}{2}), (0, \frac{-5 \pm \sqrt{19}}{2}), (\frac{-5 \pm \sqrt{19}}{2}, 0), (\frac{-5 + \sqrt{19}}{2}, \frac{-5 - \sqrt{19}}{2}), (\frac{-5 - \sqrt{19}}{2}, \frac{-5 + \sqrt{19}}{2})$, and three-dimensional otherwise. Let us study the triviality of this 1-cocycle.

A straightforward computation shows that

$$\begin{aligned}
L_X A_6^{\lambda, \nu} = & - (a_{6,0}(15 + 6\lambda) + a_{5,1}\nu)Y'' f^{(5)}g - (a_{5,1}(10 + 5\lambda) + a_{4,2}(1 + 2\nu))Y'' f^{(4)}g' \\
& - (a_{4,2}(6 + 4\lambda) + a_{3,3}(3 + 3\nu))Y'' f'''g'' - (a_{3,3}(3 + 3\lambda) + a_{2,4}(6 + 4\nu))Y'' f''g''' \\
& - (a_{2,4}(6 + 4\lambda) + a_{1,5}(10 + 5\nu))Y'' f'g^{(4)} - (a_{1,5}\lambda + a_{0,6}(15 + 6\nu))Y'' fg^{(5)} \\
& - (a_{6,0}(20 + 15\lambda) + a_{4,2}\nu)Y''' f^{(4)}g - (a_{5,1}(10 + 10\lambda) + a_{3,3}(1 + 3\nu))Y''' f'''g' \\
& - (a_{4,2}(4 + 6\lambda) + a_{2,4}(4 + 6\nu))Y''' f''g'' - (a_{3,3}(1 + 3\lambda) + a_{1,5}(10 + 10\nu))Y''' f'g''' \\
& - (a_{2,4}\lambda + a_{0,6}(20 + 15\nu))Y''' fg^{(4)} - (a_{3,3}(15 + 20\lambda) + a_{6,0}\nu)Y^{(4)} f'''g \\
& - (a_{5,1}(5 + 10\lambda) + a_{2,4}(1 + 4\nu))Y^{(4)} f''g' - (a_{1,5}(1 + 4\lambda) + a_{4,2}(5 + 10\nu))Y^{(4)} f'g'' \\
& - (a_{3,3}\lambda + a_{0,6}(15 + 20\nu))Y^{(4)} fg''' - (a_{2,4}(6 + 15\lambda) + a_{6,0}\nu)Y^{(5)} f''g \\
& - (a_{1,5}(1 + 5\lambda) + a_{5,1}(1 + 5\nu))Y^{(5)} f'g' - (a_{4,2}\lambda + a_{0,6}(6 + 15\nu))Y^{(5)} fg'' \\
& - (a_{1,5}(1 + 6\lambda) + a_{6,0}\nu)Y^{(6)} f'g - (a_{5,1}\lambda + a_{0,6}(1 + 6\nu))Y^{(6)} fg' \\
& - (a_{6,0}\lambda + a_{0,6}\nu)Y^{(7)} fg
\end{aligned}$$

- (1) If $(\lambda, \nu) = (0, \frac{-5-\sqrt{19}}{2})$, then $c_{1,6,0} = c_{1,5,1} = c_{1,4,2} = c_{1,3,3} = c_{1,2,4} = c_{1,1,5} = c_{1,0,6} = c_{2,5,0} = c_{2,4,1} = c_{2,3,2} = c_{2,2,3} = c_{2,1,4} = c_{2,0,5} = 0$,
 $c_{3,4,0} = (\frac{101+23\sqrt{19}}{1,0})c_{3,1,3} - \frac{5+\sqrt{19}}{1,5}c_{4,2,1} + \frac{25+6\sqrt{19}}{1,5}c_{4,1,2}$,
 $c_{6,1,0} = (\frac{38+9\sqrt{19}}{100})c_{3,1,3} - \frac{5+\sqrt{19}}{5,0}c_{4,2,1} + \frac{40+9\sqrt{19}}{75}c_{4,1,2}$,
 $c_{5,2,0} = -(\frac{9(38+9\sqrt{19})}{100})c_{3,1,3} + \frac{2(5+\sqrt{19})}{2,5}c_{4,2,1} - \frac{9(5+\sqrt{19})}{5,0}c_{4,1,2}$,
 $c_{5,2,0} = \frac{3(38+9\sqrt{19})}{2,0}c_{3,1,3} + \frac{(5+\sqrt{19})}{3,0}c_{4,2,1} + \frac{3(5+\sqrt{19})}{1,0}c_{4,1,2}$,
 $c_{7,0,0} = \frac{1}{840}(3 + \sqrt{19})(4 + \sqrt{19})(5 + \sqrt{19})c_{4,0,3}$,
 $c_{3,3,1} = \frac{1}{2}(3 + \sqrt{19})(4 + \sqrt{19})c_{3,1,3} - \frac{2}{3}c_{4,2,1} + \frac{2(4+\sqrt{19})}{3}c_{4,1,2}$,
 $c_{3,2,2} = \frac{3}{2}(3+\sqrt{19})c_{3,1,3} + 2c_{4,1,2}$, $c_{5,1,1} = -\frac{1}{5}c_{4,2,1} - \frac{1}{5}(-4-\sqrt{19})c_{4,1,2}$,
 $c_{6,0,1} = \frac{1}{3,0}(3 + \sqrt{19})(4 + \sqrt{19})c_{4,0,3}$, $c_{5,0,2} = \frac{3}{1,0}(3 + \sqrt{19})c_{4,0,3}$,
 $c_{3,0,4} = \frac{1}{(2+\sqrt{19})}c_{4,0,3}$

$$\dim Z_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 4$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 1, \text{ where}$$

$$\begin{aligned}
a_{0,6} = 0, \quad a_{1,5} = -\frac{4}{45}(-35 + 8\sqrt{19})a_{5,1}, \quad a_{2,4} = 2(-13 + 3\sqrt{19})a_{5,1}, \\
a_{3,3} = -\frac{4}{3}(-31 + 7\sqrt{19})a_{5,1}, \quad a_{4,2} = \frac{1,0}{3}(-4 + \sqrt{19})a_{5,1}, \quad a_{6,0} = \\
-\frac{1}{3,0}(5 + \sqrt{19})a_{5,1}.
\end{aligned}$$

The results hold true for $(\lambda, \nu) = (0, \frac{-5+\sqrt{19}}{2})$, $(\frac{-5\pm\sqrt{19}}{2}, 0)$,
 $(\frac{-5+\sqrt{19}}{2}, \frac{-5-\sqrt{19}}{2})$, $(\frac{-5-\sqrt{19}}{2}, \frac{-5+\sqrt{19}}{2})$.

- (2) If $(\lambda, \nu) = (0, 0)$, then the constant $a_{3,3}$ and $a_{2,4}$ are arbitrary. On the Other hand,

$$\begin{aligned}
a_{6,0} = a_{0,6} = 0 \\
a_{5,1} = (\frac{1}{4}a_{3,3} - \frac{1}{2}a_{2,4}) \\
a_{4,2} = (-a_{3,3} - a_{2,4}) \\
a_{1,5} = (\frac{3}{4,0}a_{3,3} + \frac{1}{2,0}a_{2,4})
\end{aligned}$$

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2$$

The results hold true for $(\lambda, \nu) = (-\frac{5}{2}, -\frac{5}{2}), (-\frac{5}{2}, 0), (0, -\frac{5}{2})$.

- (3) If $(\lambda, \nu) = (-\frac{1}{2}, 0)$ then the constant $a_{3,3}$ and $a_{2,4}$ are arbitrary. On the other hand, $a_{1,5} = 6a_{0,6}$, $a_{2,4} = \frac{5}{2}a_{5,1}$, $a_{3,3} = \frac{1,0}{3}a_{5,1}$, $a_{4,2} = \frac{5}{2}a_{5,1}$, $a_{6,0} = \frac{1}{6}a_{5,1}$.

$$\dim B_{\text{diff}}^1(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda, \nu; \mu}) = 2$$

The results hold true for $(\lambda, \nu) = (-2, -\frac{1}{2}), (-\frac{1}{2}, -\frac{5}{2}), (-\frac{5}{2}, -\frac{1}{2}), (-1, -\frac{3}{2}), (-\frac{3}{2}, -1), (-1, -2), (-2, -1), (-\frac{5}{2}, -1), (-1, -\frac{5}{2}), (-\frac{3}{2}, -\frac{3}{2}), (-2, -\frac{3}{2}), (-\frac{3}{2}, -2), (-\frac{3}{2}, -\frac{5}{2}), (-\frac{5}{2}, -\frac{3}{2}), (-2, -2), (-2, -\frac{5}{2})$ and $(-\frac{5}{2}, -2)$.

- (4) If $(\lambda, \nu) = (0, -\frac{1}{4})$, then the constant $a_{6,0}$ is arbitrary. On the other hand,

$$\begin{aligned} a_{0,6} &= 0, & a_{6,0} &= \frac{1}{6,0}a_{5,1}, & a_{5,1} &= -\frac{1}{2,0}a_{4,2}, \\ a_{4,2} &= -\frac{3}{8}a_{3,3} & a_{3,3} &= -\frac{5}{3}a_{2,4}, & a_{2,4} &= -\frac{35}{4}a_{1,5} \end{aligned}$$

The results hold true for otherwise.

(viii) The case when $\mu - \lambda - \nu = 7$

$$\begin{aligned} c(f, g) &= c_{1,7,0}Y'f^{(7)}g + c_{1,6,1}Y'f^{(6)}g' + c_{1,5,2}Y'f^{(5)}g'' + c_{1,4,3}Y'f^{(4)}g''' + c_{1,3,4}Y'f'''g^{(4)} \\ &+ c_{1,2,5}Y'f''g^{(5)} + c_{1,1,6}Y'f'g^{(6)} + c_{1,0,7}Y'fg^{(7)} + c_{2,6,0}Y''f^{(6)}g + c_{2,5,1}Y''f^{(5)}g' \\ &+ c_{2,4,2}Y''f^{(4)}g'' + c_{2,3,3}Y''f'''g''' + c_{2,2,4}Y''f''g^{(4)} + c_{2,1,5}Y''f'g^{(5)} + c_{2,0,6}Y''fg^{(6)} \\ &+ c_{3,5,0}Y'''f^{(5)}g + c_{3,4,1}Y'''f^{(4)}g' + c_{3,3,2}Y'''f'''g'' + c_{3,2,3}Y'''f''g''' + c_{3,1,4}Y'''f'g^{(4)} \\ &+ c_{3,0,5}Y'''fg^{(5)} + c_{4,4,0}Y^{(4)}f^{(4)}g + c_{4,3,1}Y^{(4)}f'''g' + c_{4,2,2}Y^{(4)}f''g'' + c_{4,1,3}Y^{(4)}fg''' \\ &+ c_{4,0,4}Y^{(4)}fg''' + c_{5,3,0}Y^{(5)}f''g + c_{5,2,1}Y^{(5)}f'g' + c_{5,1,2}Y^{(5)}f'g'' + c_{5,0,3}Y^{(5)}fg''' \\ &+ c_{6,2,0}Y^{(6)}f''g + c_{6,1,1}Y^{(6)}f'g' + c_{6,0,2}Y^{(6)}fg'' \\ &+ c_{7,1,0}Y^{(7)}f'g + c_{7,0,1}Y^{(7)}fg' + c_{8,0,0}Y^{(8)}fg \end{aligned}$$

A straightforward computation shows that the above 1-cocycle condition reads

$\mu - \lambda - \nu = 7$ and

$$\begin{aligned}
c_{1,7,0}(21 + 7\lambda) + c_{1,6,1}\nu &= 0, & c_{1,7,0}(35 + 21\lambda) + c_{1,5,2}\nu &= 0, \\
c_{1,7,0}(35 + 355\lambda) + c_{1,4,3}\nu &= 0, & c_{1,7,0}(21 + 35\lambda) + c_{1,3,4}\nu &= 0, \\
c_{1,7,0}(7 + 21\lambda) + c_{1,2,5}\nu &= 0, & c_{1,7,0}(1 + 7\lambda) + c_{1,1,6}\nu &= 0, \\
c_{1,7,0}\lambda + c_{1,0,7}\nu &= 0, & c_{1,6,1}(15 + 6\lambda) + c_{1,5,2}(1 + 2\nu) &= 0, \\
c_{1,6,1}(20 + 15\lambda) + c_{1,4,3}(1 + 3\nu) &= 0, & c_{1,6,1}(15 + 20\lambda) + c_{1,3,4}(1 + 4\nu) &= 0, \\
c_{1,6,1}(6 + 15\lambda) + c_{1,2,5}(1 + 5\nu) &= 0, & c_{1,6,1}(1 + 6\lambda) + c_{1,1,6}(1 + 6\nu) &= 0, \\
c_{1,6,1}\lambda + c_{1,0,7}(1 + 7\nu) &= 0, & c_{1,5,2}(10 + 5\lambda) + c_{1,4,3}(3 + 3\nu) &= 0, \\
c_{1,5,2}(10 + 10\lambda) + c_{1,3,4}(4 + 6\nu) &= 0, & c_{1,5,2}(5 + 10\lambda) + c_{1,2,5}(5 + 10\nu) &= 0, \\
c_{1,5,2}(1 + 5\lambda) + c_{1,1,6}(6 + 15\nu) &= 0, & c_{1,5,2}\lambda + c_{1,0,7}(7 + 21\nu) &= 0, \\
c_{1,4,3}(6 + 4\lambda) + c_{1,3,4}(6 + 4\nu) &= 0, & c_{1,4,3}(4 + 6\lambda) + c_{1,2,5}(10 + 10\nu) &= 0, \\
c_{1,4,3}(1 + 4\lambda) + c_{1,1,6}(15 + 20\nu) &= 0, & c_{1,4,3}\lambda + c_{1,0,7}(21 + 35\nu) &= 0, \\
c_{1,3,4}(3 + 3\lambda) + c_{1,2,5}(10 + 5\nu) &= 0, & c_{1,3,4}(1 + 3\lambda) + c_{1,1,6}(20 + 15\nu) &= 0, \\
c_{1,3,4}\lambda + c_{1,0,7}(35 + 35\nu) &= 0, & c_{1,2,5}(1 + 2\lambda) + c_{1,1,6}(15 + 6\nu) &= 0, \\
c_{1,2,5}\lambda + c_{1,0,7}(35 + 21\nu) &= 0, & c_{1,1,6}\lambda + c_{1,0,7}(21 + 7\nu) &= 0,
\end{aligned}$$

$c_{1,7,0} = c_{1,6,1} = c_{1,5,2} = c_{1,4,3} = c_{1,3,4} = c_{1,2,5} = c_{1,1,6} = c_{1,0,7} = 0$ for all λ and ν .

$$\begin{aligned}
c_{2,6,0}(20 + 15\lambda) + c_{2,4,2}\nu &= 0, & c_{2,6,0}(15 + 20\lambda) + c_{2,3,3}\nu &= 0, \\
c_{2,6,0}(6 + 15\lambda) + c_{2,2,4}\nu &= 0, & c_{2,6,0}(1 + 6\lambda) + c_{2,1,5}\nu &= 0, \\
c_{2,6,0}\lambda + c_{2,0,6}\nu &= 0, & c_{2,5,1}(10 + 10\lambda) + c_{2,3,3}(1 + 3\nu) &= 0, \\
c_{2,5,1}(5 + 10\lambda) + c_{2,2,4}(1 + 4\nu) &= 0, & c_{2,5,1}(1 + 5\lambda) + c_{2,1,5}(1 + 5\nu) &= 0, \\
c_{2,5,1}\lambda + c_{2,0,6}(1 + 6\nu) &= 0, & c_{2,4,2}(4 + 6\lambda) + c_{2,2,4}(4 + 6\nu) &= 0, \\
c_{2,4,2}(1 + 4\lambda) + c_{2,1,5}(5 + 10\nu) &= 0, & c_{2,4,2}\lambda + c_{2,0,6}(6 + 15\nu) &= 0, \\
c_{2,3,3}(1 + 3\lambda) + c_{2,1,5}(10 + 10\nu) &= 0, & c_{2,3,3}\lambda + c_{2,0,6}(15 + 20\nu) &= 0, \\
c_{2,2,4}\lambda + c_{2,0,6}(20 + 15\nu) &= 0,
\end{aligned}$$

$c_{2,6,0} = c_{2,5,1} = c_{2,4,2} = c_{2,3,3} = c_{2,2,4} = c_{2,1,5} = c_{2,0,6} = 0$ for all λ and ν .

$$\begin{aligned}
c_{3,5,0}(10 + 5\lambda) + c_{3,4,1}\nu + 2c_{4,4,0} &= 0, & c_{3,5,0}(5 + 10\lambda) + c_{3,2,3}\nu &= 0, \\
c_{3,5,0}(1 + 5\lambda) + c_{3,1,4}\nu &= 0, & c_{3,5,0}\lambda + c_{3,0,5}\nu &= 0, \\
c_{3,4,1}(6 + 4\lambda) + c_{3,3,2}(1 + 2\nu) + 2c_{4,3,1} &= 0, & c_{3,4,1}(1 + 4\lambda) + c_{3,1,4}(1 + 4\nu) &= 0, \\
c_{3,4,1}\lambda + c_{3,0,5}(1 + 5\nu) &= 0, & c_{3,3,2}(3 + 3\lambda) + c_{3,2,3}(3 + 3\nu) + 2c_{4,2,2} &= 0, \\
c_{3,3,2}\lambda + c_{3,0,5}(5 + 10\nu) &= 0, & c_{3,2,3}(1 + 2\lambda) + c_{3,1,4}(6 + 4\nu) + 2c_{4,1,3} &= 0, \\
c_{3,1,4}\lambda + c_{3,0,5}(10 + 5\nu) + 2c_{4,0,4} &= 0, & c_{4,4,0}(4 + 6\lambda) + c_{4,2,2}\nu + 5c_{6,2,0} &= 0, \\
c_{4,4,0}(6 + 4\lambda) + c_{4,3,1}\nu + 5c_{5,3,0} &= 0, & c_{4,3,1}(3 + 3\lambda) + c_{4,2,2}(1 + 2\nu) + 5c_{5,2,1} &= 0, \\
c_{4,4,0}\lambda + c_{4,1,3}\nu &= 0, & c_{4,2,2}(1 + 2\lambda) + c_{4,1,3}(3 + 3\nu) + 5c_{5,1,2} &= 0, \\
c_{4,3,1}(1 + 3\lambda) + c_{4,1,3}(1 + 3\nu) + 5c_{6,1,1} &= 0, & c_{4,1,3}\lambda + c_{4,0,4}(6 + 4\nu) + 5c_{5,0,3} &= 0, \\
c_{4,2,2}\lambda + c_{4,0,4}(4 + 6\nu) + 5c_{6,0,2} &= 0, & c_{5,3,0}(3 + 3\lambda) + c_{5,2,1}\nu + 9c_{6,2,0} &= 0, \\
c_{5,3,0}\lambda + c_{5,0,3}\nu + 14c_{8,0,0} &= 0, & c_{5,2,1}(1 + 2\lambda) + c_{5,1,2}(1 + 2\nu) + 9c_{6,1,1} &= 0, \\
c_{5,3,0}(1 + 3\lambda) + c_{5,1,2}\nu + 14c_{7,1,0} &= 0, & c_{5,1,2}\lambda + c_{5,0,3}(3 + 3\nu) + 9c_{6,0,2} &= 0, \\
c_{5,2,1}\lambda + c_{5,0,3}(1 + 3\nu) + 14c_{7,0,1} &= 0, & c_{6,2,0}\lambda + c_{6,0,2}\nu + 28c_{8,0,0} &= 0, \\
c_{6,2,0}(1 + 2\lambda) + c_{6,1,1}\nu + 14c_{7,1,0} &= 0, & c_{7,1,0}\lambda + c_{7,0,1}\nu + 20c_{8,0,0} &= 0, \\
c_{6,1,1}\lambda + c_{6,0,2}(1 + 2\nu) + 14c_{7,0,1} &= 0,
\end{aligned}$$

Let us study the triviality of this 1-cocycle. A straightforward computation

shows that

$$\begin{aligned}
L_X A_7^{\lambda, \nu} = & - (a_{7,0}(21 + 7\lambda) + a_{6,1}\nu)Y'' f^{(6)}g - (a_{6,1}(15 + 6\lambda) + a_{5,2}(1 + 2\nu))Y'' f^{(5)}g' \\
& - (a_{5,2}(10 + 5\lambda) + a_{4,3}(3 + 3\nu))Y'' f^{(4)}g'' - (a_{4,3}(6 + 4\lambda) + a_{3,4}(6 + 4\nu))Y'' f'''g''' \\
& - (a_{3,4}(3 + 3\lambda) + a_{2,5}(10 + 5\nu))Y'' f''g^{(4)} - (a_{2,5}(1 + 2\lambda) + a_{1,6}(15 + 6\nu))Y'' f'g^{(5)} \\
& - (a_{1,6}\lambda + a_{0,7}(21 + 7\nu))Y'' fg^{(6)} - (a_{7,0}(35 + 21\lambda) + a_{5,2}\nu)Y''' f^{(5)}g \\
& - (a_{6,1}(20 + 15\lambda) + a_{4,3}(1 + 3\nu))Y''' f^{(4)}g' - (a_{5,2}(10 + 10\lambda) + a_{3,4}(4 + 6\nu))Y''' f'''g'' \\
& - (a_{4,3}(4 + 6\lambda) + a_{2,5}(10 + 10\nu))Y''' f''g''' - (a_{3,4}(1 + 3\lambda) + a_{1,6}(20 + 15\nu))Y''' f'g^{(4)} \\
& - (a_{2,5}\lambda + a_{0,7}(35 + 21\nu))Y''' fg^{(5)} - (a_{7,0}(35 + 35\lambda) + a_{4,3}\nu)Y^{(4)} f(4)g \\
& - (a_{6,1}(15 + 20\lambda) + a_{3,4}(1 + 4\nu))Y^{(4)} f'''g' - (a_{5,2}(5 + 10\lambda) + a_{2,5}(5 + 10\nu))Y^{(4)} f''g'' \\
& - (a_{4,3}(1 + 4\lambda) + a_{1,6}(15 + 20\nu))Y^{(4)} f'g''' - (a_{3,4}\lambda + a_{0,7}(35 + 35\nu))Y^{(4)} fg^{(4)} \\
& - (a_{7,0}(21 + 35\lambda) + a_{3,4}\nu)Y^{(5)} f'''g - (a_{6,1}(6 + 15\lambda) + a_{2,5}(1 + 5\nu))Y^{(5)} f''g' \\
& - (a_{5,2}(1 + 5\lambda) + a_{1,6}(6 + 15\nu))Y^{(5)} f'g'' - (a_{4,3}\lambda + a_{0,7}(21 + 35\nu))Y^{(5)} fg''' \\
& - (a_{7,0}(7 + 21\lambda) + a_{2,5}\nu)Y^{(6)} f''g - (a_{6,1}(1 + 6\lambda) + a_{1,6}(1 + 6\nu))Y^{(6)} f'g' \\
& - (a_{5,2}\lambda + a_{0,7}(7 + 21\nu))Y^{(6)} fg'' - (a_{7,0}(1 + 7\lambda) + a_{1,6}\nu)Y^{(7)} f'g \\
& - (a_{6,1}(1 + 7\lambda) + a_{0,7}\nu)Y^{(7)} fg' - (a_{7,0}\lambda + a_{0,7}\nu)Y^{(8)} fg
\end{aligned}$$

The proof here is the same as in the previous section. We just point out that the space of solution of the 1-cocycle is four-dimensional for $(\lambda, \nu) = (0, -3), (-3, 0)$ and $(-3, -3)$, three-dimensional for $(\lambda, \nu) = (\frac{-5+\sqrt{19}}{2}, \frac{-5+\sqrt{19}}{2}), (-2, -2), (0, \nu), (\lambda, 0), (\lambda, -6-\lambda), (\sqrt{19}-1, \frac{-\sqrt{19}-5}{2}), (\frac{-\sqrt{19}-5}{2}, \sqrt{19}-1), (-\sqrt{19}-1, \frac{\sqrt{19}-5}{2}), (\frac{\sqrt{19}-5}{2}, -\sqrt{19}-1), (-3, -\frac{5}{2}), (-\frac{5}{2}, -3), (-\frac{1}{2}, -3), (-3, -\frac{1}{2}), (-2, -\frac{5}{2}), (-\frac{5}{2}, -2), (-\frac{5}{2}, -\frac{5}{2})$, and two-dimensional otherwise.

(ix) The case when $k \geq 8$

For $k \geq 8$, the number of variables generating any 1-cocycle is "much" smaller than the number of equations. For instance, the 1-cocycle condition for $k = 8$ involves 45 variables for 90 equations. For generic λ and ν the number of equations will generates a one-dimensional space, which give a unique cohomology class. This cohomology class is indeed trivial because the expression $L_X A_k^{\lambda, \nu}$ is also a 1-cocycle.

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(First Author) UNIVERSITÉ EVRY VAL D'ESSONNE ÉVRY 91000, FRANCE

E-mail address: issambartouli@yahoo.fr

(Second Author) DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES DE SFAX, BP 802, SFAX 3038, TUNISIE

E-mail address: ammar.derbali1@yahoo.fr

(Second Author) DÉPARTEMENT DE MATHÉMATIQUES, FACULTÉ DES SCIENCES DE SFAX, BP 802, SFAX 3038, TUNISIE

E-mail address: chraygui@gmail.com

(Second Author) UNIVERSITÉ EVRY VAL D'ESSONNE ÉVRY 91000, FRANCE

E-mail address: jean.lerbet@univ-evry.fr