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Lions' maximal regularity problem with $H^{\frac{1}{2}}$ -regularity in time

Mahdi Achache, El Maati Ouhabaz *

Abstract

We consider the problem of maximal regularity for non-autonomous Cauchy problems

$$\begin{cases} u'(t) + A(t)u(t) &= f(t), t \in (0, \tau] \\ u(0) &= u_0. \end{cases}$$

The time dependent operators $A(t)$ are associated with (time dependent) sesquilinear forms on a Hilbert space \mathcal{H} . We are interested in J.L. Lions's problem concerning maximal regularity of such equations. We give a positive answer to this problem under minimal regularity assumptions on the forms. Our main assumption is that the forms are piecewise $H^{\frac{1}{2}}$ with respect to the variable t . This regularity assumption is optimal and our results are the most general ones on this problem.

keywords: Maximal regularity, non-autonomous evolution equations, Sobolev regularity

Mathematics Subject Classification (2010): 35K90, 35K45, 47D06.

1 Introduction

Let $(\mathcal{H}, (\cdot, \cdot), \|\cdot\|)$ be a separable Hilbert space over \mathbb{R} or \mathbb{C} . We consider another separable Hilbert space \mathcal{V} which is densely and continuously embedded into \mathcal{H} . We denote by \mathcal{V}' the (anti-) dual space of \mathcal{V} so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

Hence there exists a constant $C > 0$ such that

$$\|u\| \leq C\|u\|_{\mathcal{V}} \quad (u \in \mathcal{V}),$$

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where $\|\cdot\|_{\mathcal{V}}$ denotes the norm of \mathcal{V} . Similarly,

$$\|\psi\|_{\mathcal{V}'} \leq C\|\psi\| \quad (\psi \in \mathcal{H}).$$

We denote by $\langle \cdot, \cdot \rangle$ the duality \mathcal{V}' - \mathcal{V} and note that $\langle \psi, v \rangle = (\psi, v)$ if $\psi \in \mathcal{H}$ and $v \in \mathcal{V}$.

We consider a family of sesquilinear forms

$$\mathbf{a} : [0, \tau] \times \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}.$$

We assume throughout this paper the following usual properties.

- [H1]: $D(\mathbf{a}(t)) = \mathcal{V}$ (constant form domain),
- [H2]: $|\mathbf{a}(t, u, v)| \leq M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ for $t \in [0, \tau]$, $u, v \in \mathcal{V}$ and some constant $M > 0$ (uniform boundedness),
- [H3]: $\operatorname{Re} \mathbf{a}(t, u, u) + \nu\|u\|^2 \geq \delta\|u\|_{\mathcal{V}}^2$ for $u \in \mathcal{V}$ and some $\delta > 0$ and $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

We denote by $A(t), \mathcal{A}(t)$ the usual associated operators with $\mathbf{a}(t)$ as operators on \mathcal{H} and \mathcal{V}' , respectively. In particular, $\mathcal{A}(t) : \mathcal{V} \rightarrow \mathcal{V}'$ as a bounded operator and

$$\mathbf{a}(t, u, v) = \langle \mathcal{A}(t)u, v \rangle \text{ for all } u, v \in \mathcal{V}.$$

The operator $A(t)$ is the part of $\mathcal{A}(t)$ on \mathcal{H} .

We consider the non-homogeneous Cauchy problem

$$\begin{cases} u'(t) + A(t)u(t) &= f(t), \quad t \in (0, \tau] \\ u(0) &= u_0. \end{cases} \quad (\text{P})$$

Definition 1.1. *The Cauchy problem (P) has maximal L^2 -regularity in \mathcal{H} if for every $f \in L^2(0, \tau; \mathcal{H})$, there exists a unique $u \in H^1(0, \tau; \mathcal{H})$ with $u(t) \in D(A(t))$ for a.e. $t \in [0, \tau]$ and u is a solution of (P) in the L^2 -sense.*

By a very well known result of J.L. Lions, maximal L^2 -regularity always holds in the space \mathcal{V}' . That is, for every $f \in L^2(0, \tau; \mathcal{V}')$ and $u_0 \in \mathcal{H}$ there exists a unique $u \in H^1(0, \tau; \mathcal{V}') \cap L^2(0, \tau; \mathcal{V})$ which solves the equation

$$\begin{cases} u'(t) + \mathcal{A}(t)u(t) &= f(t), \quad t \in (0, \tau] \\ u(0) &= u_0. \end{cases} \quad (\text{P}')$$

In applications one needs however maximal regularity in \mathcal{H} (for example for elliptic boundary value problems one has to work on \mathcal{H} rather than \mathcal{V}' in order to identify the boundary conditions). Maximal regularity in \mathcal{H} differs considerably from the same property in \mathcal{V}' . Before we recall known results and explain our main contribution in this paper, we recall that one

of the reasons why maximal regularity (both in the autonomous and non-autonomous cases) was intensively studied is due to the fact that it is a very useful tool to prove existence results for non-linear evolution equations.

For symmetric forms $\mathfrak{a}(t)$, Lions [15] (IV Sec. 6, Théorème 6.1) proved that if $t \mapsto \mathfrak{a}(t, u, v)$ is C^1 and $u_0 = 0$, then maximal L^2 -regularity in \mathcal{H} is satisfied. For general $u_0 \in D(A(0))$, Lions imposes the stronger regularity property that $t \mapsto \mathfrak{a}(t, u, v)$ is C^2 . Bardos [8] improves the latter result for forms satisfying the uniform Kato square root property (see Definition 3.4 below) by assuming that $A(\cdot)^{\frac{1}{2}}$ is continuously differentiable with values in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$ and $u_0 \in \mathcal{V}$. Ouhabaz and Spina [18] proved maximal regularity in \mathcal{H} if $t \mapsto \mathfrak{a}(t, u, v)$ is C^α for some $\alpha > \frac{1}{2}$ when $u_0 = 0$. This result was extended in Haak and Ouhabaz [11] who prove maximal L^p -regularity under a slightly better regularity condition and allowing $u_0 \in D(A(0)^{\frac{1}{2}})$. Dier [9] proved maximal L^2 -regularity for symmetric forms such that $t \mapsto \mathfrak{a}(t, u, v)$ is of bounded variations. Fackler [12] proved that the order $\alpha > \frac{1}{2}$ in [18] or [11] is optimal in the sense that there exist $\mathfrak{a}(\cdot)$ symmetric and $C^{\frac{1}{2}}$ for which maximal regularity in \mathcal{H} fails. A counter-example already appeared in Dier [9] and it is based on a form which does not satisfy the Kato square root property. Dier and Zacher [10] proved that if $t \mapsto \mathcal{A}(t)$ is in the fractional Sobolev space $H^{\frac{1}{2}+\delta}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ for some $\delta > 0$ then maximal L^2 -regularity in \mathcal{H} holds. For a Banach space version of this result, see Fackler [13].

The example in [12] is not a differential operator. For elliptic operators in divergence form on \mathbb{R}^n , Auscher and Egert [7] proved maximal regularity if the coefficients satisfy a certain BMO- $H^{\frac{1}{2}}$ condition. The example from [12] also shows that $\mathcal{A}(\cdot) \in W^{\frac{1}{2}, p}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ for $p > 2$ is not enough to obtain maximal regularity. The example in [9] shows that $\mathcal{A}(\cdot) \in W^{\frac{1}{2}, p}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ for $p < 2$ does not imply maximal regularity, at least for form which does not satisfy Kato's square root property. For a discussion on these negative results, see the review paper of Arendt, Dier and Fackler [5]. As pointed in [5], the remaining problem is the case of fractional regularity $H^{\frac{1}{2}}$. We solve this problem in the present paper. Our main result shows that for forms satisfying the uniform Kato square root property and an integrability condition (see (2.1) below), if $t \mapsto \mathcal{A}(t)$ is piecewise in the Sobolev space $H^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ then maximal L^2 -regularity in \mathcal{H} is satisfied. The initial data u_0 is arbitrary in \mathcal{V} . This result is optimal. The required Sobolev regularity cannot be smaller than $\frac{1}{2}$ since $C^{\frac{1}{2}} \subset H^\alpha$ for $\alpha < \frac{1}{2}$. In the case where $\mathcal{A}(t) - \mathcal{A}(s)$ maps into the dual space of $[H, \mathcal{V}]_\gamma$ we allow the fractional Sobolev regularity to be $\frac{\gamma}{2}$. This extends related results in Ouhabaz [17] and Arendt and Monniaux [4].

We give the precise statements of the main results in the next section. In Sections 3 and 4 we prove several key estimates and develop the necessary tools for the proofs of the main results. Some of these tools are quadratic

estimates and $L^\infty(0, \tau; \mathcal{V})$ -estimates for the solution of the Cauchy problem. The main results are proved in Section 5 and several examples are given in Section 6.

2 Main results

In this section we state explicitly our main results. For clarity of exposition we consider separately the cases $\gamma = 1$, $\gamma \in (0, 1)$ and $\gamma = 0$.

We start by recalling the definition of vector-valued fractional Sobolev spaces.

Definition 2.1. *Let X be a Banach space, $\alpha \in (0, 1)$ and I an open subset of \mathbb{R} . A function $f \in L^2(I; X)$ is in the fractional Sobolev space $H^\alpha(I; X)$ if*

$$\|f\|_{H^\alpha(I; X)}^2 := \|f\|_{L^2(I; X)}^2 + \int_{I \times I} \frac{\|f(t) - f(s)\|_X^2}{|t - s|^{2\alpha+1}} ds dt < \infty.$$

We say that f is in the homogeneous Sobolev space $\dot{H}^\alpha(I; X)$ if

$$\|f\|_{\dot{H}^\alpha(I; X)}^2 := \int_{I \times I} \frac{\|f(t) - f(s)\|_X^2}{|t - s|^{2\alpha+1}} ds dt < \infty.$$

We shall say that f is piecewise in $H^\alpha(I; X)$ (resp. $\dot{H}^\alpha(I; X)$) if there exists $t_0 < t_1 < \dots < t_n$ such that $I = \cup_i [t_i, t_{i+1}]$ and the restriction of f to each sub-interval (t_i, t_{i+1}) is in $H^\alpha(t_i, t_{i+1}; X)$ (resp. $\dot{H}^\alpha(t_i, t_{i+1}; X)$).

Let $\mathbf{a}(t) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{C}$ for $0 \leq t \leq \tau$ be a family of forms satisfying [H1]-[H3] and let $A(t)$ and $\mathcal{A}(t)$ be the associated operators on \mathcal{H} and \mathcal{V}' , respectively. We shall need the following property.

Given $\varepsilon > 0$, there exists $\tau_0 = 0 < \tau_1 < \dots < \tau_n = \tau$ such that

$$\sup_{t \in (\tau_{i-1}, \tau_i)} \int_{\tau_{i-1}}^{\tau_i} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')}^2}{|t - s|} ds < \varepsilon. \quad (2.1)$$

Note that this assumption is satisfied in many cases. Suppose for example that $t \mapsto \mathbf{a}(t, u, v)$ is C^α for some $\alpha > 0$ in the sense that

$$|\mathbf{a}(t, u, v) - \mathbf{a}(s, u, v)| \leq M|t - s|^\alpha \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} \quad (2.2)$$

for some positive constant M and all $u, v \in \mathcal{V}$. Then clearly

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \leq M|t - s|^\alpha$$

and this implies (2.1). More generally, if ω_i denotes the modulus of continuity of \mathcal{A} on the interval (τ_{i-1}, τ_i) then (2.1) is satisfied if

$$\int_{|r| \leq \tau_i - \tau_{i-1}} \frac{\omega_i(r)^2}{r} dr < \varepsilon. \quad (2.3)$$

Theorem 2.2. *Suppose that [H1]-[H3] and the uniform Kato square property are satisfied. If $t \mapsto \mathcal{A}(t)$ is piecewise in $H^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ and satisfies (2.1) then (P) has maximal L^2 -regularity in \mathcal{H} for all $u_0 \in \mathcal{V}$. In addition, there exists a positive constant C independent of u_0 and f such that*

$$\|A(\cdot)u(\cdot)\|_{L^2(0,\tau;\mathcal{H})} + \|u\|_{H^1(0,\tau;\mathcal{H})} \leq C \left[\|u_0\|_{\mathcal{V}} + \|f\|_{L^2(0,\tau;\mathcal{H})} \right]. \quad (2.4)$$

We refer to the next section for the definition of the uniform Kato square root property and for few more details on such property.

We have the following corollary which recovers the maximal regularity result proved [10] under the assumption that $t \mapsto \mathcal{A}(t)$ is in $H^{\frac{1}{2}+\delta}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$.

Corollary 2.3. *Suppose that [H1]-[H3] and the uniform Kato square property are satisfied. Suppose that $t \mapsto \mathcal{A}(t)$ is piecewise in $H^{\frac{1}{2}+\delta}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ for some $\delta > 0$. Then (P) has maximal L^2 -regularity in \mathcal{H} for all $u_0 \in \mathcal{V}$. In addition, there exists a positive constant C independent of u_0 and f such that*

$$\|A(\cdot)u(\cdot)\|_{L^2(0,\tau;\mathcal{H})} + \|u\|_{H^1(0,\tau;\mathcal{H})} \leq C \left[\|u_0\|_{\mathcal{V}} + \|f\|_{L^2(0,\tau;\mathcal{H})} \right]. \quad (2.5)$$

Proof. It follows from [19], Corollary 26 that $H^{\frac{1}{2}+\delta}(\tau_{i-1}, \tau_i; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ is continuously embedded into $C^\delta(\tau_{i-1}, \tau_i; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$. As explained above this implies condition (2.1). We then apply Theorem 2.2. \square

Let $\gamma \in [0, 1]$ and $\mathcal{V}_\gamma := [\mathcal{H}, \mathcal{V}]_\gamma$ be the usual complex interpolation space between \mathcal{H} and \mathcal{V} . We denote by $\mathcal{V}'_\gamma := (\mathcal{V}_\gamma)'$ its (anti-) dual. In some situations, one may have $\mathcal{A}(t) - \mathcal{A}(s) : \mathcal{V} \rightarrow \mathcal{V}'_\gamma$ for some $\gamma \in [0, 1]$ (see Section 3 for some additional details). For example, this happens for forms $\mathbf{a}(t)$ associated with differential operators such that the difference $\mathbf{a}(t) - \mathbf{a}(s)$ has only terms of smaller order or boundary terms. In this case the required regularity in the previous theorem can be improved. Before we state the results we introduce the following assumption

- [H4]: $\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)} \leq M_\gamma$

for some positive constant M_γ and all $t, s \in [0, \tau]$.

Theorem 2.4. *Suppose [H1]-[H3] and that $D(A(t_0)^{\frac{1}{2}}) = \mathcal{V}$ for some $t_0 \in [0, \tau]$. Let $\gamma \in (0, 1)$ and suppose [H4]. If $t \mapsto \mathcal{A}(t)$ is piecewise in the homogeneous Sobolev space $\dot{H}^{\frac{\gamma}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma))$, then (P) has maximal L^2 -regularity in \mathcal{H} for all $u_0 \in \mathcal{V}$. In addition, there exists a positive constant C independent of u_0 and f such that*

$$\|A(\cdot)u(\cdot)\|_{L^2(0,\tau;\mathcal{H})} + \|u\|_{H^1(0,\tau;\mathcal{H})} \leq C \left[\|u_0\|_{\mathcal{V}} + \|f\|_{L^2(0,\tau;\mathcal{H})} \right]. \quad (2.6)$$

Note that if [H4] holds with $\gamma = 0$ then

$$|\langle \mathcal{A}(t)u - \mathcal{A}(s)u, v \rangle| \leq M_0 \|u\|_{\mathcal{V}} \|v\|$$

for all $u, v \in \mathcal{V}$. In that case $\mathcal{A}(t) - \mathcal{A}(s)$ defines a bounded operator from \mathcal{V} into \mathcal{H} . This implies in particular that the operators $A(t)$ have the same domain $D(A(t)) = D(A(0))$. For operators satisfying the later property maximal regularity of (P) holds under the assumption that $t \mapsto A(t)v$ is relatively continuous for all $v \in D(A(0))$, see [3], Theorem 3.3. See also [14] where the later regularity assumption is replaced by a certain commutation property. We prove here that maximal regularity holds without requiring any property on the operators (or the forms). More precisely, we have

Proposition 2.5. *Suppose [H1]-[H3]. Suppose that [H4] holds for $\gamma = 0$ and that $D(A(t_0)^{\frac{1}{2}}) = \mathcal{V}$ for some $t_0 \in [0, \tau]$. Then (P) has maximal L^2 -regularity in \mathcal{H} for all $u_0 \in \mathcal{V}$. In addition, there exists a positive constant C independent of u_0 and f such that*

$$\|A(\cdot)u(\cdot)\|_{L^2(0,\tau;\mathcal{H})} + \|u\|_{H^1(0,\tau;\mathcal{H})} \leq C \left[\|u_0\|_{\mathcal{V}} + \|f\|_{L^2(0,\tau;\mathcal{H})} \right]. \quad (2.7)$$

3 Preparatory lemmas

In this section we prove several estimates which will play an important role in the proofs of the main results. We emphasize that one of the important points here is to prove estimates with constants which are independent of t .

Before we start let us point out that we may assume without loss of generality that assumption [H3] is satisfied with $\nu = 0$, that is the forms are coercive with constant $\delta > 0$ independent of t . The reason is that the maximal regularity of (P) is equivalent to the same property for

$$v'(t) + (A(t) + \nu)v(t) = g(t), \quad v(0) = u_0. \quad (3.1)$$

This can be seen by observing that for $g(t) := f(t)e^{-\nu t}$, then $v(t) = u(t)e^{-\nu t}$ and clearly $v \in H^1(0, \tau; \mathcal{H})$ if and only if $u \in H^1(0, \tau; \mathcal{H})$ (and obviously $f \in L^2(0, \tau; \mathcal{H})$ if and only if $g \in L^2(0, \tau; \mathcal{H})$).

When [H3] holds with $\nu = 0$ then clearly the operators $A(t)$ are invertible on \mathcal{H} . In addition, one has the resolvent estimate

$$\|(\mu + A(t))^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{1 + \mu} \quad (3.2)$$

for all $\mu \geq 0$. The constant is independent of $t \in [0, \tau]$ (see e.g., [2], Proposition 2.1). The same estimate holds for $\mathcal{A}(t)$ on \mathcal{V}' .

Recall that $\mathcal{V}_\gamma = [\mathcal{H}, \mathcal{V}]_\gamma$ (for $\gamma \in [0, 1]$) is the complex interpolation space between \mathcal{H} and \mathcal{V} and $\mathcal{V}'_\gamma := (\mathcal{V}_\gamma)'$ denotes its (anti-) dual space.

Lemma 3.1. *Suppose that [H1]-[H3] are satisfied with $\nu = 0$. Then there exists a constant $C > 0$ such that the following estimates hold for all $\mu \geq 0$, $r > 0$ and all $t \in [0, \tau]$.*

- 1- $\|(\mu + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}'_\gamma, \mathcal{V})} \leq \frac{C}{(\mu+1)^{\frac{1-\gamma}{2}}}$,
- 2- $\|(\mu + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}'_\gamma, \mathcal{H})} \leq \frac{C}{(\mu+1)^{1-\frac{\gamma}{2}}}$,
- 3- $\|e^{-r\mathcal{A}(t)}\|_{\mathcal{L}(\mathcal{V}'_\gamma, \mathcal{H})} \leq \frac{C}{r^{\frac{\gamma}{2}}}$.

Proof. The arguments are classical but we write them here for clarity of the exposition.

We have for $w \in \mathcal{V}'$

$$\begin{aligned} \delta \|(\mu + \mathcal{A}(t))^{-1}w\|_{\mathcal{V}}^2 &\leq \operatorname{Re} \langle \mathcal{A}(t)(\mu + \mathcal{A}(t))^{-1}w, (\mu + \mathcal{A}(t))^{-1}w \rangle \\ &= \operatorname{Re} \langle w - \mu(\mu + \mathcal{A}(t))^{-1}w, (\mu + \mathcal{A}(t))^{-1}w \rangle \\ &\leq \|w\|_{\mathcal{V}'} \|(\mu + \mathcal{A}(t))^{-1}w\|_{\mathcal{V}} + C \|(\mu + \mathcal{A}(t))^{-1}w\|_{\mathcal{V}}, \end{aligned}$$

which gives $\|(\mu + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}', \mathcal{V})} \leq C'$. A similar argument gives the estimate $\|(\mu + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{V})} \leq \frac{C}{(\mu+1)^{\frac{\gamma}{2}}}$. The first assertion follows by interpolation. For the second one we use

$$\begin{aligned} &\|(\mu + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}'_\gamma, \mathcal{H})} \\ &\leq \|(\mu + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}', \mathcal{H})}^\gamma \|(\mu + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{H})}^{1-\gamma} \\ &\leq \frac{C}{(\mu + 1)^{1-\frac{\gamma}{2}}}. \end{aligned}$$

We have $\|e^{r\mathcal{A}(t)}\|_{\mathcal{L}(\mathcal{H})} \leq 1$ and $\|e^{-r\mathcal{A}(t)}\|_{\mathcal{L}(\mathcal{V}', \mathcal{H})} \leq \frac{C}{\sqrt{r}}$ (see e.g., [11], Proposition 6). Since $\mathcal{V}'_\gamma = [\mathcal{V}', \mathcal{H}]_{1-\gamma}$ we use interpolation and obtain the third estimate. \square

We make some comments on property [H4]. Let $\gamma \in [0, 1]$ and consider for fixed $t, s \in [0, \tau]$

$$\mathcal{A}(t) - \mathcal{A}(s) \in \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma). \quad (3.3)$$

Obviously, (3.3) holds for all $t, s \in [0, \tau]$ if $\gamma = 1$ since each operator $\mathcal{A}(t)$ is bounded from \mathcal{V} into \mathcal{V}' .

Observe that (3.3) is equivalent to

$$|\mathfrak{a}(t, u, v) - \mathfrak{a}(s, u, v)| \leq C_{t,s} \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}'_\gamma} \quad (3.4)$$

for some positive constant $C_{t,s}$ and all $u, v \in \mathcal{V}$. Moreover, one can take $C_{t,s} = \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}$. In order to see this, one writes for $u, v \in \mathcal{V}$

$$\mathfrak{a}(t, u, v) - \mathfrak{a}(s, u, v) = \langle \mathcal{A}(t)u - \mathcal{A}(s)u, v \rangle, \quad (3.5)$$

and obtains immediately that (3.3) implies (3.4). For the converse, we note that by (3.5), $v \mapsto \langle \mathcal{A}(t)u - \mathcal{A}(s)u, v \rangle$ extends to a (anti-) linear continuous functional on \mathcal{V}_γ (for fixed $u \in \mathcal{V}$). The rest of the claim is easy to check. Similarly to the previous remark, $\mathcal{A}(t) - \mathcal{A}(s)$ extends to a bounded operator from \mathcal{V}_γ to \mathcal{V}' if and only if

$$|\mathfrak{a}(t, u, v) - \mathfrak{a}(s, u, v)| \leq C_{t,s} \|u\|_{\mathcal{V}_\gamma} \|v\|_{\mathcal{V}} \quad (3.6)$$

for all $u, v \in \mathcal{V}$.

Our next lemma shows stability of the Kato square root property if (3.4) or (3.6) holds for some $\gamma \in [0, 1)$.

Lemma 3.2. *Suppose the assumptions of the previous lemma. Given $t, s \in [0, \tau]$. Suppose that there exists $\gamma \in [0, 1)$ such that either (3.4) or (3.6) holds for t, s . If $D(A(s)^{\frac{1}{2}}) = \mathcal{V}$ then $D(A(t)^{\frac{1}{2}}) = \mathcal{V}$.*

Proof. Suppose that (3.4) is satisfied. Recall that

$$A(t)^{-\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty \mu^{-\frac{1}{2}} (\mu + A(t))^{-1} d\mu.$$

Hence

$$A(t)^{-\frac{1}{2}} - A(s)^{-\frac{1}{2}} = \frac{-1}{\pi} \int_0^\infty \mu^{-\frac{1}{2}} (\mu + \mathcal{A}(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\mu + A(s))^{-1} d\mu.$$

Using the previous lemma, we estimate the $\mathcal{H} - \mathcal{V}$ norm of the term in the integral by

$$\begin{aligned} & \mu^{-\frac{1}{2}} \|(\mu + A(t))^{-1}\|_{\mathcal{L}(\mathcal{V}'_\gamma, \mathcal{V})} \|(\mu + A(s))^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{V})} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)} \\ & \leq CC_{t,s} \mu^{-\frac{1}{2}} (1 + \mu)^{\frac{\gamma}{2} - 1}. \end{aligned}$$

This implies that $A(t)^{-\frac{1}{2}} - A(s)^{-\frac{1}{2}}$ is bounded from \mathcal{H} to \mathcal{V} and hence $D(A(t)^{\frac{1}{2}}) \subseteq \mathcal{V}$. In addition, for $u \in D(A(t)^{\frac{1}{2}})$

$$\begin{aligned} \|u\|_{\mathcal{V}} & \leq \| [A(t)^{-\frac{1}{2}} - A(s)^{-\frac{1}{2}}] A(t)^{\frac{1}{2}} u \|_{\mathcal{V}} + \| A(s)^{-\frac{1}{2}} A(t)^{\frac{1}{2}} u \|_{\mathcal{V}} \\ & \leq \left(\| A(t)^{-\frac{1}{2}} - A(s)^{-\frac{1}{2}} \|_{\mathcal{L}(\mathcal{H}, \mathcal{V})} + \| A(s)^{-\frac{1}{2}} \|_{\mathcal{L}(\mathcal{H}, \mathcal{V})} \right) \| A(t)^{\frac{1}{2}} u \|_{\mathcal{H}} \\ & \leq C'(1 + C_{t,s}) \| A(t)^{\frac{1}{2}} u \|_{\mathcal{H}}. \end{aligned}$$

Suppose now that $u \in \mathcal{V} = D(A(s)^{\frac{1}{2}})$. Then we write as before

$$A(t)^{\frac{1}{2}} u = \frac{1}{\pi} \int_0^\infty \mu^{-\frac{1}{2}} A(t) (\mu + A(t))^{-1} u d\mu$$

so that

$$\begin{aligned}
(\mathcal{A}(t)^{\frac{1}{2}} - \mathcal{A}(s)^{\frac{1}{2}})u &= \frac{1}{\pi} \int_0^\infty \mu^{-\frac{1}{2}} [A(t)(\mu + A(t))^{-1}u - A(s)(\mu + A(s))^{-1}u] d\mu \\
&= \frac{-1}{\pi} \int_0^\infty \mu^{\frac{1}{2}} [(\mu + A(t))^{-1}u - (\mu + A(s))^{-1}u] d\mu \\
&= \frac{1}{\pi} \int_0^\infty \mu^{\frac{1}{2}} (\mu + \mathcal{A}(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\mu + A(s))^{-1} u d\mu.
\end{aligned}$$

We estimate the norm in \mathcal{H} of the term inside the integral by

$$\begin{aligned}
&\mu^{\frac{1}{2}} \|(\mu + \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}, \mathcal{H})} \|(\mu + A(s))^{-1}\|_{\mathcal{L}(\mathcal{V})} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}_\gamma)} \|u\|_{\mathcal{V}} \\
&\leq CC_{t,s} \mu^{\frac{1}{2}} (1 + \mu)^{\frac{\gamma}{2} - 2} \|u\|_{\mathcal{V}}.
\end{aligned}$$

This gives $u \in D(A(t)^{\frac{1}{2}})$ and

$$\|A(t)^{\frac{1}{2}}u\| \leq CC_{t,s} \|u\|_{\mathcal{V}} + \|A(s)^{\frac{1}{2}}u\|$$

which proves the lemma. Note that if we assume (3.6) then we argue by duality and prove the lemma for $A(t)^{* \frac{1}{2}}$. It is well known that the equality $D(A(t)^{* \frac{1}{2}}) = \mathcal{V}$ is equivalent to $D(A(t)^{\frac{1}{2}}) = \mathcal{V}$. \square

Proposition 3.3. *Suppose that [H1]-[H3] are satisfied with $\nu = 0$. Fix $s \in [0, \tau]$ and suppose that either (3.4) or (3.6) holds for some $\gamma \in [0, 1]$ with a constant C independent of $t \in [0, \tau]$ (i.e., $C_{t,s} \leq C$ for all t). If $D(A(s)^{\frac{1}{2}}) = \mathcal{V}$, then $D(A(t)^{\frac{1}{2}}) = \mathcal{V}$ for all $t \in [0, \tau]$ and there exist positive constants C_1, C_2 such that*

$$C_1 \|u\|_{\mathcal{V}} \leq \|A(t)^{\frac{1}{2}}u\| \leq C_2 \|u\|_{\mathcal{V}} \text{ for all } u \in \mathcal{V} \text{ and all } t \in [0, \tau]. \quad (3.7)$$

Proof. All the details are already given in the proof of the previous lemma. \square

Definition 3.4. *We say that $(A(t))$ (or the corresponding forms $\mathfrak{a}(t)$) satisfy the uniform Kato square root property if $D(A(t)^{\frac{1}{2}}) = \mathcal{V}$ for all $t \in [0, \tau]$ and there exist constants $C_1, C_2 > 0$ such that (3.7) is satisfied.*

The uniform Kato square root property is obviously satisfied for symmetric forms. It is also satisfied for uniformly elliptic operators (not necessarily symmetric)

$$A(t) = - \sum_{k,l=1}^d \partial_k (a_{kl}(t, x) \partial_l)$$

on $L^2(\mathbb{R}^d)$ since $\|\nabla u\|_2$ is equivalent to $\|A(t)^{\frac{1}{2}}u\|_2$ with constants depending only on the dimension and the ellipticity constants. See [6].

The previous proposition says that in order to have the uniform Kato square

root property one needs only to check that $D(A(s)^{\frac{1}{2}}) = \mathcal{V}$ for one $s \in [0, \tau]$ provided (3.4) or (3.6) holds for some $\gamma \in [0, 1)$.

In the next lemma we show a quadratic estimate for $A(t)$ with constant independent of t . Here we assume the uniform Kato square root property and give a short proof for the quadratic estimate. It is possible to prove the same result without the uniform Kato square root property by proving that the holomorphic functional calculus of $A(t)$ has uniform estimate with respect to t (this is the case since the resolvent have uniform estimates). It is well known that quadratic estimates in \mathcal{H} are intimately related to the holomorphic functional calculus, see [16]. Quadratic estimates are an important tool in harmonic analysis and we will use them at several places in the proofs of maximal regularity.

Lemma 3.5. *Suppose in addition to [H1]-[H3] (with $\nu = 0$) that the uniform Kato square root property is satisfied. Then there exists a constant C such that for every $t \in [0, \tau]$*

$$\int_0^\tau \|A(t)^{\frac{1}{2}} e^{-sA(t)} x\|^2 ds \leq C \|x\|^2 \quad (3.8)$$

for all $x \in \mathcal{H}$.

Proof. By the uniform Kato square root property, we have

$$\begin{aligned} \int_0^\tau \|A(t)^{\frac{1}{2}} e^{-sA(t)} x\|^2 ds &\leq C_2 \int_0^\tau \|e^{-sA(t)} x\|_{\mathcal{V}}^2 ds \\ &\leq C' \int_0^\tau \operatorname{Re} \mathfrak{a}(e^{-sA(t)} x, e^{-sA(t)} x) ds \\ &= C' \int_0^\tau \operatorname{Re} (A(t) e^{-sA(t)} x, e^{-sA(t)} x) ds \\ &= -\frac{C'}{2} \int_0^\tau \frac{d}{ds} \|e^{-sA(t)} x\|^2 ds \\ &= \frac{C'}{2} [\|x\|^2 - \|e^{-\tau A} x\|^2] \leq \frac{C'}{2} \|x\|^2. \end{aligned}$$

This proves the lemma. \square

We note that the constant C is also independent of τ . We could formulate the lemma with $\int_0^\infty \|A(t)^{\frac{1}{2}} e^{-sA(t)} x\|^2 ds$. Let us also mention the following L^p -version.

Lemma 3.6. *Suppose the assumptions of the previous lemma. Let $p \geq 2$. Then there exists a constant C_p such that*

$$\int_0^\tau \|A(t)^{1/p} e^{-sA(t)} x\|^p ds \leq C_p \|x\|^p \quad (3.9)$$

for all $x \in \mathcal{H}$.

Proof. We fix $t \in [0, \tau]$ and $s > 0$. We define

$$F(z)x := A(t)^{z/2} e^{-sA(t)} x.$$

It is a classical fact that F is a holomorphic function on \mathbb{C}^+ . In addition, each operator $A(t)$, as an accretive operator on \mathcal{H} , has bounded imaginary powers. That is $\|A^{is}\|_{\mathcal{L}(\mathcal{H})} \leq C$ for some constant C and all $s \in \mathbb{R}$. See [16]. Using this one obtains immediately that

$$\|F(is)x\|_{L^\infty(0, \tau; \mathcal{H})} \leq C\|x\|.$$

On the other hand, by Lemma 3.5 and again uniform boundedness of imaginary powers on \mathcal{H} we obtain

$$\|F(1 + is)x\|_{L^2(0, \tau; \mathcal{H})} \leq C\|x\|.$$

We apply Stein's complex interpolation theorem to obtain that for all $p \geq 2$

$$\|F(\frac{2}{p})x\|_{L^p(0, \tau; \mathcal{H})} \leq C_p\|x\|$$

and we obtain the lemma. \square

Let u be the solution of (P') by Lions' theorem. Lions also proved that $u \in C([0, \tau]; \mathcal{H})$. Since $u \in L^2(0, \tau; \mathcal{V})$ we have $u(t) \in \mathcal{V}$ for a.e. $t \in [0, \tau]$. It is very useful to know whether $u(t) \in \mathcal{V}$ for every $t \in [0, \tau]$. We prove this in the following lemma under an additional assumption that $u \in L^\infty(0, \tau; \mathcal{V})$. We shall see later that this property holds when we assume that (2.1) is satisfied.

Lemma 3.7. *Suppose [H1]-[H3] and suppose in addition that the solution u belongs to $L^\infty(0, \tau; \mathcal{V})$. Then $u(t) \in \mathcal{V}$ for every $t \in [0, \tau]$.*

Proof. Let $\Gamma = \{t \in [0, \tau] \text{ s.t. } u(t) \notin \mathcal{V}\}$. Since Γ has measure zero, $[0, \tau] \setminus \Gamma$ is dense in $[0, \tau]$. Suppose that $t \in \Gamma$. There exists a sequence $(t_n)_n \in [0, \tau] \setminus \Gamma$ such that $t_n \rightarrow t$ as $n \rightarrow \infty$. Since the sequence $(u(t_n))$ is bounded in \mathcal{V} we can extract a subsequence $u(t_{n_k})$ which converges weakly to some v in \mathcal{V} . By continuity of u in \mathcal{H} , $u(t_{n_k})$ converges (in \mathcal{H}) to $u(t)$. This gives $u(t) = v \in \mathcal{V}$. Hence Γ is empty. \square

4 Key estimates

In this section we state and prove some other estimates which we will need in the proofs of the main results.

Lemma 4.1. *Suppose in addition to [H1]-[H3] (with $\nu = 0$) that the uniform Kato square root property is satisfied. Define*

$$L_0(f)(t) := \int_0^t e^{-(t-s)A(t)} f(s) ds.$$

Then $L_0 : L^2(0, \tau; \mathcal{H}) \rightarrow L^\infty(0, \tau; \mathcal{V})$ is a bounded operator.

Proof. By the uniform Kato square root property,

$$\|L_0(f)(t)\|_{\mathcal{V}} \leq C_2 \|A(t)^{\frac{1}{2}} \int_0^t e^{-(t-s)A(t)} f(s) ds\|.$$

On the other hand, for $x \in \mathcal{H}$

$$\begin{aligned} & |(A(t)^{\frac{1}{2}} \int_0^t e^{-(t-s)A(t)} f(s) ds, x)| \\ &= \int_0^t (f(s), A(t)^{* \frac{1}{2}} e^{-(t-s)A(t)^*} x) ds \\ &\leq \|f\|_{L^2(0, \tau; \mathcal{H})} \left(\int_0^t \|A(t)^{* \frac{1}{2}} e^{-(t-s)A(t)^*} x\|^2 ds \right)^{\frac{1}{2}} \\ &\leq C \|f\|_{L^2(0, \tau; \mathcal{H})} \|x\|. \end{aligned}$$

The latest inequality follows from Lemma 3.5 applied to the adjoint operator $A(t)^*$ (note that $A(t)^*$ satisfies the same properties as $A(t)$). The constant C is independent of t . Therefore,

$$\|L_0(f)(t)\|_{\mathcal{V}} \leq C_2 C \|f\|_{L^2(0, \tau; \mathcal{H})}. \quad (4.1)$$

This implies immediately that $L_0 : L^2(0, \tau; \mathcal{H}) \rightarrow L^\infty(0, \tau; \mathcal{V})$ is bounded. \square

Now we study boundedness on $L^2(0, \tau; \mathcal{H})$ of the operator

$$L(f)(t) := \int_0^t A(t) e^{-(t-s)A(t)} f(s) ds. \quad (4.2)$$

It is proved in [11] that L is bounded on $L^p(0, \tau; \mathcal{H})$ for all $p \in (1, \infty)$ provided $t \mapsto \mathfrak{a}(t, \cdot, \cdot)$ is C^ϵ for some $\epsilon > 0$ (or similarly, $t \mapsto \mathcal{A}(t)$ is C^ϵ on $[0, \tau]$ with values in $\mathcal{L}(\mathcal{V}, \mathcal{V}')$). The proof for the case $p = 2$ is based on vector-valued pseudo-differential operators. The extension from $p = 2$ to $p \in (1, \infty)$ uses Hörmander's almost L^1 -condition for singular integral operators. Here we give a direct proof for the case $p = 2$ which does not appeal to pseudo-differential operators. It is essentially based on the quadratic estimate of Lemma 3.5.

Proposition 4.2. *Suppose [H1]-[H3] (with $\nu = 0$) and the uniform Kato square root property. Let $\gamma \in (0, 1]$. If*

$$\sup_{s \in [0, \tau]} \int_s^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t - s|^\gamma} dt < \infty, \quad (4.3)$$

then the operator L is bounded on $L^2(0, \tau; \mathcal{H})$.

Proof. Fix $\gamma \in [0, 1]$. Take $g \in L^2(0, \tau; \mathcal{H})$. We have

$$\begin{aligned} & \left| \int_0^\tau \int_0^t (A(t)e^{-(t-s)A(t)} f(s), g(t)) ds dt \right| \\ &= \left| \int_0^\tau \int_0^t \left(A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} f(s), A(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} g(t) \right) ds dt \right| \\ &\leq \int_0^\tau \int_0^t \|A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} f(s)\| \|A(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} g(t)\| ds dt \\ &\leq \int_0^\tau \left(\int_0^t \|A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} f(s)\|^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|A(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} g(t)\|^2 ds \right)^{\frac{1}{2}} dt \\ &\leq C \int_0^\tau \left(\int_0^t \|A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} f(s)\|^2 ds \right)^{\frac{1}{2}} \|g(t)\| dt \\ &\leq C \left(\int_0^\tau \int_0^t \|A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} f(s)\|^2 ds dt \right)^{\frac{1}{2}} \|g\|_{L^2(0, \tau; \mathcal{H})}. \end{aligned}$$

Here we use the quadratic estimate of Lemma 3.5 for the adjoint operator $A(t)^*$. It follows that for all $f \in L^2(0, \tau; \mathcal{H})$

$$\|L(f)\|_{L^2(0, \tau; \mathcal{H})} \leq C \left(\int_0^\tau \int_0^t \|A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} f(s)\|^2 ds dt \right)^{\frac{1}{2}}. \quad (4.4)$$

We use again Lemma 3.5 and obtain

$$\begin{aligned} & \int_0^\tau \int_0^t \|A(t)^{\frac{1}{2}} e^{-(t-s)A(t)} f(s)\|^2 ds dt \\ &\leq 2 \int_0^\tau \int_0^t \|A(t)^{\frac{1}{2}} e^{-(t-s)A(t)} f(s) - A(s)^{\frac{1}{2}} e^{-(t-s)A(s)} f(s)\|^2 ds dt \\ &+ 2 \int_0^\tau \int_s^\tau \|A(s)^{\frac{1}{2}} e^{-(t-s)A(s)} f(s)\|^2 dt ds \\ &\leq 2 \int_0^\tau \int_0^t \|A(t)^{\frac{1}{2}} e^{-(t-s)A(t)} f(s) - A(s)^{\frac{1}{2}} e^{-(t-s)A(s)} f(s)\|^2 ds dt \\ &+ 2C \int_0^\tau \|f(s)\|^2 ds. \end{aligned}$$

We choose a contour Γ in the positive half-plane and we write by the holo-

morphic functional calculus

$$\begin{aligned}
& A(t)^{\frac{1}{2}} e^{-(t-s)A(t)} f(s) - A(s)^{\frac{1}{2}} e^{-(t-s)A(s)} f(s) \\
&= \int_{\Gamma} \lambda^{\frac{1}{2}} e^{-(t-s)\lambda} [(\lambda - \mathcal{A}(t))^{-1} - (\lambda - \mathcal{A}(s))^{-1}] d\lambda f(s) \\
&= \int_{\Gamma} \lambda^{\frac{1}{2}} e^{-(t-s)\lambda} (\lambda - \mathcal{A}(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\lambda - \mathcal{A}(s))^{-1} f(s) d\lambda.
\end{aligned}$$

We estimate the norm in \mathcal{H} of the latest term. For $\lambda = |\lambda|e^{i\theta}$ we apply Lemma 3.1 and obtain

$$\begin{aligned}
& \left\| \int_{\Gamma} \lambda^{\frac{1}{2}} e^{-(t-s)\lambda} (\lambda - \mathcal{A}(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(s)) (\lambda - \mathcal{A}(s))^{-1} f(s) d\lambda \right\| \\
& \leq \int_{\Gamma} |\lambda|^{\frac{1}{2}} e^{-(t-s)|\lambda| \cos \theta} \|(\lambda - \mathcal{A}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}'_{\gamma}, \mathcal{H})} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_{\gamma})} \\
& \quad \times \|(\lambda - \mathcal{A}(s))^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{V})} |d\lambda| \|f(s)\| \\
& \leq C \int_{\Gamma} |\lambda|^{\frac{1}{2}} e^{-(t-s)|\lambda| \cos \theta} \frac{1}{(1+|\lambda|)^{1-\gamma/2}} \frac{1}{(1+|\lambda|)^{\frac{1}{2}}} \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_{\gamma})} |d\lambda| \|f(s)\| \\
& \leq C' \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_{\gamma})}}{|t-s|^{\gamma/2}} \|f(s)\|.
\end{aligned}$$

Here and at other places we use the estimate

$$\int_0^{\infty} \frac{e^{-r(t-s)}}{(1+r)^{1-\gamma/2}} dr \leq \frac{C}{(t-s)^{\frac{\gamma}{2}}} \quad (4.5)$$

for some constant C and all $s < t$. This is seen by making the change of the variable $v := r(t-s)$ in the LHS which then coincides with

$$\frac{1}{(t-s)^{\frac{\gamma}{2}}} \int_0^{\infty} \frac{e^{-v}}{(t-s+v)^{1-\frac{\gamma}{2}}} dv.$$

The latter term is bounded by

$$\frac{1}{(t-s)^{\frac{\gamma}{2}}} \int_0^{\infty} \frac{e^{-v}}{v^{1-\frac{\gamma}{2}}} dv = \frac{C}{(t-s)^{\frac{\gamma}{2}}}.$$

The previous estimates give

$$\begin{aligned}
& \int_0^{\tau} \int_0^t \|A(t)^{\frac{1}{2}} e^{-(t-s)A(t)} f(s) - A(s)^{\frac{1}{2}} e^{-(t-s)A(s)} f(s)\|^2 ds dt \\
& \leq C' \int_0^{\tau} \int_s^{\tau} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_{\gamma})}^2}{|t-s|^{\gamma}} dt \|f(s)\|^2 ds \\
& \leq C' \sup_{s \in [0, \tau]} \int_s^{\tau} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_{\gamma})}^2}{|t-s|^{\gamma}} dt \int_0^{\tau} \|f(s)\|^2 ds. \quad (4.6)
\end{aligned}$$

This proves that L is bounded on $L^2(0, \tau; \mathcal{H})$. \square

Remark 4.3. *One may argue as in the proof of Lemma 11 in [11] and obtain boundedness of L on $L^p(0, \tau; \mathcal{H})$ at least for $p \in (1, 2)$.*

Corollary 4.4. *1) If \mathcal{A} satisfies (2.1) then L is bounded on $L^2(0, \tau; \mathcal{H})$.
2) If [H4] is satisfied for some $\gamma \in (0, 1)$ then L is bounded on $L^2(0, \tau; \mathcal{H})$.*

Proof. Assertion 1) follows directly from Proposition 4.2 by noticing that (2.1) implies (4.3) with $\gamma = 1$. For assertion 2) one uses [H4] to obtain

$$\int_s^\tau \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}_\gamma)}^2}{|t - s|^\gamma} dt \leq M_\gamma^2 \int_s^\tau |t - s|^{-\gamma} dt \leq cM_\gamma^2$$

for some constant $c > 0$. The result follows from Proposition 4.2. \square

Note that if $\gamma \in (0, 1)$ we do not require any regularity property for \mathcal{A} in assertion 2) of the previous proposition.

Proposition 4.5. *Suppose [H1]-[H3] (with $\nu = 0$) and the uniform Kato square root property. Let $f \in L^2(0, \tau; \mathcal{H})$, $u_0 \in V$ and let u be the Lions' solution to the problem (P').*

1) If \mathcal{A} satisfies (2.1) then $u \in L^\infty(0, \tau; \mathcal{V})$ and there exists a constant C independent of u_0 and f such that

$$\|u\|_{L^\infty(0, \tau; \mathcal{V})} \leq C \left[\|u_0\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau; \mathcal{H})} \right]. \quad (4.7)$$

2) Let $\gamma \in [0, 1)$ and suppose that [H4] is satisfied. Then $u \in L^\infty(0, \tau; \mathcal{V})$ and (4.7) holds.

Proof. As we already mentioned above, by Lions' theorem there exists a unique solution u to the problem (P') such that $u \in H^1(0, \tau; \mathcal{V}') \cap L^2(0, \tau; \mathcal{V})$. For $0 \leq s \leq t \leq \tau$, we define $v(s) := e^{-(t-s)A(t)}u(s)$. We write $v(t) = v(0) + \int_0^t v'(s) ds$ and obtain as in [11] (Lemma 8)

$$\begin{aligned} u(t) &= e^{-tA(t)}u_0 + \int_0^t e^{-(t-s)A(t)}(\mathcal{A}(t) - \mathcal{A}(s))u(s) ds \\ &\quad + \int_0^t e^{-(t-s)A(t)}f(s) ds \\ &=: R_0u_0(t) + S_0u(t) + L_0f(t). \end{aligned} \quad (4.8)$$

Clearly there exists a constant $C > 0$ such that for all $u_0 \in \mathcal{V}$,

$$\|R_0u_0(t)\|_{\mathcal{V}} = \|e^{-tA(t)}u_0\|_{\mathcal{V}} \leq C\|u_0\|_{\mathcal{V}}. \quad (4.9)$$

By Lemma 4.1,

$$\|L_0f(t)\|_{\mathcal{V}} \leq C\|f\|_{L^2(0, \tau; \mathcal{H})}. \quad (4.10)$$

Next we prove that $S_0 \in \mathcal{L}(L^\infty(0, \tau; \mathcal{V}))$. Let $g \in L^\infty(0, \tau; \mathcal{V})$. We have by the uniform Kato square root property

$$\|S_0 g(t)\|_{\mathcal{V}} \leq C_2 \left\| \int_0^t A(t)^{\frac{1}{2}} e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s)) g(s) ds \right\|.$$

In order to estimate the term on the RHS we argue as in the proof of Lemma 4.1 and use Lemma 3.1. For $x \in \mathcal{H}$ and $\gamma \in [0, 1]$, we have

$$\begin{aligned} & \left| \left(\int_0^t A(t)^{\frac{1}{2}} e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s)) g(s) ds, x \right) \right| \\ &= \left| \int_0^t (e^{-\frac{(t-s)}{2}A(t)} (\mathcal{A}(t) - \mathcal{A}(s)) g(s), A(t)^* \frac{1}{2} e^{-\frac{(t-s)}{2}A(t)} x) ds \right| \\ &\leq \left(\int_0^t \|e^{-\frac{(t-s)}{2}A(t)} (\mathcal{A}(t) - \mathcal{A}(s)) g(s)\|^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|A(t)^* \frac{1}{2} e^{-\frac{(t-s)}{2}A(t)} x\|^2 ds \right)^{\frac{1}{2}} \\ &\leq C \|x\| \left(\int_0^t \|e^{-\frac{(t-s)}{2}A(t)}\|_{\mathcal{L}(\mathcal{V}_\gamma, \mathcal{H})}^2 \|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}_\gamma)}^2 ds \right)^{\frac{1}{2}} \|g\|_{L^\infty(0, t; \mathcal{V})} \\ &\leq C \|x\| \left(\int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}_\gamma)}^2}{(t-s)^\gamma} ds \right)^{\frac{1}{2}} \|g\|_{L^\infty(0, t; \mathcal{V})}. \end{aligned}$$

Therefore,

$$\|S_0 g(t)\|_{\mathcal{V}} \leq CC_2 \left(\int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}_\gamma)}^2}{(t-s)^\gamma} ds \right)^{\frac{1}{2}} \|g\|_{L^\infty(0, \tau; \mathcal{V})}. \quad (4.11)$$

Suppose $\gamma = 1$. It follows from the assumption (2.1) that S_0 is a bounded operator on $L^\infty(0, \tau; \mathcal{V})$ with

$$\|S_0\|_{\mathcal{L}(L^\infty(0, \tau; \mathcal{V}))} \leq CC_2 \left(\sup_{t \in [0, \tau]} \int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V})}^2}{t-s} ds \right)^{\frac{1}{2}}. \quad (4.12)$$

In order to continue we wish to take the inverse of $I - S_0$. Let $\varepsilon > 0$ and τ_1 be as (2.1). We work on the interval $[0, \tau_1]$. We have

$$\sup_{t \in [0, \tau_1]} \int_0^{\tau_1} \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V})}^2}{t-s} ds < \varepsilon.$$

It follows from (4.12) that $\|S_0\|_{\mathcal{L}(L^\infty(0, \tau_1; \mathcal{V}))} < 1$.

Therefore, $u = (I - S_0)^{-1}(R_0 u_0 + L_0 f)$ and we obtain from (4.9) and (4.10) that $u \in L^\infty(0, \tau_1; \mathcal{V})$ and (4.7) is satisfied on $[0, \tau_1]$. Now repeat the same strategy. We use (2.1), we work on $[\tau_{i-1}, \tau_i]$ and argue exactly as before. We obtain (4.7) on each sub-intervals $[\tau_{i-1}, \tau_i]$. This implies (4.7)

on $[0, \tau]$ for arbitrary $\tau > 0$ and finishes the proof of assertion 1).
 In order to prove assertion 2), we use [H4] and (4.11). We obtain

$$\|S_0 g(t)\|_{\mathcal{V}} \leq CC_2 M_\gamma \left(\int_0^t \frac{1}{(t-s)^\gamma} ds \right)^{\frac{1}{2}} \|g\|_{L^\infty(0,\tau;\mathcal{V})} \leq C' \tau^{\frac{1-\gamma}{2}} \|g\|_{L^\infty(0,\tau;\mathcal{V})}.$$

We see that $\|S_0\|_{\mathcal{L}(L^\infty(0,\tau;\mathcal{V}))} < 1$ for small $\tau > 0$. We split $[0, \tau]$ into a finite number of intervals with small sizes and then argue as previously. \square

Proposition 4.6. *Suppose the assumptions of the previous proposition. Then $u(t) \in \mathcal{V}$ for every $t \in [0, \tau]$.*

Proof. This is an application of Lemma 3.7 and Proposition 4.5. Note that $u(t)$ is well defined for every t since $u \in C([0, \tau], \mathcal{H})$ by Lions' theorem. \square

The following lemma was first proved in [11] under the assumption that $\mathcal{A}(\cdot) \in C^\alpha(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$ for some $\alpha > \frac{1}{2}$. See also [17]. We prove it here in the case where \mathcal{A} satisfies (2.1) and for arbitrary \mathcal{A} if [H4] is satisfied for some $\gamma \in (0, 1)$.

Lemma 4.7. *Suppose [H1]-[H3] (with $\nu = 0$) and the uniform Kato square root property. Define the operator*

$$Ru_0(t) := A(t)e^{-tA(t)}u_0$$

for $u_0 \in \mathcal{V}$ and $t \in [0, \tau]$.

- 1) Suppose (2.1). Then R is bounded from \mathcal{V} into $L^2(0, \tau; \mathcal{H})$.
- 2) If [H4] is satisfied for some $\gamma \in (0, 1)$ then R is bounded from \mathcal{V} into $L^2(0, \tau; \mathcal{H})$.

Proof. We write

$$\begin{aligned} Ru_0(t) &= [A(t)e^{-tA(t)} - A(0)e^{-tA(0)}]u_0 + A(0)e^{-tA(0)}u_0 \\ &=: R_1u_0(t) + R_2u_0(t). \end{aligned}$$

We use Lemma 3.5 to obtain

$$\begin{aligned} \|R_2u_0\|_{L^2(0,\tau;\mathcal{H})}^2 &= \int_0^\tau \|A(0)^{\frac{1}{2}}e^{-tA(0)}A(0)^{\frac{1}{2}}u_0\|^2 dt \\ &\leq C\|A(0)^{\frac{1}{2}}u_0\|^2 \leq C'\|u_0\|_{\mathcal{V}}^2. \end{aligned}$$

We estimate R_1u_0 . We argue as in the proof of Proposition 4.2. By the holomorphic functional calculus for the sectorial operators $A(t)$ and $A(0)$ we have

$$R_1u_0(t) = \int_{\Gamma} \lambda e^{-\lambda t} (\lambda - A(t))^{-1} (\mathcal{A}(t) - \mathcal{A}(0)) (\lambda - A(0))^{-1} u_0 d\lambda.$$

Now taking the norm in \mathcal{H} we have

$$\begin{aligned} \|R_1 u_0(t)\| &\leq \int_{\Gamma} |\lambda| e^{-t \operatorname{Re} \lambda} \|(\lambda - A(t))^{-1}\|_{\mathcal{L}(\mathcal{V}'_\gamma, \mathcal{H})} \\ &\quad \times \|\mathcal{A}(0) - \mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)} \|(\lambda - A(0))^{-1}\|_{\mathcal{L}(\mathcal{V})} |d\lambda| \|u_0\|_{\mathcal{V}} \\ &\leq C \frac{\|\mathcal{A}(0) - \mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}}{t^{\frac{\gamma}{2}}} \|u_0\|_{\mathcal{V}}. \end{aligned}$$

The last estimate follows as in the proof of Proposition 4.2 in which we also use (4.5). It is valid for $\gamma \in (0, 1]$.

Therefore

$$\|R_1 u_0\|_{L^2(0, \tau; \mathcal{H})}^2 \leq C^2 \int_0^\tau \frac{\|\mathcal{A}(0) - \mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{t^\gamma} dt \|u_0\|_{\mathcal{V}}^2. \quad (4.13)$$

For $\gamma = 1$ we use the assumption (2.1) and obtain

$$\|R_1 u_0\|_{L^2(0, \tau; \mathcal{H})}^2 \leq C' \|u_0\|_{\mathcal{V}}^2$$

for some constant $C' > 0$. This proves assertion 1) of the proposition. Assertion 2) follows directly from (4.13) when $\gamma \in (0, 1)$. \square

5 Proofs of the main results

After the auxiliary results of the last two sections we are now ready to give the proofs of the main results of this paper. Note that we may assume without loss of generality that [H3] holds with $\nu = 0$, see the beginning of Section 3.

Proof of Theorems 2.2 and 2.4. Let $\gamma \in (0, 1]$. We give the proof for the two theorems without considering separately the cases $\gamma = 1$ and $\gamma \in (0, 1)$. If $\gamma = 1$ we assume (2.1).

Suppose first that $\mathcal{A} \in \dot{H}^{\frac{\gamma}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma))$.

Let $f \in L^2(0, \tau; \mathcal{H})$ and $u_0 \in \mathcal{V}$. Let $u \in L^2(0, \tau; \mathcal{V}) \cap H^1(0, \tau; \mathcal{V}')$ be the solution of (P') given by Lions' theorem. Our aim is to prove that $u \in H^1(0, \tau; \mathcal{H})$ or equivalently that $A(\cdot)u(\cdot) \in L^2(0, \tau; \mathcal{H})$. Using (4.8) we have

$$\begin{aligned} A(t)u(t) &= A(t)e^{-tA(t)}u_0 + A(t) \int_0^t e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s))u(s) ds \\ &\quad + A(t) \int_0^t e^{-(t-s)A(t)} f(s) ds \\ &=: Ru_0(t) + (Su)(t) + (Lf)(t). \end{aligned}$$

By Lemma 4.7 the operator R is bounded from \mathcal{V} into $L^2(0, \tau; \mathcal{H})$ and by Corollary 4.4 the operator L is bounded on $L^2(0, \tau; \mathcal{H})$. Concerning the operator S , we have

$$\|Su\|_{L^2(0, \tau; \mathcal{H})} \leq C \|u\|_{L^\infty(0, \tau; \mathcal{V})} \quad (5.1)$$

for some constant C independent of f and u_0 .

Suppose for a moment that (5.1) is proved. Then we apply Proposition 4.5 together with the properties of R and L we just mentioned above and obtain $A(\cdot)u(\cdot) \in L^2(0, \tau; \mathcal{H})$ with

$$\begin{aligned} \|A(\cdot)u(\cdot)\|_{L^2(0, \tau; \mathcal{H})} &\leq \|Ru_0\|_{L^2(0, \tau; \mathcal{H})} + \|Su\|_{L^2(0, \tau; \mathcal{H})} + \|Lf\|_{L^2(0, \tau; \mathcal{H})} \\ &\leq C \left(\|u_0\|_{\mathcal{V}} + \|u\|_{L^\infty(0, \tau; \mathcal{V})} + \|f\|_{L^2(0, \tau; \mathcal{H})} \right) \\ &\leq C' \left(\|u_0\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau; \mathcal{H})} \right). \end{aligned}$$

This proves the two theorems in the case where $\mathcal{A} \in \dot{H}^{\frac{\gamma}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma))$ for some $\gamma \in (0, 1]$.

Now we prove (5.1). We have

$$\begin{aligned} &\|Su(t)\| \\ &= \|A(t) \int_0^t e^{-(t-s)A(t)} (\mathcal{A}(t) - \mathcal{A}(s))u(s) ds\| \\ &\leq \sup_{x \in \mathcal{H}, \|x\|=1} \int_0^t |(A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} (\mathcal{A}(t) - \mathcal{A}(s))u(s), A(t)^{* \frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)^*} x)| ds \\ &\leq C \left(\int_0^t \|A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} (\mathcal{A}(t) - \mathcal{A}(s))u(s)\|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Here we use again the quadratic estimate of Lemma 3.5. By analyticity of the semigroup together with Lemma 3.1 we have

$$\begin{aligned} \|A(t)^{\frac{1}{2}} e^{-\frac{(t-s)}{2}A(t)} (\mathcal{A}(t) - \mathcal{A}(s))u(s)\|^2 &\leq \frac{C}{t-s} \|e^{-\frac{(t-s)}{4}A(t)} (\mathcal{A}(t) - \mathcal{A}(s))u(s)\|^2 \\ &\leq C' \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t-s|^{1+\gamma}} \|u(s)\|_{\mathcal{V}}^2. \end{aligned}$$

Therefore,

$$\|Su(t)\| \leq C'' \|u\|_{L^\infty(0, \tau; \mathcal{V})} \left(\int_0^t \frac{\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma)}^2}{|t-s|^{1+\gamma}} ds \right)^{\frac{1}{2}}.$$

This gives

$$\|Su\|_{L^2(0, \tau; \mathcal{H})} \leq C'' \|u\|_{L^\infty(0, \tau; \mathcal{V})} \|\mathcal{A}\|_{\dot{H}^{\frac{\gamma}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma))}$$

and finishes the proof of (5.1).

Suppose now that \mathcal{A} is piecewise in $\dot{H}^{\frac{\gamma}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'_\gamma))$. Then $[0, \tau] = \cup_{i=1}^n [\tau_{i-1}, \tau_i]$ and the restriction of \mathcal{A} to each sub-interval is in $\dot{H}^{\frac{\gamma}{2}}$. We apply the previous proof to each sub-interval and obtain a solution u_i in the sub-interval $[\tau_{i-1}, \tau_i]$ which has maximal regularity and satisfies apriori estimates. By Proposition 4.6, $u_i(\tau_i) \in \mathcal{V}$ and hence we can solve $u'_{i+1}(t) + A(t)u_{i+1}(t) = f(t)$ on $[\tau_i, \tau_{i+1}]$ with $u_{i+1}(\tau_i) = u_i(\tau_i)$ and u_{i+1} has maximal regularity and apriori estimate on $[\tau_i, \tau_{i+1}]$. Now we "glue" the solutions u_i and obtain a solution u of (P) on $[0, \tau]$ such that $u \in H^1(0, \tau; \mathcal{H})$. The apriori estimate (2.5) on $[0, \tau]$ follows by summing the corresponding apriori estimates on each sub-interval and by using Proposition 4.5. The uniqueness of the solution of (P) follows from this apriori estimate.

Note that in Theorem 2.4 we assume merely that $D(A(t_0)^{\frac{1}{2}}) = \mathcal{V}$ for some $t_0 \in [0, \tau]$. This assumption implies the uniform Kato square root property by Proposition 3.3. \square

Proof of Proposition 2.5. Set $B(t) := \mathcal{A}(t) - \mathcal{A}(0)$ and $E := H^1(0, \tau; \mathcal{H}) \cap L^\infty(0, \tau; \mathcal{V})$. Then E is a Banach space for the norm

$$\|u\|_E := \|u\|_{H^1(0, \tau; \mathcal{H})} + \|u\|_{L^\infty(0, \tau; \mathcal{V})}.$$

Let $f \in L^2(0, \tau; \mathcal{H})$ and $u_0 \in \mathcal{V}$. For $v \in E$, there exists a unique solution $u \in E$ to the problem

$$\begin{cases} u'(t) + A(0)u(t) &= f(t) - B(t)v(t), \quad t \in (0, \tau] \\ u(0) &= u_0. \end{cases}$$

In addition, there exists a constant C , independent of u_0, f and v such that

$$\|u\|_E \leq C \left[\|u_0\|_{\mathcal{V}} + \|f\|_{L^2(0, \tau; \mathcal{H})} + \|Bv\|_{L^2(0, \tau; \mathcal{H})} \right]. \quad (5.2)$$

Bounding $\|u\|_{H^1(0, \tau; \mathcal{H})}$ by the term on the RHS follows from the classical maximal regularity for the operator $A(0)$ in the Hilbert space \mathcal{H} . For the bound of $\|u\|_{L^\infty(0, \tau; \mathcal{V})}$ by the same term we use either Proposition 4.5 or the classical embedding

$$H^1(0, \tau; \mathcal{H}) \cap L^2(0, \tau; D(A(0))) \hookrightarrow C([0, \tau]; D(A(0)^{\frac{1}{2}}).$$

Define the operator K on E by $K(v) := u$. We prove that for $\tau > 0$ small, K is a contraction operator. Indeed, let $v_1, v_2 \in E$. Then $w := K(v_1) - K(v_2)$ satisfies

$$\begin{cases} w'(t) + A(0)w(t) &= -B(t)(v_1(t) - v_2(t)), \quad t \in (0, \tau] \\ w(0) &= 0. \end{cases}$$

Hence by (5.2) and [H4] with $\gamma = 0$

$$\begin{aligned} \|w\|_E &\leq C\|B(v_1 - v_2)\|_{L^2(0,\tau;\mathcal{H})} \\ &= C\left(\int_0^\tau \|(A(t) - A(0))(v_1 - v_2)(t)\|^2 dt\right)^{\frac{1}{2}} \\ &\leq CM_0\sqrt{\tau}\|v_1 - v_2\|_{L^\infty(0,\tau;\mathcal{V})} \\ &\leq CM_0\sqrt{\tau}\|v_1 - v_2\|_E. \end{aligned}$$

This shows that for $CM_0\sqrt{\tau} < 1$ the operator K is a contraction. Hence there exists a unique $u \in E$ such that $K(u) = u$. This gives that u satisfies (P) on $[0, \tau]$ for $\tau < \frac{1}{(CM_0)^2}$ and it follows from (5.2) that u satisfies the apriori estimate

$$\|u\|_E \leq C'[\|u_0\|_{\mathcal{V}} + \|f\|_{L^2(0,\tau;\mathcal{H})}].$$

Finally, for arbitrary $\tau > 0$, we split $[0, \tau]$ into a finite number of sub-intervals with small sizes and proceed exactly as in the previous proof. \square

6 Applications

In this section we give some applications of the previous results to concrete differential operators.

– *Elliptic operators on \mathbb{R}^n .* Let $\mathcal{H} = L^2(\mathbb{R}^n)$ and $\mathcal{V} = H^1(\mathbb{R}^n)$ and define the sesquilinear forms

$$\mathfrak{a}(t, u, v) = \sum_{k,l=1}^n \int_{\mathbb{R}^n} c_{kl}(t, x) \partial_k u \overline{\partial_l v} dx, \quad u, v \in \mathcal{V}.$$

We assume that the matrix $C(t, x) = (c_{kl}(t, x))_{1 \leq k, l \leq n}$ satisfies the usual ellipticity condition, that is, there exists $\alpha, M > 0$ such that

$$\alpha|\xi|^2 \leq \operatorname{Re}(C(t, x)\xi \cdot \bar{\xi}) \text{ and } |C(t, x)\xi \cdot \nu| \leq M|\xi||\nu|$$

for all $\xi, \nu \in \mathbb{C}^n$ and a.e $t \in [0, \tau]$, $x \in \mathbb{R}^n$. The forms $\mathfrak{a}(t)$ satisfy the assumptions [H1]-[H3]. For each t , the corresponding operator is formally given by $A(t) = -\sum_{k,l=1}^n \partial_l(c_{kl}(t, x)\partial_k)$. Next we assume that $C \in H^{\frac{1}{2}}(0, \tau; L^\infty(\mathbb{C}^{n^2}))$. We note that

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V}, \mathcal{V}')} \leq M'\|C(t, \cdot) - C(s, \cdot)\|_{L^\infty(\mathbb{C}^{n^2})}$$

for some constant M' . This implies that $\mathcal{A} \in H^{\frac{1}{2}}(0, \tau; \mathcal{L}(\mathcal{V}, \mathcal{V}'))$. We assume in addition that each c_{kl} is Hölder continuous of order α for some $\alpha > 0$ with

$$|c_{kl}(t, x) - c_{kl}(s, x)| \leq c|t - s|^\alpha$$

for a.e. $x \in \mathbb{R}^n$. This assumption implies in particular (2.1). We could also weaken this assumption by formulating it in terms of the modulus of continuity, see (2.3).

We are now allowed to apply Theorem 2.2. We obtain maximal L^2 -regularity and apriori estimate for the parabolic problem

$$\begin{cases} u'(t) + A(t)u(t) = f(t) \\ u(0) = u_0 \in H^1(\mathbb{R}^n). \end{cases}$$

That is, for every $f \in L^2(0, \tau; L^2(\mathbb{R}^n))$ and $u_0 \in H^1(\mathbb{R}^n)$ there is unique solution $u \in H^1(0, \tau; L^2(\mathbb{R}^n))$. Note that we also have from Proposition 4.5 that $u \in L^\infty(0, \tau; H^1(\mathbb{R}^n))$. As we already mentioned before, the uniform Kato square root property required in Theorem 2.2 is satisfied in this setting, see [6]. As we mentioned in the introduction, maximal L^2 -regularity for these elliptic operators was proved recently in [7] under the slightly stronger assumption that the coefficients satisfy a BMO- $H^{\frac{1}{2}}$ regularity with respect to t .

The maximal L^2 -regularity we proved here holds also in the case of elliptic operators on Lipschitz domains with Dirichlet or Neumann boundary conditions. The arguments are the same. One define the previous forms $\mathbf{a}(t)$ with domain $\mathcal{V} = H_0^1(\Omega)$ (for Dirichlet boundary conditions) or $\mathcal{V} = H^1(\Omega)$ (for Neumann boundary conditions).

– *Robin boundary conditions.* Let Ω be a bounded domain of \mathbb{R}^d with Lipschitz boundary $\partial\Omega$. We denote by Tr the classical trace operator. Let $\beta : [0, \tau] \times \partial\Omega \rightarrow [0, \infty)$ be bounded and such that

$$\int_0^\tau \int_0^\tau \frac{\|\beta(t, \cdot) - \beta(s, \cdot)\|_{L^\infty(\partial\Omega)}^2}{|t - s|^{1+2\alpha}} ds dt < \infty \quad (6.1)$$

for some $\alpha > \frac{1}{4}$. In particular, $\beta(\cdot, x) \in H^\alpha$. We define the forms

$$\mathbf{a}(t, u, v) := \int_\Omega \nabla u \cdot \nabla v dx + \int_{\partial\Omega} \beta(t, \cdot) \text{Tr}(u) \text{Tr}(v) d\sigma,$$

for all $u, v \in \mathcal{V} := H^1(\Omega)$. Formally, the associated operator $A(t)$ is (minus) the Laplacian with the time dependent Robin boundary condition

$$\frac{\partial u}{\partial n} + \beta(t, \cdot)u = 0 \text{ on } \partial\Omega.$$

Here $\frac{\partial u}{\partial n}$ denotes the normal derivative in the weak sense. Note that for any $\varepsilon > 0$

$$\begin{aligned} & |\mathbf{a}(t; u, v) - \mathbf{a}(s; u, v)| \\ &= \left| \int_{\partial\Omega} [\beta(t, \cdot) - \beta(s, \cdot)] \text{Tr}(u) \text{Tr}(v) d\sigma \right| \\ &\leq \|\beta(t, \cdot) - \beta(s, \cdot)\|_{L^\infty(\partial\Omega)} \|u\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)} \|v\|_{H^{\frac{1}{2}+\varepsilon}(\Omega)}, \end{aligned}$$

where we used the fact that the trace operator is bounded from $H^{\frac{1}{2}+\varepsilon}(\Omega)$ into $L_2(\partial\Omega)$. Now assumption (6.1) allows us to apply Theorem 2.4 with $\gamma = \frac{1}{2}+\varepsilon$ and obtain maximal L^2 -regularity for the corresponding evolution equation with initial data $u_0 \in H^1(\Omega)$. The forms considered here are symmetric and therefore the uniform Kato square root property can be checked easily. Maximal L^2 -regularity for time dependent Robin boundary condition with $\beta(\cdot, x) \in C^\alpha$ for some $\alpha > \frac{1}{4}$ was previously proved in [4] and [17]. In [17] maximal L^p -regularity is proved for all $p \in (1, \infty)$ is proved.

– *Operators with terms of lower order.* Let Ω be a domain of \mathbb{R}^n and let $b_k, m : [0, \tau] \times \Omega \rightarrow \mathbb{R}$ be a bounded measurable function for each $k = 1, \dots, n$. We define the forms

$$\mathfrak{a}(t, u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx + \sum_{k=1}^n \int_{\Omega} b_k(t, x) \partial_k u v dx + \int_{\Omega} m(t, x) u v dx,$$

with domain \mathcal{V} , a closed subset of $H^1(\Omega)$ which contains $H_0^1(\Omega)$. It is clear that

$$|\mathfrak{a}(t, u, v) - \mathfrak{a}(s, u, v)| \leq M_0 \|u\|_{\mathcal{V}} \|v\|_2$$

for some constant M_0 . This means that assumption [H4] is satisfied with $\gamma = 0$. We apply Proposition 2.5 and obtain maximal L^2 -regularity for the corresponding evolution equation.

As we mentioned in the introduction, the domains of the corresponding operators are independent of t and one may apply the results from [3] to obtain maximal regularity. Doing so, one needs to assume some regularity with respect to t for the coefficients $b_k(t, x)$ and $m(t, x)$ whereas the result we obtain from Proposition 2.5 does not require any regularity.

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