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Induced Saturation of Graphs

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Abstract

A graph G is H-saturated for a graph H, if G does not contain a copy of H but adding any new edge to G results in such a copy. An H-saturated graph on a given number of vertices always exists and the properties of such graphs, for example their highest density, have been studied intensively.

A graph G is H-induced-saturated if G does not have an induced subgraph isomorphic to H, but adding an edge to G from its complement or deleting an edge from G results in an induced copy of H. It is not immediate anymore that H-induced-saturated graphs exist. In fact, Martin and Smith [9] showed that there is no P_4-induced-saturated graph. Behrens *et al.*[1] proved that if H belongs to a few simple classes of graphs such as a class of odd cycles of length at least 5, stars of size at least 2, or matchings of size at least 2, then there is an H-induced-saturated graph.

This paper addresses the existence question for H-induced-saturated graphs. It is shown that Cartesian products of cliques are H-induced-saturated graphs for H in several infinite families, including large families of trees. A complete characterization of all connected graphs H for which a Cartesian product of two cliques is an H-induced-saturated graph is given. Finally, several results on induced saturation for prime graphs and families of graphs are provided.

Keywords: Saturation, Induced Subgraphs, Hamming graphs
Mathematics Subject Classifications: 05C35, 05C75, 05C76

1 Introduction

The notion of induced saturation is relatively new. A more studied subgraph saturation problem does not impose the "induced" condition. A graph G is called *H-saturated* if G has no subgraph isomorphic to H but adding any edge to G from its complement creates a subgraph isomorphic to H. The problem of finding dense H-saturated graphs has a

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long history dating back to 1907, when Mantel [8] determined the maximum number of edges in a triangle-free graph of a fixed order. In 1941, Turán [11] determined maximum number of edges in a K_k-saturated graph on n vertices, which gave rise to the classical extremal function $\mathrm{ex}(n, H)$, that is the largest number of edges in an H-saturated graph on n vertices. The study of H-saturated graphs with smallest possible number of edges was carried out later by Erdős, Hajnal and Moon [4], who determined the minimum number of edges in a K_k-saturated graph on n vertices. Since then, graph and hypergraph saturation became an attractive area of research, see the extensive dynamic survey of Faudree, Faudree and Schmitt [5]. Note that H-saturated graphs always exist and can be constructed greedily from an empty graph by adding edges as long as no copy of H appears.

A graph H is an *induced subgraph* of a graph G if $V(H) \subseteq V(G)$ and for any $x, y \in V(H)$, $xy \in E(H)$ if and only if $xy \in E(G)$. We say that G contains an *induced copy* of H and write $H \subseteq_I G$, if G contains an induced subgraph isomorphic to H.

The notion of induced extremal [10] and induced saturation [9] numbers were first introduced for generalized graphs, called trigraphs, that allow for "flexible" edges which can be added or omitted when one looks for an induced copy of the desired graph. In particular, the induced saturation and induced extremal numbers were defined as the minimum and maximum number of "flexible" edges in an H-induced-saturated trigraph.

In this paper, we work with the following definition of induced saturation. We say that G is *H-induced-saturated*, and write $G \hookrightarrow\!\!\!\!\rightarrow H$, if G does not have an induced subgraph isomorphic to H, but deleting any edge of G or adding any edge to G from its complement creates an induced copy of H. Note that if $G \hookrightarrow\!\!\!\!\rightarrow H$, then $\overline{G} \hookrightarrow\!\!\!\!\rightarrow \overline{H}$.

We use the following notation: K_n, C_n, and P_n for a complete graph, a cycle, and a path on n vertices, respectively; $K_{n,m}$ for a complete bipartite graph with parts of sizes n and m; C'_n for a union of C_n and K_2, that share exactly one vertex; $\widetilde{C}_n$ for a union of C_n and an edge e where the endpoints of e are distance 2 apart in C_n.

Martin and Smith [9] introduced the notion of induced saturation number and as a by-product they proved that there is no graph G such that $G \hookrightarrow\!\!\!\!\rightarrow P_4$. We provide a short proof of this result in Section 6.

All previously known examples of graphs H for which there exists an H-induced-saturated graph were given by Behrens, Erbes, Santana, Yager, and Yaeger [1].

Theorem 1 (Behrens *et al.*[1])**.** *Let $n \geq 2$, $k \geq 3$. If H is one of the following graphs: C'_3, $K_{1,n}$, nK_2, C_4, C_{2k-1}, C'_{2k}, $\widetilde{C}_{2k}$, then there is a graph G such that $G \hookrightarrow\!\!\!\!\rightarrow H$.*

In this paper, we introduce two large classes of graphs, $\mathcal{H}(n)$ and $\mathcal{F}(n)$, including a family of trees, such that for any graph H in these classes, there is an H-induced-saturated graph G, which is a Cartesian product of cliques. We need a few definitions to state our results.

The *Cartesian product* of n cliques of order k is denoted by $\overset{n}{\square} K_k$ and is formally defined as a graph whose vertex set is a set of all n-tuples with entries from a k-element set such

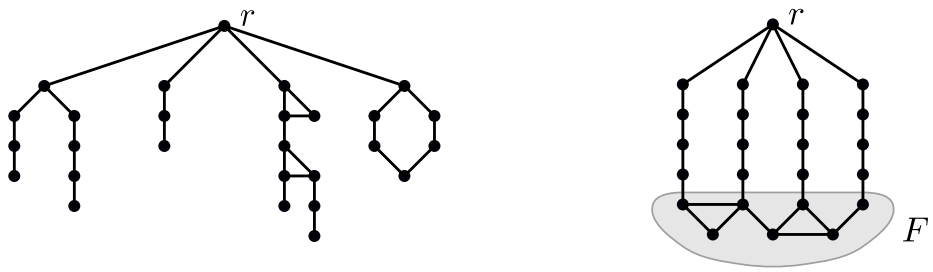
that two tuples are adjacent if and only if they differ in exactly one coordinate. We say that a graph is *Hamming* if it is isomorphic to $\overset{n}{\square} K_k$ for some $n, k \in \mathbb{N}$. We write $K_k \square K_k$ for $\overset{2}{\square} K_k$. We define the *dimension*, $\dim(G)$ of a graph G as the smallest integer n such that $G \subseteq_I \overset{n}{\square} K_k$ holds for some $k \in \mathbb{N}$. If no such n exists, the dimension is defined to be infinity.

We say that a graph is a balanced *p-legged spider* of height q if it is a union of p paths, $L_1, \ldots, L_p$ each of length q such that they share exactly one vertex, r, called the *head* of the spider, that is, $V(L_i) \cap V(L_j) = \{r\}$ for $1 \leq i < j \leq p$ such that r is the endpoint of each of the paths. The vertices of degree 1 are called the *leaves* and the paths $L_1, \ldots, L_p$ are called the legs of the spider. A connected subgraph of a balanced spider is a spider. A spider is a p-legged spider of height at least q if it contains a balanced p-legged spider of height q as a subgraph and has maximum degree p.

Now we define the classes $\mathcal{H}(n)$ and $\mathcal{F}(n)$.

- For any $n \geq 3$, let $\mathcal{H}(n)$ be the class of connected graphs that have a cut vertex r of degree $n+1$ so that $H - r$ has $n+1$ connected components, each of dimension at most $n-1$. See Figure 1a.

- For any $n \geq 3$, let $\mathcal{F}(n)$ be the class of graphs H that can be obtained as the union of graphs F and T such that $\dim(F) \leq n-1$, T is a balanced $(n+1)$-legged spider of height $n+2$, and $|V(F) \cap V(T)| = V'$, where V' is the set of leaves of T. See Figure 1b.



(a) A graph from $\mathcal{H}(3)$.

(b) A graph from $\mathcal{F}(3)$.

Figure 1: Examples of graphs from classes $\mathcal{H}(n)$ and $\mathcal{F}(n)$.

Theorem A. *For any $n \geq 3$ and $H \in \mathcal{H}(n)$, there is $k \in \mathbb{N}$ such that $\overset{n}{\square} K_k \hookrightarrow\!\!\!\!\rightarrow H$.*

Corollary 2. *For any $n \geq 3$, let $\mathcal{T}(n)$ be the class of trees with a unique vertex of maximum degree $n+1$ such that its removal results in components that have maximum degree at most $n-1$. Then for any $n \geq 3$ and $T \in \mathcal{T}(n)$, there is $k \in \mathbb{N}$ so that $\overset{n}{\square} K_k \hookrightarrow\!\!\!\!\rightarrow T$.*

Theorem B. *For any $n \geq 3$ and $H \in \mathcal{F}(n)$, there is $k \in \mathbb{N}$ such that $\overset{n}{\square} K_k \hookrightarrow\!\!\!\!\rightarrow H$.*

We say that a graph H' is obtained from a graph H by a *k-subdivision* if H' can be obtained from H by subdividing each edge of the graph H with k new vertices. Formally, the vertex set of H' can be given as $V(H') = V(H) \cup \left(\bigcup_{xy \in E(H)} V_{xy}\right)$, where the sets V_{xy}, $xy \in E(H)$ are pairwise disjoint and each V_{xy} is disjoint from $V(H)$, moreover for any edge $xy \in E(H)$, $V_{xy} \cup \{x, y\}$ induces a path on $k+2$ vertices with endpoints x and y and there are no other edges in $E(H')$.

Corollary 3. *Let $n \geq 4$ and let H be a graph with a unique vertex v of maximum degree $n+1$ such that $H - v$ has maximum degree at most $n-1$. Let H' be the graph obtained from H by an $(n+1)$-subdivision. Then there is $k \in \mathbb{N}$ such that $\overset{n}{\square} K_k \hookrightarrow\!\!\!\!\rightarrow H'$.*

Note that the same result can be easily proved when H' is obtained from H by subdividing each edge with at least $n+1$ vertices.

In Section 5, we give a characterization of connected graphs H for which there exists an integer k so that $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$.

Our results give infinite classes of graphs H for which H-induced-saturated graphs exist. These classes include some easily describable families of graphs that complement the list of Theorem 1:

Proposition 4. *Let $k \geq 3$ and let H be one of the following graphs: a spider with k legs, $\widetilde{C}_{2k+1}$, or the union of a path P and a cycle C such that P has length at least 1 and shares its endpoint and no other vertex with C. Then there is a graph G such that $G \hookrightarrow\!\!\!\!\rightarrow H$.*

The paper is organized as follows. In Section 2, we give some basic properties of Hamming graphs. In Sections 3 and 4, we prove Theorems A and B, respectively. In Section 5, we describe those connected graphs H for which there is an H-induced-saturated graph that is a Cartesian product of two cliques of equal size. We also prove Proposition 4 in Section 5. In the last two sections, we give some general observations about induced saturation of prime graphs and state some concluding remarks.

2 Preliminaries and properties of Hamming graphs

Let G and H be graphs. An injective mapping $\sigma : V(H) \to V(G)$ is an *induced embedding* of H into G if $uv \in E(H)$ holds if and only if $\sigma(u)\sigma(v) \in E(G)$. Note that G contains an induced copy of H if and only if H has an induced embedding into G.

Let G be a connected graph and $u, v \in V(G)$. The *distance* between u and v in G, denoted by $\mathrm{dist}_G(u, v)$ is the length of a shortest u, v-path in G. We shall denote the maximum degree of a graph G by $\Delta(G)$ and its chromatic index by $\chi'(G)$.

The *Hamming distance* between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, denoted by $\mathrm{dist}_{\mathrm{Ham}}(\mathbf{u}, \mathbf{v})$, is the number of coordinates where $\mathbf{u}$ and $\mathbf{v}$ differ.

Let $\mathbf{0}$ denote a vector with all zero entries and $\mathbf{e}_i$ denote a vector with all entries zero, except for the ith which is 1. We denote the ith coordinate of a vector $\mathbf{v}$ by $[\mathbf{v}]_i$. Recall that the vertex set of a Hamming graph $\overset{n}{\square} K_k$ is given by X^n with some set $|X| = k$. Throughout this work, we denote the vertices of a Hamming graph by bold letters (e.g. $\mathbf{u}, \mathbf{v}$) and operate with them as with vectors in X^n. Note that the distance of two vertices $\mathbf{u}, \mathbf{v}$ in $\overset{n}{\square} K_k$ is equal to $\mathrm{dist}_{\mathrm{Ham}}(\mathbf{u}, \mathbf{v})$.

Let $[n] = \{1, 2, \ldots, n\}$ and $[n, m] = \{k \in \mathbb{N} : n \leq k \leq m\}$.

Definition 5. *Let G be a graph, $n \in \mathbb{N}$. We call a function $c : E(G) \to X$ a* ***nice coloring*** *with n colors if $|X| = n$,*

i) all triangles are monochromatic, and

ii) for any two distinct, nonadjacent vertices u, v of G there exist two different colors i and j such that i and j appear on each induced u, v-path.

The next observation follows from the second property of a nice coloring and the fact that for adjacent vertices u, v, the edge uv is the only induced u, v-path.

Observation 6. *Let G be a graph and c be a nice coloring of $E(G)$. Then for any two distinct vertices u, v of G, there exists a color i such that i appears on each induced u, v-path.*

The class of graphs with finite dimension was characterized by Klavžar and Peterin [7]:

Theorem 7 (Coloring criterion [7])**.** *Let G be a graph and $n \in \mathbb{N}$. Then G has dimension at most n if and only if G has a nice coloring with n colors.*

Note that the smallest graph with infinite dimension is $K_4 - e$ and that if $H \subseteq_I G$, then $\dim(H) \leq \dim(G)$.

Lemma 8. *Let $n \geq 2$ and F be a forest with $\Delta(F) = n$. Then $\dim(F) = \chi'(F) = n$, in particular, $F \not\subseteq_I \overset{n-1}{\square} K_k$ for any $k \in \mathbb{N}$.*

Proof. Let F be a forest with $\Delta(F) = n$. By the coloring criterion (Theorem 7), it is enough to show that F has a nice coloring with n colors and cannot be nicely colored with a smaller number of colors. Observe that an edge-coloring of a forest is nice if and only if it is proper. Hence $\dim(F) = \chi'(F)$. It is easy to see that F does not have a proper edge-coloring with less than n colors since the n edges incident to a vertex of maximum degree need to receive different colors. On the other hand, $E(F)$ can be greedily properly colored with n colors. Thus $\dim(F) = \chi'(F) = n$. $\square$

The following lemma is an easy consequence of the fact that Hamming graphs are edge-transitive and distance-transitive.

Lemma 9. *Let $k, n \geq 2$ and $G = \overset{n}{\square} K_k$. Then*

- *for any $e, f \in E(G)$, the graphs $G - e$ and $G - f$ are isomorphic;*
- *for any $e, f \in E(\overline{G})$ with $e = \mathbf{uv}$, $f = \mathbf{xy}$ such that $\mathrm{dist}(\mathbf{u}, \mathbf{v}) = \mathrm{dist}(\mathbf{x}, \mathbf{y})$, the graphs $G + e$ and $G + f$ are isomorphic.*

Now, we give sufficient and necessary conditions for $G \hookrightarrow\!\!\!\!\rightarrow H$ in case when G is a Hamming graph. We shall use this criterion to prove Theorems A and B.

Lemma 10. *Let $n, k \in \mathbb{N}$, $G = \overset{n}{\square} K_k$, and H be a graph. Then $G \hookrightarrow\!\!\!\!\rightarrow H$ if and only if*

(S1) $H \not\subseteq_I G$,

(S2) *for any edge $e \in E(\overline{G})$, $H \subseteq_I G + e$, and*

(S3) *there is and edge $f \in E(\overline{H})$ such that $H + f \subseteq_I G$.*

Proof. By definition, $G \hookrightarrow\!\!\!\!\rightarrow H$ if and only if
(i) $H \not\subseteq_I G$,
(ii) for any $e \in E(\overline{G})$, $H \subseteq_I G + e$, and
(iii) for any $e \in E(G)$, $H \subseteq_I G - e$.
Thus it is enough to prove that (iii) is equivalent to (S3).

Assume that $H \not\subseteq_I G$ and (iii) holds. Let $e = uv \in E(G)$ and let $\sigma : V(H) \to V(G)$ be an induced embedding of H into $G - e$. Since $H \not\subseteq_I G$, the vertex set $\sigma(V(H))$ has to contain both u and v. Since σ is an induced embedding, $\sigma^{-1}(u)\sigma^{-1}(v) \notin E(H)$. Let $f = \sigma^{-1}(u)\sigma^{-1}(v)$, then σ is an induced embedding of $H + f$ into G. Thus (S3) holds.

Let (S3) hold. Let $f = u'v' \notin E(H)$ so that $H + f \subseteq_I G$ and let $\sigma : V(H) \to V(G)$ be an induced embedding of $H + f$ into G. Since σ is an induced embedding, $\sigma(u')\sigma(v') \in E(G)$. Let $e = \sigma(u')\sigma(v')$, then σ is an induced embedding of H into $G - e$. Thus $H \subseteq_I G - e$. By Lemma 9, $H \subseteq_I G - e'$ for any $e' \in E(G)$ and so (iii) holds. $\square$

Lemma 11. *Let $k, n \in \mathbb{N}$, $G = \overset{n}{\square} K_k$, and H be a graph with $H \subseteq_I G$. Then for any $u \in V(H)$ and for any $\mathbf{v} \in V(G)$, there exists an induced embedding $\sigma : V(H) \to V(G)$ so that $\sigma(u) = \mathbf{v}$.*

Proof. We assume that $V(G) = \{0, 1, \ldots, k-1\}^n$. Let $\tau : V(H) \to V(G)$ be an arbitrary induced embedding of H into G and let $\mathbf{w} = \tau(u)$. If $\mathbf{w} = \mathbf{v}$, then $\sigma = \tau$ is a desired embedding. If $\mathbf{w} \neq \mathbf{v}$, define $\mu : V(G) \to V(G)$ as $\mu(\mathbf{x}) = \mathbf{x} - \mathbf{w} + \mathbf{v} \pmod{k}$. Then μ is a bijection that preserves Hamming distance, and thus μ is an automorphism of G. Therefore $\sigma = \mu \circ \tau$ is an induced embedding of H into G and $\sigma(u) = \mathbf{w} - \mathbf{w} + \mathbf{v} = \mathbf{v}$. $\square$

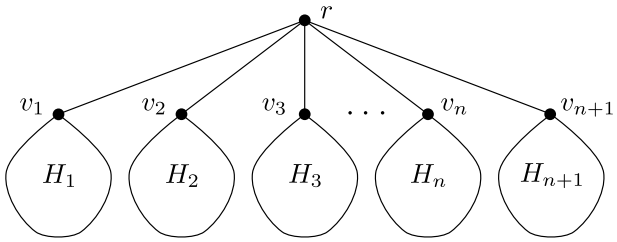


Figure 2: The structure of graphs in $\mathcal{H}(n)$.

3 Proof of Theorem A

Let $H \in \mathcal{H}(n)$, that is, H has a cut vertex r of degree $n+1$ such that $H - r$ has $n+1$ connected components, each of dimension at most $n-1$. We denote the components of $H - r$ by $H_1, H_2, \dots, H_{n+1}$ and define $\{v_i\} = V(H_i) \cap N(r)$ for all $i \in [n+1]$, see Figure 2. Let $\ell \in \mathbb{N}$ be the smallest integer so that $H_i \subseteq_I \overset{n-1}{\square} K_\ell$ for all $i \in [n+1]$.

Let $G = \overset{n}{\square} K_k$ be the Hamming graph on a vertex set $\{0, 1, \dots, k-1\}^n$, where $k \in \mathbb{N}$ is to be specified later. We shall show that $G \hookrightarrow\!\!\!\!\rightarrow H$. By Lemma 10, $G \hookrightarrow\!\!\!\!\rightarrow H$ if and only if
(S1) $H \not\subseteq_I G$,
(S2) for any edge $e \in E(\overline{G})$, $H \subseteq_I G + e$, and
(S3) there is an edge $f \in E(\overline{H})$ such that $H + f \subseteq_I G$.

We see that (S1) holds since the vertex set $\{r\} \cup N(r)$ induces a star $K_{1,n+1}$ in H, hence by Lemma 8, $H \not\subseteq_I G$.

Now we prove (S2). Let $e \in E(\overline{G})$ and $G^+ = G + e$. We need to show that $H \subseteq_I G^+$. Denote the endpoints of e by $\mathbf{u}$ and $\mathbf{v}$, and let $q = \mathrm{dist}_G(\mathbf{u}, \mathbf{v})$, $2 \le q \le n$. By Lemma 9, we can suppose that $\mathbf{u} = \mathbf{0}$ and $\mathbf{v}$ has the form $\mathbf{v} = (\underbrace{2, \dots, 2}_{q}, 0, \dots, 0)$. We shall provide an explicit embedding of H into G^+ using some auxiliary embeddings σ_i of H_i into suitable Hamming graphs F_i. We start by defining the graphs $F_1, \dots, F_{n+1}$. Consider the sets $W_1, \dots, W_{n+1}, X$ with $W_i \subseteq \{0, 2, 3, \dots, k-1\}$, $X \subseteq \{2, 3, \dots, k-1\}$ such that $|W_i| = |X| = \ell$, $W_i \cap W_j = \{0\}$ for all $j \neq i$, $2 \in X \cap W_{n+1}$ and for any $i \in [n]$, $X \cap W_i = \emptyset$. Clearly, such sets W_i and X exist if $k \ge (\ell - 1)(n+1) + 3$. Let F_i be the Hamming graph whose vertex set is the set of vectors with $n-1$ components such that if $i \in [n]$, then each coordinate has value from W_i and if $i = n+1$, then the first coordinate is from X and all other coordinates are from W_{n+1}. Since F_i is isomorphic to $\overset{n-1}{\square} K_\ell$ for any $i \in [n+1]$, there is an induced embedding σ_i of H_i into F_i. By Lemma 11, we can suppose that $\sigma_i(v_i) = \mathbf{0}$ for all $i \in [n]$ and that $\sigma_{n+1}(v_{n+1}) = (\underbrace{2, \dots, 2}_{q-1}, 0, \dots, 0)$. Note that $q - 1 \ge 1$, which implies that the first coordinate of $\sigma_{n+1}(v_{n+1})$ is equal to 2.

Define an embedding $\sigma : V(H) \to V(G^+)$ such that $\sigma(r) = \mathbf{0}$ and for $u \in V(H_i)$, $\sigma(u)$ is obtained by fixing its value in one coordinate and using the values of $\sigma_i(u)$ in the other coordinates. Formally,

(i) if $u \in V(H_1)$, then let $[\sigma(u)]_1 = 1$ and $[\sigma(u)]_j = [\sigma_1(u)]_{j-1}$ for all $2 \le j \le n$;

(ii) if $u \in V(H_i)$ with $2 \le i \le n$, then let $[\sigma(u)]_i = 1$, $[\sigma(u)]_j = [\sigma_i(u)]_j$ for all $1 \le j < i$, and $[\sigma(u)]_j = [\sigma_i(u)]_{j-1}$ for all $i < j \le n$;

(iii) if $u \in V(H_{n+1})$, then let $[\sigma(u)]_1 = 2$ and $[\sigma(u)]_j = [\sigma_{n+1}(u)]_{j-1}$ for all $2 \le j \le n$.

See Figure 3 for an illustration. Note that $\sigma(v_i) = \mathbf{e}_i$ for $i \in [n]$ and $\sigma(v_{n+1}) = \mathbf{v}$.

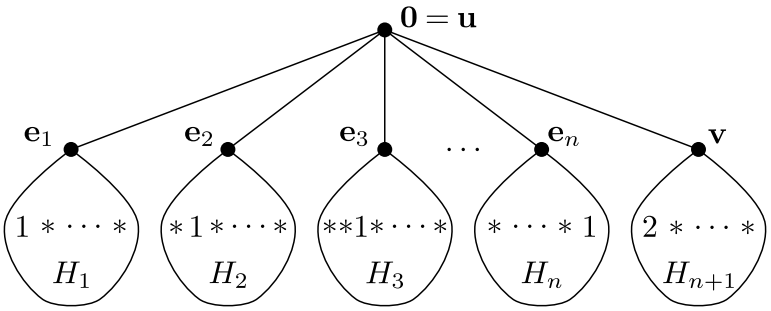


Figure 3: The scheme of the embedding σ of H in G^+, where $* \in \{0, 2, 3, \ldots, k-1\}$ correspond to the values of σ_i's.

Claim. The mapping σ is an induced embedding of H in G^+.

It follows from (i), (ii), (iii), and the injectivity of the mappings σ_i that σ is injective. We shall show that for any pair x, y of vertices of $V(H)$, $xy \in E(H)$ if and only if $\sigma(x)\sigma(y) \in E(G^+)$.

By (i), (ii), and (iii), for any $i \in [n+1]$ and $x, y \in V(H_i)$, $\sigma_i(x)$ and $\sigma_i(y)$ have Hamming distance 1 if and only if $\sigma(x)$ and $\sigma(y)$ have Hamming distance 1. Since the σ_i's are induced embeddings, $G^+[\sigma(V(H_i))]$ is isomorphic to H_i for all $i \in [n+1]$.
Also, for any $i \in [n+1]$, $\sigma(r)$ is adjacent to $\sigma(v_i)$ and for any vertex x of $H_i - v_i$, $\sigma(x)$ has at least two nonzero coordinates, hence it is not adjacent to $\sigma(r)$ in G^+.
It remains to check that two vertices from distinct H_i's are mapped to two non-adjacent vertices of G^+. Consider $x \in V(H_i)$ and $y \in V(H_j)$ with $1 \le i < j \le n+1$. Since $\sigma(x)\sigma(y) \ne \mathbf{uv}$, it sufficies to show that $\sigma(x)$ and $\sigma(y)$ differ in at least 2 coordinates.
If $j \ne n+1$, then by (i) and (ii), $[\sigma(x)]_i = 1 \ne [\sigma(y)]_i$ and $[\sigma(y)]_j = 1 \ne [\sigma(x)]_j$ with $i \ne j$. If $j = n+1$ and $2 \le i \le n$, then by (ii) and (iii), $[\sigma(x)]_i = 1 \ne [\sigma(y)]_i$, and $[\sigma(y)]_1 = 2 \ne [\sigma(x)]_1$, where $i \ne 1$. If $j = n+1$ and $i = 1$, then by (i) and (iii), $[\sigma(x)]_1 = 1 \ne 2 = [\sigma(y)]_1$ and $[\sigma(x)]_2 \ne [\sigma(y)]_2$ since $[\sigma(x)]_2 \in W_1$, $[\sigma(y)]_2 \in X$, and $W_1 \cap X = \emptyset$.

This proves the Claim and (S2).

Finally, we prove (S3). Let $f = v_n v_{n+1}$ and $H^+ = H + f$. We shall show that $H^+ \subseteq_I G$, using the coloring criterion (Theorem 7).

Since $H_i \subseteq_I \overset{n-1}{\square} K_\ell$, by Theorem 7, there is a nice coloring of H_i with $n-1$ colors. For $i \in [n]$, let c_i be a nice coloring of H_i with colors from $[n] \setminus \{i\}$ and let c_{n+1} be a nice

coloring of H_{n+1} with colors from $[n-1]$. Define a coloring $c : E(H^+) \to [n]$ as follows:

$$c(e) = \begin{cases} c_i(e): & e \in E(H_i), \\ i: & e = rv_i,\ 1 \le i \le n, \\ n: & e = rv_{n+1}, \\ n: & e = v_n v_{n+1}. \end{cases}$$

Now we verify that c is a nice coloring. Let T be a triangle in H^+. Then either $V(T) = \{r, v_n, v_{n+1}\}$ and so each edge of T has color n or T is contained in H_i for some $i \in [n+1]$ and so T is monochromatic by the niceness of c_i.
Let x, y be two distinct, nonadjacent vertices of H^+. We need to show that there are two distinct colors that appear on each induced x, y-path.
If $x, y \in V(H_i)$ for some $i \in [n+1]$, then all x, y-paths are contained in H_i and thus niceness of c_i implies that there is a pair of distinct colors present on all these paths.
If $x \in V(H_i)$, $y \in V(H_j)$, $1 \le i < j \le n+1$, $\{i, j\} \ne \{n, n+1\}$, then any x, y-path contains the edges rv_i and rv_j, thus contains the distinct colors of these edges.
If $x \in V(H_n)$ and $y \in V(H_{n+1})$, then all induced x, y-paths contain $v_n v_{n+1}$, so they contain color n. Since x and y are not adjacent in H^+ and $v_n v_{n+1} \in E(H^+)$, $x \ne v_n$ or $y \ne v_{n+1}$. Assume, without loss of generality, that $x \ne v_n$. By the niceness of c_n an Observation 6, there is a color $i \in [n-1]$ that appears on every induced x, v_n-path, thus all induced x, y-paths contain the colors i and n.
Finally, if $x = r$ and $y \in V(H_i) \setminus \{v_i\}$ with $i \in [n+1]$, then any induced x, y-path contains the edge rv_i and an induced v_i, y-path in H_i. Hence any induced x, y-path contains the color $c(rv_i)$ and since c_i is a nice coloring with colors from $[n] \setminus \{c(rv_i)\}$, by Observation 6, all induced v_i, y-paths contain a color $j \in [n] \setminus \{c(rv_i)\}$. Thus all induced x, y-paths contain the colors j and $c(rv_i)$.
Therefore c is a nice coloring of H^+ with n colors and so by Theorem 7, $\dim(H^+) \le n$, that is, there exists $m \in \mathbb{N}$ so that $H^+ \subseteq_I \overset{n}{\square} K_m$.

Letting $k = \max\{(\ell-1)(n+1)+3,\ m\}$, we see that both (S2) and (S3) hold for $G = \overset{n}{\square} K_k$.

This concludes the proof of Theorem A. $\square$

Proof of Corollary 2

Let $T \in \mathcal{T}(n)$. Then $T - r$ has $n+1$ connected components, otherwise T contains a cycle, a contradiction. It follows from Lemma 8 that each component of $T - r$ has dimension at most $n-1$ and thus $T \in \mathcal{H}(n)$. Hence Theorem A implies that there exists $k \in \mathbb{N}$ so that $\overset{n}{\square} K_k \hookrightarrow\!\!\!\!\rightarrow T$. $\square$

4 Proof of Theorem B

Let $H \in \mathcal{F}(n)$, that is, $H = F \cup T$, where F is a graph with $\dim(F) \leq n-1$, T is an $(n+1)$-legged spider of height $n+2$ such that $V' := V(F) \cap V(T)$ is the set of leaves of T. We denote the head of T by r and legs of T by $L_1, L_2, \ldots, L_{n+1}$ and for each $i \in [n+1]$ we let $L_i = p_0(i), p_1(i), \ldots, p_{n+2}(i)$ with $p_0(i) = r$ and $p_{n+2}(i) \in V(F)$. We shall write $p_j = p_j(i)$ when the leg in which p_j is contained is clear from the context.

Let $G = \overset{n}{\square} K_k$ be the Hamming graph on a vertex set $\{0, 1, \ldots, k-1\}^n$, where $k \in \mathbb{N}$ is to be specified later. We shall show that $G \hookrightarrow\!\!\!\!\rightarrow H$. By Lemma 10, $G \hookrightarrow\!\!\!\!\rightarrow H$ if and only if
(S1) $H \not\subseteq_I G$,
(S2) for any edge $e \in E(\overline{G})$, $H \subseteq_I G + e$, and
(S3) there is an edge $f \in E(\overline{H})$ such that $H + f \subseteq_I G$.

We see that (S1) holds since $\{r\} \cup N(r)$ induces a star $K_{1,n+1}$ and therefore by Lemma 8, $H \not\subseteq_I G$.

Now we prove (S2). Let $e \in E(\overline{G})$ and $G^+ = G + e$. We need to show that $H \subseteq_I G^+$. Denote the endpoints of e by $\mathbf{u}$ and $\mathbf{v}$, and let $q = \mathrm{dist}_G(\mathbf{u}, \mathbf{v})$, $2 \leq q \leq n$. By Lemma 9, we can suppose that $\mathbf{u} = \mathbf{0}$ and $\mathbf{v}$ has the form $\mathbf{v} = (\underbrace{2, \ldots, 2}_{q}, 0, \ldots, 0)$.

We shall construct an embedding σ of $H = F \cup T$ in G^+, by first defining σ on $V(F)$, then on r, and then on the legs of T. Let σ map $V(F)$ into $\{n+4, n+5, \ldots, k-1\}^n$ such that $[\sigma(v)]_n = k-1$ for all $v \in V(F)$ and such that $\sigma|_{V(F)}$ is an induced embedding of F. Such an embedding exists for sufficiently large k, since $\dim(F) \leq n-1$. Let $\sigma(r) = \mathbf{0}$. Now, we define the embedding σ on the legs $L_1, \ldots, L_{n+1}$. First, we give an embedding of $L_1, \ldots, L_{n-1}$, then of L_n, and finally of L_{n+1}.

Let $i \in [n-1]$ and let $p_j = p_j(i)$ for all $j \in [n+2]$, that is, $L_i = p_0, p_1, \ldots, p_{n+2}$. Recall that $p_0 = r$, $p_{n+2} \in V(F)$, thus $\sigma(p_{n+2}) = (y_1, y_2, \ldots, y_{n-1}, k-1)$, where $y_j \in \{n+4, \ldots, k-1\}$. We define σ on $V(L_i)$ as

$$
\begin{aligned}
\sigma(p_1) &= (0,\ 0,\ 0, \ldots,\ 0,\ 1,\ 0,\ 0,\ 0, \ldots,\ 0,\ 0\ ) = \mathbf{e}_i,\\
\sigma(p_2) &= (y_1, 0,\ 0, \ldots,\ 0,\ 1,\ 0,\ 0,\ 0, \ldots,\ 0,\ 0\ ),\\
\sigma(p_3) &= (y_1, y_2, 0, \ldots,\ 0,\ 1,\ 0,\ 0,\ 0, \ldots,\ 0,\ 0\ ),\\
&\vdots\\
\sigma(p_i) &= (y_1, y_2, y_3, \ldots, y_{i-1},\ 1,\ 0,\ 0,\ 0, \ldots,\ 0,\ 0\ ),\\
\sigma(p_{i+1}) &= (y_1, y_2, y_3, \ldots, y_{i-1},\ 1,\ y_{i+1},\ 0,\ 0, \ldots,\ 0,\ 0\ ),\\
\sigma(p_{i+2}) &= (y_1, y_2, y_3, \ldots, y_{i-1},\ 1,\ y_{i+1},\ y_{i+2},\ 0, \ldots,\ 0,\ 0\ ),\\
&\vdots\\
\sigma(p_{n-1}) &= (y_1, y_2, y_3, \ldots, y_{i-1},\ 1,\ y_{i+1},\ y_{i+2}\ , \ldots,\ y_{n-1},\ 0\ ),\\
\sigma(p_n) &= (y_1, y_2, y_3, \ldots, y_{i-1},\ 1,\ y_{i+1},\ y_{i+2}\ , \ldots,\ y_{n-1},\ i+2),\\
\sigma(p_{n+1}) &= (y_1, y_2, y_3, \ldots, y_{i-1},\ y_i,\ y_{i+1},\ y_{i+2}\ , \ldots,\ y_{n-1},\ i+2),
\end{aligned}
$$

$$\sigma(p_{n+2}) = (y_1, y_2, y_3, \ldots, y_{i-1},\ y_i, y_{i+1},\ y_{i+2}\ \ , \ldots,\ y_{n-1},\ k-1).$$

Note that $2 < i+2 < n+4 \leq k-1$ for $i \in [n-1]$.

For $i = n$, let $p_j = p_j(n)$ for all $j \in [n+2]$, that is, $L_n = p_0, p_1, \ldots, p_{n+2}$ and let $\sigma(p_{n+2}) = (y_1, y_2, \ldots, y_{n-1}, k-1)$, where $y_j \in \{n+4, \ldots, k-1\}$. We define σ on $V(L_n)$ as

$$\begin{aligned}
\sigma(p_1) &= (0,\ 0,\ \ldots,\ 0,\ 0,\ 1\ ) = \mathbf{e}_n,\\
\sigma(p_2) &= (y_1,\ 0,\ \ldots,\ 0,\ 0,\ 1\ ),\\
\sigma(p_3) &= (y_1, y_2, 0, \ldots,\ 0,\ 0,\ 1\ ),\\
&\vdots\\
\sigma(p_{n-1}) &= (y_1, y_2,\ \ldots,\ y_{n-2},\ 0,\ 1\ ),\\
\sigma(p_n) &= (y_1, y_2,\ \ldots,\ y_{n-2},\ 0,\ n+2),\\
\sigma(p_{n+1}) &= (y_1, y_2,\ \ldots,\ y_{n-2},\ y_{n-1},\ n+2),\\
\sigma(p_{n+2}) &= (y_1, y_2,\ \ldots,\ y_{n-2},\ y_{n-1},\ k-1).
\end{aligned}$$

For $i = n+1$, let $p_j = p_j(n+1)$ for all $j \in [n+2]$, that is, $L_{n+1} = p_0, p_1, \ldots, p_{n+2}$ and let $\sigma(p_{n+2}) = (y_1, y_2, \ldots, y_{n-1}, k-1)$, where $y_j \in \{n+4, \ldots, k-1\}$. We define σ on $V(L_{n+1})$ as

$$\begin{aligned}
\sigma(p_1) &= (2,\ 2,\ \ldots, 2,\ 0 \ldots,\ 0,\ 0\ ) = \mathbf{v},\\
\sigma(p_2) &= (2,\ 2,\ \ldots, 2,\ 0, \ldots,\ 0,\ n+3),\\
\sigma(p_3) &= (2,\ 2,\ \ldots, 2,\ 0, \ldots, y_{n-1},\ n+3),\\
&\vdots\\
\sigma(p_n) &= (2,\ y_2,\ \ldots,\ y_{n-2},\ y_{n-1},\ n+3),\\
\sigma(p_{n+1}) &= (y_1, y_2,\ \ldots,\ y_{n-2},\ y_{n-1},\ n+3),\\
\sigma(p_{n+2}) &= (y_1, y_2,\ \ldots,\ y_{n-2},\ y_{n-1},\ k-1).
\end{aligned}$$

Claim. σ is an induced embedding of H in G^+

It is clear from the way we defined σ that for any distinct $u, v \in V(H)$, $\sigma(u) \neq \sigma(v)$ and if $uv \in E(H)$, then $\sigma(u)\sigma(v) \in E(G^+)$.

Thus we only need to verify that nonadjacent pairs of vertices of H are mapped to nonadjacent pairs of vertices of G^+. Let $x, y \in V(H)$ such that $xy \notin E(H)$ and denote $\sigma(x)$ by $\mathbf{x}$ and $\sigma(y)$ by $\mathbf{y}$. To prove that $\mathbf{xy} \notin E(G^+)$, it sufficies to show that $\mathbf{x}$ and $\mathbf{y}$ differ in at least two coordinates.

Let $x, y \in V(F)$, then $\mathbf{xy} \notin E(G^+)$ since $\sigma|_{V(F)}$ is an induced embedding of F.

Let $x \notin V(F)$, $y \in V(F)$ and x has a neighbor x' in $V(F)$. By construction, $\mathbf{x}$ differs from $\mathbf{x}' := \sigma(x')$ in exactly the last coordinate. Moreover, $\mathbf{y}$ differs from $\mathbf{x}'$ in at least one

coordinate, denote it by j. Note that $j \neq n$ since $[\mathbf{x}']_n = [\mathbf{y}]_n = k-1$. Thus $\mathbf{x}$ differs from $\mathbf{y}$ in the jth and the nth coordinates.

Let $x \notin V(F)$, $y \in V(F)$ and x does not have any neighbors in $V(F)$. Let $i \in [n+1]$ such that $x \in V(L_i)$. If $i \in [n-1]$, then $[\mathbf{x}]_i = 1 \neq [\mathbf{y}]_i$, if $i = n$, then $[\mathbf{x}]_{n-1} = 0 \neq [\mathbf{y}]_{n-1}$, and if $i = n+1$, then $[\mathbf{x}]_1 = 2 \neq [\mathbf{y}]_1$. Moreover for any value of i, $[\mathbf{x}]_n \in \{0, 1, i+2\}$ and $[\mathbf{y}]_n \notin \{0, 1, i+2\}$, thus $\mathbf{x}$ and $\mathbf{y}$ differ in at least two coordinates.

Let $x, y \notin V(F)$ and x, y are in the same leg. It is clear that $\mathbf{xy} \notin E(G^+)$.

Let $x \in V(L_i) \setminus V(F)$ and $y \in V(L_j) \setminus V(F)$ with $1 \leq i < j \leq n+1$, $x \neq r$, $y \neq r$. If one of them, say x, has a neighbor in $V(F)$, that is, $x = p_{n+1}(i)$, then $[\mathbf{x}]_n = i+2 \neq [\mathbf{y}]_n$. Thus if $\mathbf{xy} \in E(G^+)$, then $\mathbf{x}$ and $\mathbf{y}$ agree on the first $n-1$ coordinates, from which $p_{n+2}(i) = p_{n+2}(j)$ follows, a contradiction. Hence $\mathbf{xy} \notin E(G^+)$. Finally, assume that neither x nor y has a neighbor in $V(F)$. If $j \leq n$, then $[\mathbf{x}]_i = 1 \neq [\mathbf{y}]_i$, $[\mathbf{y}]_j \in \{1, n+2\}$, and $[\mathbf{x}]_j \notin \{1, n+2\}$. If $j = n+1$, then $[\mathbf{y}]_1 = 2 \neq [\mathbf{x}]_1$ and either $[\mathbf{y}]_2 = 2 \neq [\mathbf{x}]_2$ or $[\mathbf{y}]_n = n+3 \neq [\mathbf{x}]_n$.

This proves the claim and (S2).

To prove (S3), we let $f = xy$, where $x = p_1(n)$ and $y = p_1(n+1)$, that is, x and y are neighbors of r on legs L_n and L_{n+1} respectively. We shall show that G contains an induced copy of $H^+ = H + f$, using the coloring criterion (Theorem 7).
Since $\dim(F) \leq n-1$, there is a nice coloring c' of $E(F)$ with colors from $[n-1]$. We shall define a coloring c of $E(H^+)$ with colors from $[n]$. Let $c(e) = c'(e)$ for all $e \in E(F)$.

Let $c(L_i)$ be the ordered list of colors on the leg L_i starting with an edge incident to r, that is, $c(L_i) = \Big(c\big(e_1(i)\big), c\big(e_2(i)\big), \ldots, c\big(e_{n+2}(i)\big)\Big)$ where $e_j(i) = p_{j-1}(i)p_j(i)$ for $j \in [n+2]$ are the edges of L_i. For $n = 3$, we illustrate the coloring of the legs L_1, L_2, L_3, L_4 of T in Figure 4. For $n \geq 4$, define the coloring c on the legs as follows:

$$\begin{aligned}
c(L_1) &= (\ 1, \quad 2, \quad \ldots, n-1, \quad n, \quad 1, \quad n), \\
c(L_2) &= (\ 2, \quad 3, \quad \ldots, \quad n, \quad 1, \quad 2, \quad n), \\
&\vdots \\
c(L_{n-1}) &= (n-1,\ n,\ 1, \ldots,\ n-3,\ n-2,\ n-1,\ n), \\
c(L_n) &= (\ n, \quad 1,\ 2, \ldots, n-2, \quad n, \quad n-1,\ n), \\
c(L_{n+1}) &= (\ n, \quad 1,\ 2, \ldots, n-2,\ n-1, \quad 1, \quad n).
\end{aligned}$$

Color xy with n, so, we have $c(rx) = c(ry) = c(xy) = n$. Note that each leg contains all the colors from $[n]$ and ends with the same color n.

Claim. The coloring c is a nice coloring.

The vertex set $\{r, x, y\}$ induces a triangle of color n. Any other triangle is in F and is monochromatic as c' is a nice coloring.
Let u, v be any two distinct, non-adjacent vertices of H^+. We need to show that there are two distinct colors that appear on each induced u, v-path.

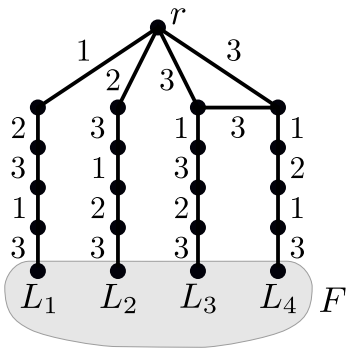


Figure 4: The coloring c of the legs L_1, L_2, L_3, L_4 when $H \in \mathcal{F}(3)$.

Case 1. $u, v \in V(F)$.
Since c' is a nice coloring, there is a pair of distinct colors i, j appearing on all induced u, v-paths in F. Any other induced u, v-path in H^+ contains all the edges of two legs of T (except for maybe rx, ry) and so it contains all the colors from $[n]$. Thus the colors i and j appear on such a path as well.

Case 2. $u, v \in V(L_j)$, $j \in [n+1]$.
Any induced u, v-path is either a subpath P of L_j or contains all the edges of a leg L_i, $i \neq j$ (except for maybe rx, ry) and thus it contains all the colors from $[n]$. Since P contains at least two distinct colors, these two colors are present on each induced u, v-path.

Case 3. $u, v \notin V(F)$, $u \in V(L_i)$, $v \in V(L_j) \setminus V(L_i)$, $1 \leq i < j \leq n+1$.
Then $v \neq r$ and we can assume that $u \neq r$, otherwise we are in Case 2. Let $u' \in V(L_i) \cap V(F)$, $v' \in V(L_j) \cap V(F)$ be leaves of L_i and L_j, respectively. Let P'_u be the subpath of L_i from u to r and P''_u be subpath of L_i from u to u'. Similarly, let P'_v be the subpath of L_j from v to r and P''_v, subpath of L_j from v to v'. See Figure 5.
If an induced u, v-path contains an edge of a leg that is not in $E(L_i \cup L_j)$, then it is easy to see that this path contains all the colors from $[n]$. So, we can consider only u, v-paths that are either equal to $P'_u \cup P'_v$ (or $((P'_u \cup P'_v) - r) \cup \{xy\}$ if $i = n$, $j = n+1$), or a union of $P''_u \cup P''_v$ and a u', v'-path in F.

Assume first that u is adjacent to u' or v is adjacent to v'.
Without loss of generality, u is adjacent to u'. Then P'_u contains all the colors from $[n]$ and P''_u contains the color n. By the niceness of c' and Observation 6, there is a color $\ell \in [n-1]$ that appears on each induced u', v'-path in F and so the union of $P''_u \cup P''_v$ and any u', v'-path in F contains the colors ℓ, n. Thus all induced u, v-paths contain the colors ℓ and n.

Assume that u is not adjacent to u' and v is not adjacent to v'.
If $j < n+1$, then P'_u, P''_u contain color i and P'_v, P''_v contain color j, thus all induced u, v-paths contain the colors i and j.
If $j = n+1$ and v is not adjacent to r, then each of P'_v and P''_v contains the colors $1, n$ and so the colors $1, n$ appear on any induced u, v-path.
If $j = n+1$ and v is adjacent to r, we distinguish two cases based on i. If $i < n$, then each

of $P'_u \cup P'_v$ and $P''_u \cup P''_v$ contains the colors i, n and thus all induced u, v-paths contain the colors i and n. If $i = n$, then u is not adjacent to r, otherwise $uv \in E(H^+)$, a contradiction. Hence the colors $1, n$ appear on each of $P'_u \cup P'_v$ and $P''_u \cup P''_v$ and so all induced u, v-paths contain the colors 1 and n.

Case 4. $v \in V(L_j)$, $j \in [n+1]$ and $u \in V(F)$.
We can assume that $v \notin V(F)$ and $u \notin V(L_j)$ otherwise we are in Case 1 or in Case 2, respectively. Let $v' \in V(L_j) \cap V(F)$ be a leaf of L_j and P be the subpath of L_j from v to v'. Then P contains the color n. By the niceness of c' and Observation 6, there is a color $\ell \in [n-1]$ that appears on each induced u, v'-path in F. Let P' be an induced u, v-path that is not the union of P and an induced u, v'-path in F. Then P contains all the edges of a leg L_i, $i \neq j$ (except for maybe rx, ry) and thus it contains all the colors from $[n]$. Hence the colors ℓ and n appear on all induced u, v-paths.

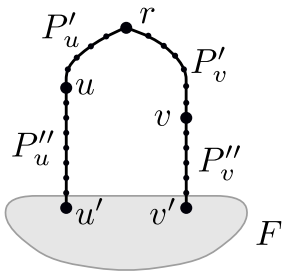


Figure 5: Illustration for the proof of Theorem B.

This concludes the proof of Claim, (S3), and the proof of Theorem B. □

Proof of Corollary 3. Let $v \in V(H)$ be the (unique) vertex of maximum degree $n+1$, $F = H - v$ and $F' \subseteq H'$ be an $(n+1)$-subdivision of F, so $V(F) \subseteq V(F')$. To apply Theorem B, we need to verify that $H' \in \mathcal{F}(n)$. For that it suffices to show that F' has dimension at most $n-1$. We prove this by defining a nice coloring $c : E(F') \to [n-1]$ and using the coloring criterion (Theorem 7). The maximum degree of F' is at most $n-1$. Denote the set of paths of length $n+2$ that have both endpoints in $V(F)$ by $\mathcal{P}$, that is, $\mathcal{P}$ is the set of paths that arise from the subdivision of an edge of F. Color all edges of F' incident to vertices corresponding to vertices of F with at most $n-1$ colors properly. This gives a coloring of the end-edges of each path in $\mathcal{P}$. For a path $P = v_1, v_2, \ldots, v_{n+2}$ and an edge-coloring c, we use the notation $c(P) = \big(c(v_1v_2), c(v_2v_3), \ldots, c(v_{n+1}v_{n+2})\big)$. Let $P \in \mathcal{P}$. If the end-edges of P have different colors i, j with $i > j$, then let $c(P) = (i, 1, 2, 3, \ldots, n-1, x, j)$, where $x \neq j$ and $x \neq n-1$ (such an x exists, since $n-1 \geq 3$). If the end-edges of P have the same color i, then let $c(P) = (i, 1, 2, 3, \ldots, n-1, 1, i)$ if $i \neq 1$ and $c(P) = (1, 2, 1, 3, 4, \ldots, n-1, 2, 1)$ if $i = 1$. Note that any path in $\mathcal{P}$ contains all the colors from $[n-1]$.
Now we verify that c is a nice coloring. Since F' is triangle-free, we only need to check that for any two nonadjacent vertices u, v, there are distinct colors i, j that appear in any induced u, v-path. Let P_u, P_v be some paths in $\mathcal{P}$ such that $u \in V(P_u)$ and $v \in V(P_v)$.
If $V(P_u) \cap V(P_v) = \emptyset$, then any u, v-path contains an entire path from $\mathcal{P}$ and thus all the

colors from $[n-1]$.
If $V(P_u) \cap V(P_v) = \{w\}$, then w is an endpoint of both paths, that is, it corresponds to a vertex of F and so the end-edges of P_u and P_v that are incident to w are colored with distinct colors, i, j. Any other u, v-path contains an entire path from $\mathcal{P}$ and thus all the colors from $[n-1]$.
If $|V(P_u) \cap V(P_u)| \geq 2$, then $P_u = P_v$ and since c is a proper coloring, there are distinct colors i, j that appear in the shortest u, v-path (which is a subpath of $P_u = P_v$). Any other u, v-path contains an entire path from $\mathcal{P}$ and thus all the colors from $[n-1]$. $\square$

5 Graphs induced-saturated in the Cartesian product of two cliques.

The aim of this section is to give a complete characterization of connected graphs H so that $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$ for some $k \in \mathbb{N}$:

Theorem 12. *Let H be a connected graph. There is $k \in \mathbb{N}$ such that $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$ if and only if H is either chipped or $H \in \mathcal{A} \cap \mathcal{B}$.*

To define the notion of chipped graphs and classes $\mathcal{A}$ and $\mathcal{B}$ we need some definitions.

5.1 Definitions

We use the notation $|H|$ for the number of vertices of a graph H and $\{\{X\}\}$ for the complete graph on the vertex set X. We call a graph H 2-*Hamming* if there exists $k \in \mathbb{N}$ so that $H \subseteq_I K_k \square K_k$. We say that $\{F_1, F_2\}$ is a 2-*Hamming decomposition* of a graph H if $V(F_1) = V(F_2) = V(H)$, $\{E(F_1), E(F_2)\}$ is a partition of $E(H)$, and F_i is a pairwise vertex-disjoint union of cliques for $i \in \{1, 2\}$. Note that a clique in F_1 intersects any clique in F_2 in at most one vertex. The coloring criterion (Theorem 7) implies that a graph is 2-Hamming if and only if it has a nice coloring with two colors. It is easy to see that a 2-Hamming decomposition corresponds to a nice coloring with colors 1 and 2, such that the edges of F_1 have color 1 and the edges of F_2 have color 2. Thus a graph is 2-Hamming if and only if it has a 2-Hamming decomposition. One can show that $\min\{k : H \subseteq_I K_k \square K_k\}$ is equal to the smallest integer ℓ such that H has a 2-Hamming decomposition $\{F_1, F_2\}$ such that each of F_1 and F_2 has at most ℓ connected components. We prove this fact formally in Lemma 17. For a graph G, let $\mathcal{C}(G)$ denote the number of connected components of G.

Definition 13. *Let H be a 2-Hamming graph. We say that a 2-Hamming decomposition $\{F_1, F_2\}$ of H is* ***optimal****, if $\min\{k : H \subseteq_I K_k \square K_k\} = \max_{i \in \{1,2\}} \mathcal{C}(F_i)$. We call H* ***balanced*** *if $\mathcal{C}(F_1) = \mathcal{C}(F_2)$ for an optimal 2-Hamming decomposition $\{F_1, F_2\}$ of H and* ***unbalanced*** *otherwise.*

Definition of chipped graphs

We call a graph H *chipped* if $H = H' - e$, where H' is a connected, 2-Hamming graph with a 2-Hamming decomposition $\{F_1, F_2\}$, such that F_1 has a component K of order 3 or 4 that contains the endpoints u, v of e and one of the following holds:

(ch1) $V(K) = \{u, v, w\}$, $\deg_{H'}(w) \geq 3$;

(ch2) $V(K) = \{u, v, w\}$ and $H' - E(K)$ has two components, one of which is $\{\{w\}\}$;

(ch3) $V(K) = \{u, v, w, x\}$, $\deg_{H'}(w) = 3$ and $H' - E(K)$ has a connected component $\widetilde{H}$ so that $V(K) \cap V(\widetilde{H}) = \{u\}$.

See Figure 6 for an illustration.

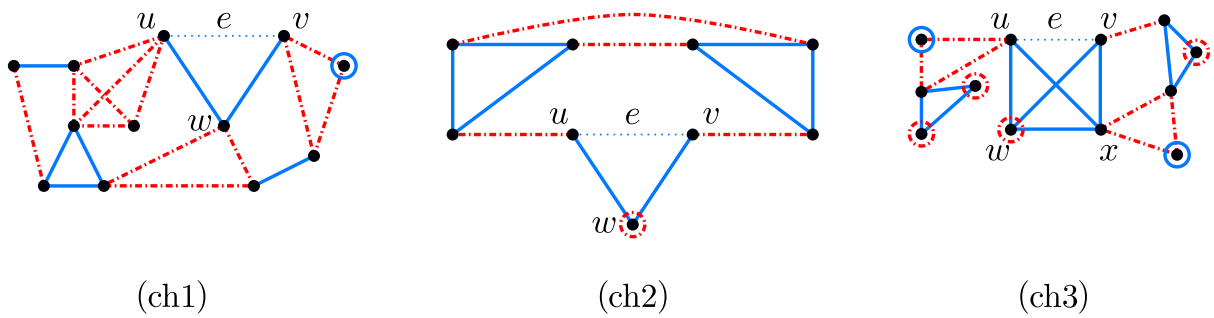


Figure 6: Examples of chipped graphs. Colors indicate a nice 2-coloring of H', each color also corresponds to edges and singleton cliques of F_i, $i \in \{1, 2\}$.

Definition of the graph classes $\mathcal{A}$ and $\mathcal{B}$

Let H be a 2-Hamming graph with 2-Hamming decomposition $\{F_1, F_2\}$. For a subgraph H' of H and $i \in \{1, 2\}$, let $F_i|H'$ denote the subgraph of F_i that is the union of those components of F_i whose vertex sets are completely contained in $V(H')$.

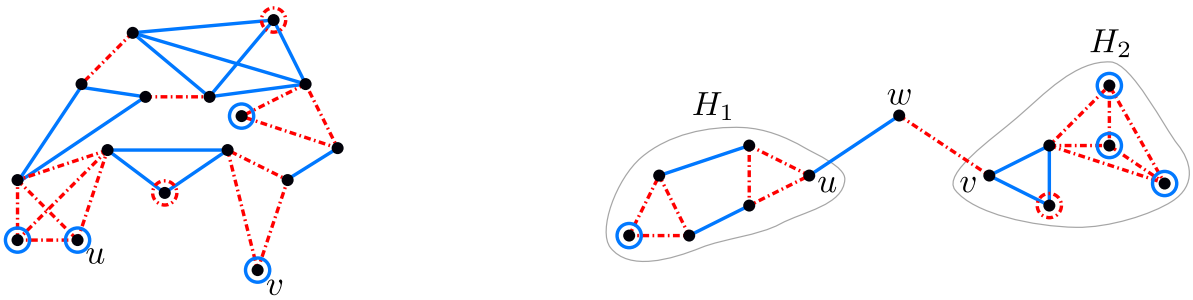
Let $\mathcal{A}$ be the class of connected, unbalanced, 2-Hamming graphs H that have an optimal 2-Hamming decomposition $\{F_1, F_2\}$ with $\mathcal{C}(F_1) > \mathcal{C}(F_2)$ such that H has two nonadjacent vertices u, v, for which one of the following holds:

(a1) $\deg_{F_1}(u) = \deg_{F_1}(v) = 0$;

(a2) there is a vertex $w \in N(u) \cap N(v)$ such that $H - \{uw, wv\}$ has 3 components H_1, H_2, H_3 with $\mathcal{C}(F_1|H_1) > \mathcal{C}(F_2|H_1)$, $\mathcal{C}(F_1|H_2) > \mathcal{C}(F_2|H_2)$, and $H_3 = \{\{w\}\}$.

See Figure 7 for an illustration.

Let $\mathcal{B}$ be the class of connected, unbalanced, 2-Hamming graphs H that have an optimal 2-Hamming decomposition $\{F_1, F_2\}$ with $\mathcal{C}(F_1) > \mathcal{C}(F_2)$ such that H has two adjacent vertices u, v for which one of the following holds:



(a1) $\mathcal{C}(F_1) = 8$ and $\mathcal{C}(F_2) = 7$.

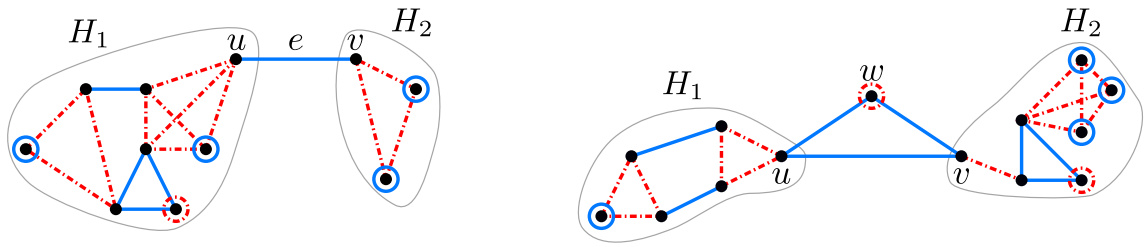
(a2) $\mathcal{C}(F_1) = 8$, $\mathcal{C}(F_2) = 5$, $\mathcal{C}(F_1|H_1) = 3$, $\mathcal{C}(F_2|H_1) = 2$, $\mathcal{C}(F_1|H_2) = 4$, $\mathcal{C}(F_2|H_2) = 2$.

Figure 7: Examples of graphs from class $\mathcal{A}$.

(b1) the edge $e = uv$ is a bridge with $e \in E(F_i)$ so that the components H_1, H_2 of $H - e$ satisfy $\mathcal{C}(F_1|H_1) \geq \mathcal{C}(F_2|H_1) + i$ and $\mathcal{C}(F_1|H_2) \geq \mathcal{C}(F_2|H_2) + i$.

(b2) there is a vertex $w \in N(u) \cap N(v)$, that is, $\{u, v, w\}$ induce a triangle T in H, and $H - E(T)$ has 3 components H_1, H_2, H_3 with $\mathcal{C}(F_1|H_1) > \mathcal{C}(F_2|H_1)$, $\mathcal{C}(F_1|H_2) > \mathcal{C}(F_2|H_2)$, and $H_3 = \{\{w\}\}$.

See Figure 8 for an illustration.



(b1) $\mathcal{C}(F_1) = 7$, $\mathcal{C}(F_2) = 4$, $\mathcal{C}(F_1|H_1) = 4$, $\mathcal{C}(F_2|H_1) = 3$, $\mathcal{C}(F_1|H_2) = 2$, $\mathcal{C}(F_2|H_2) = 1$.

(b2) $\mathcal{C}(F_1) = 8$, $\mathcal{C}(F_2) = 6$, $\mathcal{C}(F_1|H_1) = 3$, $\mathcal{C}(F_2|H_1) = 2$, $\mathcal{C}(F_1|H_2) = 4$, $\mathcal{C}(F_2|H_2) = 3$.

Figure 8: Examples of graphs from class $\mathcal{B}$.

5.2 Preliminary Lemmas

First we give a criterion for $G \hookrightarrow\!\!\!\!\rightarrow H$ in the special case when G is the Cartesian product of two cliques.

Lemma 14. *Let $k \in \mathbb{N}$, $G = K_k \square K_k$ and H be a graph. Then $G \hookrightarrow\!\!\!\!\rightarrow H$ if and only if*

(s1) $H \not\subseteq_I G$,

(s2) *there is an edge $e \in E(H)$ such that $H - e \subseteq_I G$ and*

(s3) *there is an edge $f \in E(\overline{H})$ such that $H + f \subseteq_I G$.*

Proof. By Lemma 10, $G \hookrightarrow\!\!\!\!\rightarrow H$ if and only if
(S1) $H \not\subseteq_I G$,
(S2) for any $e \in E(\overline{G})$, $\quad H \subseteq_I G + e$, and
(S3) there is an edge $f \in E(\overline{H})$ such that $H + f \subseteq_I G$.
Thus it is sufficient to prove that if $H \not\subseteq_I G$, then (s2) is equivalent to (S2).

Assume that $H \not\subseteq_I G$ and (S2) holds. Let $e = uv \in E(\overline{G})$ and let $\sigma : V(H) \to V(G)$ be an induced embedding of H into $G + e$. Since $H \not\subseteq_I G$, the vertex set $\sigma(V(H))$ has to contain both u and v. Since σ is an induced embedding, $\sigma^{-1}(u)\sigma^{-1}(v) \in E(H)$. Let $e' = \sigma^{-1}(u)\sigma^{-1}(v)$, then σ is an induced embedding of $H - e'$ into G, that is, (s2) holds.

Now assume that $H \not\subseteq_I G$ and (s2) holds. Let $e' = u'v' \in E(H)$ so that $H - e' \subseteq_I G$ and let $\sigma : V(H) \to V(G)$ be an induced embedding of $H - e'$ into G. Since σ is an induced embedding, $\sigma(u')\sigma(v') \notin E(G)$. Let $f = \sigma(u')\sigma(v')$, then σ is an induced embedding of H into $G + f$, that is, there exists an edge $f \in E(\overline{G})$ such that $H \subseteq_I G + f$. Since the distance of two distinct vertices in G is either 1 or 2, any pair of nonadjacent vertices have distance 2. Thus Lemma 9 implies that if there is an edge $f \in E(\overline{G})$ such that $H \subseteq_I G + f$, then $H \subseteq_I G + e$ for any $e \in E(\overline{G})$, that is, (S2) holds. □

Lemma 15. *Let H be a 2-Hamming graph, $\{F_1, F_2\}$ be a 2-Hamming decomposition of H and K be a component in F_1. Then for any $v \in V(H) \setminus V(K)$, v is adjacent to at most one vertex of K. In particular, for any vertex/edge of H, any maximal clique in H containing this vertex/edge is a component of either F_1 or F_2.*

Proof. Suppose that $v \in V(H) \setminus V(K)$ is adjacent to two distinct vertices $u, u' \in V(K)$. Then $uu' \in E(F_1)$ and $uv, u'v \in E(F_2)$. Thus u and u' are in the same connected component of F_2. Since each component of F_2 is a clique, $uu' \in E(F_2)$, a contradiction since $uu' \in E(F_1)$. □

Lemma 16. *Let $k \geq 2$. A graph that contains an induced copy of $K_{1,3}$, $K_4 - e$, or C_{2k+1} is not 2-Hamming.*

Proof. It is easy to see that the listed graphs do not have a nice coloring with 2 colors, thus by the coloring criterion (Theorem 7) their dimension is at least 3. Therefore, any graph that contains an induced copy of them has dimension at least 3, that is, is not 2-Hamming. □

Lemma 17. *For any 2-Hamming graph H,*

$$\min\{k : H \subseteq_I K_k \square K_k\} = \min_{\{F_1,F_2\}} \max_{i\in\{1,2\}} \mathcal{C}(F_i),$$

where the minimum on the right hand side is taken over all $\{F_1, F_2\}$ 2-Hamming decompositions of H.

Proof. Let $c = \min\big\{\max\{\mathcal{C}(F_1), \mathcal{C}(F_2)\} : \{F_1, F_2\}$ is a 2-Hamming decomposition of $H\big\}$ and k be the smallest integer such that $H \subseteq_I K_k \square K_k$.

We show that $k \leq c$ by proving that $H \subseteq_I K_c \square K_c$. Let $\{F_1, F_2\}$ be a 2-Hamming decomposition of G so that $c = \max_{i\in\{1,2\}} \mathcal{C}(F_i)$. Without loss of generality, $c = \mathcal{C}(F_1) \geq \mathcal{C}(F_2)$. Let $C_1^1, C_2^1, \ldots, C_c^1$ be the components of F_1 and $C_1^2, C_2^2, \ldots, C_d^2$ be the components of F_2 with $d = \mathcal{C}(F_2) \leq c$. Since F_1 and F_2 are spanning subgraphs, each vertex of G belongs to exactly one component of F_1 and exactly one component of F_2.
We define an embedding σ of H into $K_c \square K_c$ as follows: for any $v \in V$, let $\sigma(v) = (i, j)$ if $v \in C_i^1$ and $v \in C_j^2$. Since any component of F_1 intersects a component of F_2 in at most one vertex, σ is injective. Moreover, σ is an induced embedding, since

$$\begin{aligned} uv \in E(H) &\iff uv \in E(F_1) \text{ or } uv \in E(F_2) \\ &\iff u \text{ and } v \text{ are in the same component either in } F_1 \text{ or in } F_2 \\ &\iff \mathrm{dist}_{\mathrm{Ham}}(\sigma(u), \sigma(v)) = 1 \iff \sigma(u)\sigma(v) \in E(K_c \square K_c). \end{aligned}$$

We show that $k \geq c$ by defining a 2-Hamming decomposition $\{F_1, F_2\}$ of $H = (V, E)$ such that $\max\{\mathcal{C}(F_1), \mathcal{C}(F_2)\} \leq k$. Let σ be an induced embedding of H into $K_k \square K_k$. Let $E = E_1 \cup E_2$ such that $uv \in E_i$ if $\sigma(u)$ and $\sigma(v)$ differ exactly in the ith coordinate, $i \in \{1, 2\}$. Then $F_1 = (V, E_1)$ and $F_2 = (V, E_2)$ form a 2-Hamming decomposition of H. We see that two distinct vertices are nonadjacent in F_i if and only if they differ in the $(3 - i)^{\text{th}}$ coordinate. Therefore there are at most k pairwise nonadjacent vertices in F_i, that is the size of the maximal independent set in F_i is at most k. It implies that F_i has at most k connected components, thus $\max_{i\in\{1,2\}} \mathcal{C}(F_i) \leq k$ and so $c \leq k$. □

The following lemma is a consequence of Lemma 15 and the coloring criterion (Theorem 7).

Lemma 18. *The 2-Hamming decomposition of a connected graph is unique. In particular, if a 2-Hamming graph H has maximal cliques K, K' and two 2-Hamming decompositions $\{F_1, F_2\}$, $\{F_1^*, F_2^*\}$ such that K and K' belong to the same family in one and to different families in the other, that is, $K, K' \subseteq F_i$ and $K \subseteq F_i^*$, $K' \subseteq F_{3-i}^*$ then H is disconnected and K and K' belong to distinct connected components of H.*

5.3 Proof of Theorem 12

We call a graph H *good* if $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$ for some $k \in \mathbb{N}$.

Recall that Theorem 12 claims that a connected graph H is good if and only if it is either chipped or contained in $\mathcal{A} \cap \mathcal{B}$.

Let H be a connected graph. We shall prove a more precise statement: if $\dim(H) > 2$, then H is good if and only if H is chipped; if $\dim(H) \leq 2$, then H is good if and only if $H \in \mathcal{A} \cap \mathcal{B}$.

Claim 1. If H is chipped then it is good.
Let H be chipped with H', e, K, u, v, w, x and $\widetilde{H}$ as in the definition of chipped graphs. Let $\{F_1', F_2'\}$ be a 2-Hamming decomposition of H' and assume, without loss of generality, that $uv \in E(F_1')$. To show that H is good, it is sufficient to prove that H is not 2-Hamming and that there are suitable edges $e \in E(H)$, $f \in E(\overline{H})$ so that $\dim(H - e) \leq 2$ and $\dim(H + f) \leq 2$. Indeed, if $\dim(H - e) \leq 2$ and $\dim(H + f) \leq 2$, then there are integers $l, m \in \mathbb{N}$ so that $H - e \subseteq_I K_l \square K_l$, $H + f \subseteq_I K_m \square K_m$, hence for any $k \geq l, m$ we have $H \not\subseteq_I K_k \square K_k$, $H - e \subseteq_I K_k \square K_k$, and $H + f \subseteq_I K_k \square K_k$, thus by Lemma 14, $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$. First we show that H is not 2-Hamming by finding an induced copy of a not 2-Hamming graph in H and applying Lemma 16.

If (ch1) holds, then by Lemma 15, H contains an induced a star $K_{1,3}$ centered at w. If (ch2) holds, consider a shortest u, v-path P in $H' - E(K)$. Then P is an induced path and so it is F_1'-F_2'-alternating with its first and last edge in $E(F_2')$. Hence P is an induced odd u, v-path and thus H contains an induced odd cycle of length at least 5 formed by the edges vw, wu and the edges of P. If (ch3) holds, then $V(K)$ induces a graph isomorphic to K_4 minus an edge. We see by Lemma 16, that in any of the above cases H is not 2-Hamming.

By the definition of chipped graphs, adding the edge $f = uv$ to H results the 2-Hamming graph H'.

If (ch1) or (ch2) holds, then deleting the edge uw results a 2-Hamming graph. Indeed, $F_1^- = (F_1' - K) \cup \{\{v, w\}\} \cup \{\{u\}\}$, $F_2^- = F_2'$ form a 2-Hamming decomposition of $H - uw$. If (ch3) holds, then deleting ux results a 2-Hamming graph since $F_1^- = (F_1' - K - (F_1'|\widetilde{H})) \cup (\{\{w, v, x\}\} \cup (F_2'|\widetilde{H}))$, $F_2^- = (F_2' - \{\{w\}\} - (F_2'|\widetilde{H})) \cup (\{\{u, w\}\} \cup (F_1'|\widetilde{H}))$ form a 2-Hamming decomposition of $H - ux$, see Figure 9.

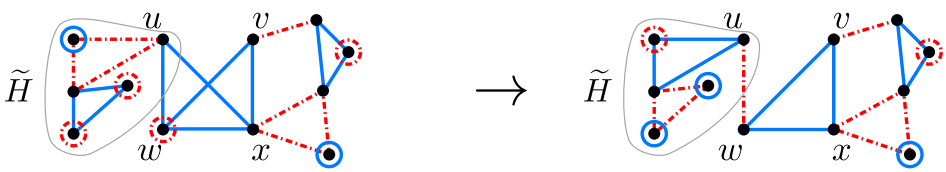


Figure 9: Illustration for the proof of Theorem 12 (Claim 1, case (ch3)).

Thus H is good.

Claim 2. If $H \in \mathcal{A} \cap \mathcal{B}$ then H is good.
If $H \in \mathcal{A} \cap \mathcal{B}$, then H is an unbalanced 2-Hamming graph. Let k be the smallest integer such that $H \subseteq_I K_{k+1} \square K_{k+1}$.
We shall show that $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$ by using Lemma 14. In particular, we shall prove that
(s1) $H \not\subseteq_I K_k \square K_k$,
(s2) there is an edge $e \in E(H)$ such that $H - e \subseteq_I K_k \square K_k$, and
(s3) there is an edge $f \in E(\overline{H})$ such that $H + f \subseteq_I K_k \square K_k$.

By the choice of k, $H \not\subseteq_I K_k \square K_k$, so (s1) follows. We shall prove that (s2) is implied

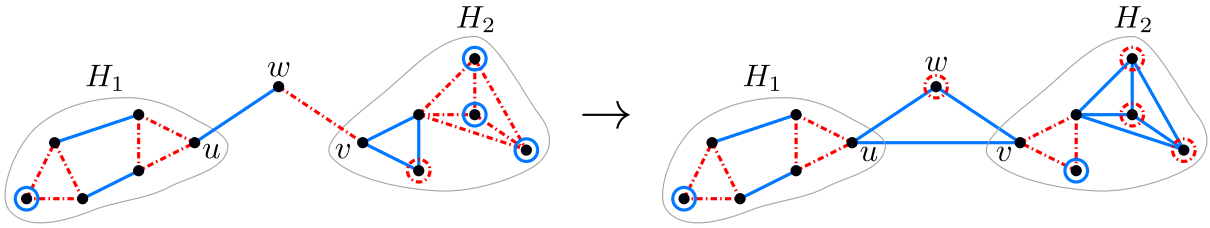


Figure 10: Illustration for the proof of Theorem 12 (Claim 2, case of H^+, (a2)).

by the fact that $H \in \mathcal{B}$ and (s3) is implied by the fact that $H \in \mathcal{A}$. Let $\{F_1, F_2\}$ be an optimal 2-Hamming decomposition of H with $\mathcal{C}(F_1) = k+1$ and $\mathcal{C}(F_2) \leq k$, see Lemma 17.

Let u, v, H_2 be as in the definition of the classes $\mathcal{A}$ and $\mathcal{B}$. The rough idea of the proof of this claim is that if we add or delete the edge uv, then we can "swap the decomposition in H_2", that is, exchange $F_1|H_2$ and $F_2|H_2$. Then using the unbalancedness assumptions on $\{F_1|H_2, F_2|H_2\}$, we show that the resulting decomposition has at most k components in both families and thus by Lemma 17, $K_k \square K_k$ contains an induced copy of $H+f$ or $H-e$ respectively.

Let $H \in \mathcal{A}$. Let $f = uv$, where u, v are vertices as in the definition of class $\mathcal{A}$ and let $H^+ = H + f$. We shall construct a 2-Hamming decomposition $\{F_1^+, F_2^+\}$ of H^+.

If (a1) holds, then let $F_2^+ = F_2$ and let F_1^+ be obtained from F_1 by replacing the singleton components induced by u and v with $\{\{u, v\}\}$. Then $\mathcal{C}(F_1^+) = \mathcal{C}(F_1) - 1 = k$ and $\mathcal{C}(F_2^+) = \mathcal{C}(F_2) \leq k$, implying that $H + f \subseteq_I K_k \square K_k$.

Now assume that (a2) holds and let w, H_1 and H_2 be as in the definition of class $\mathcal{A}$. Then by Lemma 15, $\{\{u, w\}\}$ and $\{\{v, w\}\}$ are a components of F_1 or F_2. Without loss of generality, $\{\{u, w\}\}$ is a component of F_1 and $\{\{v, w\}\}$ is a component of F_2. Define $F_1^+ = (F_1|H_1) \cup (F_2|H_2) \cup \{\{u, v, w\}\}$ and $F_2^+ = (F_2|H_1) \cup (F_1|H_2) \cup \{\{w\}\}$, see Figure 10. Then

$$\mathcal{C}(F_1^+) = \mathcal{C}(F_1|H_1) + \mathcal{C}(F_2|H_2) + 1 < \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) + 1 = \mathcal{C}(F_1) = k+1 \text{ and}$$
$$\mathcal{C}(F_2^+) = \mathcal{C}(F_2|H_1) + \mathcal{C}(F_1|H_2) + 1 < \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) + 1 = \mathcal{C}(F_1) = k+1$$

Thus $\max\{\mathcal{C}(F_1^+), \mathcal{C}(F_2^+)\} \leq k$, implying that $H + f \subseteq_I K_k \square K_k$.

Let $H \in \mathcal{B}$. Let $e = uv$, where u, v are vertices as in the definition of class $\mathcal{B}$ and let $H^- = H - e$. We shall construct a 2-Hamming decomposition $\{F_1^-, F_2^-\}$ of H^-.

Let (b1) hold for H and let H_1, H_2 be the components of $H - e$. Then without loss of generality, $u \in V(H_1)$ and $v \in V(H_2)$.

If $e \in E(F_1)$, we have $\mathcal{C}(F_1|H_1) \geq \mathcal{C}(F_2|H_1) + 1$ and $\mathcal{C}(F_1|H_2) \geq \mathcal{C}(F_2|H_2) + 1$. Let $F_1^- = (F_1|H_1) \cup (F_2|H_2) \cup \{\{u\}\}$ and $F_2^- = (F_2|H_1) \cup (F_1|H_2) \cup \{\{v\}\}$, see Figure 11. Then

$$\mathcal{C}(F_1^-) = \mathcal{C}(F_1|H_1) + \mathcal{C}(F_2|H_2) + 1 < \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) + 1 = \mathcal{C}(F_1) = k+1 \text{ and}$$

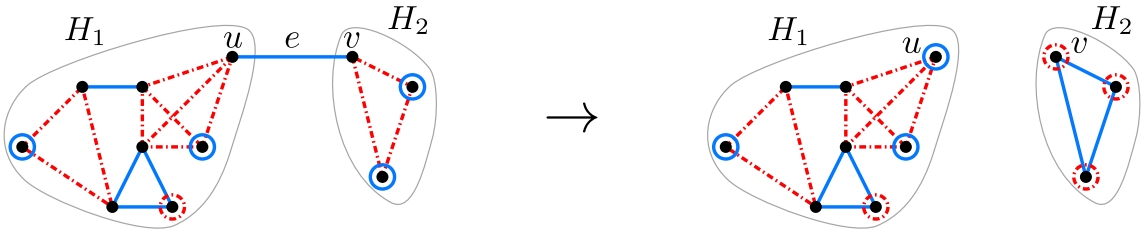


Figure 11: Illustration for the proof of Theorem 12 (Claim 2, case of H^-, (b1)).

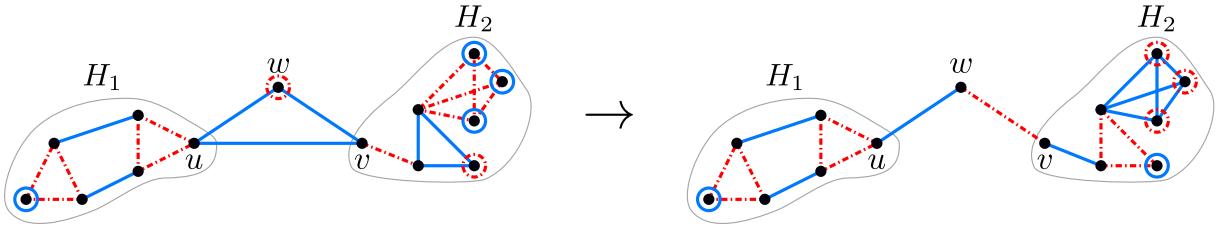


Figure 12: Illustration for the proof of Theorem 12 (Claim 2, case of H^-, (b2)).

$$\mathcal{C}(F_2^-) = \mathcal{C}(F_2|H_1) + \mathcal{C}(F_1|H_2) + 1 < \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) + 1 = \mathcal{C}(F_1) = k + 1.$$

If $e \in E(F_2)$, we have $\mathcal{C}(F_1|H_1) \geq \mathcal{C}(F_2|H_1) + 2$ and $\mathcal{C}(F_1|H_2) \geq \mathcal{C}(F_2|H_2) + 2$. Let $F_1^- = (F_1|H_1) \cup (F_2|H_2) \cup \{\{v\}\}$ and $F_2^- = (F_2|H_1) \cup (F_1|H_2) \cup \{\{u\}\}$. Then

$$\mathcal{C}(F_1^-) = \mathcal{C}(F_1|H_1) + \mathcal{C}(F_2|H_2) + 1 < \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) = \mathcal{C}(F_1) = k + 1 \text{ and}$$
$$\mathcal{C}(F_2^-) = \mathcal{C}(F_2|H_1) + \mathcal{C}(F_1|H_2) + 1 < \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) = \mathcal{C}(F_1) = k + 1.$$

In both cases, $\max\{\mathcal{C}(F_1^-), \mathcal{C}(F_2^-)\} \leq k$, thus $H - e \subseteq_I K_k \square K_k$.

Let (b2) hold for H and let w, T, H_1, H_2 be as in the definition of class $\mathcal{B}$.

If T is a component of F_1, then let $F_1^- = (F_1|H_1) \cup (F_2|H_2) \cup \{\{u, w\}\}$ and $F_2^- = (F_2|H_1) \cup (F_1|H_2) \cup \{\{v, w\}\}$, see Figure 12.

If T is a component of F_2, then let $F_1^- = (F_1|H_1) \cup (F_2|H_2) \cup \{\{v, w\}\}$ and $F_2^- = (F_2|H_1) \cup (F_1|H_2) \cup \{\{u, w\}\}$.
In both cases,

$$\mathcal{C}(F_1^-) = \mathcal{C}(F_1|H_1) + \mathcal{C}(F_2|H_2) + 1 < \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) + 1 = \mathcal{C}(F_1) = k + 1 \text{ and}$$
$$\mathcal{C}(F_2^-) = \mathcal{C}(F_2|H_1) + \mathcal{C}(F_1|H_2) + 1 < \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) + 1 = \mathcal{C}(F_1) = k + 1.$$

Thus $\max\{\mathcal{C}(F_1^-), \mathcal{C}(F_2^-)\} \leq k$, implying that $H - e \subseteq_I K_k \square K_k$.

To summarize, we have that $H \not\subseteq_I K_k \square K_k$, $H - e \subseteq_I K_k \square K_k$, and $H + f \subseteq_I K_k \square K_k$. Thus by Lemma 14, H is good. This completes the proof of Claim 2.

Claim 3. If H is good then H is either chipped or $H \in \mathcal{A} \cap \mathcal{B}$.

Let H be a good graph. By Lemma 14, there exists $k \in \mathbb{N}$ so that $H \not\subseteq_I K_k \square K_k$ and there are edges $e \in E(H)$ and $f \in E(\overline{H})$ such that $H - e \subseteq_I K_k \square K_k$ and $H + f \subseteq_I K_k \square K_k$. In

particular, $H-e$ and $H+f$ are 2-Hamming. Since $H \not\subseteq_I K_k \square K_k$, we have that either H is not 2-Hamming, that is, $\dim(H) > 2$, or in any 2-Hamming decomposition of H, one of the families has at least $k+1$ components, see Lemma 17.
Let $\{F_1^+, F_2^+\}$ be an optimal 2-Hamming decomposition of $H+f$ and $\{F_1^-, F_2^-\}$ be an optimal 2-Hamming decomposition of $H-e$. By Lemma 17, $\max\{\mathcal{C}(F_1^+), \mathcal{C}(F_2^+)\} \leq k$ and $\max\{\mathcal{C}(F_1^-), \mathcal{C}(F_2^-)\} \leq k$.
Let $K(e)$ be the maximal clique of H that contains e and $K(f)$ be the maximal clique of $H+f$ that contains f. Recall that by Lemma 16, $H-e$ cannot contain an induced copy of K_4 minus an edge. It implies that $|K(e)| \leq 3$, and that $|K(f)| \leq 4$ since otherwise H contains a copy of K_5 minus an edge and thus $H-e'$ contains an induced K_4 minus an edge for any $e' \in E(H)$.

Claim 3.1. If H is good and $\dim(H) > 2$ then H is chipped.
Assume that H is good and $\dim(H) > 2$. We shall show that H is chipped.

Without loss of generality, $f \in E(F_1^+)$ and thus $K = K(f)$ is a component of F_1^+.
If $|K| = 2$ then H is 2-Hamming, a contradiction.
If $|K| = 3$, let $K = \{\{u, v, w\}\}$ and $f = uv$. If $\deg(w) \geq 3$, then H is chipped of type (ch1). If $\deg(w) = 2$, then $\{\{w\}\}$ is a component of $H - \{uw, wv\}$ and u, v are in the same connected component of $H - \{uw, wv\}$, otherwise H would be 2-Hamming. Thus H is chipped of type (ch2).

Recall that by Lemma 16, $H-e$ cannot contain an induced copy $K_{1,3}$ or K_4 minus an edge.

Finally let $|K| = 4$. Then e has to be induced by $V(K)$, otherwise $H-e$ contains an induced copy of K_4 minus an edge, a contradiction. Hence $V(K)$ induces exactly 4 edges in $H-e$, that is, $V(K)$ induces either a C_4 or a triangle with a pendant edge in $H-e$.
Let $V(K)$ induce C_4 in $H-e$. We claim that $H-e$ is isomorphic to C_4 then. Otherwise, since H is connected, there is a vertex not in $V(K)$ adjacent to a vertex in $V(K)$. By Lemma 15, it is adjacent to exactly one vertex of K and thus $H-e$ contains an induced copy of $K_{1,3}$, a contradiction. Hence $H-e$ is isomorphic to C_4 and thus H is isomorphic to K_4 minus an edge, so H is chipped of type (ch3).

Let $V(K)$ induce a triangle with a pendant edge in $H-e$. Let $K = \{\{u, v, w, x\}\}$ such that w has degree 3 and u has degree 1 in the graph induced by $V(K)$ in $H-e$. If there is a vertex not in $V(K)$ adjacent to w, then by Lemma 15 (applied to $H+f$), it is not adjacent to any vertex in $V(K) \setminus \{w\}$ and thus there is an induced $K_{1,3}$ in $H-e$ centered in w, a contradiction. Thus $\deg_H(w) = 3$. Let $\{F_1^-, F_2^-\}$ be a 2-Hamming decomposition of $H-e$. By Lemma 15, $\{\{u, w\}\}$ and $\{\{v, w, x\}\}$ are components of F_1^- or F_2^-. Then without loss of generality, $\{\{v, w, x\}\}$ is a component of F_1^- and $\{\{u, w\}\}$ is a component of F_2^-. Let H'' be the graph obtained from H by deleting the edges induced by $V(K)$ and let $K(u), K(v)$ and $K(x)$ denote the maximal cliques in H'' that contain the vertices u, v and x respectively (they might be singletons). Then in $H+f$, $K(u), K(v)$, and $K(x)$

must all belong to F_2^+ but in $H-e$, $K(u)$ belongs to F_1^-, and both $K(v)$, $K(x)$ belong to F_2^-. By Lemma 18, $K(u)$ is not in the same connected component of H'' as $K(v)$ or $K(x)$. Therefore the connected component $\widetilde{H}$ of H'' that contains u satisfies $V(\widetilde{H}) \cap K = \{u\}$ and thus H is chipped of type (ch3).
This proves that H is chipped.

Claim 3.2. If $\dim(H) = 2$ and H is good then $H \in \mathcal{A} \cap \mathcal{B}$.

If $\dim(H) = 2$ and H is good, there exists $k \in \mathbb{N}$ so that $H \not\subseteq_I K_k \square K_k$, $H \subseteq_I K_{k+1} \square K_{k+1}$ and there are edges $e \in E(H)$ and $f \in E(\overline{H})$ such that $H - e \subseteq_I K_k \square K_k$ and $H + f \subseteq_I K_k \square K_k$.

By Lemma 17, it means that in any 2-Hamming decomposition of H, one of the families has at least $k+1$ components, but after deleting or adding a suitable edge, we can decompose H with at most k components in each family. Let $\{F_1, F_2\}$ be a 2-Hamming decomposition of H.

Recall that $K(e)$ denotes the maximal clique of H that contains e and $K(f)$ denotes the maximal clique of $H+f$ that contains f. To avoid redundant arguments for e and f, we introduce a variable $g \in \{e, f\}$. Denote $K(g)$ by K and the endpoints of g by u, v. Let H' be the graph obtained from H by deleting the edges of K. Since H and $H-e$ cannot contain an induced copy of K_4 minus an edge, $2 \leq |K| \leq 3$. If $|K| = 3$, then let $K = \{u, v, w\}$ and observe that $\deg_{H'}(w) = 0$, otherwise it follows from Lemma 15, that H or $H-e$ would contain an induced $K_{1,3}$ centered at w. Let $K(u)$ and $K(v)$ denote the maximal cliques of H' containing u and v respectively. Observe that $N[u] = V(K \cup K(u))$ and $N[v] = V(K \cup K(v))$, otherwise H would contain an induced $K_{1,3}$. For a 2-Hamming decomposition $\{Q_1, Q_2\}$ of H, $H-e$, or $H+f$, let $\{Q_1', Q_2'\}$ denote the 2-Hamming decomposition of $H' = H - E(K)$ such that Q_1' contains $Q_1|H'$ and Q_2' contains $Q_2|H'$ as subgraphs. Note that Q_i' is obtained from $Q_i|H'$ by possibly adding the singleton vertices $\{\{u\}\}$ or $\{\{v\}\}$.

Verification of $H \in \mathcal{A}$.

Let $g = f$. We shall show that H is unbalanced and that (a1) or (a2) holds.

If $|K(f)| = 2$, then since the 2-Hamming decomposition of H is unique, the endpoints of f have to be singletons in F_1 or F_2, say in F_1. Since all the components of $\{F_1, F_2\}$, except for the ones containing the endpoints of f, are the same as in $\{F_1^+, F_2^+\}$, we have $\mathcal{C}(F_1^+) = \mathcal{C}(F_1) - 1$ and $\mathcal{C}(F_2^+) = \mathcal{C}(F_2)$. We also know that $\max\{\mathcal{C}(F_1), \mathcal{C}(F_2)\} = k+1$ and $\max\{\mathcal{C}(F_1^+), \mathcal{C}(F_2^+)\} = k$, therefore it follows that $\mathcal{C}(F_2) \leq k$ and $\mathcal{C}(F_1) = k+1$. Thus H is unbalanced, (a1) holds and so $H \in \mathcal{A}$.

If $|K(f)| = 3$, then without loss of generality, $K(f)$ is a component of F_1^+. Recall that in this case $\deg_{H'}(w) = 0$, that is, $\deg_H(w) = 2$. If follows that $K(u)$ and $K(v)$ are components in $(F_2^+)'$, but on the other hand, they belong to different families in the 2-Hamming decomposition $\{F_1', F_2'\}$ of H'. By Lemma 18, $K(u)$ and $K(v)$ are in different

components of H'. Let $H_1, H_2, \{\{w\}\}$ be the components of H' with $u \in V(H_1)$ and $v \in V(H_2)$. By the uniqueness of the 2-Hamming decomposition of connected graphs, the only 2-Hamming decomposition of H is $\{\widetilde{F}_1, \widetilde{F}_2\}$, where $\widetilde{F}_1 = (F_1^+|H_1) \cup (F_2^+|H_2) \cup \{\{u, w\}\}$ and $\widetilde{F}_2 = (F_2^+|H_1) \cup (F_1^+|H_2) \cup \{\{v, w\}\}$.

Let $n_i = \mathcal{C}(F_i^+|H_1)$ and $l_i = \mathcal{C}(F_i^+|H_2)$. Then

$$\mathcal{C}(\widetilde{F}_i|H_1) = \mathcal{C}(F_i^+|H_1) = n_i \quad \text{and} \quad \mathcal{C}(\widetilde{F}_i|H_2) = \mathcal{C}(F_{3-i}^+|H_2) = l_{3-i}. \tag{1}$$

We see that $\mathcal{C}(F_i^+) = n_i + l_i + 1$ and $\mathcal{C}(\widetilde{F}_i) = n_i + l_{3-i} + 1$.
By Lemma 17, $\max\{\mathcal{C}(F_1^+), \mathcal{C}(F_2^+)\} \leq k$ and $\max\{\mathcal{C}(\widetilde{F}_1), \mathcal{C}(\widetilde{F}_2)\} = k + 1$, which gives

$$\max\{n_1 + l_1 + 1, n_2 + l_2 + 1\} < \max\{n_1 + l_2 + 1, n_2 + l_1 + 1\}. \tag{2}$$

Let $i \in \{1, 2\}$ such that $\mathcal{C}(\widetilde{F}_i) \geq \mathcal{C}(\widetilde{F}_{3-i})$, that is, $n_i + l_{3-i} + 1 \geq n_{3-i} + l_i + 1$. Then (2) imples that $\max\{n_1 + l_1 + 1, n_2 + l_2 + 1\} < n_i + l_{3-i} + 1$ and thus $n_i > n_{3-i}$ and $l_{3-i} > l_i$. Let $F_1 = \widetilde{F}_i$ and $F_2 = \widetilde{F}_{3-i}$. Then by (1), $n_i > n_{3-i}$ means that $\mathcal{C}(F_1|H_1) > \mathcal{C}(F_2|H_1)$ and $l_{3-i} > l_i$ means that $\mathcal{C}(F_1|H_2) > \mathcal{C}(F_2|H_2)$. Finally, we have that $\mathcal{C}(F_1) = \mathcal{C}(F_1|H_1) + \mathcal{C}(F_1|H_2) + 1 > \mathcal{C}(F_2|H_1) + \mathcal{C}(F_2|H_1) + 1 = \mathcal{C}(F_2)$. Thus H is unbalanced, (a2) holds and so $H \in \mathcal{A}$.

Verification of $H \in \mathcal{B}$.

Let $g = e$. We shall show that H is unbalanced and that (b1) or (b2) holds.

Without loss of generality, $K(u)$ is a component of F_1^-. It follows that $K(v)$ has to be a component of F_2^-, indeed, if $|K(e)| = 3$, it is clear and if $|K(e)| = 2$ and $K(v)$ is a component of F_1^-, then F_1^- and $(F_2^- - \{\{u\}\} - \{\{v\}\}) \cup \{\{u, v\}\}$ would form a 2-Hamming decomposition of H with at most k components in both families, a contradiction. On the other hand, Lemma 15 implies that $K(e)$ is a component in one of $\{F_1, F_2\}$, therefore $K(u)$ and $K(v)$ belong the same family in $\{F_1', F_2'\}$. By Lemma 18, $K(u)$ and $K(v)$ are in different connected components of H'. Let H_1, H_2 denote the components of H' with $u \in V(H_1)$ and $v \in V(H_2)$.

As observed before, $N[u] = V(K(e) \cup K(u))$ and $N[v] = V(K(e) \cup K(v))$. Thus if $|K(e)| = 2$, then $\{\{u\}\}$ and $\{\{v\}\}$ are singleton components of F_2^- and F_1^- respectively.

By the uniqueness of the 2-Hamming decomposition of connected graphs, the unique 2-Hamming decomposition $\{\widetilde{F}_1, \widetilde{F}_2\}$ of H is

$$\widetilde{F_1} = \begin{cases} (F_1^-|H_1) \cup (F_2^-|H_2) & \text{if } |K(e)| = 2, \\ (F_1^-|H_1) \cup (F_2^-|H_2) \cup \{\{w\}\} & \text{if } |K(e)| = 3, \end{cases}$$

$$\widetilde{F_2} = \begin{cases} ((F_2^-|H_1) - \{\{u\}\}) \cup ((F_1^-|H_2) - \{\{v\}\}) \cup \{\{u, v\}\} & \text{if } |K(e)| = 2, \\ (F_2^-|H_1) \cup (F_1^-|H_2) \cup \{\{u, v, w\}\} & \text{if } |K(e)| = 3. \end{cases}$$

Let $n_i = \mathcal{C}(F_i^-|H_1)$ and $l_i = \mathcal{C}(F_i^-|H_2)$, then

$$\mathcal{C}(\widetilde{F}_1|H_1) = n_1, \qquad \mathcal{C}(\widetilde{F}_1|H_2) = l_2, \tag{3}$$

$$\mathcal{C}(\widetilde{F}_2|H_1) = n_2 - 1, \qquad \mathcal{C}(\widetilde{F}_2|H_2) = l_1 - 1 \qquad \text{if } |K(e)| = 2, \tag{4}$$
$$\mathcal{C}(\widetilde{F}_2|H_1) = n_2, \qquad \mathcal{C}(\widetilde{F}_2|H_2) = l_1 \qquad \text{if } |K(e)| = 3. \tag{5}$$

It follows that for $i \in \{1, 2\}$,

$$\mathcal{C}(F_i^-) = n_i + l_i, \quad \mathcal{C}(\widetilde{F}_1) = n_1 + l_2, \quad \text{and} \quad \mathcal{C}(\widetilde{F}_2) = n_2 + l_1 - 1 \quad \text{if } |K(e)| = 2, \tag{6}$$
$$\mathcal{C}(F_i^-) = n_i + l_i + 1, \quad \mathcal{C}(\widetilde{F}_1) = n_1 + l_2 + 1, \quad \text{and} \quad \mathcal{C}(\widetilde{F}_2) = n_2 + l_1 + 1 \quad \text{if } |K(e)| = 3. \tag{7}$$

By Lemma 17, $\max\{\mathcal{C}(F_1^-), \mathcal{C}(F_2^-)\} \leq k$ and $\max\{\mathcal{C}(\widetilde{F}_1), \mathcal{C}(\widetilde{F}_2)\} = k + 1$, which gives

$$\max\{n_1 + l_1, n_2 + l_2\} < \max\{n_1 + l_2, n_2 + l_1 - 1\} \qquad \text{if } |K(e)| = 2, \tag{8}$$
$$\max\{n_1 + l_1 + 1, n_2 + l_2 + 1\} < \max\{n_1 + l_2 + 1, n_2 + l_1 + 1\} \qquad \text{if } |K(e)| = 3. \tag{9}$$

$$\text{Let } i \in \{1, 2\} \text{ such that } \mathcal{C}(\widetilde{F}_{3-i}) \leq \mathcal{C}(\widetilde{F}_i) \text{ and let } F_1 := \widetilde{F}_i,\ F_2 := \widetilde{F}_{3-i}. \tag{10}$$

Assume that $|K(e)| = 2$. By the definition of $\{\widetilde{F}_1, \widetilde{F}_2\}$, $e \in E(\widetilde{F}_2)$. We shall show that H is unbalanced and that (b1) holds, that is, that if $e \in E(F_j)$, then $\mathcal{C}(F_2|H_1) + j \leq \mathcal{C}(F_1|H_1)$ and $\mathcal{C}(F_2|H_2) + j \leq \mathcal{C}(F_1|H_2)$. It follows from (10) and the fact that $e \in E(\widetilde{F}_2)$ that $e \in E(F_j)$ if and only if $F_j = \widetilde{F}_2$ and $F_{3-j} = \widetilde{F}_1$.

If $j = 1$, then

$$F_1 = \widetilde{F}_2,\ F_2 = \widetilde{F}_1, \text{ and} \tag{11}$$

$$n_1 + l_2 \overset{(6)}{=} \mathcal{C}(\widetilde{F}_1) \overset{(11)}{=} \mathcal{C}(F_2) \overset{(10)}{\leq} \mathcal{C}(F_1) \overset{(11)}{=} \mathcal{C}(\widetilde{F}_2) \overset{(6)}{=} n_2 + l_1 - 1. \tag{12}$$

From (8) and (12), we obtain $\max\{n_1+l_1, n_2+l_2\} < n_2+l_1-1$, which implies $n_1 < n_2-1$ and $l_2 < l_1 - 1$. Then H is unbalanced since

$$\mathcal{C}(F_2) \overset{(11)}{=} \mathcal{C}(\widetilde{F}_1) \overset{(6)}{=} n_1 + l_2 < n_2 + l_2 - 1 < n_2 + l_1 - 1 \overset{(6)}{=} \mathcal{C}(\widetilde{F}_2) \overset{(11)}{=} \mathcal{C}(F_1).$$

Moreover,

$$\mathcal{C}(F_2|H_1) + 1 \overset{(11)}{=} \mathcal{C}(\widetilde{F}_1|H_1) + 1 \overset{(3)}{=} n_1 + 1 \leq n_2 - 1 \overset{(4)}{=} \mathcal{C}(\widetilde{F}_2|H_1) \overset{(11)}{=} \mathcal{C}(F_1|H_1),$$
$$\mathcal{C}(F_2|H_2) + 1 \overset{(11)}{=} \mathcal{C}(\widetilde{F}_1|H_2) + 1 \overset{(3)}{=} l_2 + 1 \ \leq\ l_1 - 1 \overset{(4)}{=} \mathcal{C}(\widetilde{F}_2|H_2) \overset{(11)}{=} \mathcal{C}(F_1|H_2).$$

Thus (b1) holds for H.

If $j = 2$, then

$$F_1 = \widetilde{F}_1,\ F_2 = \widetilde{F}_2, \text{ and} \tag{13}$$

$$n_2 + l_1 - 1 \overset{(6)}{=} \mathcal{C}(\widetilde{F}_2) \overset{(13)}{=} \mathcal{C}(F_2) \overset{(10)}{\leq} \mathcal{C}(F_1) \overset{(13)}{=} \mathcal{C}(\widetilde{F}_1) \overset{(6)}{=} n_1 + l_2. \tag{14}$$

From (8) and (14), we obtain $\max\{n_1 + l_1, n_2 + l_2\} < n_1 + l_2$, which implies $n_2 < n_1$ and $l_1 < l_2$. Then H is unbalanced since

$$\mathcal{C}(F_2) \overset{(13)}{=} \mathcal{C}(\widetilde{F}_2) \overset{(6)}{=} n_2 + l_1 - 1 < n_2 + l_1 < n_1 + l_2 \overset{(6)}{=} \mathcal{C}(\widetilde{F}_1) \overset{(13)}{=} \mathcal{C}(F_1).$$

Moreover,

$$\mathcal{C}(F_2|H_1) + 2 \overset{(13)}{=} \mathcal{C}(\widetilde{F}_2|H_1) + 2 \overset{(4)}{=} n_2 + 1 \leq n_1 \overset{(3)}{=} \mathcal{C}(\widetilde{F}_1|H_1) \overset{(13)}{=} \mathcal{C}(F_1|H_1),$$
$$\mathcal{C}(F_2|H_2) + 2 \overset{(13)}{=} \mathcal{C}(\widetilde{F}_2|H_2) + 2 \overset{(4)}{=} l_1 + 1 \ \leq \ l_2 \overset{(3)}{=} \mathcal{C}(\widetilde{F}_1|H_2) \overset{(13)}{=} \mathcal{C}(F_1|H_2).$$

Thus (b1) holds for H.

Assume that $|K(e)| = 3$. We shall show that H is unbalanced and that (b2) holds, that is that $\mathcal{C}(F_2|H_1) < \mathcal{C}(F_1|H_1)$, $\mathcal{C}(F_2|H_2) < \mathcal{C}(F_1|H_2)$ and $H_3 = \{\{w\}\}$. We showed in the beginning of the proof of Claim 3.2, that w has degree 0 in $H' = H - K(e)$, thus $\{\{w\}\}$ is a component of $H - K(e)$. By the choice of i,

$$n_{3-i} + l_i + 1 \overset{(7)}{=} \mathcal{C}(\widetilde{F}_{3-i}) \overset{(10)}{\leq} \mathcal{C}(\widetilde{F}_i) \overset{(7)}{=} n_i + l_{3-i} + 1. \tag{15}$$

From (9) and (15), we get $\max\{n_1 + l_1 + 1, n_2 + l_2 + 1\} < n_i + l_{3-i} + 1$, which implies that $n_i > n_{3-i}$ and $l_{3-i} > l_i$. Then H is unbalanced since

$$\mathcal{C}(F_2) \overset{(10)}{=} \mathcal{C}(\widetilde{F}_{3-i}) \overset{(7)}{=} n_{3-i} + l_i + 1 < n_i + l_{3-i} + 1 \overset{(7)}{=} \mathcal{C}(\widetilde{F}_i) \overset{(10)}{=} \mathcal{C}(F_1).$$

Moreover,

$$\mathcal{C}(F_2|H_1) \overset{(10)}{=} \mathcal{C}(\widetilde{F}_{3-i}|H_1) \overset{(3),(5)}{=} n_{3-i} < n_i \overset{(3),(5)}{=} \mathcal{C}(\widetilde{F}_i|H_1) \overset{(10)}{=} \mathcal{C}(F_1|H_1),$$
$$\mathcal{C}(F_2|H_2) \overset{(10)}{=} \mathcal{C}(\widetilde{F}_{3-i}|H_2) \overset{(3),(5)}{=} l_i < l_{3-i} \overset{(3),(5)}{=} \mathcal{C}(\widetilde{F}_i|H_2) \overset{(10)}{=} \mathcal{C}(F_1|H_2).$$

Thus (b2) holds for H.

Therefore, H is unbalanced, (b1) holds if $|K(e)| = 2$, (b2) holds if $|K(e)| = 3$ and thus $H \in \mathcal{B}$. This concludes the proof of Claim 3.2.

To prove the statement of Theorem 12, we need to show that for any connected graph H,

$$\exists k \in \mathbb{N} \text{ such that } K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H \iff H \text{ is either chipped or } H \in \mathcal{A} \cap \mathcal{B}.$$

($\Leftarrow$)
By Claim 1, if H is chipped, then H is good, that is, $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$ for some $k \in \mathbb{N}$.
By Claim 2, if $H \in \mathcal{A} \cap \mathcal{B}$ then H is good, that is, $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$ for some $k \in \mathbb{N}$, which proves the first implication.

($\Rightarrow$)
Assume that H is such that $K_k \square K_k \hookrightarrow\!\!\!\!\rightarrow H$ for some $k \in \mathbb{N}$, that is, H is good. Consider two cases based on the dimension of H:
if $\dim(H) > 2$, then by Claim 3.1 H is chipped;
if $\dim(H) \leq 2$, then by Claim 3.2 $H \in \mathcal{A} \cap \mathcal{B}$.

This concludes the proof of Theorem 12.

□

Proof of Proposition 4. We shall show that H is either in the class $\mathcal{T}(n)$ or chipped and so by Corollary 2 and Theorem 12, there is a graph G so that $G \hookrightarrow\!\!\!\!\rightarrow H$. If H is a spider with at least 4 legs, then $H \in \mathcal{T}(n)$. If H is a spider with 3 legs, then H is chipped of type (ch1). If $H \cong \widetilde{C}_{2k+1}$, then H is chipped of type (ch2). Finally, if H is a union of a cycle C and a path P such that P has length at least 1 and shares exactly one vertex with C which is an endpoint of it, then H is chipped of type (ch1). □

6 Further results

For a family $\mathcal{F}$ of graphs, we use $\mathrm{Forb}(\mathcal{F})$ to denote a set of all graphs that do not contain an induced copy of any graph from $\mathcal{F}$. When $\mathcal{F} = \{H\}$, we write $\mathrm{Forb}(\{H\}) = \mathrm{Forb}(H)$. We say that G is *$\mathcal{F}$-induced-saturated* and write $G \hookrightarrow\!\!\!\!\rightarrow \mathcal{F}$ if $G \in \mathrm{Forb}(\mathcal{F})$, but deleting any edge from G or adding any edge from $\overline{G}$ to G results in an induced copy of a member of $\mathcal{F}$.

Using the structural descriptions of the classes $\mathrm{Forb}(C_4, \overline{C}_4, P_4)$ (due to Chvátal and Hammer [2]) and $\mathrm{Forb}(C_4, \overline{C}_4, C_5)$ (due to Földes and Hammer [6]), Behrens *et al.*[1] showed that there is no graph G so that $G \hookrightarrow\!\!\!\!\rightarrow \{\overline{C}_4, C_4, P_4\}$ or $G \hookrightarrow\!\!\!\!\rightarrow \{\overline{C}_4, C_4, C_5\}$. The later example shows that even though there are graphs G_1, G_2, G_3 so that $G_1 \hookrightarrow\!\!\!\!\rightarrow \overline{C}_4$, $G_2 \hookrightarrow\!\!\!\!\rightarrow C_4$ and $G_3 \hookrightarrow\!\!\!\!\rightarrow C_5$, there is no induced-saturated graph for the family $\{\overline{C}_4, C_4, C_5\}$.

In this section, we give a short proof of the fact, first proved by Martin and Smith [9], that there is no P_4-induced-saturated graph and we further extend the list of families for which there is no induced-saturated graph, see Theorem 19. We generalize the ideas used in these proofs and provide some saturation properties of prime graphs, see Theorem 20.

First we need a few definitions. We say that a vertex of a graph is *full* if it is adjacent to all other vertices of the graph. A *complement reducible graph*, shortly *cograph*, is defined recursively from a single vertex graph by taking disjoint unions and complements, that is, K_1 is a cograph and for cographs G_1, G_2, the graphs $G_1 \dot\cup G_2$, $\overline{G}_1$, $\overline{G}_2$ are also cographs. Corneil *et al.*[3] showed that a graph G is a cograph if and only if $G \in \mathrm{Forb}(P_4)$. A *homogeneous set* of an n-vertex graph G is a set $X \subsetneq V(G)$ with $2 \leq |X| \leq n-1$, such that every vertex $v \in V(G) \setminus X$ is either complete to X, that is, $X \subseteq N(v)$ or anticomplete to X, that is, $X \cap N(v) = \emptyset$. We say that a graph H is *prime* if it does not have a homogeneous set, that is, for any proper subset X of vertices with $|X| \geq 2$, there is a vertex $v \notin X$ such that v sends both an edge and a non-edge to X. Note that if H is a prime graph on at least 2 vertices, then H and $\overline{H}$ are connected, in particular H does not have an isolated or a full vertex.

Theorem 19. *There is no graph G such that $G \hookrightarrow\!\!\!\!\rightarrow P_4$ and there is no graph G such that $G \hookrightarrow\!\!\!\!\rightarrow \{P_4, C_4\}$.*

Proof. Assume that G is a graph on the smallest number of vertices such that $G \hookrightarrow\!\!\!\!\rightarrow P_4$. Then $|G| \geq 4$ and since $P_4 \not\subseteq_I G$, G is a cograph. In particular, G or its complement is disconnected. Assume without loss of generality that G is disconnected and let G' be a union of some connected components of G such that $G' \neq G$ and $|G'| \geq 2$. Deleting an edge of G' or adding an edge from the complement of G' creates a copy of P_4 in G. Since P_4 is connected, this copy cannot contain vertices from both $V(G')$ and $V(G) \setminus V(G')$, that is, it lies completely in G'. Thus $G' \hookrightarrow\!\!\!\!\rightarrow P_4$, which contradicts the minimality of G.

Assume that $G = (V, E)$ is a graph on the smallest number of vertices so that $G \hookrightarrow\!\!\!\!\rightarrow \{P_4, C_4\}$. Then G is $\{P_4, C_4\}$-free and by the minimality of G, G is connected. Wolk [12] proved that every finite, connected $\{P_4, C_4\}$-free graph has a full vertex. Thus G has a full vertex, denote it by v and let $G' = G - v$. Since neither P_4 nor C_4 contains a full vertex, deleting an edge of G' or adding an edge to G' from its complement results in an induced copy of P_4 or C_4 that does not use v, that is it lies in G'. Therefore $G' \hookrightarrow\!\!\!\!\rightarrow \{P_4, C_4\}$ which contradicts the minimality of G. □

For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the *blowup product* of G_1 and G_2, denoted by $G_1 \star G_2$ is a graph on vertex set $V = V_1 \times V_2$ where two vertices $(u_1, v_1), (u_2, v_2) \in V$ are adjacent if either $u_1u_2 \in E_1$ or $u_1 = u_2$ and $v_1v_2 \in E_2$. For a vertex $u \in V(G_1)$, the set $B_u = \{(u, v) : v \in V(G_2)\}$ is called a *blob* of $G_1 \star G_2$. Notice that the subgraph of $G_1 \star G_2$ induced by any blob is isomorphic to G_2. Moreover, for any two distinct blobs $B_u, B_{u'}$ of $G_1 \star G_2$, the bipartite graph between B_u and $B_{u'}$ induced by the edges of $G_1 \star G_2$ is complete if $uu' \in E(G_1)$ and empty otherwise. Informally, $G_1 \star G_2$ is obtained by blowing up G_1 and inserting a copy of G_2 in each blob.

Theorem 20. *Let H be a prime graph for which there exists an H-induced-saturated graph G. Then $G \star G \hookrightarrow\!\!\!\!\rightarrow H$ and there is a prime graph G' so that $G' \hookrightarrow\!\!\!\!\rightarrow H$.*

Proof. Let $G \hookrightarrow\!\!\!\!\rightarrow H$.

We shall show first that $G \star G \hookrightarrow\!\!\!\!\rightarrow H$. We need to verify that $H \not\subseteq_I G \star G$ and that adding or deleting any edge results in an induced copy of H. Let $X \subseteq V(G \star G)$ of size $|H|$. If X intersects each blob of $G \star G$ in at most one vertex or X is contained in one blob of $G \star G$, then X induces a subgraph of G and thus $(G \star G)[X]$ is not isomorphic to H. If there exists a blob B so that $X \not\subseteq V(B)$ and $X \cap B = Y$ with $|Y| \geq 2$, then X cannot induce a prime graph since a vertex of X that is not in B is either complete or anticomplete to Y. If we add or delete an edge between vertices that are contained in the same blob of $G \star G$, then since the blob induces G and $G \hookrightarrow\!\!\!\!\rightarrow H$, this change results in an induced copy of H within the blob. If we add or delete an edge between vertices that are contained in different blobs of $G \star G$, then this pair belongs to a copy $\widetilde{G}$ of G obtained by taking a single vertex from each blob. Thus this change corresponds to an edge addition or deletion in $\widetilde{G}$ that results in an induced copy of H.

Next we shall show that there is a prime graph G' so that $G' \hookrightarrow\!\!\!\!\rightarrow H$. Suppose that G is not a prime graph and consider a minimal homogeneous set S in G. Let $G' = G[S]$.

Observe that if $X \subsetneq V(G')$ is a homogeneous set in G' then it is also a homogeneous set in G. By the minimality of S, we conclude that G' does not have a homogeneous set, that is, G' is a prime graph. Since $G \hookrightarrow\!\!\!\!\rightarrow H$, deleting an edge in G' or adding an edge from the complement of G' creates an induced copy H' of H. Since H is a prime graph, H' cannot contain vertices from both S and $V(G) \setminus S$ therefore $H' \subseteq G'$ and hence $G' \hookrightarrow\!\!\!\!\rightarrow H$. $\square$

Corollary 21. *If H is a prime graph such that each $G \in \mathrm{Forb}(H)$ is nonprime, then there is no H-induced-saturated graph. In particular there is no P_4-induced-saturated graph.*

Corollary 22. *If H is a prime graph and there is a graph G such that $G \hookrightarrow\!\!\!\!\rightarrow H$, then there is an arbitrarily large graph G' so that $G' \hookrightarrow\!\!\!\!\rightarrow H$.*

7 Concluding remarks

We have shown that graphs from certain families of graphs have Hamming graphs as induced saturating graphs. Our families do not include paths, hence it remains open whether there is an induced saturating graph for any path of length at least 4.

Note that the blowup product results in a nonprime graph. It is interesting to ask whether for all prime graphs H for which there is an H-induced-saturated graph, there exists an arbitrarily large prime graph G so that $G \hookrightarrow\!\!\!\!\rightarrow H$.

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