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## EIGENVALUE AND GAP ESTIMATES OF ISOMETRIC IMMERSIONS FOR THE DIRICHLET-TO-NEUMANN OPERATOR ACTING ON $p$ -FORMS

DEBORAH MICHEL

**ABSTRACT.** In this paper, we study the first eigenvalue of the Dirichlet-to-Neumann operator acting on differential forms of a Riemannian manifold with boundary isometrically immersed in some Euclidean space. We give a lower bound of the integral energy of  $p$ -forms in terms of its first eigenvalue associated with  $(p - 1)$ -forms. We also find a lower bound for the gap between two consecutive first eigenvalues in terms of the curvature of the boundary.

### Estimations de valeurs propres et du gap de l'opérateur de Dirichlet-à-Neumann agissant sur les $p$ -formes pour les immersions isométriques

**RÉSUMÉ.** Dans cet article, nous étudions la première valeur propre de l'opérateur de Dirichlet-à-Neumann agissant sur les formes différentielles d'une variété riemannienne à bord plongée isométriquement dans un espace euclidien. Nous obtenons une borne inférieure de l'énergie des  $p$ -formes en termes de sa première valeur propre associée aux  $(p - 1)$ -formes. Nous trouvons aussi une borne inférieure pour l'écart entre deux premières valeurs propres consécutives par rapport à la courbure de la frontière.

### 1. INTRODUCTION

Let  $(M, g)$  be an  $(n + 1)$ -dimensional compact oriented Riemannian manifold with smooth boundary  $\partial M$  isometrically immersed in some Euclidean space  $\mathbb{R}^d$ . In [?], P. Guerini and A. Savo studied the first eigenvalue of the Hodge Laplacian acting on  $p$ -differential forms of a manifold with boundary, as well as the gap for consecutive values of the degree  $p$ . For all  $p = 0, \dots, n + 1$ , the Laplacian  $\Delta_p$  (often denoted  $\Delta$  when no confusion is possible) is defined on the space of  $p$ -forms of  $M$ ,  $\Lambda^p(M)$ , by:

$$\Delta_p \omega := (d\delta + \delta d)\omega, \quad \forall \omega \in \Lambda^p(M),$$

where  $d$  is the exterior derivative and  $\delta$  the co-differential. They considered the absolute boundary condition problem:

$$\begin{cases} \Delta \omega = \mu \omega \\ J^* i_{\tilde{\nu}} \omega = J^* i_{\tilde{\nu}} d\omega = 0, \end{cases}$$

where  $J : \partial M \hookrightarrow M$  is the natural inclusion and  $\tilde{\nu}$  is the inward unit normal vector field at each point of  $\partial M$ . Among other things, they obtained (see [?]) a general lower bound for the integral energy of a co-closed  $p$ -form on  $M$  and a lower bound for the gap  $\mu''_{1,p} - \mu'_{1,p}$  in terms of the curvature term  $W_p$  of Bochner formula and

the shape operator of the immersion  $T_p$  (see section 2). Here,  $\mu''_{1,p}$  (resp.  $\mu'_{1,p}$ ) is the first positive eigenvalue of the absolute problem restricted to co-closed (resp. closed)  $p$ -forms. In this paper, we apply the method developed in [?] to the Dirichlet-to-Neumann operator defined by Raulot and Savo in [?].

## 2. DEFINITIONS AND BASIC FACTS

First, let us recall some facts about the Dirichlet-to-Neumann operator. Originally, the Dirichlet-to-Neumann operator  $T$ , also called Steklov operator, acts on smooth functions on  $\partial M$ . It is defined for all  $f \in C^\infty(\partial M)$  by :

$$Tf := -\frac{\partial \hat{f}}{\partial \bar{\nu}}$$

where  $\hat{f}$  is the harmonic extension of  $f$  on  $M$ . Steklov operator has been widely studied, initially because of its applications. In fact, if we consider a steady state distribution of temperature in a body for given temperature values on the body surface, then the resulting heat flux is a Steklov operator. It is also used to solve inverse boundary problems such as electrical impedance tomography problems. Especially, its first eigenvalues have already been estimated, for example in [?] and [?].

We extend this operator on functions to an operator  $T^{[p]}$  acting on the bundle of  $p$ -forms,  $\Lambda^p(\partial M)$  for  $0 \leq p \leq n$ . The following definition is the one developed in [?].

For  $\omega$  a  $p$ -form on  $\partial M$ , there exists a unique  $p$ -form (see [?])  $\hat{\omega}$  on  $M$  such that:

$$\begin{cases} \Delta \hat{\omega} = 0 \text{ on } M, \\ J^* \hat{\omega} = \omega, \quad i_{\bar{\nu}} \hat{\omega} = 0. \end{cases} \quad (1)$$

If we let :

$$T^{[p]} \omega = -i_{\bar{\nu}} d\hat{\omega}, \quad (2)$$

then  $T^{[p]} : \Lambda^p(\partial M) \rightarrow \Lambda^p(\partial M)$  defines a pseudo-differential linear operator which is elliptic, self-adjoint and positive. It possesses a discrete spectrum denoted by:

$$0 \leq \nu_{1,p}(M) \leq \nu_{2,p}(M) \leq \dots \nearrow \infty.$$

Moreover, one has the variational characterization for the eigenvalues :

$$\nu_{1,p}(M) = \inf \left\{ \frac{\int_M \|d\omega\|^2 + \|\delta\omega\|^2}{\int_{\partial M} \|\omega\|^2} \mid \omega \in \Lambda^p(M), \quad i_{\bar{\nu}} \omega = 0 \text{ on } \partial M \right\}. \quad (3)$$

We note that  $\nu_{1,p}$  could be zero and its multiplicity is given by the  $p$ -th Betti number (see [?]). In particular, one has  $\nu_{1,0} = 0$ , with multiplicity one and associated eigenfunction given by the constants. Thus, the Dirichlet-to-Neumann is closely related to the shape of the boundary and encodes some of its curvature properties.

The main result in this paper is Theorem 4 which extends previously mentioned estimates to arbitrary compact manifolds with boundary immersed in  $\mathbb{R}^n$ , provided that a suitable curvature condition holds. Namely, let  $M$  be a  $(n+1)$  dimensional

compact oriented Riemannian manifold with smooth boundary  $\partial M$ , and fix  $x \in \partial M$  and  $p = 0, \dots, n$ . We denote :

$$\begin{aligned}\sigma_p(x) &:= \text{sum of the } p \text{ smallest principal curvatures of } \partial M, \\ \sigma_p &:= \inf_{x \in \partial M} \sigma_p(x).\end{aligned}$$

Then, one has :

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p},$$

provided that  $W_p \geq T_p$ , where  $W_p$  is the curvature term in the Bochner-Weitzenbock formula (see Proposition 1) and  $T_p$  is a symmetric endomorphism acting on  $p$ -forms and defined in (6) below.

This estimate depends itself on the estimate of Theorem 3, which is of its own independent interest and extends to the Dirichlet-to-Neumann case a corresponding result obtained for the Hodge Laplacian in [?].

### 3. GAP OF ISOMETRIC IMMERSIONS

From now on, we consider an isometric immersion of the Riemannian manifold  $M^{n+1}$  into some Euclidean space  $\mathbb{R}^d$ ,  $d \geq n + 1$ .

If  $\nu$  is a vector normal to  $M$ , we introduce the shape operator  $S_\nu$  characterized by:

$$\langle S_\nu(X), Y \rangle = \langle L(X, Y), \nu \rangle, \quad (4)$$

for all  $X, Y \in TM$ , where  $L$  is the second fundamental form of the immersion.

Then we can extend  $S_\nu$  to a selfadjoint endomorphism of  $\Lambda^p(M)$ , denoted by  $S_\nu^p$  and given by:

$$S_\nu^p(\omega)(X_1, \dots, X_p) = \sum_{i=1}^p \omega(X_1, \dots, S_\nu(X_i), \dots, X_p) \quad (5)$$

If  $(\nu_1, \dots, \nu_m)$  is an orthonormal basis of the normal bundle of  $M$  at any fixed point (i.e.  $m = d - n - 1$ ), then

$$T_p = \sum_{\alpha=1}^m (S_{\nu_\alpha}^p)^2 \quad (6)$$

defines a selfadjoint nonnegative endomorphism of  $\Lambda^p(M)$  which does not depend on the orthonormal basis chosen. In particular, for all  $\omega \in \Lambda^p(M)$ :

$$\langle T_p(\omega), \omega \rangle = \sum_{\alpha=1}^m \|S_{\nu_\alpha}^p(\omega)\|^2. \quad (7)$$

Last but not least are the very usefull Bochner-Weitzenbock and Reilly formulas, which can be found in [?] and [?] respectively.

**Proposition 1.** *If  $\omega$  is a  $p$ -form, then:*

$$\langle \Delta\omega, \omega \rangle = \|\nabla\omega\|^2 + \frac{1}{2}\Delta\|\omega\|^2 + \langle W_p\omega, \omega \rangle.$$

Here,  $W_p$  is a symmetric endomorphism acting on  $\Lambda^p(M)$ , called the Bochner curvature term. For  $p = 1$ , one has  $W_1 = Ric$ , the Ricci tensor ; hence  $W_1$  is non-negative provided that  $M$  has non-negative Ricci curvature. Moreover, from

the work of Gallot and Mayer (in [?]), we know that if  $\gamma$  is a lower bound of the eigenvalues of the Riemann curvature operator, then:

$$W_p \geq p(n+1-p)\gamma. \quad (8)$$

We deduce from this that, if the curvature operator of  $M$  is non-negative, then also  $W_p \geq 0$  for all degrees  $p$ .

**Proposition 2.** *Let  $\omega$  be a  $p$ -form on  $M$ , then:*

$$\int_M \|d\omega\|^2 + \|\delta\omega\|^2 = \int_M \|\nabla\omega\|^2 + \langle W_p\omega, \omega \rangle + 2 \int_{\partial M} \langle i_{\bar{\nu}}\omega, \delta^{\partial M}(J^*\omega) \rangle + \int_{\partial M} B(\omega, \omega),$$

where :

$$B(\omega, \omega) = \langle S^p(J^*\omega), (J^*\omega) \rangle + nH\|i_{\bar{\nu}}\omega\|^2 - \langle S^{p-1}(i_{\bar{\nu}}\omega), i_{\bar{\nu}}\omega \rangle,$$

$\delta^{\partial M}$  is the co-derivative on  $\partial M$  and  $\nabla$  the Levi-Civita connection on  $M$  and that  $S$  denotes the shape operator of the immersion of  $\partial M$  in  $M$ .

On the immersed manifold  $M$ , we focus on the family of all vector fields which are the orthogonal projection of unit parallel vector fields on the ambient Euclidean space  $\mathbb{R}^d$ . This family is naturally parametrized by the sphere  $S^{d-1}$ . Its typical elements will be denoted by  $\bar{V}$ .

At any point of  $M$ , we can split :

$$\bar{V} = V + V^\perp, \quad (9)$$

where  $V \in TM$  is the orthogonal projection of  $\bar{V}$  onto  $TM$  and  $V^\perp \in TM^\perp$ . Hence, any  $\bar{V} \in S^{d-1}$  gives rise to a vector field  $V$  on  $M$ .

**Remark 1.** *As in [?], it can be easily proved that if  $\omega, \phi \in \Lambda^p(M)$ , then at any point of  $M$ , we have:*

$$\int_{S^{d-1}} \langle i_V\omega, i_V\phi \rangle d\bar{V} = c_d p \langle \omega, \phi \rangle. \quad (10)$$

Here,  $c_d = \frac{|S^{d-1}|}{d}$ .

In order to prove our main result, we wish to integrate some inequalities with respect to  $\bar{V}$  and the canonical measure of  $S^{d-1}$  which will be denoted by  $d\bar{V}$ . To this scope, we complete lemma 4.8 in [?].

**Lemma 1.** *Let  $\omega \in \Lambda^p(M)$ ,  $p = 1, \dots, n+1$ . At any point of  $M$  :*

$$\begin{aligned} \int_{S^{d-1}} \|i_V\omega\|^2 d\bar{V} &= c_d p \|\omega\|^2, \\ \int_{S^{d-1}} \|di_V\omega\|^2 d\bar{V} &= c_d \{ \|\nabla\omega\|^2 + \langle T_p(\omega), \omega \rangle + (p-1)\|d\omega\|^2 \}, \\ \int_{S^{d-1}} \|\delta i_V\omega\|^2 d\bar{V} &= c_d (p-1) \|\delta\omega\|^2. \end{aligned}$$

*Proof.* (1) The first and the last equations can be found in [?]. They are proved using remark 1 and the Cartan formula :

$$L_V\omega = di_V\omega + i_Vd\omega. \quad (11)$$

- (2) Now, remarking that for any vector field  $V$ , which is the gradient of a smooth function, one has  $\delta i_V \omega = -i_V \delta$ . Indeed, the projected fields  $V$  are gradients of restrictions to  $M$  of suitable distance functions to hyperplanes in  $R^{n+1}$ . Thus,

$$\delta i_V = -i_V \delta. \quad (12)$$

Using remark 1, we obtain:

$$\begin{aligned} \int_{S^{d-1}} \|\delta i_V \omega\|^2 d\bar{V} &= \int_{S^{d-1}} \|i_V \delta \omega\|^2 d\bar{V} \\ &= c_d (p-1) \|\delta \omega\|^2. \end{aligned}$$

□

**Remark 2.** *We have:*

$$\begin{aligned} \nabla_{e_{i_j}} V &= \nabla_{e_{i_j}} \left( \sum_{k=1}^d \langle \bar{V}, e_k \rangle e_k \right) \\ &= \sum_{k=1}^n (e_j \langle \bar{V}, e_k \rangle) e_k + \sum_{k=1}^n \langle \bar{V}, e_k \rangle \nabla_{e_{i_j}} e_k. \end{aligned}$$

Let  $\tilde{\nabla}$  be the Levi-Civita connection on  $\mathbb{R}^d$ . By its compatibility with the metric and the fact that  $\bar{V}$  is parallel, we get:

$$\nabla_{e_{i_j}} V = \left( \sum_{k=1}^n \langle \bar{V}, \tilde{\nabla}_{e_j} e_k \rangle e_k \right).$$

Moreover, if we choose  $(e_1, \dots, e_n)$  to be geodesic at  $x \in M$ ,

$$\tilde{\nabla}_{e_i} e_j = \nabla_{e_i} e_j + (\nabla_{e_i} e_j)^\perp = (\nabla_{e_i} e_j)^\perp.$$

Thus,

$$\begin{aligned} \langle \bar{V}, \tilde{\nabla}_{e_j} e_k \rangle &= \left\langle \bar{V}, \left( \tilde{\nabla}_{e_j} e_k \right)^\perp \right\rangle \\ &= \left\langle V^\perp, \left( \tilde{\nabla}_{e_j} e_k \right)^\perp \right\rangle = \left\langle V^\perp, \left( \tilde{\nabla}_{e_j} e_k \right) \right\rangle \\ &= \langle S_{V^\perp} e_j, e_k \rangle \end{aligned}$$

Eventually, we obtain:

$$\nabla_{e_{i_j}} V = S_{V^\perp} e_j.$$

**Lemma 2.** *Let  $\omega \in \Lambda^p(M)$  such that  $i_{\tilde{\nu}} \omega = 0$  on  $\partial M$ , with  $\tilde{\nu}$  the unit inner vector field normal to the boundary. Then, for all  $\bar{V} \in S^{d-1}$ :*

$$\nu_{1,p-1}(M) \int_{\partial M} \|i_V \omega\|^2 \leq \int_M \|di_V \omega\|^2 + \|\delta i_V \omega\|^2. \quad (13)$$

*Proof.* If  $p \geq 1$  and  $\omega$  is tangential, then  $i_{\tilde{\nu}} i_V \omega = -i_V i_{\tilde{\nu}} \omega = 0$ . So,  $i_V \omega$  is a tangential  $(p-1)$ -form. Hence, for all  $\bar{V} \in S^{d-1}$ ,  $i_V \omega$  is a relevant choice as a test form for  $\nu_{1,p-1}(M)$ . We apply the min-max principle and get (13). □

Integrating these inequality on  $S^{d-1}$  and using lemma 1, we obtain the following theorem :

**Theorem 3.** *Let  $M^{n+1} \rightarrow \mathbb{R}^d$  an isometric immersion, with  $M$  a Riemannian compact and oriented manifold with smooth boundary  $\partial M$ . Let also  $\omega$  be a  $p$ -form on  $M$ ,  $p = 1, \dots, n+1$ , satisfying  $i_{\bar{\nu}}\omega = 0$  on  $\partial M$ . Then:*

$$\int_M \{ \|\nabla\omega\|^2 + \langle T_p\omega, \omega \rangle + (p-1)(\|d\omega\|^2 + \|\delta\omega\|^2) \} \geq p \nu_{1,p-1}(M) \int_{\partial M} \|\omega\|^2.$$

*The inequality is sharp for any harmonic extension on  $B^{n+1}$  of an eigenform associated with  $\nu_{1,p}(B^{n+1})$  for  $\frac{n+3}{2} \leq p \leq n$ .*

*Proof.* Integrating (13) on  $S^{d-1}$  gives:

$$\int_{S^{d-1}} \nu_{1,p-1}(M) \int_{\partial M} \|i_V\omega\|^2 d\bar{V} \leq \int_{S^{d-1}} \int_M \|di_V\omega\|^2 + \|\delta i_V\omega\|^2 d\bar{V}.$$

Then, using lemma 1 as well as Fubini's theorem, we get theorem 3.

Concerning the sharpness, let  $\omega \in \Lambda^p(S^n)$  an eigenform of unit  $L^2$  norm associated with  $\nu_{1,p}$  and  $\hat{\omega}$  its harmonic extension on the ball. Applying theorem 3 to  $\hat{\omega}$ , one gets:

$$\int_{B^{n+1}} \{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega} \rangle + (p-1)(\|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2) \} \geq p \nu_{1,p-1}(M) \int_{S^n} \|\hat{\omega}\|^2. \quad (14)$$

Clearly, we have :

- $\int_{B^{n+1}} \|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2 = \nu_{1,p}$  by the variational characterization;
- $\langle T_p\hat{\omega}, \hat{\omega} \rangle = 0$  since  $B^{n+1}$  is open in  $\mathbb{R}^{n+1}$ ;
- $\int_{S^n} \|\hat{\omega}\|^2 = \int_{S^n} \|\omega\|^2 = 1$ .

Now, by Reilly formula in Proposition 2 applied to  $\hat{\omega}$ , and the fact that in the given range  $\nu_{1,p}(B^{n+1}) = p+1$ .

$$\int_{B^{n+1}} \|\nabla\hat{\omega}\|^2 = (p+1) - \int_{S^n} \langle S^p\omega, \omega \rangle,$$

because  $\hat{\omega}$  is tangential and  $W_p = 0$ . Since  $S^n$  is totally umbilical with constant mean curvature equals to 1, we obtain:

$$S^p\omega = p\omega.$$

Finally:  $\int_{B^{n+1}} \|\nabla\hat{\omega}\|^2 = 1$ . Next, by the results in [?]:

$$\nu_{1,p}(B^{n+1}) = \begin{cases} \frac{n+3}{n+1}p & \text{if } 1 \leq p \leq \frac{n+1}{2} \\ p+1 & \text{if } \frac{n+1}{2} \leq p \leq n. \end{cases}$$

Calculating both sides of (14) in function of  $p$ , we see that the equality is attained for  $\frac{n+3}{2} \leq p \leq n$ .  $\square$

**Theorem 4.** *Let  $M^{n+1} \rightarrow \mathbb{R}^d$  an isometric immersion with  $M$  having  $p$ -convex boundary. We also suppose that  $W_p - T_p \geq 0$  at all points of  $M$ . For all  $p = 1, \dots, n$ , one has:*

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p}. \quad (15)$$

*Equality is achieved when  $M$  is the unit Euclidean ball of  $\mathbb{R}^{n+1}$  and  $\frac{n+3}{2} \leq p \leq n$ .*

*Proof.* Let  $\omega$  an eigenform associated with  $\nu_{1,p}(M)$  (which we suppose has unit  $L^2$  norm on  $\partial M$  by normalization) and let  $\hat{\omega}$  its harmonic extension. Then,

$$\begin{aligned}\nu_{1,p}(M) &= \nu_{1,p}(M) \int_{\partial M} \|\omega\|^2 \\ &= \nu_{1,p}(M) \int_{\partial M} \|\hat{\omega}\|^2 \\ &= \int_M \|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2.\end{aligned}$$

Applying theorem 3 to  $\hat{\omega}$  leads to:

$$\begin{aligned}p \nu_{1,p-1}(M) \int_{\partial M} \|\hat{\omega}\|^2 &\leq \int_M \{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega} \rangle + (p-1) (\|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2) \} \\ &= \int_M \{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega} \rangle \} + (p-1)\nu_{1,p}(M);\end{aligned}$$

which reads as

$$p\nu_{1,p-1}(M) - (p-1)\nu_{1,p}(M) \leq \int_M \{ \|\nabla\hat{\omega}\|^2 + \langle T_p\hat{\omega}, \hat{\omega} \rangle \}.$$

But, by Reilly formula in proposition 2 and since  $\hat{\omega}$  is tangential, we have

$$\begin{aligned}\int_M \|\nabla\hat{\omega}\|^2 &= \int_M \{ \|d\hat{\omega}\|^2 + \|\delta\hat{\omega}\|^2 \} - \int_M \langle W_p\hat{\omega}, \hat{\omega} \rangle - \int_{\partial M} \langle S_p\omega, \omega \rangle \\ &= \nu_{1,p} - \int_M \langle W_p\hat{\omega}, \hat{\omega} \rangle - \int_{\partial M} \langle S_p\omega, \omega \rangle,\end{aligned}$$

hence :

$$\nu_{1,p-1}(M) \leq \nu_{1,p}(M) + \int_M \frac{1}{p} \langle (T_p - W_p)\hat{\omega}, \hat{\omega} \rangle - \frac{1}{p} \int_{\partial M} \langle S_p\omega, \omega \rangle.$$

Now, by hypothesis,  $\partial M$  is  $p$ -convex, so that  $\sigma_p \geq 0$  and since we assumed  $W_p - T_p \geq 0$ , we get:

$$\nu_{1,p}(M) \geq \nu_{1,p-1}(M) + \frac{\sigma_p}{p}.$$

Concerning the equality case, let  $M = B^{n+1}$  the unit ball and its boundary  $S^n$ . As  $B^{n+1}$  is flat and open in  $\mathbb{R}^{n+1}$ ,  $W_p = 0$  and  $T_p = 0$ , so that  $W_p - T_p = 0$  is satisfied. Moreover, the ball is convex, so it is  $p$  convex for all  $p \geq 1$ . Now, for all  $p$ , we have  $\sigma_p = p$  and

- If  $1 \leq p \leq \frac{n+1}{2}$ :

$$\begin{cases} \nu_{1,p}(B^{n+1}) = \frac{n+3}{n+1}p \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = \frac{n+3}{n+1}(p-1) + 1. \end{cases}$$

- If  $\frac{n+1}{2} \leq p \leq n$  :

$$- \text{ if } \frac{n-1}{2} \leq p-1 \leq \frac{n+1}{2},$$

$$\begin{cases} \nu_{1,p}(B^{n+1}) = p + 1 \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = \frac{n+3}{n+1}(p-1) + 1; \end{cases}$$

$$\begin{aligned}
& - \text{ if } \frac{n+1}{2} \leq p-1 \leq n-1, \\
& \quad \begin{cases} \nu_{1,p}(B^{n+1}) = p+1 \\ \nu_{1,p-1}(B^{n+1}) + \frac{\sigma_p}{p} = p+1. \end{cases}
\end{aligned}$$

And we see as announced that the equality is obtained for  $\frac{n+3}{2} \leq p \leq n$ .  $\square$

We apply theorem 4 to Euclidean domains, as done in [?] (see Theorem 4).

**Corollary 5.** (1) *Assume that the Euclidean domain  $M$  is  $p$ -convex for some  $p = 1, \dots, n-1$ . Then:*

$$\nu_{1,p-1}(M) \leq \nu_{1,p-1}(M) \leq \dots \leq \nu_{1,n}(M). \quad (16)$$

(2) *If  $M$  is indeed convex, then the sequence  $\{\nu_{1,p}(M)\}$  is non-decreasing with respect to the degree  $p$  :*

$$\nu_{1,0}(M) \leq \nu_{1,1}(M) \leq \dots \leq \nu_{1,n}(M). \quad (17)$$

(3) *If  $M$  is strictly  $p$ -convex, so that  $\sigma_p > 0$ , then  $\nu_{1,q}(M) < \nu_{1,q+1}(M)$  for all  $q \geq p$ . Thus, these inequalities in (1) and (2) are strict when starting from  $\nu_{1,p}(M)$  and  $\nu_{1,1}(M)$  respectively.*

*Proof.* Since  $M$  is an Euclidean domain of  $\mathbb{R}^{n+2}$  (so that  $T_p = W_p = 0$ ), we get by theorem 4 :

$$\nu_{1,p} \geq \nu_{1,p-1}. \quad (18)$$

As a  $p$ -convex domain is  $q$ -convex for all  $q \in [p, n]$ , (1) follows. Then (2) and (3) are consequences of (1).  $\square$

**Corollary 6.** *Let  $M$  be a  $p$ -convex spherical domain, isometrically immersed in  $\mathbb{R}^{n+2}$ . Then, for all  $p \leq \frac{n+1}{2}$ , we obtain an estimation of the  $p$ -gap :*

$$\nu_{1,p} - \nu_{1,p-1} \geq 0. \quad (19)$$

*Proof.* We have  $T_p = p^2 \cdot Id$  and  $W_p = p(n+1-p) \cdot Id$  so that :

$$W_p - T_p \geq 0 \iff p \leq \frac{n+1}{2}.$$

So, the hypothesis of theorem 4 are satisfied for all  $p \leq \frac{n+1}{2}$ . Moreover, in this case, we have for all  $p$ ,  $\sigma_p \geq 0$ . Applying theorem 4, the claim follows.  $\square$

**Remark 3.** *Note that  $M = S_+^{n+1}$ , the upper hemisphere, is a particular case of corollary 6 for  $p \leq \frac{n+1}{2}$ . Indeed, its boundary is the equator so that  $\sigma_p = 0$ . Thus, for all  $p \leq \frac{n+1}{2}$ , we get an estimation of the  $p$ -gap of the hemisphere:*

$$\nu_{1,p}(S_+^{n+1}) - \nu_{1,p-1}(S_+^{n+1}) \geq 0. \quad (20)$$

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