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COHOMOLOGY OF Vect($\mathbb{R}$) ACTING ON THE SPACE OF BILINEAR DIFFERENTIAL OPERATORS

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Abstract. We consider the Vect($\mathbb{R}$)-module structure on the spaces of bilinear differential operators acting on the spaces of weighted densities. We compute the first differential cohomology of the vector fields Lie algebra Vect($\mathbb{R}$) with coefficients in space $\mathcal{D}_{\lambda,\nu;\mu}$ of bilinear differential operators acting on weighted densities.

1. Introduction

Let $\mathbb{M}$ be a smooth manifold and Vect($\mathbb{M}$) the Lie algebra of vector fields on $\mathbb{M}$. The main purpose of this article is to study the cohomology of Vect($\mathbb{M}$) with coefficients in the space of linear differential operators acting on tensor fields. This cohomology is, actually, a natural generalization of the Gelfand-Fuchs cohomology (i.e., of Vect($\mathbb{M}$)- cohomology with coefficients in the modules of tensor fields on $\mathbb{M}$).
Let $\mathfrak{g}$ be a Lie algebra and let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathfrak{g}-$modules. It is well-known that nontrivial extensions of $\mathfrak{g}-$modules:

$$0 \to \mathcal{M} \to . \to \mathcal{N} \to 0$$

are classified by the first cohomology group $\mathrm{H}^1(\mathfrak{g}, \mathrm{Hom}(N, M))$ (see, e.g., [9]). Any $1-$cocycle Λ generates a new action on $\mathcal{M} \oplus \mathcal{N}$ as follows: for all $g \in \mathfrak{g}$ and for all $(a, b) \in \mathcal{M} \oplus \mathcal{N}$, we define $g * (a, b) := (g * a + \Lambda(b), g * b)$.

Although our results and their applications are essentially geometric, the proofs are algebraic. The considerations are local, so we work in the Euclidean case, replacing $\mathbb{M}$ by $\mathbb{R}$.

The space of weighted densities of weight λ on $\mathbb{R}$ (or λ-densities for short), denoted by:

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in C^\infty(\mathbb{R}) \right\}, \ \lambda \in \mathbb{R},$$

is the space of sections of the line bundle $(\mathrm{T}^*\mathbb{R})^{\otimes \lambda}$ for positive integer λ. The Lie algebra Vect($\mathbb{R}$) of vector fields $X_h = h\frac{d}{dx}$, where $h \in C^\infty(\mathbb{R})$, acts by the Lie derivative. Alternatively, this action can be written as follows:

$$X_h.(f dx^\lambda) = L^\lambda_{X_h}(f) dx^\lambda \ \textit{with} \ L^\lambda_{X_h}(f) = hf' + \lambda h' f, \tag{1.1}$$

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where f', h' are $\frac{df}{dx}$, $\frac{dh}{dx}$. Each bilinear differential operator A from $C^\infty(\mathbb{R})\otimes C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ thus gives rise to a morphism from $\mathcal{F}_\lambda\otimes\mathcal{F}_\nu$ to $\mathcal{F}_\mu$, for any $(\lambda,\nu,\mu)\in\mathbb{R}^3$, by $fdx^\lambda\otimes gdx^\nu\mapsto A(f\otimes g)dx^\mu$. The Lie algebra $\mathrm{Vect}(\mathbb{R})$ acts on the space $\mathrm{Hom}(\mathcal{F}_\lambda\otimes\mathcal{F}_\nu,\mathcal{F}_\mu)=\mathcal{D}_{\lambda,\nu;\mu}$ of these differential operators by:

$$X_h.A = L^{\mu}_{X_h}\circ A - A\circ L^{(\lambda,\nu)}_{X_h}, \tag{1.2}$$

where, $L^{(\lambda,\nu)}_{X_h}$ is the Lie derivative on $\mathcal{F}_\lambda\otimes\mathcal{F}_\nu$ defined by the Leibniz rule:

$$L^{(\lambda,\nu)}_{X_h}(f\otimes g) = L^{\lambda}_{X_h}(f)\otimes g + f\otimes L^{\nu}_{X_h}(g).$$

According to Nijenhuis-Richardson [12],the space $\mathrm{H}^1(\mathfrak{g};\mathrm{End}(V))$ classifies the infinitesimal deformations of a $\mathfrak{g}$-module V and the obstructions to integrability of a given infinitesimal deformation of V are elements of $\mathrm{H}^2(\mathfrak{g};\mathrm{End}(V))$ while the spaces $\mathrm{H}^1(\mathfrak{g};L(\otimes^n_s V,V))$ appear naturally in the problem of normalization of nonlinear representations of $\mathfrak{g}$ in V. To be more precise, let

$$\mathrm{T}:\mathfrak{g}\to\sum_{n\geq 0}L(\otimes^n_s V,V), X\mapsto \mathrm{T}_X=\sum \mathrm{T}^n_X,$$

be a nonlinear representation of $\mathfrak{g}$ in V,that is, $\mathrm{T}_{[X,Y]}=[\mathrm{T}_X,\mathrm{T}_Y]$. In [1], it is proved that the representation T is normalized if T^n_X is in a supplementary of $\mathrm{B}^1(\mathfrak{g};L(\otimes^n_s V,V))$ in $\mathrm{Z}^1(\mathfrak{g};L(\otimes^n_s V,V))$.
In fact if A is a differential operator on the line, A can be viewed as an homomorphism from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$. If A is with order n, we can define its symbol as an element in $\mathcal{S}_\delta=\bigoplus_{k=0}^n\mathcal{F}_{\delta-k}$, for $\delta=\mu-\lambda$. If n goes to $+\infty$,the space $\mathcal{S}_\delta=\bigoplus_{k\geq 0}\mathcal{F}_{\delta-k}$, appears as the space symbols for all differential operators.The space $\mathrm{H}^1\left(\mathrm{Vect}(\mathbb{R}),L(\otimes^2_s\mathcal{S}_\delta,\mathcal{S}_\delta)\right)$ can be decomposed as a sum of spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})$.Thus, the computation of the spaces $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})$ is the first step to normalize any nonlinear representation of $\mathrm{Vect}(\mathbb{R})$ in $\mathcal{S}_\delta$.
For the space of tensor densities of weight $\lambda\in\mathbb{R}$, $\mathcal{F}_\lambda$, viewed as a module over the Lie algebra of smooth vector fields $\mathrm{Vect}(\mathbb{R})$, the classification of nontrivial extensions

$$0\to\mathcal{F}_\mu\to .\to\mathcal{F}_\lambda\to 0.$$

leads Feigin and Fuks in [10] to compute the cohomology group $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathrm{Hom}(\mathcal{F}_\lambda,\mathcal{F}_\mu))$. Later, Ovsienko and Bouarroudj in [3] have computed the corresponding relative cohomology group with respect to $\mathfrak{sl}(2)$, namely

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2);\mathrm{Hom}(\mathcal{F}_\lambda,\mathcal{F}_\mu)).$$

Later, Bouarroudj in [2] has computed the corresponding relative cohomology group with respect to $\mathfrak{sl}(2)$, namely

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathfrak{sl}(2);\mathrm{Hom}(\mathcal{F}_\lambda\otimes\mathcal{F}_\nu,\mathcal{F}_\mu)).$$

In this paper, we will compute the first differential cohomology group $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu})$, where, $\mathcal{D}_{\lambda,\nu;\mu}$ is the space of bilinear differential operators from $\mathcal{F}_\lambda \otimes \mathcal{F}_\nu$ to $\mathcal{F}_\mu$.

This work allows us to study the first cohomology of the Lie superalgebra $\mathcal{K}(1)$ with coefficients in the superspace of bilinear differential operators.

2. Definitions and Notations

In this section, we recall the main definitions and facts related to the geometry of the space $\mathbb{R}$. We also recall some fundamental concepts from cohomology theory (see, e.g.,[8, 9]).

2.1. **Cohomology theory.** Let us first recall some fundamental concepts from cohomology theory (see, e.g., [6, 7, 8, 9]). Let $\mathfrak{g}$ be a Lie algebra acting on a vector space V. The space of n-cochains of $\mathfrak{g}$ with values in V is the $\mathfrak{g}$-module

$$C^n(\mathfrak{g}, V) := \mathrm{Hom}(\Lambda^n \mathfrak{g}; V).$$

The *coboundary operator* $\delta_n : C^n(\mathfrak{g}, V) \longrightarrow C^{n+1}(\mathfrak{g}, V)$ is a $\mathfrak{g}$-map satisfying $\delta_n \circ \delta_{n-1} = 0$. The kernel of δ_n, denoted $Z^n(\mathfrak{g}, V)$, is the space of *n-cocycles*, among them, the elements in the range of δ_{n-1} are called *n-coboundaries*. We denote $B^n(\mathfrak{g}, V)$ the space of n-coboundaries.

By definition, the n^{th} cohomolgy space is the quotient space

$$\mathrm{H}^n(\mathfrak{g}, V) = Z^n(\mathfrak{g}, V)/B^n(\mathfrak{g}, V).$$

We will only need the formula of δ_n (which will be simply denoted δ) in degrees 0 , 1 and 2: for $\Xi \in C^0(\mathfrak{g}, V) = V$, $\delta\Xi(x) := x \cdot \Xi$, and for $\Lambda \in C^1(\mathfrak{g}, V)$,

$$\delta(\Lambda)(x,\, y) := g \cdot \Lambda(y) - y \cdot \Lambda(x) - \Lambda([x,\; y]) \quad \text{for any} \quad x, y \in \mathfrak{g}, \tag{2.1}$$

2.2. **The space of tensor densities on $\mathbb{R}$.** The Lie algebra, $\mathrm{Vect}(\mathbb{R})$, of vector fields on $\mathbb{R}$ naturally acts, by the Lie derivative, on the space

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \;:\; f \in C^\infty(\mathbb{R}) \right\},$$

of weighted densities of degree λ. The Lie derivative L^λ_X of the space $\mathcal{F}_\lambda$ along the vector field $X\frac{d}{dx}$ is defined by

$$L^\lambda_X = X\partial_x + \lambda X', \tag{2.2}$$

where $X, f \in C^\infty(\mathbb{R})$ and $X' := \frac{dX}{dx}$. More precisely, for all $f dx^\lambda \in \mathcal{F}_\lambda$, we have

$$L^\lambda_X(f dx^\lambda) = (Xf' + \lambda f X')dx^\lambda.$$

2.3. **The space of bilinear differential operators as a $\mathrm{Vect}(\mathbb{R})$-module.** The space of bilinear differential operators is a $\mathrm{Vect}(\mathbb{R})$-module, denoted

$$\mathcal{D}_{\underline{\lambda},\mu} := \mathrm{Hom}_{\mathrm{diff}}(\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}, \mathcal{F}_{\mu}).$$

The $\mathrm{Vect}(\mathbb{R})$ action is:

$$L_X^{\underline{\lambda},\mu}(A) = L_X^{\mu} \circ A - A \circ L_X^{\underline{\lambda}}, \tag{2.3}$$

where $\underline{\lambda} = (\lambda_1, \lambda_2)$ and $L_X^{(\lambda_1,\lambda_2)}$ is the Lie derivative on $\mathcal{F}_{\lambda_1} \otimes \mathcal{F}_{\lambda_2}$ defined by the Leibnitz rule:

$$L_X^{\underline{\lambda}}(f_1 dx^{\lambda_1} \otimes f_2 dx^{\lambda_2}) = L_X^{\lambda_1}(f_1) \otimes f_2 dx^{\lambda_2} + f_1 dx^{\lambda_1} \otimes L_X^{\lambda_2}(f_2 dx^{\lambda_2}).$$

3. Cohomology of $\mathrm{Vect}(\mathbb{R})$ acting on $\mathcal{D}_{\lambda,\nu;\mu}$

In this section, we compute the "differentiable" cohomology of the Lie algebra $\mathrm{Vect}(\mathbb{R})$ with coefficients in the space of bilinear differential operators $\mathcal{D}_{\lambda,\nu;\mu}$. Namely, we consider only cochains that are given by differentiable maps.

Theorem 3.1. *(i) If $\mu - \lambda - \nu = 0$, then*

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = \mathbb{R} \qquad \text{for all } \lambda \text{ and } \nu.$$

(ii) If $\mu - \lambda - \nu = 1$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = \begin{cases} \mathbb{R}^3 & \text{if} \quad (\lambda,\nu) = (0,0), \\ \mathbb{R} & \quad \text{otherwise.} \end{cases}$$

(iii) If $\mu - \lambda - \nu = 2$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = \begin{cases} \mathbb{R}^2 & \text{if} \quad (\lambda,\nu) = (0,0), \\ \mathbb{R} & \text{if} \quad (\lambda,\nu) = (0,-1), (0,\nu), (\lambda,0), (\lambda,-1-\lambda), \\ 0 & \quad \text{otherwise.} \end{cases}$$

(iv) If $\mu - \lambda - \nu = 3$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = \begin{cases} \mathbb{R}^3 & \text{if} \quad (\lambda,\nu) = (0,0), \\ \mathbb{R}^2 & \text{if} \quad (\lambda,\nu) = (-2,0), (0,-2), (-\frac{2}{3}, -\frac{2}{3}), \\ \mathbb{R} & \quad \text{otherwise.} \end{cases}$$

(v) If $\mu - \lambda - \nu = 4$, then

$$\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = \mathbb{R}^2 \qquad \text{for all } \lambda \text{ and } \nu.$$

(vi) If $\mu-\lambda-\nu=5$, *then*

$$\mathrm{H}^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=\begin{cases}\mathbb{R}^3 & \text{if}\quad (\lambda,\nu)=(0,0),(0,-4),(-4,0),\\ \mathbb{R}^2 & \text{if}\quad (\lambda,\nu)\neq(-\frac{3}{2},-\frac{1}{2}),(-\frac{1}{2},-2),(-1,-\frac{3}{2}),(-1,-2),\\ & (-\frac{1}{2},-\frac{3}{2}),(-\frac{3}{2},-1),(-\frac{3}{2},-\frac{3}{2}),(-\frac{3}{2},-2),\\ & (-2,-\frac{1}{2}),(-1,-1),(-2,-1),(-2,-\frac{3}{2})\\ \mathbb{R} & \text{otherwise.}\end{cases}$$

(vii) If $\mu-\lambda-\nu=6$, *then*

$$\mathrm{H}^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=\begin{cases}\mathbb{R}^3 & \text{if}\quad (\lambda,\nu)=(\frac{-5\pm\sqrt{19}}{2},0),(0,\frac{-5\pm\sqrt{19}}{2}),\\ & (\frac{\sqrt{19}-5}{2},\frac{-\sqrt{19}-5}{2}),(-\frac{\sqrt{19}-5}{2},\frac{\sqrt{19}-5}{2}),\\ \mathbb{R}^2 & \text{if}\quad (\lambda,\nu)\neq(-\frac{1}{2},-2),(-2,-\frac{1}{2}),(-\frac{1}{2},-\frac{5}{2}),\\ & (-\frac{5}{2},-\frac{1}{2}),(-1,-\frac{3}{2}),(-\frac{3}{2},-1),(-1,-2),\\ & (-\frac{5}{2},-1),(-1,-\frac{5}{2}),(-\frac{3}{2},-\frac{3}{2}),(-2,-\frac{3}{2}),\\ & (-\frac{3}{2},-2),(-\frac{3}{2},-\frac{5}{2}),(-\frac{5}{2},-\frac{3}{2}),(-2,-1),\\ & (-2,-2),(-2,-\frac{5}{2}),(-\frac{5}{2},-2),\\ \mathbb{R} & \text{otherwise.}\end{cases}$$

(viii) If $\mu-\lambda-\nu=7$, *then*

$$\mathrm{H}^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=\begin{cases}\mathbb{R}^2 & \text{if}\quad (\lambda,\nu)=(\frac{-5-\sqrt{19}}{2},\frac{-5-\sqrt{19}}{2}),(\frac{-5+\sqrt{19}}{2},\frac{-5+\sqrt{19}}{2}),\\ & (0,\nu),(\lambda,0),(\lambda,-6-\lambda),(\sqrt{19}-1,\frac{-\sqrt{19}-5}{2}),\\ & (\frac{-\sqrt{19}-5}{2},\sqrt{19}-1),(-\sqrt{19}-1,\frac{\sqrt{19}-5}{2}),\\ & (\frac{\sqrt{19}-5}{2},-\sqrt{19}-1),\\ \mathbb{R} & \text{if}\quad (\lambda,\nu)\neq(-\frac{1}{2},-3),(-3,-\frac{1}{2}),(-3,-\frac{5}{2}),\\ & (-\frac{5}{2},-3),(-1,-3),(-3,-1),(-1,-2),\\ & (-2,-1),(-2,-3),(-3,-2),(-2,-\frac{3}{2}),\\ & (-\frac{1}{2},-\frac{5}{2}),(-\frac{5}{2},-\frac{1}{2}),(-\frac{3}{2},-2),(-\frac{3}{2},-\frac{5}{2}),\\ & (-\frac{5}{2},-\frac{3}{2}),(-2,-2),(-2,-\frac{5}{2}),(-\frac{5}{2},-2),\\ 0 & \text{otherwise.}\end{cases}$$

(ix) If $\mu-\lambda-\nu\geq 8$ *but with* λ *and* ν *generic then*

$$\mathrm{H}^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=0.$$

Remark 3.2. The values $-\frac{2}{3}$ can be deduced from the work of Grozman [13] and the values $\sqrt{19}$ from the work of Feigin-Fuchs [10].

4. **Proof of Theorem 3.1**

To prove Theorem 3.1 we proceed bye following the three steps:

- We first study the dimension of the space of operators that satisfy the 1-cocycle condition.

- We then study all trivial 1-cocycles, namely, operators of the form $L_Y A_k^{\lambda,\nu}$ where $A_k^{\lambda,\nu}$ is a bilinear operator given by (4.2)
- Lastly, by taking into account Part 1 and Part 2 and depending on λ and ν-the dimension of the cohomology group $\mathrm{H}^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu})$ will be equal to

$$\dim(\text{operators that are } 1-\text{cocycles}) - \dim(\text{operators of the form } L_Y A_k^{\lambda,\nu}).$$

We need also the following two Lemmas:

Lemma 4.1. *Let Y be a vector fields in $\mathrm{Vect}(\mathbb{R})$ and let $c : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \to \mathcal{F}_\mu$ be a bilinear differential operator defined as follows. For all $f \in \mathcal{F}_\lambda$ and for all $g \in \mathcal{F}_\nu$:*

$$c(f,g) = \sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} f^{(j)} g^{(l)} \tag{4.1}$$

where $c_{i,j,l}$ are constants. Then, for all $X \in \mathrm{Vect}(\mathbb{R})$, we have

$$L_X^{\lambda,\nu;\mu} c(f,g) = \sum_{i+j+l=k+2} c_{i,j,l} X Y^{(i)} f^{(j)} g^{(l)} + \sum_{i+j+l=k+1} (\mu-\lambda-\nu-j-l) c_{i,j,l} X' Y^{(i)} f^{(j)} g^{(l)} -$$
$$\sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} (\sum_{s=2}^{j} C_j^s X^{(s)} f^{(j-s+1)}) g^{(l)} - \lambda \sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} (\sum_{s=1}^{j} C_j^s X^{(s+1)} f^{(j-s)}) g^{(l)} -$$
$$\sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} f^{(j)} (\sum_{t=2}^{l} C_l^t X^{(t)} g^{(l-t+1)}) - \nu \sum_{i+j+l=k+1} c_{i,j,l} Y^{(i)} f^{(j)} (\sum_{t=1}^{l} C_l^t X^{(t+1)} g^{(l-t)})$$

Proof. Straightforward computation. By using (2.3). □

Lemma 4.2. *Let $A_k^{\lambda,\nu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\nu \to \mathcal{F}_\mu$ be a bilinear differential operator defined as follows. For all $f \in \mathcal{F}_\lambda$ and for all $g \in \mathcal{F}_\nu$:*

$$A_k^{\lambda,\nu}(f,g) = \sum_{i+j=k} a_{i,j} f^{(i)} g^{(j)} \tag{4.2}$$

where $a_{i,j}$ are constants. Then, for all $Y \in \mathrm{Vect}(\mathbb{R})$, we have

$$L_Y^{\lambda,\nu;\mu} A_k^{\lambda,\nu}(f,g) = \sum_{i+j=k} (\mu-\lambda-\nu-i-j) a_{i,j} Y' f^{(i)} g^{(j)} - \sum_{i+j=k} a_{i,j} (\sum_{s=2}^{i} (C_i^s +$$
$$\lambda C_i^{s-1}) Y^{(s)} f^{(i-s+1)}) g^{(l)} - \sum_{i+j=k} a_{i,j} f^{(i)} (\sum_{t=2}^{j} (C_j^t + \nu C_j^{t-1}) Y^{(t)} g^{(j-t+1)})$$

Proof. Straightforward computation. By using (2.3). □

Any 1-cocycle on Vect(ℝ) should retains the following general form:

$$c(f,g)=\sum_{i+j+l=k+1} c_{i,j,l}Y^{(i)}f^{(j)}g^{(l)}, \tag{4.3}$$

where $c_{i,j,l}$ are constants. The fact that this 1-cocycle implies that

$$c_{0,j,l}=0$$

the 1-cocycle condition reads as follows: for all $f\in\mathcal{F}_\lambda$, for all $g\in\mathcal{F}_\nu$ and for all $X,Y\in\mathrm{Vect}(\mathbb{R})$, we have

$$c([X,Y],f,g)-L_X^{\lambda,\nu;\mu}c(Y,f,g)+L_Y^{\lambda,\nu;\mu}c(X,f,g)=0.$$

(i) The case when $\mu-\lambda-\nu=0$

$$c(f,g)=c_{1,0,0}Y'fg$$

By a straightforward computation, the 1-cocycle condition above reads

$$\mu-\lambda-\nu=0$$

(ii) The case when $\mu-\lambda-\nu=1$

$$c(f,g)=c_{1,1,0}Y'f'g+c_{1,0,1}Y'fg'+c_{2,0,0}Y''fg$$

Again, by a straightforward computation, the above 1-cocycle condition reads

$$\mu-\lambda-\nu=1 \text{ and } c_{1,1,0}\lambda+c_{1,0,1}\nu=0$$

Let us study now the triviality of this 1-cocycle. A straightforward computation shows that

$$L_YA_1^{\lambda,\nu}=-(a_{1,0}\lambda+a_{0,1}\nu)Y''fg$$

(1) If $(\lambda,\nu)=(0,0)$, then

$$\mathrm{dim}Z^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=3$$

$$\mathrm{dim}B^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=0$$

(2) If $\lambda=0$ and $\nu\neq0$, then $c_{1,1,0}=0$

$$\mathrm{dim}Z^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=2$$

$\mathrm{dim}B^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=1$, generated by

$$\Lambda=-a_{0,1}\nu Y''fg.$$

The results hold true for ($\lambda\neq0$ together with $\nu=0$) and $(\lambda,\nu)\neq(0,0)$.

(iii) The case when $\mu-\lambda-\nu=2$

$$\begin{array}{rcl} c(f,g) & = & c_{1,2,0}Y'f''g+c_{1,1,1}Y'f'g'+c_{1,0,2}Y'fg'' \\ & + & c_{2,1,0}Y''f'g+c_{2,0,1}Y''fg'+c_{3,0,0}Y'''fg \end{array}$$

Thanks to a straightforward computation, the above 1-cocycle condition reads

$$\mu-\lambda-\nu=2 \text{ and } \left\{ \begin{array}{c} c_{1,2,0}(1+2\lambda)+c_{1,1,1}\nu=0 \\ c_{1,2,0}\lambda+c_{1,0,2}\nu=0 \\ c_{1,1,1}\lambda+c_{1,0,2}(1+2\nu)=0 \end{array} \right.$$

Let us study the triviality of this 1-cocycle. a straightforward computation shows that

$$L_X A_2^{\lambda,\nu}=-(a_{2,0}(1+2\lambda)+a_{1,1}\nu)Y''f'g-(a_{1,1}\lambda+a_{0,2}(1+2\nu))Y''fg'-(a_{2,0}\lambda+a_{0,2}\nu)Y'''fg$$

(1) If $(\lambda,\nu)=(0,0)$, then $c_{1,2,0}=c_{1,0,2}=0$

$$\dim Z^1_{\text{diff}}(\text{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=4$$

$\dim B^1_{\text{diff}}(\text{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=2$, generated by

$$\Lambda_1=-a_{2,0}Y''f'g,$$

$$\Lambda_2=-a_{0,2}Y''fg'.$$

(2) If $(\lambda,\nu)=(0,-1)$, then $c_{1,2,0}=c_{1,0,2}$ and $c_{1,0,2}=0$

$$\dim Z^1_{\text{diff}}(\text{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=4$$

$\dim B^1_{\text{diff}}(\text{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=3$, generated by

$$\begin{array}{l} \Lambda_1=-a_{2,0}Y''f'g, \\ \Lambda_2=a_{1,1}Y''f'g, \\ \Lambda_3=-a_{0,2}(Y''fg'+Y'''fg). \end{array}$$

The results hold true for ($\lambda\neq 0$ together with $\nu=0$), ($\lambda=0$ together with $\nu\neq 0$) and $(\lambda,\nu)=(\lambda,-1-\lambda)$.

(3) for other cases for λ and ν, then $c_{1,2,0}=c_{1,1,1}=c_{1,0,2}=0$. Thus, the 1-cocycle is trivial.

(iv) The case when $\mu-\lambda-\nu=3$

$$\begin{array}{rcl} c(f,g) & = & c_{1,0,3}Y'fg'''+c_{1,1,2}Y'f'g''+c_{1,2,1}Y'f''g'+c_{1,3,0}Y'f'''g+c_{2,0,2}Y''fg'' \\ & + & c_{2,1,1}Y''f'g'+c_{2,2,0}Y''f''g+c_{3,1,0}Y'''f'g+c_{3,0,1}Y'''fg'+c_{4,0,0}Y^{(4)}fg \end{array}$$

By a straightforward computation, the above 1-cocycle condition reads
$\mu-\lambda-\nu=3$ and

$$\begin{array}{ll} c_{1,0,3}(3+3\nu)+c_{1,1,2}\lambda=0, & c_{1,0,3}(1+3\nu)+c_{1,2,1}\lambda=0, \\ c_{1,0,3}\nu+c_{1,3,0}\lambda=0, & c_{1,1,2}(1+2\nu)+c_{1,2,1}(1+2\lambda)=0, \\ c_{1,1,2}\nu+c_{1,3,0}(1+3\lambda)=0, & c_{1,2,1}\nu+c_{1,3,0}(3+3\lambda)=0, \\ c_{2,0,2}\nu+c_{2,2,0}\lambda=0, & c_{3,0,1}\nu+c_{3,1,0}\lambda+2c_{4,0,0}=0. \end{array}$$

Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{array}{rcl} L_X A_3^{\lambda,\nu} = & - & (a_{0,3}(3+3\nu)+a_{1,2}\lambda)Y''fg'' - (a_{1,2}(1+2\nu)+a_{2,1}(1+2\lambda))Y''f'g' \\ & - & (a_{2,1}\nu+a_{3,0}(3+3\lambda))Y''f''g - (a_{1,2}\nu+a_{3,0}(1+3\lambda))Y'''f'g \\ & - & (a_{0,3}(1+3\nu)+a_{2,1}\lambda)Y'''fg' - (a_{0,3}\nu+a_{3,0}\lambda)Y^{(4)}fg \end{array}$$

(1) If $(\lambda,\nu)=(0,0)$, then $c_{1,0,3}=c_{1,3,0}=c_{4,0,0}=0$ and $c_{1,1,2}=-c_{1,2,1}$

$$\dim Z^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=6$$

$\dim B^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=3$, generated by

$$\begin{array}{l} \Lambda_1=-a_{0,3}(3Y''fg''+Y'''fg'), \\ \Lambda_2=-a_{1,2}Y''f'g', \\ \Lambda_3=-a_{2,1}Y''f'g'. \end{array}$$

(2) If $(\lambda,\nu)=(-2,0)$, then $c_{1,0,3}=c_{2,0,2}=0$, $c_{1,3,0}=2c_{1,1,2}$ and $c_{4,0,0}=c_{3,1,0}$

$$\dim Z^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=5$$

$\dim B^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=3$, generated by

$$\begin{array}{l} \Lambda_1=-a_{0,3}(3Y''fg''+Y'''fg'), \\ \Lambda_2=-a_{1,2}(-2Y''fg''+Y''f'g'), \\ \Lambda_3=a_{2,1}(3Y''f'g'+2Y'''fg'). \end{array}$$

The results hold true for $(0,-2)$ and $(-\frac{2}{3},-\frac{2}{3})$.

(3) If λ and ν are not like above, then $c_{1,0,3}=c_{1,1,2}=c_{1,2,1}=c_{1,3,0}=0$, $c_{2,0,2}=-\frac{\lambda}{\nu}c_{2,2,0}$ and $c_{4,0,0}=\frac{1}{2}-\nu c_{3,0,1}-\lambda c_{3,1,0}$

$$\dim Z^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=4$$

$\dim B^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=3$, generated by

$$\Lambda_1=a_{0,3}\big(-\tfrac{(1+2\nu)(3+3\nu)}{\lambda}Y''f'g'-\tfrac{\nu(3+3\nu)}{\lambda}Y'''f'g+(1+3\nu)Y'''fg'+\nu Y^{(4)}fg\big),$$

$$\Lambda_2=a_{2,1}\big(-\lambda Y''fg''+2(\lambda-\nu)Y''f'g'+\nu Y''f''g-\nu Y'''f'g+\lambda Y'''fg'\big),$$

$$\Lambda_3=-a_{3,0}\big(-\tfrac{\lambda(3+3\lambda)}{\nu}Y''fg''-\tfrac{(1+2\nu)(3+3\lambda)}{\nu}Y''f'g'+(3+3\lambda)Y''f''g+(1+3\lambda)Y'''f'g+\lambda Y^{(4)}fg\big).$$

(v) The case when $\mu-\lambda-\nu=4$

$$\begin{array}{rcl} c(f,g) & = & c_{1,4,0}Y'f^{(4)}g+c_{1,3,1}Y'f'''g'+c_{1,2,2}Y'f''g''+c_{1,1,3}Y'f'g'''+c_{1,0,4}Y'fg^{(4)} \\ & + & c_{2,3,0}Y''f'''g+c_{2,2,1}Y''f''g'+c_{2,1,2}Y''f'g''+c_{2,0,3}Y''fg'''+c_{3,2,0}Y'''f''g \\ & + & c_{3,1,1}Y'''f'g'+c_{3,0,2}Y'''fg''+c_{4,1,0}Y^{(4)}f'g+c_{4,0,1}Y^{(4)}fg'+c_{5,0,0}Y^{(5)}fg \end{array}$$

By a straightforward computation, the above 1-cocycle condition reads $\mu-\lambda-\nu=4$ and

$$\begin{array}{ll}
c_{1,4,0}(6+4\lambda)+c_{1,3,1}\nu=0, & c_{1,4,0}(4+6\lambda)+c_{1,2,2}\nu=0,\\
c_{1,4,0}(1+4\lambda)+c_{1,1,3}\nu=0, & c_{1,4,0}\lambda+c_{1,0,4}\nu=0,\\
c_{1,3,1}(3+3\lambda)+c_{1,2,2}(1+2\nu)=0, & c_{1,3,1}(1+3\lambda)+c_{1,1,3}(1+3\nu)=0,\\
c_{1,3,1}\lambda+c_{1,0,4}(1+4\nu)=0, & c_{1,2,2}(1+2\lambda)+c_{1,1,3}(3+3\nu)=0,\\
c_{1,2,2}\lambda+c_{1,0,4}(4+6\nu)=0, & c_{1,1,3}\lambda+c_{1,0,4}(6+4\nu)=0,\\
c_{2,3,0}(1+3\lambda)+c_{2,1,2}\nu=0, & c_{2,3,0}\lambda+c_{2,0,3}\nu=0,\\
c_{2,2,1}\lambda+c_{2,0,3}(1+3\nu)=0, & c_{3,2,0}(1+2\lambda)+c_{3,1,1}\nu+2c_{4,1,0}=0,\\
c_{3,1,1}\lambda+c_{3,0,2}(1+2\nu)+2c_{4,0,1}=0, & c_{4,1,0}\lambda+c_{4,0,1}\nu+5c_{5,0,0}=0.
\end{array}$$

The space of solutions of the above system is five-dimentional for $(\lambda,\nu)=(0,0),(\lambda,0),(0,\nu)$, and four-dimentional otherwise.
Let us study the triviality of this 1-cocycle.A straightforward computation shows that

$$\begin{array}{rcl}
L_XA_4^{\lambda,\nu} & = & -\ (a_{4,0}(6+4\lambda)+a_{3,1}\nu)Y''f'''g-(a_{3,1}(3+3\lambda)+a_{2,2}(1+2\nu))Y''f''g'\\
& - & (a_{2,2}(1+2\lambda)+a_{1,3}(3+3\nu))Y''f'g''-(a_{1,3}\lambda+a_{0,4}(6+4\nu))Y''fg'''\\
& - & (a_{4,0}(4+6\lambda)+a_{2,2}\nu)Y'''f''g-(a_{3,1}(1+3\lambda)+a_{1,3}(1+3\nu))Y'''f'g'\\
& - & (a_{2,2}\lambda+a_{0,4}(4+6\nu))Y'''fg''-(a_{4,0}(1+4\lambda)+a_{1,3}\nu)Y^{(4)}f'g\\
& - & (a_{3,1}\lambda+a_{0,4}(1+4\nu))Y^{(4)}fg'-(a_{4,0}\lambda+a_{0,4}\nu)Y^{(4)}fg
\end{array}$$

(1) If $(\lambda,\nu)=(0,0)$, then $c_{1,4,0}=c_{1,3,1}=c_{1,2,2}=c_{1,1,3}=c_{1,0,4}=c_{2,3,0}=c_{2,0,3}=c_{5,0,0}=0$, $c_{3,2,0}=-2c_{4,1,0}$ and $c_{3,0,2}=-2c_{4,0,1}$

$$\dim Z^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=5$$

$\dim B^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=3$, generated by

$$\begin{array}{l}
\Lambda_1=-a_{3,1}(3Y''f''g'+Y'''f'g'),\\
\Lambda_2=-a_{2,2}(Y''f''g'+Y''f'g''),\\
\Lambda_3=-a_{1,3}(3Y''f'g''+Y'''f'g').
\end{array}$$

The results hold true for $(\lambda,\nu)=(\lambda,0),(0,\nu)$.

(2) for the other values of λ and ν, then the space $B^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})$ is two-dimentional.
For example, if $(\lambda,\nu)=(-\frac{1}{2},-\frac{1}{2})$, then $c_{1,4,0}=c_{1,3,1}=c_{1,2,2}=c_{1,1,3}=c_{1,0,4}=0$, $c_{2,2,1}=-c_{2,0,3}$, $c_{2,1,2}=c_{2,0,3}$, $c_{4,1,0}=\frac{1}{4}c_{3,1,1}$, $c_{4,0,1}=\frac{1}{4}c_{3,1,1}$ and $c_{5,0,0}=\frac{1}{2,0}c_{3,1,1}$

$$\dim Z^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=4$$

$\dim B^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=2$, generated by

$$\Lambda_1=a_{4,0}(-3Y''f'''g-3Y''f''g'+3Y''f'g''+3Y''fg'''-Y'''f''g+Y'''fg''-Y^{(4)}f'g),$$
$$\Lambda_2=\tfrac{1}{2}a_{3,1}(Y'''f''g+Y'''fg'')$$

(vi) The case when $\nu-\lambda-\nu=5$

$$\begin{array}{rcl} c(f,g) &=& c_{1,5,0}Y'f^{(5)}g+c_{1,4,1}Y'f^{(4)}g'+c_{1,3,2}Y'f'''g''+c_{1,2,3}Y'f''g'''+c_{1,1,4}Y'f'g^{(4)}\\ &+& c_{1,0,5}Y'fg^{(5)}+c_{2,4,0}Y''f^{(4)}g+c_{2,3,1}Y''f'''g'+c_{2,2,2}Y''f''g''+c_{2,1,3}Y''f'g'''\\ &+& c_{2,0,4}Y''fg^{(4)}+c_{3,3,0}Y'''f'''g+c_{3,2,1}Y'''f''g'+c_{3,1,2}Y'''f'g''+c_{3,0,3}Y'''fg'''\\ &+& c_{4,2,0}Y^{(4)}f''g+c_{4,1,1}Y^{(4)}f'g'+c_{4,0,2}Y^{(4)}fg''+c_{5,1,0}Y^{(5)}f'g\\ &+& c_{5,0,1}Y^{(5)}fg'+c_{6,0,0}Y^{(6)}fg \end{array}$$

A straightforward computation shows that the above 1-cocycle condition reads
$\mu-\lambda-\nu=5$ and

$$\begin{array}{ll}
c_{1,5,0}(10+5\lambda)+c_{1,4,1}\nu=0, & c_{1,5,0}(10+10\lambda)+c_{1,3,2}\nu=0,\\
c_{1,5,0}(5+10\lambda)+c_{1,2,3}\nu=0, & c_{1,5,0}(1+5\lambda)+c_{1,1,4}\nu=0,\\
c_{1,5,0}\lambda+c_{1,0,5}\nu=0, & c_{1,4,1}(6+4\lambda)+c_{1,3,2}(1+2\nu)=0,\\
c_{1,4,1}(4+6\lambda)+c_{1,2,3}(1+3\nu)=0, & c_{1,4,1}(1+4\lambda)+c_{1,1,4}(1+4\nu)=0,\\
c_{1,4,1}\lambda+c_{1,0,5}(1+5\nu)=0, & c_{1,3,2}(3+3\lambda)+c_{1,2,3}(3+3\nu)=0,\\
c_{1,3,2}(1+3\lambda)+c_{1,1,4}(4+6\nu)=0, & c_{1,3,2}\lambda+c_{1,0,5}(5+10\nu)=0,\\
c_{1,2,3}(1+2\lambda)+c_{1,1,4}(6+4\nu)=0, & c_{1,2,3}\lambda+c_{1,0,5}(10+10\nu)=0,\\
c_{1,1,4}\lambda+c_{1,0,5}(10+5\nu)=0, & \\
c_{2,4,0}(4+6\lambda)+c_{2,2,2}\nu=0, & c_{2,4,0}(1+4\lambda)+c_{2,1,3}\nu=0,\\
c_{2,4,0}\lambda+c_{2,0,4}\nu=0, & c_{2,3,1}(1+3\lambda)+c_{2,1,3}(1+3\nu)=0,\\
c_{2,3,1}\lambda+c_{2,0,4}(1+4\nu)=0, & c_{2,3,1}\lambda+c_{2,0,4}(4+6\nu)=0,\\
(c_{4,2,0}-c_{3,3,0})\lambda+(c_{4,0,2}-c_{3,0,3})\nu+5c_{6,0,0}=0, & c_{3,3,0}(3+3\lambda)+c_{3,2,1}\nu+2c_{4,2,0}=0,\\
c_{3,2,1}(1+2\lambda)+c_{3,1,2}(1+2\nu)+2c_{4,1,1}=0, & c_{3,1,2}\lambda+c_{3,0,3}(3+3\nu)+2c_{4,0,2}=0,\\
c_{4,2,0}(1+2\lambda)+c_{4,1,1}\nu+5c_{5,1,0}=0, & \\
c_{4,1,1}\lambda+c_{4,0,2}(1+2\nu)+5c_{5,0,1}=0, & c_{5,1,0}\lambda+c_{5,0,1}\nu+9c_{6,0,0}=0.
\end{array}$$

The space of solutions of the above system is six-dimentional for $(\lambda,\nu)=(0,0)$, five-dimentional for $(\lambda,\nu)=(0,-2),(-2,0),(0,-4),(-4,0)$, four-dimentional for $(\lambda,\nu)=(-\frac{2}{3},-\frac{2}{3}),(-2,-2),(0,\nu),(\lambda,0)$, and three-dimentional otherwise. Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$\begin{array}{rcl} L_XA_5^{\lambda,\nu} &=& -\ (a_{5,0}(10+5\lambda)+a_{4,1}\nu)Y''f^{(4)}g-(a_{4,1}(6+4\lambda)+a_{3,2}(1+2\nu))Y''f'''g'\\ &&-\ (a_{3,2}(3+3\lambda)+a_{2,3}(3+3\nu))Y''f''g''-(a_{2,3}(1+2\lambda)+a_{1,4}(6+4\nu))Y''f'g'''\\ &&-\ (a_{1,4}\lambda+a_{0,5}(10+5\nu))Y''fg^{(4)}-(a_{5,0}(10+10\lambda)+a_{3,2}\nu)Y'''f'''g\\ &&-\ (a_{4,1}(4+6\lambda)+a_{2,3}(1+3\nu))Y'''f''g'-(a_{3,2}(1+3\lambda)+a_{1,4}(4+6\nu))Y'''f'g''\\ &&-\ (a_{2,3}\lambda+a_{0,5}(10+10\nu))Y'''fg'''-(a_{5,0}(5+10\lambda)+a_{2,3}\nu)Y^{(4)}f''g\\ &&-\ (a_{4,1}(1+4\lambda)+a_{1,4}(1+4\nu))Y^{(4)}f'g'-(a_{3,2}\lambda+a_{0,5}(5+10\nu))Y^{(4)}fg''\\ &&-\ (a_{5,0}(1+5\lambda)+a_{1,4}\nu)Y^{(5)}f'g-(a_{4,1}\lambda+a_{0,5}(1+5\nu))Y^{(5)}fg'\\ &&-\ (a_{5,0}\lambda+a_{0,5}\nu)Y^{(6)}fg \end{array}$$

(1) If $(\lambda,\nu)=(0,0)$, then $c_{1,5,0}=c_{1,4,1}=c_{1,3,2}=c_{1,2,3}=c_{1,1,4}=c_{1,0,5}=c_{2,4,0}=c_{2,0,4}=c_{6,0,0}=0$, $c_{2,3,1}=-c_{2,1,3}$, $c_{3,3,0}=-\frac{1,0}{3}c_{5,1,0}$, $c_{3,2,1}=-c_{3,1,2}-2c_{4,1,1}$, $c_{3,0,3}=-\frac{1,0}{3}c_{5,0,1}$, $c_{4,2,0}=-5c_{5,1,0}$, $c_{4,0,2}=-5c_{5,0,1}$

$$\dim Z^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=6$$

$\dim B^1_{\text{diff}}(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = 3$, generated by

$$\begin{aligned}
\Lambda_1 &= -a_{4,1}(-6Y''f'''g' + 6Y''f'g''' - 4Y'''f''g' + 4Y'''f'g''),\\
\Lambda_2 &= -a_{3,2}(-Y''f'''g' - 3Y''f''g'' + Y''f'g''' - \tfrac{1}{3}Y'''f'g'' + \tfrac{1}{6}Y^{(4)}f'g'),\\
\Lambda_3 &= -a_{2,3}(-3Y''f''g'' - Y'''f''g' + \tfrac{2}{3}Y'''f'g'' + \tfrac{1}{6}Y^{(4)}f'g').
\end{aligned}$$

(2) If $(\lambda, \nu) = (0, -4)$, then the constant $a_{2,3}$ and $a_{1,4}$ are arbitrary. On the Other hand,

$$\begin{aligned}
a_{0,5} &= 0\\
a_{5,0} &= \tfrac{1,4,2}{45}a_{2,3} - \tfrac{158}{9}a_{1,4}\\
a_{4,1} &= \tfrac{62}{9}a_{2,3} - \tfrac{3,0,5}{9}a_{1,4}\\
a_{3,2} &= \tfrac{1,3}{3}a_{2,3} - \tfrac{4,0}{3}a_{1,4}
\end{aligned}$$

$$\dim B^1_{\text{diff}}(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = 2$$

The results hold true for $(\lambda, \nu) = (-4, 0)$.

(3) If $(\lambda, \nu) = (0, -2)$, then the constant $a_{2,3}$, $a_{1,4}$ and $a_{0,5}$ are arbitrary. On the other hand,

$$\begin{aligned}
a_{5,0} &= \tfrac{3}{5}a_{2,3} - a_{1,4}\\
a_{4,1} &= -2a_{2,3} - 3a_{1,4}\\
a_{3,2} &= \tfrac{7}{3}a_{2,3} - \tfrac{8}{3}a_{1,4}\\
a_{0,5}
\end{aligned}$$

$$\dim B^1_{\text{diff}}(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = 2$$

The results hold true for $(\lambda, \nu) = (-2, 0)$.

(4) If $(\lambda, \nu) = (0, \nu)$, then the constant $a_{2,3}$ and $a_{1,4}$ are arbitrary. On the other hand,

$$\begin{aligned}
a_{0,5} &= 0\\
a_{3,2} &= \tfrac{1-3\nu}{3}a_{2,3} + \tfrac{24+16\nu}{3}a_{1,4}\\
a_{4,1} &= \tfrac{3\nu^2-4\nu-2}{9}a_{2,3} + \tfrac{(17\nu+7)(2\nu+3)}{9}a_{1,4}\\
a_{5,0} &= \tfrac{-3\nu^2-5\nu+2}{90}a_{2,3} + \tfrac{(-\nu+7)(2\nu+3)}{90}a_{1,4}
\end{aligned}$$

$$\dim B^1_{\text{diff}}(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = 2$$

The results hold true for $(\lambda, \nu) = (\lambda, 0)$.

(5) If $(\lambda, \nu) = (-1, -1)$, then the constant $a_{3,2}$ and $a_{0,5}$ are arbitrary. On the other hand,

$$\begin{aligned}
a_{5,0} &= \tfrac{1}{1,0}a_{3,2}\\
a_{4,1} &= \tfrac{1}{2}a_{3,2}\\
a_{2,3} &= 10a_{0,5}\\
a_{1,4} &= 5a_{0,5}
\end{aligned}$$

$$\dim B^1_{\text{diff}}(\text{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu}) = 2$$

The results hold true for $(\lambda, \nu) = (-\frac{3}{2}, -\frac{1}{2}), (-\frac{1}{2}, -2), (-1, -\frac{3}{2}), (-1, -2),$ $(-\frac{1}{2}, -\frac{3}{2}), (-\frac{3}{2}, -1), (-\frac{3}{2}, -\frac{3}{2}), (-\frac{3}{2}, -2), (-2, -\frac{1}{2}), (-2, -1), (-2, -\frac{3}{2})$.

(6) If λ and ν are not like above, then the space $B^1_{\mathrm{diff}}(\mathrm{Vect}(\mathbb{R}), \mathcal{D}_{\lambda,\nu;\mu})$ is one-dimentional.
For example, if $(\lambda, \nu) = (-\frac{1}{2}, -\frac{1}{2})$, then the constant $a_{2,3}$ is arbitrary.
On the other hand,

$$a_{5,0} = a_{4,1} = a_{1,4} = a_{0,5} = 0$$
$$a_{3,2} = -a_{2,3}$$

(vii) The case when $\mu - \lambda - \nu = 6$

$$\begin{array}{rcl} c(f,g) &=& c_{1,6,0}Y'f^{(6)}g + c_{1,5,1}Y'f^{(5)}g' + c_{1,4,2}Y'f^{(4)}g'' + c_{1,3,3}Y'f'''g''' + c_{1,2,4}Y'f''g^{(4)} \\ &+& c_{1,1,5}Y'f'g^{(5)} + c_{1,0,6}Y'fg^{(6)} + c_{2,5,0}Y''f^{(5)}g + c_{2,4,1}Y''f^{(4)}g' + c_{2,3,2}Y''f'''g'' \\ &+& c_{2,2,3}Y''f''g''' + c_{2,1,4}Y''f'g^{(4)} + c_{2,0,5}Y''fg^{(5)} + c_{3,4,0}Y'''f^{(4)}g + c_{3,3,1}Y'''f'''g' \\ &+& c_{3,2,2}Y'''f''g'' + c_{3,1,3}Y'''f'g''' + c_{3,0,4}Y'''fg^{(4)} + c_{4,3,0}Y^{(4)}f'''g + c_{4,2,1}Y^{(4)}f''g' \\ &+& c_{4,1,2}Y^{(4)}fg'' + c_{4,0,3}Y^{(4)}fg''' + c_{5,2,0}Y^{(5)}f''g + c_{5,1,1}Y^{(5)}f'g' + c_{5,0,2}Y^{(5)}fg'' \\ &+& c_{6,1,0}Y^{(6)}f'g + c_{6,0,1}Y^{(6)}fg' + c_{7,0,0}Y^{(7)}fg \end{array}$$

A straightforward computation shows that the above 1-cocycle condition reads
$\mu - \lambda - \nu = 6$ and

$c_{1,6,0}(15+6\lambda) + c_{1,5,1}\nu = 0,$
$c_{1,6,0}(20+15\lambda) + c_{1,4,2}\nu = 0,$
$c_{1,6,0}(15+20\lambda) + c_{1,3,3}\nu = 0,$
$c_{1,6,0}(6+15\lambda) + c_{1,2,4}\nu = 0,$
$c_{1,6,0}(1+6\lambda) + c_{1,1,5}\nu = 0,$
$c_{1,6,0}\lambda + c_{1,0,6}\nu = 0,$
$c_{1,5,1}(10+5\lambda) + c_{1,4,2}(1+2\nu) = 0,$
$c_{1,5,1}(10+10\lambda) + c_{1,3,3}(1+3\nu) = 0,$
$c_{1,5,1}(5+10\lambda) + c_{1,2,4}(1+4\nu) = 0,$
$c_{1,5,1}(1+5\lambda) + c_{1,1,5}(1+5\nu) = 0,$
$c_{1,5,1}\lambda + c_{1,0,6}(1+6\nu) = 0,$
$c_{1,4,2}(6+4\lambda) + c_{1,3,3}(3+3\nu) = 0,$
$c_{1,4,2}(4+6\lambda) + c_{1,2,4}(4+6\nu) = 0,$
$c_{1,4,2}(1+4\lambda) + c_{1,1,5}(5+10\nu) = 0,$
$c_{1,4,2}\lambda + c_{1,0,6}(6+15\nu) = 0,$
$c_{1,3,3}(3+3\lambda) + c_{1,2,4}(6+4\nu) = 0,$
$c_{1,3,3}(1+3\lambda) + c_{1,1,5}(10+10\nu) = 0,$
$c_{1,3,3}\lambda + c_{1,0,6}(15+20\nu) = 0,$
$c_{1,2,4}(1+2\lambda) + c_{1,1,5}(10+5\nu) = 0,$
$c_{1,2,4}\lambda + c_{1,0,6}(20+15\nu) = 0,$
$c_{1,1,5}\lambda + c_{1,0,6}(15+6\nu) = 0,$
$c_{2,5,0}(10+10\lambda) + c_{2,3,2}\nu = 0,$
$c_{2,5,0}(5+10\lambda) + c_{2,2,3}\nu = 0,$
$c_{2,5,0}(1+5\lambda) + c_{2,1,4}\nu = 0,$
$c_{2,5,0}\lambda + c_{2,0,5}\nu = 0,$
$c_{2,4,1}(4+6\lambda) + c_{2,2,3}(1+3\nu) = 0,$
$c_{2,4,1}(1+4\lambda) + c_{2,1,4}(1+4\nu) = 0,$
$c_{2,4,1}\lambda + c_{2,0,5}(1+5\nu) = 0,$
$c_{2,3,2}(1+3\lambda) + c_{2,1,4}(4+6\nu) = 0,$
$c_{2,3,2}\lambda + c_{2,0,5}(5+10\nu) = 0,$
$c_{2,2,3}\lambda + c_{2,0,5}(10+10\nu) = 0,$
$c_{3,4,0}(6+4\lambda) + c_{3,3,1}\nu + 2c_{4,3,0} = 0,$
$c_{3,3,1}(3+3\lambda) + c_{3,2,2}(1+2\nu) + 2c_{4,2,1} = 0,$
$c_{6,1,0}\lambda + c_{6,0,1}\nu + 14c_{7,0,0} = 0,$
$c_{3,2,2}(1+2\lambda) + c_{3,1,3}(3+3\nu) + 2c_{4,1,2} = 0,$
$c_{3,1,3}\lambda + c_{3,0,4}(6+4\nu) + 2c_{4,0,3} = 0,$
$c_{4,3,0}(1+3\lambda) - c_{3,4,0}(1+4\lambda) + (c_{4,1,2} - c_{3,1,3})\nu + 5c_{6,1,0} = 0,$
$c_{4,3,0}(3+3\lambda) + c_{4,2,1}\nu + 5c_{4,0,3} = 0,$
$(c_{3,3,1} - c_{4,2,1})\lambda - c_{3,0,4}(1+4\nu) + c_{4,0,3}(1+3\nu) + 5c_{6,0,1} = 0,$
$c_{4,1,2}\lambda + c_{4,0,3}(3+3\nu) + 5c_{5,0,2} = 0,$
$c_{4,2,1}(1+2\lambda) + c_{4,1,2}(1+2\nu) + 5c_{5,1,1} = 0,$
$c_{5,2,0}(1+2\lambda) + c_{5,1,1}\nu + 9c_{6,1,0} = 0,$
$c_{5,2,0}\lambda + c_{5,0,2}\nu - c_{3,4,0}\lambda - c_{3,0,4}\nu + 14c_{7,0,0} = 0,$
$c_{5,1,1}\lambda + c_{5,0,2}(1+2\nu) + 9c_{6,0,1} = 0.$

The space of solutions of the above system is four-dimentional for $(\lambda, \nu) = (0,0), (-\frac{5}{2}, -\frac{5}{2}), (-\frac{5}{2}, 0), (0, -\frac{5}{2}), (0, \frac{-5\pm\sqrt{19}}{2}), (\frac{-5\pm\sqrt{19}}{2}, 0), (\frac{-5+\sqrt{19}}{2}, \frac{-5-\sqrt{19}}{2}), (\frac{-5-\sqrt{19}}{2}, \frac{-5+\sqrt{19}}{2})$, and three-dimentional otherwise. Let us study the triviality of this 1-cocycle.

A straightforward computation shows that

$$\begin{array}{rcl} L_X A_6^{\lambda,\nu} & = & -\ (a_{6,0}(15+6\lambda)+a_{5,1}\nu)Y''f^{(5)}g-(a_{5,1}(10+5\lambda)+a_{4,2}(1+2\nu))Y''f^{(4)}g' \\ & & -\ (a_{4,2}(6+4\lambda)+a_{3,3}(3+3\nu))Y''f'''g''-(a_{3,3}(3+3\lambda)+a_{2,4}(6+4\nu))Y''f''g''' \\ & & -\ (a_{2,4}(6+4\lambda)+a_{1,5}(10+5\nu))Y''f'g^{(4)}-(a_{1,5}\lambda+a_{0,6}(15+6\nu))Y''fg^{(5)} \\ & & -\ (a_{6,0}(20+15\lambda)+a_{4,2}\nu)Y'''f^{(4)}g-(a_{5,1}(10+10\lambda)+a_{3,3}(1+3\nu))Y'''f'''g' \\ & & -\ (a_{4,2}(4+6\lambda)+a_{2,4}(4+6\nu))Y'''f''g''-(a_{3,3}(1+3\lambda)+a_{1,5}(10+10\nu))Y'''f'g''' \\ & & -\ (a_{2,4}\lambda+a_{0,6}(20+15\nu))Y'''fg^{(4)}-(a_{3,3}(15+20\lambda)+a_{6,0}\nu)Y^{(4)}f'''g \\ & & -\ (a_{5,1}(5+10\lambda)+a_{2,4}(1+4\nu))Y^{(4)}f''g'-(a_{1,5}(1+4\lambda)+a_{4,2}(5+10\nu))Y^{(4)}f'g'' \\ & & -\ (a_{3,3}\lambda+a_{0,6}(15+20\nu))Y^{(4)}fg'''-(a_{2,4}(6+15\lambda)+a_{6,0}\nu)Y^{(5)}f''g \\ & & -\ (a_{1,5}(1+5\lambda)+a_{5,1}(1+5\nu))Y^{(5)}f'g'-(a_{4,2}\lambda+a_{0,6}(6+15\nu))Y^{(5)}fg'' \\ & & -\ (a_{1,5}(1+6\lambda)+a_{6,0}\nu)Y^{(6)}f'g-(a_{5,1}\lambda+a_{0,6}(1+6\nu))Y^{(6)}fg' \\ & & -\ (a_{6,0}\lambda+a_{0,6}\nu)Y^{(7)}fg \end{array}$$

(1) If $(\lambda,\nu)=(0,\frac{-5-\sqrt{19}}{2})$, then $c_{1,6,0}=c_{1,5,1}=c_{1,4,2}=c_{1,3,3}=c_{1,2,4}=c_{1,1,5}=c_{1,0,6}=c_{2,5,0}=c_{2,4,1}=c_{2,3,2}=c_{2,2,3}=c_{2,1,4}=c_{2,0,5}=0$,
$c_{3,4,0}=(\frac{101+23\sqrt{19}}{1,0})c_{3,1,3}-\frac{5+\sqrt{19}}{1,5}c_{4,2,1}+\frac{25+6\sqrt{19}}{1,5}c_{4,1,2}$,
$c_{6,1,0}=(\frac{38+9\sqrt{19}}{100})c_{3,1,3}-\frac{5+\sqrt{19}}{5,0}c_{4,2,1}+\frac{40+9\sqrt{19}}{75}c_{4,1,2}$,
$c_{5,2,0}=-(\frac{9(38+9\sqrt{19})}{100})c_{3,1,3}+\frac{2(5+\sqrt{19})}{2,5}c_{4,2,1}-\frac{9(5+\sqrt{19})}{5,0}c_{4,1,2}$,
$c_{5,2,0}=\frac{3(38+9\sqrt{19})}{2,0}c_{3,1,3}+\frac{(5+\sqrt{19})}{3,0}c_{4,2,1}+\frac{3(5+\sqrt{19})}{1,0}c_{4,1,2}$,
$c_{7,0,0}=\frac{1}{840}(3+\sqrt{19})(4+\sqrt{19})(5+\sqrt{19})c_{4,0,3}$,
$c_{3,3,1}=\frac{1}{2}(3+\sqrt{19})(4+\sqrt{19})c_{3,1,3}-\frac{2}{3}c_{4,2,1}+\frac{2(4+\sqrt{19})}{3}c_{4,1,2}$,
$c_{3,2,2}=\frac{3}{2}(3+\sqrt{19})c_{3,1,3}+2c_{4,1,2}$, $c_{5,1,1}=-\frac{1}{5}c_{4,2,1}-\frac{1}{5}(-4-\sqrt{19})c_{4,1,2}$,
$c_{6,0,1}=\frac{1}{3,0}(3+\sqrt{19})(4+\sqrt{19})c_{4,0,3}$, $c_{5,0,2}=\frac{3}{1,0}(3+\sqrt{19})c_{4,0,3}$,
$c_{3,0,4}=\frac{1}{(2+\sqrt{19})}c_{4,0,3}$

$$\dim Z^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=4$$

$$\dim B^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=1, \text{ where}$$

$a_{0,6}=0$, $a_{1,5}=-\frac{4}{45}(-35+8\sqrt{19})a_{5,1}$, $a_{2,4}=2(-13+3\sqrt{19})a_{5,1}$, $a_{3,3}=-\frac{4}{3}(-31+7\sqrt{19})a_{5,1}$, $a_{4,2}=\frac{1,0}{3}(-4+\sqrt{19})a_{5,1}$, $a_{6,0}=-\frac{1}{3,0}(5+\sqrt{19})a_{5,1}$.
The results hold true for $(\lambda,\nu)=(0,\frac{-5+\sqrt{19}}{2}),(\frac{-5\pm\sqrt{19}}{2},0)$, $(\frac{-5+\sqrt{19}}{2},\frac{-5-\sqrt{19}}{2}),(\frac{-5-\sqrt{19}}{2},\frac{-5+\sqrt{19}}{2})$.

(2) If $(\lambda,\nu)=(0,0)$, then the constant $a_{3,3}$ and $a_{2,4}$ are arbitrary. On the Other hand,

$$\begin{array}{l} a_{6,0}=a_{0,6}=0 \\ a_{5,1}=(\frac{1}{4}a_{3,3}-\frac{1}{2}a_{2,4}) \\ a_{4,2}=(-a_{3,3}-a_{2,4}) \\ a_{1,5}=(\frac{3}{4,0}a_{3,3}+\frac{1}{2,0}a_{2,4}) \end{array}$$

$$\dim B^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=2$$

The results hold true for $(\lambda,\nu)=(-\frac{5}{2},-\frac{5}{2}),(-\frac{5}{2},0),(0,-\frac{5}{2})$.

(3) If $(\lambda,\nu)=(-\frac{1}{2},0)$ then the constant $a_{3,3}$ and $a_{2,4}$ are arbitrary. On the other hand, $a_{1,5}=6a_{0,6}$, $a_{2,4}=\frac{5}{2}a_{5,1}$, $a_{3,3}=\frac{1,0}{3}a_{5,1}$, $a_{4,2}=\frac{5}{2}a_{5,1}$, $a_{6,0}=\frac{1}{6}a_{5,1}$.

$$\dim B^1_{\rm diff}(\mathrm{Vect}(\mathbb{R}),\mathcal{D}_{\lambda,\nu;\mu})=2$$

The results hold true for $(\lambda,\nu)=(-2,-\frac{1}{2}),(-\frac{1}{2},-\frac{5}{2}),(-\frac{5}{2},-\frac{1}{2})$, $(-1,-\frac{3}{2}),(-\frac{3}{2},-1),(-1,-2),(-2,-1),(-\frac{5}{2},-1),(-1,-\frac{5}{2}),(-\frac{3}{2},-\frac{3}{2})$, $(-2,-\frac{3}{2}),(-\frac{3}{2},-2),(-\frac{3}{2},-\frac{5}{2}),(-\frac{5}{2},-\frac{3}{2}),(-2,-2)$, $(-2,-\frac{5}{2})$ and $(-\frac{5}{2},-2)$.

(4) If $(\lambda,\nu)=(0,-\frac{1}{4})$, then the constant $a_{6,0}$ is arbitrary. On the other hand,

$$a_{0,6}=0,\qquad a_{6,0}=\frac{1}{6,0}a_{5,1},\qquad a_{5,1}=-\frac{1}{2,0}a_{4,2},$$
$$a_{4,2}=-\frac{3}{8}a_{3,3}\qquad a_{3,3}=-\frac{5}{3}a_{2,4},\qquad a_{2,4}=-\frac{35}{4}a_{1,5}$$

The results hold true for otherwise.

(viii)The case when $\mu-\lambda-\nu=7$

$$\begin{array}{rcl}
c(f,g) &=& c_{1,7,0}Y'f^{(7)}g+c_{1,6,1}Y'f^{(6)}g'+c_{1,5,2}Y'f^{(5)}g''+c_{1,4,3}Y'f^{(4)}g'''+c_{1,3,4}Y'f'''g^{(4)}\\
&+& c_{1,2,5}Y'f''g^{(5)}+c_{1,1,6}Y'f'g^{(6)}+c_{1,0,7}Y'fg^{(7)}+c_{2,6,0}Y''f^{(6)}g+c_{2,5,1}Y''f^{(5)}g'\\
&+& c_{2,4,2}Y''f^{(4)}g''+c_{2,3,3}Y''f'''g'''+c_{2,2,4}Y''f''g^{(4)}+c_{2,1,5}Y''f'g^{(5)}+c_{2,0,6}Y''fg^{(6)}\\
&+& c_{3,5,0}Y'''f^{(5)}g+c_{3,4,1}Y'''f^{(4)}g'+c_{3,3,2}Y'''f'''g''+c_{3,2,3}Y'''f''g'''+c_{3,1,4}Y'''f'g^{(4)}\\
&+& c_{3,0,5}Y'''fg^{(5)}+c_{4,4,0}Y^{(4)}f^{(4)}g+c_{4,3,1}Y^{(4)}f'''g'+c_{4,2,2}Y^{(4)}fg''+c_{4,1,3}Y^{(4)}fg'''\\
&+& c_{4,0,4}Y^{(4)}fg'''+c_{5,3,0}Y^{(5)}f''g+c_{5,2,1}Y^{(5)}f''g'+c_{5,1,2}Y^{(5)}f'g''+c_{5,0,3}Y^{(5)}fg'''\\
&+& c_{6,2,0}Y^{(6)}f''g+c_{6,1,1}Y^{(6)}f'g'+c_{6,0,2}Y^{(6)}fg''\\
&+& c_{7,1,0}Y^{(7)}f'g+c_{7,0,1}Y^{(7)}fg'+c_{8,0,0}Y^{(8)}fg
\end{array}$$

A straightforward computation shows that the above 1-cocycle condition reads
$\mu-\lambda-\nu=7$ and

$$\begin{array}{ll}
c_{1,7,0}(21+7\lambda)+c_{1,6,1}\nu=0, & c_{1,7,0}(35+21\lambda)+c_{1,5,2}\nu=0,\\
c_{1,7,0}(35+355\lambda)+c_{1,4,3}\nu=0, & c_{1,7,0}(21+35\lambda)+c_{1,3,4}\nu=0,\\
c_{1,7,0}(7+21\lambda)+c_{1,2,5}\nu=0, & c_{1,7,0}(1+7\lambda)+c_{1,1,6}\nu=0,\\
c_{1,7,0}\lambda+c_{1,0,7}\nu=0, & c_{1,6,1}(15+6\lambda)+c_{1,5,2}(1+2\nu)=0,\\
c_{1,6,1}(20+15\lambda)+c_{1,4,3}(1+3\nu)=0, & c_{1,6,1}(15+20\lambda)+c_{1,3,4}(1+4\nu)=0,\\
c_{1,6,1}(6+15\lambda)+c_{1,2,5}(1+5\nu)=0, & c_{1,6,1}(1+6\lambda)+c_{1,1,6}(1+6\nu)=0,\\
c_{1,6,1}\lambda+c_{1,0,7}(1+7\nu)=0, & c_{1,5,2}(10+5\lambda)+c_{1,4,3}(3+3\nu)=0,\\
c_{1,5,2}(10+10\lambda)+c_{1,3,4}(4+6\nu)=0, & c_{1,5,2}(5+10\lambda)+c_{1,2,5}(5+10\nu)=0,\\
c_{1,5,2}(1+5\lambda)+c_{1,1,6}(6+15\nu)=0, & c_{1,5,2}\lambda+c_{1,0,7}(7+21\nu)=0,\\
c_{1,4,3}(6+4\lambda)+c_{1,3,4}(6+4\nu)=0, & c_{1,4,3}(4+6\lambda)+c_{1,2,5}(10+10\nu)=0,\\
c_{1,4,3}(1+4\lambda)+c_{1,1,6}(15+20\nu)=0, & c_{1,4,3}\lambda+c_{1,0,7}(21+35\nu)=0,\\
c_{1,3,4}(3+3\lambda)+c_{1,2,5}(10+5\nu)=0, & c_{1,3,4}(1+3\lambda)+c_{1,1,6}(20+15\nu)=0,\\
c_{1,3,4}\lambda+c_{1,0,7}(35+35\nu)=0, & c_{1,2,5}(1+2\lambda)+c_{1,1,6}(15+6\nu)=0,\\
c_{1,2,5}\lambda+c_{1,0,7}(35+21\nu)=0, & c_{1,1,6}\lambda+c_{1,0,7}(21+7\nu)=0,
\end{array}$$

$c_{1,7,0}=c_{1,6,1}=c_{1,5,2}=c_{1,4,3}=c_{1,3,4}=c_{1,2,5}=c_{1,1,6}=c_{1,0,7}=0$ for all λ and ν.

$$\begin{array}{ll}
c_{2,6,0}(20+15\lambda)+c_{2,4,2}\nu=0, & c_{2,6,0}(15+20\lambda)+c_{2,3,3}\nu=0,\\
c_{2,6,0}(6+15\lambda)+c_{2,2,4}\nu=0, & c_{2,6,0}(1+6\lambda)+c_{2,1,5}\nu=0,\\
c_{2,6,0}\lambda+c_{2,0,6}\nu=0, & c_{2,5,1}(10+10\lambda)+c_{2,3,3}(1+3\nu)=0,\\
c_{2,5,1}(5+10\lambda)+c_{2,2,4}(1+4\nu)=0, & c_{2,5,1}(1+5\lambda)+c_{2,1,5}(1+5\nu)=0,\\
c_{2,5,1}\lambda+c_{2,0,6}(1+6\nu)=0, & c_{2,4,2}(4+6\lambda)+c_{2,2,4}(4+6\nu)=0,\\
c_{2,4,2}(1+4\lambda)+c_{2,1,5}(5+10\nu)=0, & c_{2,4,2}\lambda+c_{2,0,6}(6+15\nu)=0,\\
c_{2,3,3}(1+3\lambda)+c_{2,1,5}(10+10\nu)=0, & c_{2,3,3}\lambda+c_{2,0,6}(15+20\nu)=0,\\
c_{2,2,4}\lambda+c_{2,0,6}(20+15\nu)=0, &
\end{array}$$

$c_{2,6,0}=c_{2,5,1}=c_{2,4,2}=c_{2,3,3}=c_{2,2,4}=c_{2,1,5}=c_{2,0,6}=0$ for all λ and ν.

$$\begin{array}{ll}
c_{3,5,0}(10+5\lambda)+c_{3,4,1}\nu+2c_{4,4,0}=0, & c_{3,5,0}(5+10\lambda)+c_{3,2,3}\nu=0,\\
c_{3,5,0}(1+5\lambda)+c_{3,1,4}\nu=0, & c_{3,5,0}\lambda+c_{3,0,5}\nu=0,\\
c_{3,4,1}(6+4\lambda)+c_{3,3,2}(1+2\nu)+2c_{4,3,1}=0, & c_{3,4,1}(1+4\lambda)+c_{3,1,4}(1+4\nu)=0,\\
c_{3,4,1}\lambda+c_{3,0,5}(1+5\nu)=0, & c_{3,3,2}(3+3\lambda)+c_{3,2,3}(3+3\nu)+2c_{4,2,2}=0,\\
c_{3,3,2}\lambda+c_{3,0,5}(5+10\nu)=0, & c_{3,2,3}(1+2\lambda)+c_{3,1,4}(6+4\nu)+2c_{4,1,3}=0,\\
c_{3,1,4}\lambda+c_{3,0,5}(10+5\nu)+2c_{4,0,4}=0, & \\
c_{4,4,0}(6+4\lambda)+c_{4,3,1}\nu+5c_{5,3,0}=0, & c_{4,4,0}(4+6\lambda)+c_{4,2,2}\nu+5c_{6,2,0}=0,\\
c_{4,4,0}\lambda+c_{4,1,3}\nu=0, & c_{4,3,1}(3+3\lambda)+c_{4,2,2}(1+2\nu)+5c_{5,2,1}=0,\\
c_{4,3,1}(1+3\lambda)+c_{4,1,3}(1+3\nu)+5c_{6,1,1}=0, & c_{4,2,2}(1+2\lambda)+c_{4,1,3}(3+3\nu)+5c_{5,1,2}=0,\\
c_{4,2,2}\lambda+c_{4,0,4}(4+6\nu)+5c_{6,0,2}=0, & c_{4,1,3}\lambda+c_{4,0,4}(6+4\nu)+5c_{5,0,3}=0,\\
c_{5,3,0}\lambda+c_{5,0,3}\nu+14c_{8,0,0}=0, & c_{5,3,0}(3+3\lambda)+c_{5,2,1}\nu+9c_{6,2,0}=0,\\
c_{5,3,0}(1+3\lambda)+c_{5,1,2}\nu+14c_{7,1,0}=0, & c_{5,2,1}(1+2\lambda)+c_{5,1,2}(1+2\nu)+9c_{6,1,1}=0,\\
c_{5,2,1}\lambda+c_{5,0,3}(1+3\nu)+14c_{7,0,1}=0, & c_{5,1,2}\lambda+c_{5,0,3}(3+3\nu)+9c_{6,0,2}=0,\\
c_{6,2,0}(1+2\lambda)+c_{6,1,1}\nu+14c_{7,1,0}=0, & c_{6,2,0}\lambda+c_{6,0,2}\nu+28c_{8,0,0}=0,\\
c_{6,1,1}\lambda+c_{6,0,2}(1+2\nu)+14c_{7,0,1}=0, & c_{7,1,0}\lambda+c_{7,0,1}\nu+20c_{8,0,0}=0.
\end{array}$$

Let us study the triviality of this 1-cocycle. A straightforward computation shows that

$$
\begin{aligned}
L_X A_7^{\lambda,\nu} = & - (a_{7,0}(21+7\lambda)+a_{6,1}\nu)Y''f^{(6)}g - (a_{6,1}(15+6\lambda)+a_{5,2}(1+2\nu))Y''f^{(5)}g' \\
& - (a_{5,2}(10+5\lambda)+a_{4,3}(3+3\nu))Y''f^{(4)}g'' - (a_{4,3}(6+4\lambda)+a_{3,4}(6+4\nu))Y''f'''g''' \\
& - (a_{3,4}(3+3\lambda)+a_{2,5}(10+5\nu))Y''f''g^{(4)} - (a_{2,5}(1+2\lambda)+a_{1,6}(15+6\nu))Y''f'g^{(5)} \\
& - (a_{1,6}\lambda+a_{0,7}(21+7\nu))Y''fg^{(6)} - (a_{7,0}(35+21\lambda)+a_{5,2}\nu)Y'''f^{(5)}g \\
& - (a_{6,1}(20+15\lambda)+a_{4,3}(1+3\nu))Y'''f^{(4)}g' - (a_{5,2}(10+10\lambda)+a_{3,4}(4+6\nu))Y'''f'''g'' \\
& - (a_{4,3}(4+6\lambda)+a_{2,5}(10+10\nu))Y'''f''g''' - (a_{3,4}(1+3\lambda)+a_{1,6}(20+15\nu))Y'''f'g^{(4)} \\
& - (a_{2,5}\lambda+a_{0,7}(35+21\nu))Y'''fg^{(5)} - (a_{7,0}(35+35\lambda)+a_{4,3}\nu)Y^{(4)}f(4)g \\
& - (a_{6,1}(15+20\lambda)+a_{3,4}(1+4\nu))Y^{(4)}f'''g' - (a_{5,2}(5+10\lambda)+a_{2,5}(5+10\nu))Y^{(4)}f''g'' \\
& - (a_{4,3}(1+4\lambda)+a_{1,6}(15+20\nu))Y^{(4)}f'g''' - (a_{3,4}\lambda+a_{0,7}(35+35\nu))Y^{(4)}fg^{(4)} \\
& - (a_{7,0}(21+35\lambda)+a_{3,4}\nu)Y^{(5)}f'''g - (a_{6,1}(6+15\lambda)+a_{2,5}(1+5\nu))Y^{(5)}f''g' \\
& - (a_{5,2}(1+5\lambda)+a_{1,6}(6+15\nu))Y^{(5)}f'g'' - (a_{4,3}\lambda+a_{0,7}(21+35\nu))Y^{(5)}fg''' \\
& - (a_{7,0}(7+21\lambda)+a_{2,5}\nu)Y^{(6)}f''g - (a_{6,1}(1+6\lambda)+a_{1,6}(1+6\nu))Y^{(6)}f'g' \\
& - (a_{5,2}\lambda+a_{0,7}(7+21\nu))Y^{(6)}fg'' - (a_{7,0}(1+7\lambda)+a_{1,6}\nu)Y^{(7)}f'g \\
& - (a_{6,1}(1+7\lambda)+a_{0,7}\nu)Y^{(7)}fg' - (a_{7,0}\lambda+a_{0,7}\nu)Y^{(8)}fg
\end{aligned}
$$

The proof here is the same as in the previous section. We just point out that the space of solution of the 1-cocycle is four-dimensional for $(\lambda,\nu) = (0,-3), (-3,0)$ and $(-3,-3)$, three-dimentional for $(\lambda,\nu) = (\frac{-5+\sqrt{19}}{2}, \frac{-5+\sqrt{19}}{2})$, $(-2,-2), (0,\nu), (\lambda,0), (\lambda,-6-\lambda), (\sqrt{19}-1, \frac{-\sqrt{19}-5}{2}), (\frac{-\sqrt{19}-5}{2}, \sqrt{19}-1), (-\sqrt{19}-1, \frac{\sqrt{19}-5}{2}), (\frac{\sqrt{19}-5}{2}, -\sqrt{19}-1), (-3,-\frac{5}{2}), (-\frac{5}{2},-3), (-\frac{1}{2},-3), (-3,-\frac{1}{2}), (-2,-\frac{5}{2}), (-\frac{5}{2},-2), (-\frac{5}{2},-\frac{5}{2})$, and two-dimentional otherwise.

(ix)The case when $k \geq 8$

For $k \geq 8$, the number of variables generating any 1-cocycle is "much" smaller than the number of equations. For instance, the 1-cocycle condition for $k = 8$ involves 45 variables for 90 equations. For generic λ and ν the number of equations will generates a one-dimensional space, which give a unique cohomology class. This cohomology class is indeed trivial because the expression $L_X A_k^{\lambda,\nu}$ is also a 1-cocycle.

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