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Asymptotically Orthonormal Basis and Toeplitz operators

Emmanuel Fricain and Rishika Rupam

ABSTRACT. Recently, M. Mitkovski [Mit17] gave a criterion for basicity of a sequence of complex exponentials in terms of the invertibility properties of a certain naturally associated Toeplitz operator, in the spirit of work of Khrushchëv–Nikolski–Pavlov [HNP81]. In our paper, we extend the results of Mitkovski to model spaces associated with meromorphic inner functions and we also give an analogue for the property of being an asymptotically orthonormal basis.

1. Introduction

The classical theory of Fourier series says that the trigonometric system $(e^{int})_{n \in \mathbb{Z}}$ forms an orthonormal basis of $L^2(0, 2\pi)$. It is natural to ask what happens if we replace the trigonometric system by another systems of complex exponential $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$, $\Lambda = (\lambda_n)_{n \in \mathbb{Z}} \subset \mathbb{R}$. Indeed this problem has a very long and rich history and finds its roots in the work of Paley–Wiener and Levinson. There at least two directions of approach to this problem. First, we can try to find small perturbations of the trigonometric systems which retains the desired basis properties of expansion. This problem of perturbation in fact gave rise to a whole direction of research, culminating with the beautiful theorem of Ingham–Kadets. See [You01] for more details. The second direction is to find characterization of discrete sequences $\Lambda = (\lambda_n)_{n \in \mathbb{Z}} \subset \mathbb{R}$ such that $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ forms a Riesz basis/asymptotically orthonormal basis of $L^2(0, 2\pi)$. In the seventies, an approach using Toeplitz operators was developed by Douglas, Sarason and Clark. Not only did it allow the recapture of all the classical results but also permitted Khrushchëv–Nikolski–Pavlov [HNP81] to give the solution of the Riesz basis problem for complex exponentials. We briefly explain the basic idea of the Toeplitz approach. Since the map $f(t) \mapsto f(t)e^{-t}$ is an isomorphism on $L^2(0, 2\pi)$, it is clear that $(e^{i\lambda_n t})_{n \in \mathbb{Z}}$ forms a Riesz basis of $L^2(0, 2\pi)$ if and only if $(e^{i\lambda'_n t})_{n \in \mathbb{Z}}$ does, where $\lambda'_n = \lambda_n + i$, $n \in \mathbb{Z}$. If we apply the inverse of the Fourier transform, it is easy to see that $(e^{i\lambda'_n t})_{n \in \mathbb{Z}}$ is a Riesz basis for $L^2(0, 2\pi)$ if and only if $(k_{\lambda'_n}^{\Theta_{2\pi}})_{n \in \mathbb{Z}}$ is a Riesz basis for $\mathcal{K}_{\Theta_{2\pi}}$. Here $\mathcal{K}_{\Theta_{2\pi}}$ is the model space

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associated to $\Theta_{2\pi}(z) = e^{2i\pi z}$ and $k_{\chi_n}^{\Theta_{2\pi}}$ is its reproducing kernel. Then, the crucial idea is to observe that the basis properties of the reproducing kernels in $K_{\Theta_{2\pi}}$ are encoded in the invertibility property of the Toeplitz operator $T_{\overline{B_{\Lambda'}\Theta_{2\pi}}}$, where $B_{\Lambda'}$ is the Blaschke product associated to $\Lambda' = (\chi'_n)_{n \in \mathbb{Z}}$. If we want to apply the same method to study the asymptotically orthonormal basis property, we are faced with a difficulty because the latter property is not preserved by isomorphism, and consequently we cannot shift Λ to Λ' . But recently, Mitkovski [Mit17] has shown that we can adapt the approach of Khrushchëv–Nikolski–Pavlov avoiding the use of the translated sequence but instead exploiting the ideas of Makarov–Poltoratski’s Toeplitz approach to the completeness problem of exponential systems. In this paper, we will show that one can generalize Mitkovski’s results to more general model spaces and we also give an analogous result for the asymptotically orthonormal basis property.

In Section 2, we start with some general preliminaries on Riesz basis, asymptotically orthonormal basis and meromorphic inner functions. As we will see, our characterization of asymptotically orthonormal basis involves the condition that a certain Toeplitz operator is unitary modulo compact (meaning that it can be written as the sum of a unitary operator and a compact one). In Section 3, we thus revisit some classical results on Toeplitz operators. In particular, we explain how to derive a result of Douglas (characterizing the Toeplitz operators which are unitary modulo compact) from Hartman’s theorem on compactness of Hankel operators and Devinatz–Widom’s theorem on invertibility of Toeplitz operators. These results in the context of Hardy space of the unit disc can be found for instance in Peller [Pel03] and Nikolski [Nik09]. Here we need the analogue for the upper-half plane and we provide the details with an aim to make our presentation complete. In Section 4, we give a characterization for Riesz basis property and in the last section, we discuss the asymptotically orthonormal basis property for systems of reproducing kernels in the model spaces.

2. Preliminaries

2.1. Definition of Riesz sequences and AOB. Let \mathcal{H} be a Hilbert space, $\mathfrak{X} = (x_n)_{n \geq 1}$ be a sequence of vectors in \mathcal{H} . We recall that \mathfrak{X} is said to be:

- (a) *minimal* if for every $n \geq 1$,

$$x_n \notin \text{span}(x_\ell : \ell \neq n),$$

where $\text{span}(\dots)$ denotes the closure of the finite linear combination of (\dots) ;

- (b) A *Riesz sequence* (abbreviated RS) if there exists two positive constants c, C such that

$$(1) \quad c \sum_{n \geq 1} |a_n|^2 \leq \left\| \sum_{n \geq 1} a_n x_n \right\|_{\mathcal{H}}^2 \leq C \sum_{n \geq 1} |a_n|^2,$$

for every finitely supported sequence of complex numbers $(a_n)_n$;

- (c) An *asymptotically orthonormal sequence* (abbreviated AOS) if there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there are positive constants c_N, C_N verifying

$$(2) \quad c_N \sum_{n \geq N} |a_n|^2 \leq \left\| \sum_{n \geq N} a_n x_n \right\|_{\mathcal{H}}^2 \leq C_N \sum_{n \geq N} |a_n|^2,$$

for every finitely supported sequence of complex numbers $(a_n)_n$ and $\lim_{N \rightarrow \infty} c_N = 1 = \lim_{N \rightarrow \infty} C_N$;

- (d) An *asymptotically orthonormal basic sequence* (abbreviated AOB) if it is an AOS with $N_0 = 1$;
- (e) A *Riesz basis* for \mathcal{H} (abbreviated RB) if it is a complete Riesz sequence, that is a Riesz sequence satisfying

$$\text{span}(x_n : n \geq 1) = \mathcal{H}.$$

It is easy to see that $(x_n)_{n \geq 1}$ is an AOB if and only if it is an AOS as well as a RS. Also, $(x_n)_{n \geq 1}$ is an AOB if and only if it is minimal and an AOS. The reader should pay attention to the fact that AOB does not imply completeness; an AOB is a basis for its span but not necessarily for the whole space.

2.2. Meromorphic inner functions and model spaces. A function $\Theta : \mathbb{C}_+ \rightarrow \mathbb{C}$ is said to be *inner* on the upper-half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$ if Θ is analytic and bounded on \mathbb{C}_+ and if its radial limits are of modulus one almost everywhere on \mathbb{R} . We say that the inner function Θ is a meromorphic inner function (abbreviated by MIF) if Θ admits a meromorphic extension to the whole complex plane. A well-known theorem by Riesz and V. Smirnov says that all MIF functions have the form

$$\Theta(z) = B(z)e^{iaz}$$

where $a \geq 0$ and B is the Blaschke product formed with the zeros of the function Θ given by $Z = (z_n)_{n \geq 1}$, where $|z_n| \rightarrow \infty$ and satisfy the convergence criterion

$$\sum_{n \geq 1} \frac{\Im z_n}{1 + |z_n|^2} < \infty.$$

In particular, using this representation, it is not difficult to see that Θ will admit an analytic extension through the real line.

To each inner function Θ , we associate a *model space*

$$\mathcal{K}_\Theta = \mathcal{H}^2 \ominus \Theta \mathcal{H}^2 = \mathcal{H}^2 \cap \overline{\Theta \mathcal{H}^2},$$

where \mathcal{H}^2 is the Hardy space on \mathbb{C}_+ . It is well known that if Θ is a MIF, then every function $f \in \mathcal{K}_\Theta$ can be extended analytically through the real line. In the case when Θ is a MIF with $\Theta' \in L^\infty(\mathbb{R})$, we will need some uniform L^2 -estimate on this extension. We start with the following result appearing in [Dya02, Lemma 2]. For completeness, we provide a proof.

Lemma 2.1. *Let Θ be a MIF inner function such that $\Theta' \in L^\infty(\mathbb{R})$. Then, for every $\varepsilon > 0$, we have*

$$\inf_{0 \leq \Im z < \varepsilon} |\Theta(z)| \geq 1 - \varepsilon \|\Theta'\|_\infty.$$

Proof. Since Θ is analytic on the real line, we can apply the mean value theorem. Then, for all $x, h \in \mathbb{R}$,

$$|\Theta(x+h) - \Theta(x)| \leq M|h|,$$

where $M = \|\Theta'\|_\infty$. Consider the function $z \rightarrow \Theta(z+h) - \Theta(z)$. This function is the Poisson integral of the boundary function $x \rightarrow \Theta(x+h) - \Theta(x)$, which permits to extend the latter estimate in \mathbb{C}_+ . In other words, for all $h \in \mathbb{R}_+$ and $z \in \mathbb{C}_+$, we have

$$|\Theta(z+h) - \Theta(z)| \leq M|h|.$$

But then by the definition of the derivative,

$$|\Theta'(z)| \leq M,$$

for all $z \in \mathbb{C}_+$. Note that we have proved that $\Theta' \in H^\infty(\mathbb{C}_+)$. Now letting $z = x + iy$, $0 < y < \varepsilon$, we have

$$|\Theta(z) - \Theta(x)| \leq |z - x| \sup_{0 \leq \Im z < \varepsilon} |\Theta'(z)| \leq \varepsilon M,$$

which gives

$$(3) \quad |\Theta(z)| \geq 1 - \varepsilon M.$$

□

Now, let $f \in \mathcal{K}_\Theta$ and $0 < \varepsilon < \|\Theta'\|_\infty^{-1}$. We can write $f(z) = \Theta(z)\overline{h(z)}$, $z \in \mathbb{C}_+$, for some $h \in \mathcal{H}^2$. Then, if we define for $-\varepsilon < \Im(z) < 0$,

$$(4) \quad f(z) = \frac{1}{\Theta(\bar{z})}\overline{h(\bar{z})},$$

we obtain the analytic continuation of f on $\{z \in \mathbb{C} : \Im(z) > -\varepsilon\}$.

Lemma 2.2. *Let Θ be a meromorphic inner function such that $\Theta' \in L^\infty(\mathbb{R})$. Let $\varepsilon > 0$ satisfying $\varepsilon < \|\Theta'\|_\infty^{-1}$. Then, for every $f \in \mathcal{K}_\Theta$, we have*

$$\sup_{|y| < \varepsilon} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx \leq \frac{\|f\|_2^2}{(1 - \varepsilon \|\Theta'\|_\infty)^2}.$$

Proof. It is clear that

$$\sup_{0 \leq y < \varepsilon} \int_{-\infty}^{+\infty} |f(x + iy)|^2 dx \leq \|f\|_2^2.$$

Now assume that $-\varepsilon < y < 0$. According to (3) and (4), we have

$$\begin{aligned} |f(x + iy)|^2 &= \frac{|h(x - iy)|^2}{|\Theta(x - iy)|^2} \\ &\leq \frac{|h(x - iy)|^2}{(1 - \varepsilon \|\Theta'\|_\infty)^2}. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{-\infty}^{+\infty} |f(x + iy)|^2 dx &\leq \frac{1}{(1 - \varepsilon \|\Theta'\|_\infty)^2} \int_{-\infty}^{+\infty} |h(x - iy)|^2 dx \\ &\leq \frac{\|h\|_2^2}{(1 - \varepsilon \|\Theta'\|_\infty)^2} \\ &= \frac{\|f\|_2^2}{(1 - \varepsilon \|\Theta'\|_\infty)^2}. \end{aligned}$$

□

Recall that for a MIF U and $\lambda \in \mathbb{R}$, the function

$$K_\lambda^U(z) = \frac{i}{2\pi} \frac{1 - \overline{U(\lambda)}U(z)}{z - \bar{\lambda}}$$

is the reproducing kernel of the space \mathcal{K}_U and $2\pi \|K_\lambda^U\|_2^2 = |U'(\lambda)|$. Hence, the function

$$k_\lambda^U(z) = \frac{i}{\sqrt{2\pi|U'(\lambda)|}} \frac{1 - \overline{U(\lambda)}U(z)}{z - \bar{\lambda}}$$

is the normalized reproducing kernel of \mathcal{K}_U .

Lemma 2.3. *Let I and Θ be two MIF. Assume that $\{I = 1\} = \{\lambda_n : n \geq 1\}$ and denote by $\eta_n = |I'(\lambda_n)|^{1/2} |\Theta'(\lambda_n)|^{-1/2}$, $n \geq 1$. Then, for every finitely supported sequence of complex numbers $(a_n)_{n \geq 1}$, we have*

$$(5) \quad (1 - I) \sum_{n \geq 1} a_n k_{\lambda_n}^\Theta = \sum_{n \geq 1} a_n \eta_n k_{\lambda_n}^I - \Theta \sum_{n \geq 1} a_n \eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I.$$

Proof. We have

$$\begin{aligned} (1 - I(z))k_{\lambda_n}^\Theta(z) &= \frac{i}{\sqrt{2\pi}} \frac{(1 - I(z))}{|\Theta'(\lambda_n)|^{1/2}} \frac{1 - \overline{\Theta(\lambda_n)}\Theta(z)}{z - \bar{\lambda}_n} \\ &= \frac{|I'(\lambda_n)|^{1/2}}{|\Theta'(\lambda_n)|^{1/2}} (1 - \overline{\Theta(\lambda_n)}\Theta(z)) \frac{i}{\sqrt{2\pi|I'(\lambda_n)|}} \frac{1 - \overline{I(\lambda_n)}I(z)}{z - \bar{\lambda}_n} \\ &= \eta_n (1 - \overline{\Theta(\lambda_n)}\Theta(z)) k_{\lambda_n}^I(z). \end{aligned}$$

Hence

$$(1 - I)k_{\lambda_n}^\Theta = \eta_n k_{\lambda_n}^I - \Theta \eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I.$$

□

Assume that the real sequence $(\lambda_n)_{n \geq 1}$ satisfies the following two conditions:

$$(6) \quad \delta := \inf_{n \geq 1} (\lambda_{n+1} - \lambda_n) > 0,$$

and

$$(7) \quad \sup_n \left| \sum_{k \neq n} \left(\frac{1}{\lambda_n - \lambda_k} + \frac{\lambda_k}{\lambda_k^2 + 1} \right) \right| < \infty.$$

Define the measure $\mu = \sum_{k=1}^{\infty} \nu_k \delta_{\lambda_k}$, where $(\nu_k)_{k \geq 1}$ is a sequence of positive real numbers satisfying $\sum_k \frac{\nu_k}{1 + \lambda_k^2} < \infty$ and $\sup_k |\nu_k| < \infty$. Let G be the function defined by

$$G(z) = \sum_{k \geq 1} \nu_k \left(\frac{1}{\lambda_k - z} - \frac{\lambda_k}{\lambda_k^2 + 1} \right).$$

Clearly, G is analytic in \mathbb{C}_+ and $\Im(G(z)) > 0$, $z \in \mathbb{C}_+$. Therefore, the function

$$(8) \quad I = \frac{G - i}{G + i}$$

is a meromorphic inner function in the upper half-plane and it is easy to see that $\{I = 1\} = \{\lambda_n : n \geq 1\}$. The measure μ is the so-called Clark measure for I and the sequence $(k_{\lambda_n}^I)_{n \geq 1}$ is an orthonormal basis for \mathcal{K}_I , see [Cla72]. Moreover, according to [Bar06, Lemma 5.2], we have $I' \in L^\infty(\mathbb{R})$ and $|I'(\lambda_n)| = 2/\nu_n$. Recall that if I is given by $I(z) = e^{iaz} B_Z(z)$, where $a \geq 0$ and $Z = (z_n)_n$ is the zero sequence of I , and B_Z is the Blaschke product associated to Z , then for $t \in \mathbb{R}$, we have

$$|I'(t)| = a + 2 \sum_{k \geq 1} \frac{\Im(z_k)}{|t - z_k|^2}.$$

Since $I' \in L^\infty(\mathbb{R})$, Lemma 2.1 implies $\delta_1 := \inf_n (\Im(z_n)) > 0$. Now, it is not difficult to see that if $|t - \lambda_n| \leq \delta$ (where δ is the constant in (6)), then

$$\frac{1}{2(1 + \frac{\delta^2}{\delta_1^2})} \leq \frac{|I'(t)|}{|I'(\lambda_n)|} = \frac{\nu_n}{2} |I'(t)|.$$

Since $\mathbb{R} = \bigcup_n [\lambda_n - \delta, \lambda_n + \delta]$ and $\sup_n \nu_n < \infty$, we finally get

$$(9) \quad |I'(t)| \asymp 1, \quad t \in \mathbb{R}.$$

We now extend a result from [Mit17]. Part 1 is already contained in that form in [Bar06, Lemma 5.4].

Lemma 2.4. *Let $\Lambda = (\lambda_n)_{n \geq 1}$ be a real sequence satisfying (6) and (7) and let I be the meromorphic inner function defined by (8). Let Θ be a meromorphic inner function such that $\Theta' \in L^\infty(\mathbb{R})$.*

- (1) *If $f \in \mathcal{K}_\Theta$ and $f(\lambda_n) = 0$, $n \geq 1$, then $f/(1 - I) \in \mathcal{K}_\Theta$.*
- (2) *The Toeplitz operator $T_{1-I} : \mathcal{K}_\Theta \rightarrow H^2$ is bounded below.*

Here, since $1 - I$ is analytic, the Toeplitz operator T_{1-I} is just the multiplication by $1 - I$.

Proof. As already mentioned (1) is exactly Lemma 5.4 from [Bar06]. Let us now prove (2), that is

$$\inf_{f \in \mathcal{K}_\Theta, \|f\|_2=1} \|(1 - I)f\|_2 > 0.$$

Assume on the contrary that there exists a sequence $(f_k)_{k \geq 1}$ in \mathcal{K}_Θ , $\|f_k\|_2 = 1$, such that $\|(1 - I)f_k\|_2 \rightarrow 0$ as $k \rightarrow \infty$. Denote by ϕ an increasing branch of the argument of I . Since $\phi'(t) = |I'(t)|$, according to (9), there exists $c_1, c_2 > 0$ such that

$$c_1 \leq \phi'(t) \leq c_2, \quad t \in \mathbb{R}.$$

Let us fix $\varepsilon > 0$ such that

$$\varepsilon < \min \left(\frac{2\pi}{c_2}, \frac{\delta}{6}, \frac{1}{4\|\Theta'\|_\infty} \right).$$

Then, for every $n \geq 1$, we have

$$0 < c_1\varepsilon \leq \phi(\lambda_n + \varepsilon) - \phi(\lambda_n) = \int_{\lambda_n}^{\lambda_n + \varepsilon} \phi'(t) dt \leq c_2\varepsilon < 2\pi,$$

and

$$0 < c_1\varepsilon \leq \phi(\lambda_n) - \phi(\lambda_n - \varepsilon) = \int_{\lambda_n - \varepsilon}^{\lambda_n} \phi'(t) dt \leq c_2\varepsilon < 2\pi.$$

Since $\{\lambda_n\} = \{t \in \mathbb{R} : e^{i\phi(t)} = 1\}$, it is now easy to see that there exists $c_3 > 0$ such that for all $t \in \mathbb{R}$ satisfying $\text{dist}(t, \Lambda) \geq \varepsilon$, we have

$$|I(t) - 1| \geq c_3.$$

Let $E_\varepsilon = \bigcup_{n \geq 1} (\lambda_n - \varepsilon, \lambda_n + \varepsilon)$. Therefore, we get

$$\int_{\mathbb{R} \setminus E_\varepsilon} |1 - I(t)|^2 |f_k(t)|^2 dt \geq c_3^2 \int_{\mathbb{R} \setminus E_\varepsilon} |f_k(t)|^2 dt.$$

Choose now k sufficiently large so that

$$\|(1 - I)f_k\|_2 \leq \frac{c_3}{\sqrt{2}},$$

which gives

$$\int_{\mathbb{R} \setminus E_\varepsilon} |f_k(t)|^2 dt \leq \frac{1}{2}.$$

Using $1 = \|f_k\|_2$, we then get

$$\frac{1}{2} \leq \int_{E_\varepsilon} |f_k(t)|^2 dt.$$

According to Lemma 2.2, we know that f_k is analytic on $\{z \in \mathbb{C} : \Im(z) > -4\varepsilon\}$ and

$$\sup_{|y| < 4\varepsilon} \int_{-\infty}^{+\infty} |f_k(x + iy)|^2 dx \leq \frac{1}{(1 - 4\varepsilon\|\Theta'\|_\infty)^2}.$$

Now let $t \in (-3\varepsilon, 3\varepsilon)$. By subharmonicity of $|f_k|^2$, we have

$$\begin{aligned} \sum_{n \geq 1} |f_k(\lambda_n + t)|^2 &\leq \sum_{n \geq 1} \frac{1}{9\pi\varepsilon^2} \int_{B(\lambda_n + t, 3\varepsilon)} |f_k(z)|^2 dA(z) \\ &\leq \sum_{n \geq 1} \frac{1}{9\pi\varepsilon^2} \int_{-3\varepsilon}^{3\varepsilon} \int_{\lambda_n + t - 3\varepsilon}^{\lambda_n + t + 3\varepsilon} |f_k(x + iy)|^2 dx dy. \end{aligned}$$

□

Using the fact that $|\lambda_{n+1} - \lambda_n| \geq \delta > 6\varepsilon$, it follows that $[\lambda_n + t - 3\varepsilon, \lambda_n + t + 3\varepsilon]$ are disjoint intervals which gives

$$\begin{aligned} \sum_{n \geq 1} |f_k(\lambda_n + t)|^2 &\leq \frac{1}{9\pi\varepsilon^2} \int_{-3\varepsilon}^{3\varepsilon} \int_{-\infty}^{+\infty} |f_k(x + iy)|^2 dx dy \\ &\leq \frac{2}{3\pi\varepsilon} \frac{1}{(1 - 4\varepsilon\|\Theta'\|_\infty)^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2} \leq \int_{E_\varepsilon} |f_k(t)|^2 dt &= \sum_{n \geq 1} \int_{-\varepsilon}^{\varepsilon} |f_k(t + \lambda_n)|^2 dt \\ &\leq \frac{4}{3\pi} \frac{1}{(1 - 4\varepsilon\|\Theta'\|_\infty)^2}. \end{aligned}$$

Hence $(1 - 4\varepsilon\|\Theta'\|_\infty)^2 \leq 8/(3\pi)$, and letting $\varepsilon \rightarrow 0$, we get a contradiction, which proves that the operator T_{1-I} is bounded below as an operator from \mathcal{K}_Θ to H^2 .

3. Toeplitz operators of the form unitary plus compact

In this section we revisit some classical results concerning the Toeplitz operators. As we will see our characterization of AOB we get in Section 5 will involve the condition that a certain Toeplitz operator is of the form unitary plus compact. It turns out that this property of Toeplitz operators has been characterized by Douglas [Dou73]. In this section, we revisit his result and explain how to derive Douglas's result from Hartman's theorem on the compactness of Hankel operators [Har58] and the Devinatz-Widom criterion for the invertibility of Toeplitz operators [Nik86, Nik09]. Most of this can be found in [Pel03] or [Nik86] in the context of the disc but we

need the analogues for the upper half-plane. So we give some details for completeness.

First, let us recall that if $u \in L^\infty(\mathbb{R})$, then the *Toeplitz operator* T_u on the Hardy space of the upper half-plane \mathcal{H}^2 is defined by $T_u(f) = P_+(uf)$, $f \in \mathcal{H}^2$, where P_+ is the Riesz projection from $L^2(\mathbb{R})$ onto \mathcal{H}^2 . Its cousin, the *Hankel operator*, H_u is defined as an operator from \mathcal{H}^2 into $\mathcal{H}_-^2 = L^2(\mathbb{R}) \ominus \mathcal{H}^2$ by $H_u(f) = P_-(uf)$, $f \in \mathcal{H}^2$, where $P_- = Id - P_+$.

A well-known theorem of Hartman [Pel03, Corollary 8.5] says that H_u is compact if and only if $u \in H^\infty + C(\dot{\mathbb{R}})$, where H^∞ is the space of analytic and bounded functions of \mathbb{C}_+ and

$$C(\dot{\mathbb{R}}) = \{f \in C(\mathbb{R}) : \lim_{x \rightarrow \pm\infty} f(x) \text{ exist and are equal}\}.$$

It turns out that the space $H^\infty + C(\dot{\mathbb{R}})$ is indeed a closed subalgebra of $L^\infty(\mathbb{R})$ (which is not obvious at the first sight but can be deduced from Hartman's theorem). We also recall a part of the Devinatz-Widom criterion for the invertibility of the Toeplitz operator. For the complete result, we refer the reader to ([Nik09], page 250). Let us recall that if $b \in L^1_{\mathbb{H}}$, that is

$$\int_{-\infty}^{\infty} \frac{|b(t)|}{1+t^2} dt < \infty,$$

the Hilbert transform of b is defined as the singular integral

$$\tilde{b}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \left(\frac{1}{x-t} + \frac{t}{1+t^2} \right) b(t) dt.$$

Theorem 3.1 (Devinatz-Widom). *Let $u \in L^\infty(\mathbb{R})$, $|u| = 1$ a.e. on \mathbb{R} . The following are equivalent.*

- (1) *The Toeplitz operator T_u is invertible;*
- (2) *there exist real valued bounded functions a, b on \mathbb{R} where $\|a\|_\infty < \pi/2$ and a real constant c such that $u = e^{i(c+a+\tilde{b})}$.*
- (3) *There is an outer function $h \in H^\infty$ such that $\|u - h\|_\infty < 1$*

Note that in [Nik09], the result is expressed in the context of the unit disc \mathbb{D} . To transfer the result to the upper-half plane, it is sufficient to use the conformal map (Cayley transform) $\phi(z) = (z-i)/(z+i)$ from \mathbb{C}_+ onto \mathbb{D} , the map $\mathcal{M}(f)(t) = \frac{\sqrt{2}}{t+i}(f \circ \phi)(t)$ for $f \in H^2(\mathbb{D})$, which is a unitary map from the Hardy space of the unit onto the Hardy space of the upper-half plane, and note that $T_u = \mathcal{M}T_{u \circ \phi^{-1}}\mathcal{M}^{-1}$.

The following lemma gives a sufficient condition for invertibility.

Lemma 3.2. *If $u = e^{i(\alpha+\tilde{\beta})}$, for some real-valued functions $\alpha, \beta \in C(\dot{\mathbb{R}})$, then T_u is invertible.*

Proof. Since $\alpha \in C(\dot{\mathbb{R}})$, at $\pm\infty$, it admits the same limit, say ℓ . Then, we can find a real-valued function $v \in C_c^1(\mathbb{R})$, the space of compactly supported

C^1 functions on \mathbb{R} , such that

$$\|\alpha - \ell - v\|_\infty < \pi/2.$$

Note that since $v \in C_c^1(\mathbb{R})$, \tilde{v} is a real-valued continuous function which tends to zero at infinity and we have $\tilde{\tilde{v}} = -v$ (see [Mas09, Theorem 14.1 & Corollary 14.9]). Then,

$$\alpha + \tilde{\beta} = \alpha - \ell - v + \widetilde{\beta - \tilde{v}} + \ell.$$

Hence

$$u = e^{i\ell} e^{i(\alpha - \ell - v + \widetilde{\beta - \tilde{v}})}.$$

We can apply the criterion of Devinatz-Widom to conclude that T_u is invertible. \square

When we try to characterize the property for a Toeplitz operator to be unitary modulo the compacts, the class QC of *quasicontinuous functions* naturally appears:

$$QC = (H^\infty + C(\dot{\mathbb{R}})) \cap \overline{(H^\infty + C(\dot{\mathbb{R}}))}.$$

Lemma 3.3 (Douglas). *Let $u \in L^\infty$, $|u| = 1$ a.e. on \mathbb{R} . Then, T_u is unitary modulo the compacts iff $u \in QC$ and T_u is invertible.*

Proof. First, let us assume that $u \in QC$ and T_u is invertible. Using the polar decomposition of bounded invertible operators, we can write

$$T_u = U|T_u|,$$

where U is a unitary operator and $|T_u| = (T_u^* T_u)^{\frac{1}{2}}$. Furthermore, we notice the following equality

$$I - (T_u^* T_u)^{\frac{1}{2}} = (I + (T_u^* T_u)^{\frac{1}{2}})^{-1} (I - T_u^* T_u),$$

using the positivity of $(T_u^* T_u)^{\frac{1}{2}}$ to ensure that the operator $I + (T_u^* T_u)^{\frac{1}{2}}$ is invertible. We recall that $I - T_u^* T_u = H_u^* H_u$ and using Hartman's theorem, we get that $I - (T_u^* T_u)^{\frac{1}{2}}$ is compact. Hence, $|T_u|$ is identity modulo the compacts, ensuring that $T_u = U|T_u|$ is unitary modulo the compacts.

Conversely, let $T_u = U + K$, where U is unitary and K is compact. Then, $T_u^* = U^* + K^*$ and

$$\begin{aligned} T_u T_u^* &= (U + K)(U^* + K^*) \\ &= I + K_1, \end{aligned}$$

where $K_1 = UK^* + KU^* + KK^*$ is compact. Hence, $I - T_u T_u^*$ is compact. Similarly, we can show that $I - T_u^* T_u$ is compact. Thus, $H_u^* H_u = I - T_u^* T_u$ and $H_{\bar{u}}^* H_{\bar{u}} = I - T_u T_u^*$ are both compact. But we recall the fact that an operator T is compact iff $T^* T$ is compact. This means that both u and \bar{u} belong to the space $H^\infty + C(\mathbb{R})$. That is to say, $u \in QC$. It remains to prove that T_u is invertible. The representation $T_u = U + K$ allows us to reason that since U is Fredholm with index 0 and K is compact, then T_u is also

Fredholm with index 0. Similarly, T_u^* is also Fredholm with index 0. We now use Coburn's lemma to claim that either $\ker T_u = \{0\}$ or $\ker T_u^* = \{0\}$. Thus, either and hence both T_u and T_u^* are invertible. \square

We revisit the following theorem of Sarason's [Sar73] that completely characterizes the unimodular functions that belong to the space QC . We note that the Cayley transform $\phi(z) = (z - i)/(z + i)$ is also a Blaschke factor vanishing at i .

Lemma 3.4 (Sarason). *Let $u \in L^\infty(\mathbb{R})$ be a unimodular function. Then, $u \in QC$ if and only if $u = \phi^n e^{i(a+\bar{b})}$, for some $n \in \mathbb{Z}$ and two real valued functions $a, b \in C(\mathbb{R})$.*

Proof. Let us assume that u is of the form $\phi^n e^{i(a+\bar{b})}$. We can rewrite u as follows.

$$u = \phi^n e^{(b+i\bar{b})+(ia-b)},$$

where $\phi^n e^{ia-b} \in C(\dot{\mathbb{R}})$, $e^{b+i\bar{b}} \in H^\infty$. Since $H^\infty + C(\dot{\mathbb{R}})$ is an algebra, we have that $u \in H^\infty + C(\dot{\mathbb{R}})$. In a similar manner, we can show that $\bar{u} \in H^\infty + C(\dot{\mathbb{R}})$. Therefore, $u \in QC$.

Conversely, assume that $u \in QC$. We can easily check that the following relationships are true.

$$\begin{aligned} I - T_u T_{\bar{u}} &= H_{\bar{u}}^* H_u \\ I - T_{\bar{u}} T_u &= H_u^* H_{\bar{u}}. \end{aligned}$$

Since $u, \bar{u} \in H^\infty + C(\dot{\mathbb{R}})$, by Hartman's theorem H_u and $H_{\bar{u}}$ are compact. This means that $I - T_u T_{\bar{u}}$ and $I - T_{\bar{u}} T_u$ are both compact. Thus, T_u is Fredholm. Let $n = -\text{ind} T_u$. First let us assume that $n \geq 0$. Let us define a new function

$$v := \overline{\phi^n} u.$$

Then, $T_v = T_{\overline{\phi^n}} T_u$ is also Fredholm. Moreover,

$$\text{ind}(T_v) = \text{ind}(T_{\overline{\phi^n}}) + \text{ind}(T_u) = n - n = 0.$$

By Coburn's result, either $\ker T_v = \{0\}$ or $\ker T_{\bar{v}} = \{0\}$. Since the index is zero, we finally deduce that both kernels are trivial and T_v will be invertible. Thus, by the Devinatz-Widom theorem, there is an outer function $h \in H^\infty$ such that

$$\|v - h\|_\infty = \|1 - \bar{v}h\|_\infty < 1.$$

Consequently, $\bar{v}h$ has a logarithm in the Banach algebra $H^\infty + C(\dot{\mathbb{R}})$ (see Lemma 2.13, page 34 in [Dou72]). Therefore, there is a $g \in H^\infty$ and $f \in C(\dot{\mathbb{R}})$ such that

$$\frac{1}{\bar{v}h} = \frac{v}{h} = e^{f+g}.$$

Hence,

$$v = h e^{f+g} = e^{ic + \log|h| + i\widetilde{\log|h|} + f + g},$$

for some $c \in \mathbb{R}$. Since $|v| = 1$ a.e. on \mathbb{R} , it follows that $\log |h| + \Re f + \Re g = 0$. Let us denote

$$b := \log |h| + \Re(g) = -\Re(f) \in C(\dot{\mathbb{R}}).$$

Since, $g \in H^\infty$, we have

$$\tilde{b} = \widetilde{\log |h|} + \Im(g) - \Im(g(i)).$$

Denoting $a := \Im(f) + c + \Im(g(i)) \in C(\dot{\mathbb{R}})$. Finally, we have that

$$u = \phi^n v = \phi^n e^{i(a+\tilde{b})}.$$

In the case $n < 0$, we let $v = \phi^n u$ and follow the same argument as before to arrive at the same conclusion. \square

Finally, we can prove the following result.

Theorem 3.5. *Let $u \in L^\infty(\mathbb{R})$, $|u| = 1$ a.e. on \mathbb{R} . The following are equivalent.*

- (1) T_u is unitary modulo the compacts.
- (2) There exist real valued functions $a, b \in C(\dot{\mathbb{R}})$ such that $u = e^{i(a+\tilde{b})}$.

Proof. First let us assume that there exist real valued functions $a, b \in C(\dot{\mathbb{R}})$ such that $u = e^{i(a+\tilde{b})}$. Using Lemma 3.2, we can conclude that T_u is invertible. Furthermore, using Lemmas 3.4 and 3.3, we can conclude that T_u is indeed unitary modulo the compacts.

Conversely, assume that T_u is unitary modulo the compacts. It follows from Lemma 3.3 that $u \in QC$ and T_u is invertible. Next, using Lemma 3.4, we see that $u = \phi^n e^{i(a+\tilde{b})}$, for some real valued functions $a, b \in C(\dot{\mathbb{R}})$ and some $n \in \mathbb{Z}$. It remains to prove that $n = 0$. Let us denote by $u_1 = e^{i(a+\tilde{b})}$. By Lemma 3.2, the Toeplitz operator T_{u_1} is invertible. If $n > 0$, then

$$T_u = T_{u_1} \phi^n = T_{u_1} T_{\phi^n},$$

whence $T_{\phi^n} = T_{\phi^n}^* = T_u^* (T_{u_1}^{-1})^*$ should also be invertible. But this is absurd because $\ker T_{\phi^n} = K_{\phi^n}$. If $n < 0$, then $T_u = T_{\phi^n} T_{u_1}$, that is to say

$$T_{\phi^n} = T_u (T_{u_1})^{-1}$$

is invertible, which is also absurd because $\ker T_{\phi^n} = K_{\phi^{-n}}$. \square

4. Riesz bases

In this section we revisit a recent work of Mitkovski [Mit17] on systems of exponential systems which form a Riesz basis. We provide a generalization of his result, while maintaining the techniques of the proofs.

Theorem 4.1. *Let $(\lambda_n)_{n \geq 1}$ be a real sequence satisfying (6) and (7) and let I be the meromorphic inner function defined by (8). Let Θ be a meromorphic inner function such that $\Theta' \in L^\infty(\mathbb{R})$ and $\inf_{n \geq 1} |\Theta'(\lambda_n)| > 0$. Then,*

- (1) $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is complete in \mathcal{K}_Θ if and only if $T_{\Theta\bar{I}} : \mathcal{H}^2 \rightarrow \mathcal{H}^2$ has a dense range.
- (2) $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is a Riesz sequence in \mathcal{K}_Θ if and only if $T_{\Theta\bar{I}} : \mathcal{H}^2 \rightarrow \mathcal{H}^2$ is injective with closed range.
- (3) $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is a Riesz basis for \mathcal{K}_Θ if and only if $T_{\Theta\bar{I}} : \mathcal{H}^2 \rightarrow \mathcal{H}^2$ is invertible.

Proof. Let us first show that the upper inequality in Riesz basis property (1) is satisfied for $(k_{\lambda_n}^\Theta)_n$ under the hypothesis of the theorem. Indeed, according to Lemma 2.3, we have

$$(1 - I) \sum_{n \geq 1} a_n k_{\lambda_n}^\Theta = \sum_{n \geq 1} a_n \eta_n k_{\lambda_n}^I - \Theta \sum_{n \geq 1} a_n \eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I,$$

where $\eta_n = |I'(\lambda_n)|^{1/2} |\Theta'(\lambda_n)|^{-1/2} \asymp 1$ by hypothesis and (9). Using Lemma 2.4,

$$\begin{aligned} \left\| \sum_{n \geq 1} a_n k_{\lambda_n}^\Theta \right\|_2 &\lesssim \left\| (1 - I) \sum_{n \geq 1} a_n k_{\lambda_n}^\Theta \right\|_2 \\ &\leq \left\| \sum_{n \geq 1} a_n \eta_n k_{\lambda_n}^I \right\|_2 + \left\| \sum_{n \geq 1} a_n \eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I \right\|_2. \end{aligned}$$

Then, using the fact that $(k_{\lambda_n}^I)_{n \geq 1}$ is an orthonormal basis for \mathcal{K}_I and $|\Theta(\lambda_n)| = 1$, $n \geq 1$, we get

$$(10) \quad \left\| \sum_{n \geq 1} a_n k_{\lambda_n}^\Theta \right\|_2 \lesssim \left(\sum_{n \geq 1} |a_n|^2 \right)^{1/2}.$$

Let us now prove (1). First, assume that $T_{\Theta\bar{I}}$ has dense range and assume that there exists $f \in \mathcal{K}_\Theta$, $f(\lambda_n) = 0$, $n \geq 1$, and $f \neq 0$. According to Lemma 2.4, the function $g := f/(1 - I)$ belongs to \mathcal{K}_Θ and, since $\mathcal{K}_\Theta = \mathcal{H}^2 \cap \Theta \overline{\mathcal{H}^2}$, $\Theta I g = -\Theta(1 - I)g + \Theta g = -\Theta f + \Theta g \in \overline{\mathcal{H}^2}$. In particular, $T_{I\bar{\Theta}} g = 0$ and $T_{I\bar{\Theta}}$ is not injective which contradicts the fact that $T_{\Theta\bar{I}} = T_{I\bar{\Theta}}^*$ has dense range. Conversely, suppose that $T_{\Theta\bar{I}}$ does not have a dense range. Then, there is a function $g \in \ker(T_{I\bar{\Theta}})$, $g \neq 0$. Hence $I g \bar{\Theta} \in \overline{\mathcal{H}^2}$, which gives that both g and $I g$ belong to \mathcal{K}_Θ . So $f := (1 - I)g$ is also in \mathcal{K}_Θ , $f \neq 0$ and

$$f(\lambda_n) = (1 - I(\lambda_n))g(\lambda_n) = 0, \quad n \geq 1,$$

which proves that $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is not complete in \mathcal{K}_Θ .

Let us now prove (2). Assume that $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is not a Riesz sequence in \mathcal{K}_Θ . According to (10), it means that the lower inequality in (1) is not

satisfied. Then, for any $\varepsilon > 0$, there exists $\{b_n\}_{n \geq 1} \in \ell^2$ such that

$$\left\| \sum_{n \geq 1} b_n k_{\lambda_n}^\Theta \right\|^2 < \varepsilon \sum_{n \geq 1} |b_n|^2.$$

Take $a_n = b_n / \|b_n\|_{\ell^2}$. Then $\sum_{n \geq 1} |a_n|^2 = 1$ and

$$\left\| \sum_{n \geq 1} a_n k_{\lambda_n}^\Theta \right\|^2 < \varepsilon.$$

In particular, we can construct a sequence $(a_n^\ell)_{n \geq 1}$ of elements of ℓ^2 such that for every $\ell \geq 1$, $\sum_{n \geq 1} |a_n^\ell|^2 = 1$ and

$$\left\| \sum_{n \geq 1} a_n^\ell k_{\lambda_n}^\Theta \right\|_2 \rightarrow 0, \quad \text{as } \ell \rightarrow \infty.$$

Let $g_\ell := \sum_{n \geq 1} a_n^\ell k_{\lambda_n}^\Theta$. Then $g_\ell \in \mathcal{K}_\Theta$ and $\|g_\ell\|_2 \rightarrow 0$ as $\ell \rightarrow \infty$. According to Lemma 2.3, we have

$$(1 - I)g_\ell = \sum_{n \geq 1} a_n^\ell \eta_n k_{\lambda_n}^I - \Theta \sum_{n \geq 1} a_n^\ell \eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I.$$

Multiply by \bar{I} and take the Riesz projection gives

$$P_+(\bar{I}(1 - I)g_\ell) = \sum_{n \geq 1} a_n^\ell \eta_n P_+(\bar{I}k_{\lambda_n}^I) - P_+(\Theta \bar{I} \sum_{n \geq 1} a_n^\ell \eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I).$$

But $k_{\lambda_n}^I \in \mathcal{K}_I = \mathcal{H}^2 \cap \overline{I\mathcal{H}^2}$, hence $P_+(\bar{I}k_{\lambda_n}^I) = 0$ and, then

$$P_+(\bar{I}(1 - I)g_\ell) = -T_{\Theta \bar{I}} \left(\sum_{n \geq 1} a_n^\ell \eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I \right).$$

On one hand, we have

$$\|P_+(\bar{I}(1 - I)g_\ell)\|_2 \rightarrow 0, \quad \text{as } \ell \rightarrow \infty.$$

And on the other hand, since $\Theta' \in L^\infty(\mathbb{R})$, we also have

$$\begin{aligned} \left\| \sum_{n \geq 1} a_n^\ell \eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I \right\|_2^2 &= \sum_{n \geq 1} |a_n^\ell|^2 \eta_n^2 |\Theta(\lambda_n)|^2 \\ &= \sum_{n \geq 1} |a_n^\ell|^2 \eta_n^2 \\ &\gtrsim \sum_{n \geq 1} |a_n^\ell|^2 = 1, \end{aligned}$$

which proves that $T_{\Theta \bar{I}}$ is not bounded below. Conversely, assume that $T_{\Theta \bar{I}}$ is not bounded below. Hence there exists a sequence $(f_\ell)_{\ell \geq 1}$, $f_\ell \in \mathcal{H}^2$, $\|f_\ell\|_2 = 1$, such that $\|T_{\Theta \bar{I}} f_\ell\|_2 \rightarrow 0$, as $\ell \rightarrow \infty$. Let $h_\ell = \bar{T}_{\Theta \bar{I}} f_\ell$. Then,

$\|h_\ell\|_2 \rightarrow 0$ as $\ell \rightarrow \infty$, and $g_\ell := \Theta \bar{I} f_\ell - h_\ell \in \overline{\mathcal{H}^2}$ with $\|g_\ell\|_2 \asymp 1$ (because $\|f_\ell\|_2 = 1$ and $\|h_\ell\|_2 \rightarrow 0$). Note that

$$I g_\ell = \Theta f_\ell - I h_\ell \in \mathcal{H}^2 \cap I \overline{\mathcal{H}^2} = \mathcal{K}_I.$$

Since $(k_{\lambda_n}^I)_{n \geq 1}$ is an orthonormal basis for \mathcal{K}_I , there exists $(g_n^\ell)_{n \geq 1} \in \ell^2$ such that

$$I g_\ell = \sum_{n \geq 1} g_n^\ell k_{\lambda_n}^I.$$

Note also that

$$\sum_{n \geq 1} |g_n^\ell|^2 = \|I g_\ell\|_2^2 = \|g_\ell\|_2^2 \asymp 1.$$

Then

$$I h_\ell = \Theta f_\ell - I g_\ell = \Theta f_\ell - \sum_{n \geq 1} g_n^\ell k_{\lambda_n}^I,$$

and using one more time Lemma 2.3, we get

$$(11) \quad I h_\ell = \Theta f_\ell - \left((1 - I) \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta + \Theta \sum_{n \geq 1} g_n^\ell \overline{\Theta(\lambda_n)} k_{\lambda_n}^I \right).$$

Thus

$$\Theta f_\ell - \Theta \sum_{n \geq 1} g_n^\ell \overline{\Theta(\lambda_n)} k_{\lambda_n}^I - \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta = I h_\ell - I \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta.$$

The difference of the first two terms on the left side is in $\Theta \mathcal{H}^2$, while the third term is in \mathcal{K}_Θ . Using orthogonality and triangle's inequality, we obtain

$$\begin{aligned} \left\| \Theta f_\ell - \Theta \sum_{n \geq 1} g_n^\ell \overline{\Theta(\lambda_n)} k_{\lambda_n}^I \right\|_2^2 + \left\| \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta \right\|_2^2 &\leq \left(\|I h_\ell\|_2 + \left\| I \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta \right\|_2 \right)^2 \\ &= \left(\|h_\ell\|_2 + \left\| \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta \right\|_2 \right)^2. \end{aligned}$$

Since by (10), we have

$$\left\| \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta \right\|_2 \lesssim \left(\sum_{n \geq 1} \frac{|g_n^\ell|^2}{\eta_n^2} \right)^{1/2} \lesssim 1,$$

and since $\|h_\ell\|_2 \rightarrow 0$ as $\ell \rightarrow \infty$, we deduce

$$\left\| \Theta f_\ell - \Theta \sum_{n \geq 1} g_n^\ell \overline{\Theta(\lambda_n)} k_{\lambda_n}^I \right\|_2 \rightarrow 0, \quad \text{as } \ell \rightarrow \infty.$$

Using this and (11), we obtain

$$\left\| (1 - I) \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta \right\|_2 \rightarrow 0, \quad \text{as } \ell \rightarrow \infty.$$

Lemma 2.4 implies now that

$$\left\| \sum_{n \geq 1} \frac{g_n^\ell}{\eta_n} k_{\lambda_n}^\Theta \right\|_2 \rightarrow 0, \quad \text{as } \ell \rightarrow \infty,$$

which together with

$$\sum_{n \geq 1} \frac{|g_n^\ell|^2}{\eta_n^2} \asymp \sum_{n \geq 1} |g_n^\ell|^2 \asymp 1,$$

implies that $(k_{\lambda_n}^\Theta)_{n \geq 1}$ cannot be a Riesz sequence in \mathcal{K}_Θ .

The part (3) follows immediately from (1) and (2). □

5. Asymptotically orthonormal bases.

In that section, we give a result on sequences of reproducing kernels which form an AOB. We need to recall the notion of angle between two closed subspaces \mathcal{F}, \mathcal{G} of an Hilbert space \mathcal{H} . The angle $\langle \mathcal{F}, \mathcal{G} \rangle$ between \mathcal{F} and \mathcal{G} is defined by the condition:

$$\langle \mathcal{F}, \mathcal{G} \rangle \in [0, \pi/2] \quad \cos \langle \mathcal{F}, \mathcal{G} \rangle = \sup_{f \in \mathcal{F}, g \in \mathcal{G}} \frac{|\langle f, g \rangle|}{\|f\| \|g\|}.$$

It is easy to see that $\cos \langle \mathcal{F}, \mathcal{G} \rangle = \|P_{\mathcal{F}} P_{\mathcal{G}}\|$, where $P_{\mathcal{F}}$ and $P_{\mathcal{G}}$ are the corresponding orthogonal projections.

Theorem 5.1. *Let $(\lambda_n)_{n \geq 1}$ be a real sequence satisfying (6) and (7) and let I be the meromorphic inner function defined by (8). Let Θ be a meromorphic inner function such that $\Theta' \in L^\infty(\mathbb{R})$ and $\eta_n = |I'(\lambda_n)|^{1/2} |\Theta'(\lambda_n)|^{-1/2} \rightarrow 1$ as $n \rightarrow \infty$. Then, the following assertions are equivalent:*

- (1) *there exists two real-valued functions $\alpha, \beta \in C(\mathbb{R})$ satisfying*

$$\arg \Theta - \arg I = \alpha + \tilde{\beta};$$

- (2) *the operator $T_{\Theta \bar{I}}$ is of the form unitary plus compact.*
 (3) (i) *the sequence $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is a complete AOB for \mathcal{K}_Θ ;*
 (ii) *$\cos \langle \overline{\text{span}}(k_{\lambda_n}^\Theta : n \geq N), I\mathcal{H}^2 \rangle \rightarrow 0$ as $N \rightarrow \infty$.*

Proof. The equivalence between (1) and (2) is proved in Theorem 3.5.

Let us now prove (2) \implies (3). So, we assume that $T_{\Theta \bar{I}}$ is of the form unitary plus compact. We have already seen that necessarily $T_{\Theta \bar{I}}$ is invertible

(see Lemma 3.3) and by Theorem 4.1, we get that the sequence $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is a Riesz basis for \mathcal{K}_Θ . Observe now that

$$(T_{\bar{I}}|_{\mathcal{K}_\Theta})^* = P_\Theta I|_{\mathcal{K}_\Theta} = \Theta P_- \bar{\Theta} I|_{\mathcal{K}_\Theta} = \Theta H_{I\bar{\Theta}}|_{\mathcal{K}_\Theta}.$$

But Lemma 3.3 implies $I\bar{\Theta} \in QC$ and by Hartman's theorem, we deduce that $H_{I\bar{\Theta}}$ is compact. Hence $T_{\bar{I}}|_{\mathcal{K}_\Theta}$ is compact. Using Lemma 2.3, we have

$$(1 - I)k_{\lambda_n}^\Theta = \eta_n k_{\lambda_n}^I - \overline{\Theta(\lambda_n)} \eta_n k_{\lambda_n}^I,$$

where $\eta_n \rightarrow 1$ as $n \rightarrow \infty$. If we apply the operator $T_{\bar{I}}$, using the fact that $\ker T_{\bar{I}} = \mathcal{K}_I$, we obtain

$$(Id - T_{\bar{I}})k_{\lambda_n}^\Theta = T_{\Theta\bar{I}}(\eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I).$$

Since $(\eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I)_{n \geq 1}$ is an orthogonal basis for \mathcal{K}_I , since $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is a Riesz basis of \mathcal{K}_Θ and $T_{\Theta\bar{I}}$ is invertible, the last equation implies that the operator $(Id - T_{\bar{I}})|_{\mathcal{K}_\Theta}$ is also invertible. Therefore, we can write

$$k_{\lambda_n}^\Theta = ((Id - T_{\bar{I}})|_{\mathcal{K}_\Theta})^{-1} T_{\Theta\bar{I}}(\eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I).$$

Now observe that the operator $((Id - T_{\bar{I}})|_{\mathcal{K}_\Theta})^{-1} T_{\Theta\bar{I}}$ is of the form unitary plus compact and the sequence $(\eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I)_{n \geq 1}$ is a complete AOB for \mathcal{K}_I . It implies that the sequence $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is a complete AOB for \mathcal{K}_Θ . To prove (ii), note that

$$\begin{aligned} \cos\langle \overline{\text{span}}(k_{\lambda_n}^\Theta : n \geq N), I\mathcal{H}^2 \rangle &= \|P_{I\mathcal{H}^2} \overline{\text{span}}(k_{\lambda_n}^\Theta : n \geq N)\| \\ &= \|IP_+ \bar{I} \overline{\text{span}}(k_{\lambda_n}^\Theta : n \geq N)\| \\ &= \|T_{\bar{I}} \overline{\text{span}}(k_{\lambda_n}^\Theta : n \geq N)\|. \end{aligned}$$

Since $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is a complete AOB for \mathcal{K}_Θ and $T_{\bar{I}}|_{\mathcal{K}_\Theta}$ is compact, we can use [FM16, Lemma 10.31] to conclude that

$$\|T_{\bar{I}} \overline{\text{span}}(k_{\lambda_n}^\Theta : n \geq N)\| \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

It remains to prove (3) \implies (2). Let us denote by P_N^Θ the projection from \mathcal{K}_Θ onto $\overline{\text{span}}(k_{\lambda_n}^\Theta : n \geq N)$. We have

$$\begin{aligned} \left\| P_{I\mathcal{H}^2} P_N^\Theta \left(\sum_{n \geq 1} a_n k_{\lambda_n}^\Theta \right) \right\|_2^2 &= \left\| P_{I\mathcal{H}^2} \left(\sum_{n \geq N} a_n k_{\lambda_n}^\Theta \right) \right\|_2^2 \\ &\leq \varepsilon_N^2 \left\| \sum_{n \geq N} a_n k_{\lambda_n}^\Theta \right\|_2^2, \end{aligned}$$

where $\varepsilon_N = \cos(\overline{\text{span}}(k_{\lambda_n}^\Theta : n \geq N), I\mathcal{H}^2) \rightarrow 0$ as $N \rightarrow \infty$. Moreover, since $(k_{\lambda_n}^\Theta)_{n \geq 1}$ is a complete AOB for \mathcal{K}_Θ , then

$$\begin{aligned} \left\| P_{I\mathcal{H}^2} P_N^\Theta \left(\sum_{n \geq 1} a_n k_{\lambda_n}^\Theta \right) \right\|_2^2 &\lesssim \varepsilon_N^2 \sum_{n \geq N} |a_n|^2 \\ &\leq \varepsilon_N^2 \sum_{n \geq 1} |a_n|^2 \\ &\lesssim \varepsilon_N^2 \left\| \sum_{n \geq 1} a_n k_{\lambda_n}^\Theta \right\|_2^2. \end{aligned}$$

Therefore, we get $\|P_{I\mathcal{H}^2} P_N^\Theta\| \lesssim \varepsilon_N \rightarrow 0$, $N \rightarrow \infty$. Since

$$P_{I\mathcal{H}^2}|_{\mathcal{K}_\Theta} = P_{I\mathcal{H}^2} P_N^\Theta + P_{I\mathcal{H}^2}(Id - P_N^\Theta),$$

and $Id - P_N^\Theta$ is a finite rank operator, we get that $P_{I\mathcal{H}^2}|_{\mathcal{K}_\Theta}$ can be approximated in norm by a sequence of finite rank operators and hence $P_{I\mathcal{H}^2}|_{\mathcal{K}_\Theta}$ is compact. We deduce that $T_{\bar{I}}|_{\mathcal{K}_\Theta} = \overline{I} P_{I\mathcal{H}^2}|_{\mathcal{K}_\Theta}$ is also compact. Since $(k_{\lambda_n}^\Theta)_n$ is an AOB for \mathcal{K}_Θ and $(\eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I)_n$ is an AOB for \mathcal{K}_I , there is an isomorphism $U : \mathcal{K}_I \rightarrow \mathcal{K}_\Theta$ of the form unitary plus compact such that $U(\eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I) = k_{\lambda_n}^\Theta$, $n \geq 1$. Using one more time the relation

$$(Id - T_{\bar{I}})k_{\lambda_n}^\Theta = T_{\Theta\bar{I}}(\eta_n \overline{\Theta(\lambda_n)} k_{\lambda_n}^I),$$

we then deduce that the operator $T_{\Theta\bar{I}} : \mathcal{K}_I \rightarrow \mathcal{K}_\Theta$ coincide with $(Id - T_{\bar{I}})U$. In particular, it is of the form unitary plus compact. Note now that the operator $T_{\Theta\bar{I}} : I\mathcal{H}^2 \rightarrow \Theta\mathcal{H}^2$ is unitary. Hence, we get that the operator $T_{\Theta\bar{I}} : \mathcal{H}^2 \rightarrow \mathcal{H}^2$ is unitary plus compact. \square

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