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EIGENVALUE AND GAP ESTIMATES OF ISOMETRIC IMMERSIONS FOR THE DIRICHLET-TO-NEUMANN OPERATOR ACTING ON p -FORMS

DEBORAH MICHEL

ABSTRACT. In this paper, we study the first eigenvalue of the Dirichlet-to-Neumann operator acting on differential forms of a Riemannian manifold with boundary isometrically immersed in some Euclidean space. We give a lower bound of the integral energy of p -forms in terms of its first eigenvalue associated with $(p - 1)$ -forms. We also find a lower bound for the gap between two consecutive first eigenvalues in terms of the curvature of the boundary.

Estimations de valeurs propres et du gap de l'opérateur de Dirichlet-à-Neumann agissant sur les p -formes pour les immersions isométriques

RÉSUMÉ. Dans cet article, nous étudions la première valeur propre de l'opérateur de Dirichlet-à-Neumann agissant sur les formes différentielles d'une variété riemannienne à bord plongée isométriquement dans un espace euclidien. Nous obtenons une borne inférieure de l'énergie des p -formes en termes de sa première valeur propre associée aux $(p - 1)$ -formes. Nous trouvons aussi une borne inférieure pour l'écart entre deux premières valeurs propres consécutives par rapport à la courbure de la frontière.

1. INTRODUCTION

Let (M, g) be an $(n + 1)$ -dimensional compact oriented Riemannian manifold with smooth boundary ∂M isometrically immersed in some Euclidean space \mathbb{R}^d . In [?], P. Guerini and A. Savo studied the first eigenvalue of the Hodge Laplacian acting on p -differential forms of a manifold with boundary, as well as the gap for consecutive values of the degree p . For all $p = 0, \dots, n + 1$, the Laplacian Δ_p (often denoted Δ when no confusion is possible) is defined on the space of p -forms of M , $\Lambda^p(M)$, by:

$$\Delta_p \omega := (d\delta + \delta d)\omega, \quad \forall \omega \in \Lambda^p(M),$$

where d is the exterior derivative and δ the co-differential. They considered the absolute boundary condition problem:

$$\begin{cases} \Delta \omega = \mu \omega \\ J^* i_{\tilde{\nu}} \omega = J^* i_{\tilde{\nu}} d\omega = 0, \end{cases}$$

where $J : \partial M \hookrightarrow M$ is the natural inclusion and $\tilde{\nu}$ is the inward unit normal vector field at each point of ∂M . Among other things, they obtained (see [?]) a general lower bound for the integral energy of a co-closed p -form on M and a lower bound for the gap $\mu''_{1,p} - \mu'_{1,p}$ in terms of the curvature term W_p of Bochner formula and

the shape operator of the immersion T_p (see section 2). Here, $\mu''_{1,p}$ (resp. $\mu'_{1,p}$) is the first positive eigenvalue of the absolute problem restricted to co-closed (resp. closed) p -forms. In this paper, we apply the method developed in [?] to the Dirichlet-to-Neumann operator defined by Raulot and Savo in [?].

2. DEFINITIONS AND BASIC FACTS

First, let us recall some facts about the Dirichlet-to-Neumann operator. Originally, the Dirichlet-to-Neumann operator T , also called Steklov operator, acts on smooth functions on ∂M . It is defined for all $f \in C^\infty(\partial M)$ by :

$$Tf := -\frac{\partial \hat{f}}{\partial \bar{\nu}}$$

where \hat{f} is the harmonic extension of f on M . Steklov operator has been widely studied, initially because of its applications. In fact, if we consider a steady state distribution of temperature in a body for given temperature values on the body surface, then the resulting heat flux is a Steklov operator. It is also used to solve inverse boundary problems such as electrical impedance tomography problems. Especially, its first eigenvalues have already been estimated, for example in [?] and [?].

We extend this operator on functions to an operator $T^{[p]}$ acting on the bundle of p -forms, $\Lambda^p(\partial M)$ for $0 \leq p \leq n$. The following definition is the one developed in [?].

For ω a p -form on ∂M , there exists a unique p -form (see [?]) $\hat{\omega}$ on M such that:

$$\begin{cases} \Delta \hat{\omega} = 0 \text{ on } M, \\ J^* \hat{\omega} = \omega, \quad i_{\bar{\nu}} \hat{\omega} = 0. \end{cases} \quad (1)$$

If we let :

$$T^{[p]} \omega = -i_{\bar{\nu}} d\hat{\omega}, \quad (2)$$

then $T^{[p]} : \Lambda^p(\partial M) \rightarrow \Lambda^p(\partial M)$ defines a pseudo-differential linear operator which is elliptic, self-adjoint and positive. It possesses a discrete spectrum denoted by:

$$0 \leq \nu_{1,p}(M) \leq \nu_{2,p}(M) \leq \dots \nearrow \infty.$$

Moreover, one has the variational characterization for the eigenvalues :

$$\nu_{1,p}(M) = \inf \left\{ \frac{\int_M \|d\omega\|^2 + \|\delta\omega\|^2}{\int_{\partial M} \|\omega\|^2} \mid \omega \in \Lambda^p(M), \quad i_{\bar{\nu}} \omega = 0 \text{ on } \partial M \right\}. \quad (3)$$

We note that $\nu_{1,p}$ could be zero and its multiplicity is given by the p -th Betti number (see [?]). In particular, one has $\nu_{1,0} = 0$, with multiplicity one and associated eigenfunction given by the constants. Thus, the Dirichlet-to-Neumann is closely related to the shape of the boundary and encodes some of its curvature properties.

The main result in this paper is Theorem 4 which extends previously mentioned estimates to arbitrary compact manifolds with boundary immersed in \mathbb{R}^n , provided that a suitable curvature condition holds. Namely, let M be a $(n+1)$ dimensional

compact oriented Riemannian manifold with smooth boundary ∂M , and fix $x \in \partial M$ and $p = 0, \dots, n$. We denote :

$$\begin{aligned}\sigma_p(x) &:= \text{sum of the } p \text{ smallest principal curvatures of } \partial M, \\ \sigma_p &:= \inf_{x \in \partial M} \sigma_p(x).\end{aligned}$$

Then, one has :

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p},$$

provided that $W_p \geq T_p$, where W_p is the curvature term in the Bochner-Weitzenböck formula (see Proposition 1) and T_p is a symmetric endomorphism acting on p -forms and defined in (6) below.

This estimate depends itself on the estimate of Theorem 3, which is of its own independent interest and extends to the Dirichlet-to-Neumann case a corresponding result obtained for the Hodge Laplacian in [?].

3. GAP OF ISOMETRIC IMMERSIONS

From now on, we consider an isometric immersion of the Riemannian manifold M^{n+1} into some Euclidean space \mathbb{R}^d , $d \geq n + 1$.

If ν is a vector normal to M , we introduce the shape operator S_ν characterized by:

$$\langle S_\nu(X), Y \rangle = \langle L(X, Y), \nu \rangle, \quad (4)$$

for all $X, Y \in TM$, where L is the second fundamental form of the immersion.

Then we can extend S_ν to a selfadjoint endomorphism of $\Lambda^p(M)$, denoted by S_ν^p and given by:

$$S_\nu^p(\omega)(X_1, \dots, X_p) = \sum_{i=1}^p \omega(X_1, \dots, S_\nu(X_i), \dots, X_p) \quad (5)$$

If (ν_1, \dots, ν_m) is an orthonormal basis of the normal bundle of M at any fixed point (i.e. $m = d - n - 1$), then

$$T_p = \sum_{\alpha=1}^m (S_{\nu_\alpha}^p)^2 \quad (6)$$

defines a selfadjoint nonnegative endomorphism of $\Lambda^p(M)$ which does not depend on the orthonormal basis chosen. In particular, for all $\omega \in \Lambda^p(M)$:

$$\langle T_p(\omega), \omega \rangle = \sum_{\alpha=1}^m \|S_{\nu_\alpha}^p(\omega)\|^2. \quad (7)$$

Last but not least are the very usefull Bochner-Weitzenböck and Reilly formulas, which can be found in [?] and [?] respectively.

Proposition 1. *If ω is a p -form, then:*

$$\langle \Delta\omega, \omega \rangle = \|\nabla\omega\|^2 + \frac{1}{2}\Delta\|\omega\|^2 + \langle W_p\omega, \omega \rangle.$$

Here, W_p is a symmetric endomorphism acting on $\Lambda^p(M)$, called the Bochner curvature term. For $p = 1$, one has $W_1 = Ric$, the Ricci tensor ; hence W_1 is non-negative provided that M has non-negative Ricci curvature. Moreover, from

the work of Gallot and Mayer (in [?]), we know that if γ is a lower bound of the eigenvalues of the Riemann curvature operator, then:

$$W_p \geq p(n+1-p)\gamma. \quad (8)$$

We deduce from this that, if the curvature operator of M is non-negative, then also $W_p \geq 0$ for all degrees p .

Proposition 2. *Let ω be a p -form on M , then:*

$$\int_M \|d\omega\|^2 + \|\delta\omega\|^2 = \int_M \|\nabla\omega\|^2 + \langle W_p\omega, \omega \rangle + 2 \int_{\partial M} \langle i_{\bar{\nu}}\omega, \delta^{\partial M}(J^*\omega) \rangle + \int_{\partial M} B(\omega, \omega),$$

where :

$$B(\omega, \omega) = \langle S^p(J^*\omega), (J^*\omega) \rangle + nH\|i_{\bar{\nu}}\omega\|^2 - \langle S^{p-1}(i_{\bar{\nu}}\omega), i_{\bar{\nu}}\omega \rangle,$$

$\delta^{\partial M}$ is the co-derivative on ∂M and ∇ the Levi-Civita connection on M and that S denotes the shape operator of the immersion of ∂M in M .

On the immersed manifold M , we focus on the family of all vector fields which are the orthogonal projection of unit parallel vector fields on the ambient Euclidean space \mathbb{R}^d . This family is naturally parametrized by the sphere S^{d-1} . Its typical elements will be denoted by \bar{V} .

At any point of M , we can split :

$$\bar{V} = V + V^\perp, \quad (9)$$

where $V \in TM$ is the orthogonal projection of \bar{V} onto TM and $V^\perp \in TM^\perp$. Hence, any $\bar{V} \in S^{d-1}$ gives rise to a vector field V on M .

Remark 1. *As in [?], it can be easily proved that if $\omega, \phi \in \Lambda^p(M)$, then at any point of M , we have:*

$$\int_{S^{d-1}} \langle i_V\omega, i_V\phi \rangle d\bar{V} = c_d p \langle \omega, \phi \rangle. \quad (10)$$

Here, $c_d = \frac{|S^{d-1}|}{d}$.

In order to prove our main result, we wish to integrate some inequalities with respect to \bar{V} and the canonical measure of S^{d-1} which will be denoted by $d\bar{V}$. To this scope, we complete lemma 4.8 in [?].

Lemma 1. *Let $\omega \in \Lambda^p(M)$, $p = 1, \dots, n+1$. At any point of M :*

$$\begin{aligned} \int_{S^{d-1}} \|i_V\omega\|^2 d\bar{V} &= c_d p \|\omega\|^2, \\ \int_{S^{d-1}} \|di_V\omega\|^2 d\bar{V} &= c_d \{ \|\nabla\omega\|^2 + \langle T_p(\omega), \omega \rangle + (p-1)\|d\omega\|^2 \}, \\ \int_{S^{d-1}} \|\delta i_V\omega\|^2 d\bar{V} &= c_d (p-1) \|\delta\omega\|^2. \end{aligned}$$

Proof. (1) The first and the last equations can be found in [?]. They are proved using remark 1 and the Cartan formula :

$$L_V\omega = di_V\omega + i_Vd\omega. \quad (11)$$

- (2) Now, remarking that for any vector field V , which is the gradient of a smooth function, one has $\delta i_V \omega = -i_V \delta$. Indeed, the projected fields V are gradients of restrictions to M of suitable distance functions to hyperplanes in R^{n+1} . Thus,

$$\delta i_V \omega = -i_V \delta. \quad (12)$$

Using remark 1, we obtain:

$$\begin{aligned} \int_{S^{d-1}} \|\delta i_V \omega\|^2 d\bar{V} &= \int_{S^{d-1}} \|i_V \delta \omega\|^2 d\bar{V} \\ &= c_d (p-1) \|\delta \omega\|^2. \end{aligned}$$

□

Remark 2. *We have:*

$$\begin{aligned} \nabla_{e_{i_j}} V &= \nabla_{e_{i_j}} \left(\sum_{k=1}^d \langle \bar{V}, e_k \rangle e_k \right) \\ &= \sum_{k=1}^n (e_j \langle \bar{V}, e_k \rangle) e_k + \sum_{k=1}^n \langle \bar{V}, e_k \rangle \nabla_{e_{i_j}} e_k. \end{aligned}$$

Let $\tilde{\nabla}$ be the Levi-Civita connection on \mathbb{R}^d . By its compatibility with the metric and the fact that \bar{V} is parallel, we get:

$$\nabla_{e_{i_j}} V = \left(\sum_{k=1}^n \langle \bar{V}, \tilde{\nabla}_{e_j} e_k \rangle e_k \right).$$

Moreover, if we choose (e_1, \dots, e_n) to be geodesic at $x \in M$,

$$\tilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + (\nabla_{e_i} e_j)^\perp = (\nabla_{e_i} e_j)^\perp.$$

Thus,

$$\begin{aligned} \langle \bar{V}, \tilde{\nabla}_{e_j} e_k \rangle &= \left\langle \bar{V}, \left(\tilde{\nabla}_{e_j} e_k \right)^\perp \right\rangle \\ &= \left\langle V^\perp, \left(\tilde{\nabla}_{e_j} e_k \right)^\perp \right\rangle = \left\langle V^\perp, \left(\tilde{\nabla}_{e_j} e_k \right) \right\rangle \\ &= \langle S_{V^\perp} e_j, e_k \rangle \end{aligned}$$

Eventually, we obtain:

$$\nabla_{e_{i_j}} V = S_{V^\perp} e_j.$$

Lemma 2. *Let $\omega \in \Lambda^p(M)$ such that $i_{\tilde{\nu}} \omega = 0$ on ∂M , with $\tilde{\nu}$ the unit inner vector field normal to the boundary. Then, for all $\bar{V} \in S^{d-1}$:*

$$\nu_{1,p-1}(M) \int_{\partial M} \|i_V \omega\|^2 \leq \int_M \|di_V \omega\|^2 + \|\delta i_V \omega\|^2. \quad (13)$$

Proof. If $p \geq 1$ and ω is tangential, then $i_{\tilde{\nu}} i_V \omega = -i_V i_{\tilde{\nu}} \omega = 0$. So, $i_V \omega$ is a tangential $(p-1)$ -form. Hence, for all $\bar{V} \in S^{d-1}$, $i_V \omega$ is a relevant choice as a test form for $\nu_{1,p-1}(M)$. We apply the min-max principle and get (13). □

Integrating these inequality on S^{d-1} and using lemma 1, we obtain the following theorem :

Theorem 3. *Let $M^{n+1} \rightarrow \mathbb{R}^d$ an isometric immersion, with M a Riemannian compact and oriented manifold with smooth boundary ∂M . Let also ω be a p -form on M , $p = 1, \dots, n+1$, satisfying $i_{\bar{\nu}}\omega = 0$ on ∂M . Then:*

$$\int_M \{ \|\nabla\omega\|^2 + \langle T_p\omega, \omega \rangle + (p-1)(\|d\omega\|^2 + \|\delta\omega\|^2) \} \geq p \nu_{1,p-1}(M) \int_{\partial M} \|\omega\|^2.$$

The inequality is sharp for any harmonic extension on B^{n+1} of an eigenform associated with $\nu_{1,p}(B^{n+1})$ for $\frac{n+3}{2} \leq p \leq n$.

Proof. Integrating (13) on S^{d-1} gives:

$$\int_{S^{d-1}} \nu_{1,p-1}(M) \int_{\partial M} \|i_V\omega\|^2 d\bar{V} \leq \int_{S^{d-1}} \int_M \|di_V\omega\|^2 + \|\delta i_V\omega\|^2 d\bar{V}.$$

Then, using lemma 1 as well as Fubini's theorem, we get theorem 3.

Concerning the sharpness, let $\omega \in \Lambda^p(S^n)$ an eigenform of unit L^2 norm associated with $\nu_{1,p}$ and $\hat{\omega}$ its harmonic extension on the ball. Applying theorem 3 to $\hat{\omega}$, one gets:

$$\int_{B^{n+1}} \{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega} \rangle + (p-1)(\|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2) \} \geq p \nu_{1,p-1}(M) \int_{S^n} \|\hat{\omega}\|^2. \quad (14)$$

Clearly, we have :

- $\int_{B^{n+1}} \|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2 = \nu_{1,p}$ by the variational characterization;
- $\langle T_p\hat{\omega}, \hat{\omega} \rangle = 0$ since B^{n+1} is open in \mathbb{R}^{n+1} ;
- $\int_{S^n} \|\hat{\omega}\|^2 = \int_{S^n} \|\omega\|^2 = 1$.

Now, by Reilly formula in Proposition 2 applied to $\hat{\omega}$, and the fact that in the given range $\nu_{1,p}(B^{n+1}) = p+1$.

$$\int_{B^{n+1}} \|\nabla\hat{\omega}\|^2 = (p+1) - \int_{S^n} \langle S^p\omega, \omega \rangle,$$

because $\hat{\omega}$ is tangential and $W_p = 0$. Since S^n is totally umbilical with constant mean curvature equals to 1, we obtain:

$$S^p\omega = p\omega.$$

Finally: $\int_{B^{n+1}} \|\nabla\hat{\omega}\|^2 = 1$. Next, by the results in [?]:

$$\nu_{1,p}(B^{n+1}) = \begin{cases} \frac{n+3}{n+1}p & \text{if } 1 \leq p \leq \frac{n+1}{2} \\ p+1 & \text{if } \frac{n+1}{2} \leq p \leq n. \end{cases}$$

Calculating both sides of (14) in function of p , we see that the equality is attained for $\frac{n+3}{2} \leq p \leq n$. \square

Theorem 4. *Let $M^{n+1} \rightarrow \mathbb{R}^d$ an isometric immersion with M having p -convex boundary. We also suppose that $W_p - T_p \geq 0$ at all points of M . For all $p = 1, \dots, n$, one has:*

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p}. \quad (15)$$

Equality is achieved when M is the unit Euclidean ball of \mathbb{R}^{n+1} and $\frac{n+3}{2} \leq p \leq n$.

Proof. Let ω an eigenform associated with $\nu_{1,p}(M)$ (which we suppose has unit L^2 norm on ∂M by normalization) and let $\hat{\omega}$ its harmonic extension. Then,

$$\begin{aligned}\nu_{1,p}(M) &= \nu_{1,p}(M) \int_{\partial M} \|\omega\|^2 \\ &= \nu_{1,p}(M) \int_{\partial M} \|\hat{\omega}\|^2 \\ &= \int_M \|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2.\end{aligned}$$

Applying theorem 3 to $\hat{\omega}$ leads to:

$$\begin{aligned}p \nu_{1,p-1}(M) \int_{\partial M} \|\hat{\omega}\|^2 &\leq \int_M \{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega} \rangle + (p-1) (\|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2) \} \\ &= \int_M \{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega} \rangle \} + (p-1)\nu_{1,p}(M);\end{aligned}$$

which reads as

$$p\nu_{1,p-1}(M) - (p-1)\nu_{1,p}(M) \leq \int_M \{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega} \rangle \}.$$

But, by Reilly formula in proposition 2 and since $\hat{\omega}$ is tangential, we have

$$\begin{aligned}\int_M \|\nabla\hat{\omega}\|^2 &= \int_M \{ \|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2 \} - \int_M \langle W_p\hat{\omega}, \hat{\omega} \rangle - \int_{\partial M} \langle S_p\omega, \omega \rangle \\ &= \nu_{1,p} - \int_M \langle W_p\hat{\omega}, \hat{\omega} \rangle - \int_{\partial M} \langle S_p\omega, \omega \rangle,\end{aligned}$$

hence :

$$\nu_{1,p-1}(M) \leq \nu_{1,p}(M) + \int_M \frac{1}{p} \langle (T_p - W_p)\hat{\omega}, \hat{\omega} \rangle - \frac{1}{p} \int_{\partial M} \langle S_p\omega, \omega \rangle.$$

Now, by hypothesis, ∂M is p -convex, so that $\sigma_p \geq 0$ and since we assumed $W_p - T_p \geq 0$, we get:

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p}.$$

Concerning the equality case, let $M = B^{n+1}$ the unit ball and its boundary S^n . As B^{n+1} is flat and open in \mathbb{R}^{n+1} , $W_p = 0$ and $T_p = 0$, so that $W_p - T_p = 0$ is satisfied. Moreover, the ball is convex, so it is p convex for all $p \geq 1$. Now, for all p , we have $\sigma_p = p$ and

- If $1 \leq p \leq \frac{n+1}{2}$:

$$\begin{cases} \nu_{1,p}(B^{n+1}) = \frac{n+3}{n+1}p \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = \frac{n+3}{n+1}(p-1) + 1. \end{cases}$$

- If $\frac{n+1}{2} \leq p \leq n$:

$$- \text{ if } \frac{n-1}{2} \leq p-1 \leq \frac{n+1}{2},$$

$$\begin{cases} \nu_{1,p}(B^{n+1}) = p + 1 \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = \frac{n+3}{n+1}(p-1) + 1; \end{cases}$$

$$\begin{aligned}
& - \text{ if } \frac{n+1}{2} \leq p-1 \leq n-1, \\
& \quad \begin{cases} \nu_{1,p}(B^{n+1}) = p+1 \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = p+1. \end{cases}
\end{aligned}$$

And we see as announced that the equality is obtained for $\frac{n+3}{2} \leq p \leq n$. \square

We apply theorem 4 to Euclidean domains, as done in [?] (see Theorem 4).

Corollary 5. (1) *Assume that the Euclidean domain M is p -convex for some $p = 1, \dots, n-1$. Then:*

$$\nu_{1,p-1}(M) \leq \nu_{1,p-1}(M) \leq \dots \leq \nu_{1,n}(M). \quad (16)$$

(2) *If M is indeed convex, then the sequence $\{\nu_{1,p}(M)\}$ is non-decreasing with respect to the degree p :*

$$\nu_{1,0}(M) \leq \nu_{1,1}(M) \leq \dots \leq \nu_{1,n}(M). \quad (17)$$

(3) *If M is strictly p -convex, so that $\sigma_p > 0$, then $\nu_{1,q}(M) < \nu_{1,q+1}(M)$ for all $q \geq p$. Thus, these inequalities in (1) and (2) are strict when starting from $\nu_{1,p}(M)$ and $\nu_{1,1}(M)$ respectively.*

Proof. Since M is an Euclidean domain of \mathbb{R}^{n+2} (so that $T_p = W_p = 0$), we get by theorem 4 :

$$\nu_{1,p} \geq \nu_{1,p-1}. \quad (18)$$

As a p -convex domain is q -convex for all $q \in [p, n]$, (1) follows. Then (2) and (3) are consequences of (1). \square

Corollary 6. *Let M be a p -convex spherical domain, isometrically immersed in \mathbb{R}^{n+2} . Then, for all $p \leq \frac{n+1}{2}$, we obtain an estimation of the p -gap :*

$$\nu_{1,p} - \nu_{1,p-1} \geq 0. \quad (19)$$

Proof. We have $T_p = p^2 \cdot Id$ and $W_p = p(n+1-p) \cdot Id$ so that :

$$W_p - T_p \geq 0 \iff p \leq \frac{n+1}{2}.$$

So, the hypothesis of theorem 4 are satisfied for all $p \leq \frac{n+1}{2}$. Moreover, in this case, we have for all p , $\sigma_p \geq 0$. Applying theorem 4, the claim follows. \square

Remark 3. *Note that $M = S_+^{n+1}$, the upper hemisphere, is a particular case of corollary 6 for $p \leq \frac{n+1}{2}$. Indeed, its boundary is the equator so that $\sigma_p = 0$. Thus, for all $p \leq \frac{n+1}{2}$, we get an estimation of the p -gap of the hemisphere:*

$$\nu_{1,p}(S_+^{n+1}) - \nu_{1,p-1}(S_+^{n+1}) \geq 0. \quad (20)$$

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