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# A BASIS FOR THE KAUFFMAN SKEIN MODULE OF THE PRODUCT OF A SURFACE AND A CIRCLE

RENAUD DETCHERRY AND MAXIME WOLFF

ABSTRACT. The Kauffman bracket skein module  $\mathcal{S}(M)$  of a 3-manifold  $M$  is a  $\mathbb{Q}(A)$ -vector space spanned by links in  $M$  modulo the so-called Kauffman relations. In this article, for any closed oriented surface  $\Sigma$  we provide an explicit spanning family for the skein modules  $\mathcal{S}(\Sigma \times S^1)$ . Combined with earlier work of Gilmer and Masbaum [6], we answer their question about the dimension of  $\mathcal{S}(\Sigma \times S^1)$  being  $2^{2g+1} + 2g - 1$ .

## 1. INTRODUCTION

The Kauffman bracket [10] skein module is an invariant of compact oriented 3-manifolds. It was first introduced independently by Przytycki [16] and Turaev [17] as a way to generalize the Jones polynomial of links in  $S^3$ . It can be thought as a module over any ring  $R$  containing an invertible element  $A \in R$ . The skein module  $\mathcal{S}(M, R)$  with coefficients in  $R$  is the  $R$ -module:

$$\mathcal{S}(M, R) = \text{Span}_R(L \subset M \text{ framed link}) /_{\text{isotopy}, K_1, K_2}$$

spanned by isotopy classes of framed links in  $M$ , modulo the two (local) Kauffman relations  $K_1$  and  $K_2$  :

$$\begin{aligned} K1: & \quad \text{Diagram 1} = A \text{ Diagram 2} + A^{-1} \text{ Diagram 3} \\ K2: & \quad L \cup \text{Diagram 4} = (-A^2 - A^{-2})L \end{aligned}$$

where the above relations identify linear combinations of framed links that are identical except in a small ball in  $M$ . In the significant case where  $R = \mathbb{Q}(A)$  is the field of rational functions in the variable  $A$ , we will write  $\mathcal{S}(M)$  for  $\mathcal{S}(M, \mathbb{Q}(A))$  for simplicity.

The skein modules of 3-manifolds have long been mysterious and notoriously hard to compute, and for a long time, very little was known about the structure of skein modules of general 3-manifolds. They were partially or completely computed for an increasing number of closed 3-manifolds ( $S^3$  and lens spaces by Hoste and Przytycki [8][9], integer Dehn surgeries on the trefoil by Bullock [1], some prism manifolds by Mroczkowski [15], the quaternionic manifold by Gilmer and Harris [5], the 3-torus by Carrega [2] and Gilmer [4], and some infinite family of hyperbolic manifolds by the first author [3]). Finally, Witten conjectured that  $\mathcal{S}(M)$  has finite dimension for every closed 3-manifold (see [6, Section 8] for a discussion). Recently, Gunningham, Jordan and Safronov posted a general proof [7] of Witten’s finiteness conjecture. Their proof, which relies on factorization algebras, the theory of  $DQ$ -modules and some careful quantization of character varieties/character stacks, is powerful and generalizes to other types of skein modules than the Kauffman bracket skein module.

However, the proof in [7] is rather non-constructive; in particular, it is still hard for a given 3-manifold  $M$  to compute the dimension  $\dim_{\mathbb{Q}(A)}(\mathcal{S}(M))$  or to find a family of framed links that is a basis of  $\mathcal{S}(M)$ . It would be rather interesting to give some general interpretation

of the dimensions  $\dim_{\mathbb{Q}(A)}(\mathcal{S}(M))$ . Another aspect of skein modules that remains unclear is the integral structure of skein modules of closed 3-manifolds. Indeed, a more precise statement, concerning the “integral” version of the skein module  $\mathcal{S}(M)$ , that is, the skein module  $\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}])$  with coefficients in  $\mathbb{Z}[A^{\pm 1}]$ , has been conjectured by Julien Marché. For  $n \in \mathbb{Z}$  let  $\{n\} = A^n - A^{-n}$ .

**Conjecture 1.1.** *Let  $M$  be a closed compact oriented 3-manifold. There exists an integer  $d \geq 0$  and finitely generated  $\mathbb{Z}[A^{\pm 1}]$ -modules  $N_k$  so that*

$$\mathcal{S}(M, \mathbb{Z}[A^{\pm 1}]) = \mathbb{Z}[A^{\pm 1}]^d \oplus \bigoplus_{k \geq 1} N_k$$

where, furthermore, the module  $N_k$  is a  $\{k\}$ -torsion module.

The two authors heard of this conjecture from Julien Marché by private communication. To the best of our knowledge, this integral version of the finiteness conjecture is not implied by Gunningham, Jordan and Safronov’s results. The integral structure of skein modules is also of interest because of its interaction with quantum invariants, as we will explain later in this introduction.

In the whole article, we focus on a single family of manifolds: products  $\Sigma \times S^1$  of a compact closed oriented surface  $\Sigma$  of genus  $g \geq 2$  and a circle. (Although, we believe that our techniques can be employed for other circle bundles over a closed surface). We note that skein elements in  $\Sigma \times S^1$  admit some particularly nice diagrammatic representations as so-called *arrowed diagrams*. Indeed, we can always put a link  $L \subset \Sigma \times S^1$  in general position with respect to the projection  $\Sigma \times S^1 \rightarrow \Sigma$ , and besides the over/under-crossing information, we only need to remember where the link  $L$  intersects  $\Sigma \times \{0\}$ . We remember this information by putting an arrow on the diagram at such intersections; the direction of the arrow gives the direction in which  $L$  is rising. This notation was first introduced by Dabkowski and Mroczkowski in [14], where it was used to compute the skein module of  $\Sigma_{0,3} \times S^1$ .

With this notation, let us introduce a family  $\mathcal{B}$  of elements of  $\mathcal{S}(\Sigma \times S^1)$  consisting of the following diagrams. For each non-zero homology class of  $H_1(\Sigma, \mathbb{Z}/2)$ , we choose an oriented, simple closed curve representing it, and consider the diagram consisting of this curve with no arrows, as well as the diagram consisting of this representant with 1 arrow. Finally, we consider a trivial curve on  $\Sigma$ , with 0 to  $2g$  arrows. Then, our main result is the following.

**Theorem 1.2.** *The family  $\mathcal{B}$  is a basis of the skein module  $\mathcal{S}(\Sigma \times S^1)$ . In particular,  $\mathcal{S}(\Sigma \times S^1)$  has dimension  $2^{2g+1} + 2g - 1$ .*

Our proof is completely elementary; it uses only skein relations and direct computation. It is independent of [7].

The skein modules of the manifolds  $\Sigma \times S^1$  were previously studied by Gilmer and Masbaum [6]. They introduced a general tool to bound below the dimension of skein modules. Given a closed compact oriented 3-manifold  $M$ , Gilmer and Masbaum defined a linear map

$$\text{ev}: \mathcal{S}(M) \rightarrow \mathbb{C}_{ae}^{\mathcal{U}}.$$

In the above,  $\mathcal{U}$  denotes the set of roots of unity of even order, and  $\mathbb{C}_{ae}^{\mathcal{U}}$  is the  $\mathbb{Q}(A)$ -vector space of functions of the variable  $A \in \mathcal{U}$  that are defined almost everywhere. For example, any function  $F(A) \in \mathbb{Q}(A)$  is an element of  $\mathbb{C}_{ae}^{\mathcal{U}}$  as it is defined as long as  $A$  is not a root of the denominator of  $F$ . Moreover, for any  $k \in \mathbb{Z}$ , there is a map  $p^k \in \mathbb{C}_{ae}^{\mathcal{U}}$  which maps any primitive root of unity of order  $2p$  to  $p^k$ . The map  $\text{ev}$  is defined using the Reshetikhin-Turaev invariants of links in  $M$ : given a  $2p$ -th root of unity  $A$  where  $p \geq 3$ , and a link  $K \subset M$ , there

is a well-defined topological invariant  $RT(M, K, A) \in \mathbb{C}$ . The invariant can be computed by choosing any surgery presentation  $L$  for  $M$ , and computing a colored Kauffman bracket of  $L \cup K$ , where  $L$  has been colored by some special polynomial  $\omega_p \in \mathbb{Z}[A^{\pm 1}][z]$  called the Kirby color. We will not give a complete definition of  $RT(M, K, A)$ , and just refer to [6] for details. The important point to us is that, with this definition, the invariant  $RT(M, K, A)$  naturally satisfies the Kauffman relations with respect to  $K$ . Thus it is possible to extend it  $\mathbb{Q}(A)$ -linearly from the set of framed links to any element of  $\mathcal{S}(M)$ , as those elements are  $\mathbb{Q}(A)$  linear combinations of links in  $M$ . The only caveat is that we may have to exclude some values of  $A \in \mathcal{U}$  if they are in denominators of the coefficients of the linear combination. Using the map  $\text{ev}$  in the special case where  $M = \Sigma \times S^1$ , Gilmer and Masbaum showed that the dimension of its image is at least  $2^{2g+1} + 2g - 1$ , and thus so is the dimension of  $\mathcal{S}(\Sigma \times S^1)$ . They conjectured that this is actually the dimension of  $\mathcal{S}(\Sigma \times S^1)$ . Theorem 1.2 answers positively to their conjecture; our contribution is to prove that the family  $\mathcal{B}$ , of cardinal  $2^{2g+1} + 2g - 1$ , generates  $\mathcal{S}(\Sigma \times S^1)$ . Our proof is constructive: given a link  $L$  in  $\Sigma \times S^1$ , it is possible to algorithmically decompose it as a linear combination of elements of  $\mathcal{B}$ .

For simplicity we stated Theorem 1.2 with the skein module with coefficients in  $\mathbb{Q}(A)$ , but in fact we only need  $\{k\} = A^k - A^{-k}$  to be invertible, for all  $k \neq 0$ . In particular, a by-product of our proof is that torsion elements of the integral skein module  $\mathcal{S}(\Sigma \times S^1, \mathbb{Z}[A^{\pm 1}])$  are always of  $\{k\}$ -torsion for some  $k \geq 1$ , in conformity with Conjecture 1.1.

For any manifold  $M$ , the skein module  $\mathcal{S}(M)$  has a natural  $H_1(M, \mathbb{Z}/2)$ -grading, as the Kauffman relations always involve links in  $M$  that have the same homology class in  $H_1(M, \mathbb{Z}/2)$ . Thanks to the basis we computed, we can answer some other questions raised in [6] about the structure of quantum invariants of links in  $\Sigma \times S^1$  :

**Corollary 1.3.** *For any  $z \in \mathcal{S}(\Sigma \times S^1)$ , the image of  $z$  by Gilmer-Masbaum's evaluation map is of the form*

$$\text{ev}(z) = \sum_{i \in I} R_i(A) p^i$$

for some rational functions  $R_i(A) \in \mathbb{Q}(A)$ , and where  $I = \{g - 1, g + 1, \dots, 3g - 3\} \cup \{g\}$ . Moreover, the Gilmer-Masbaum map is injective on each graded subspace of  $\mathcal{S}(\Sigma \times S^1)$ .

The corollary results from the fact that Gilmer and Masbaum showed that it is the case for elements that are linear combinations of non-separating simple closed curves with 0 or 1 arrows, and the trivial curves with 0 to  $2g$  arrows. In particular, the rational functions  $R_i(A)$ , as linear combinations of the coefficients in the basis  $\mathcal{B}$ , are algorithmically computable invariants of links in  $\Sigma \times S^1$ , that satisfy the Kauffman relations.

Let us stress that Theorem 1.2 does not imply Conjecture 1.1 for  $M = \Sigma \times S^1$ ; one would need to prove in particular that the  $R_i(A)$  have bounded denominators. This is related to the work of Marché and Santharoubane [13], who defined a Jones like polynomial invariant for links  $L$  in  $\Sigma \times S^1$ , considering the highest order of the asymptotics of the quantum invariants  $RT_p(\Sigma \times S^1, L)$ . Their invariant, which has value in  $\mathbb{Z}[A^{\pm 1}]$  is closely related to the invariant  $R_{3g-3}(A)$ . As our method is algorithmic, we computed the coefficients in the basis  $\mathcal{B}$  of a few arrowed diagrams. These coefficients have interesting integral properties; they seem to be in  $\mathbb{Z}[A^{\pm 1}]$  instead of  $\mathbb{Q}(A)$ , which corroborates Conjecture 1.1.

We may also hope to find more direct formulas for these coefficients: this may produce an alternative proof of the linear independence of  $\mathcal{B}$  as it would then suffice to prove invariance of these coefficients by Reidemeister moves. We hope to explore further these coefficients, which may be thought of as Jones-like polynomial invariants for links in  $\Sigma \times S^1$ , in a later work.

The article is organized as follows. In Section 2, we introduce the notion of arrowed diagrams and the elementary moves they satisfy. In Section 3.1, we introduce several important relations that we will use throughout the paper. In the remainder of Section 3, we define a notion of degree on the set of arrowed multicurves, and by expressing multicurves as linear combinations of multicurves of smaller degree, we prove that  $\mathcal{S}(\Sigma \times S^1)$  is spanned by arrowed trivial curves and arrowed non-separating curves. In Section 4, we show that one only needs up to 1 arrow on non-separating curves, and up to  $2g$  arrows on the trivial curve to span  $\mathcal{S}(\Sigma \times S^1)$ . Finally, in Section 5.2, we introduce an equivalence relation on the set of non-separating curves that is motivated by relations in the skein module. We compute the equivalence classes of this relation, and deduce that non-separating curves (with same number of arrows) that represent the same element in  $H_1(\Sigma, \mathbb{Z}/2)$  also represent the same element of  $\mathcal{S}(\Sigma \times S^1)$ , concluding the proof of Theorem 1.2.

*Acknowledgements.* The first author was supported by the Max Planck Institute for Mathematics during this research, and thanks the institute for its hospitality. The second author learned about this problem, and about TQFT more generally, from Julien Marché while he was writing his introductory text [12]. We are also both grateful to Gregor Masbaum for his constant interest in this work.

## 2. ARROWED DIAGRAMS, COMPLEXITY AND TRIVIAL COMPONENTS

In this section, we introduce the notion of arrowed multicurves and arrowed diagrams on  $\Sigma$ . While the definition of elements in the skein module  $\mathcal{S}(M)$  of a 3-manifold  $M$  is very much 3-dimensional, this notion which will give us a purely 2-dimensional way of thinking of elements of  $\mathcal{S}(\Sigma \times S^1)$ . Arrowed diagrams were first introduced by Dabkowski and Mroczkowski in [14], where they were used to compute the skein module of  $\Sigma_{0,3} \times S^1$ .

Let us view the  $S^1$  factor of  $\Sigma \times S^1$  as  $[0, 1]_{/0 \sim 1}$ , and let  $L$  be a link in  $\Sigma \times S^1$ . By a general position argument, up to isotopy  $L$  can be assumed to be transverse to  $\Sigma \times \{0\}$ , to have no vertical tangent and, furthermore, the image of  $L$  by the projection  $\Sigma \times S^1 \rightarrow \Sigma$  can be assumed to only have a finite number of double points with transverse intersection. The diagram of the projection, together with the choice of upper and lower strand at each double point/crossing, is almost enough to determine  $L$  up to isotopy. The only missing information is when does  $L$  cross the level  $\Sigma \times \{0\}$ . Thus we add an arrow on the projection at each point of the projection coming from an intersection point  $L \cap (\Sigma \times \{0\})$ . Moreover, we choose the direction of the arrow to be the direction in which  $L$  crosses  $\Sigma \times \{0\}$  positively. Conversely, any arrowed diagram gives rise to a link in  $\Sigma \times S^1$  in an obvious way. Because the Kauffman bracket skein module deals with framed links, we would like to put a canonical framing on each arrowed diagram. We do so by choosing the parallel of  $L$  to be the push-off of  $L$  along the positive direction of  $S^1$ . With this convention, any framed link  $L$  still has an arrowed diagram, as we can always correct the framing by adding curls to the components of the diagram.

Dabkowski and Mroczkowski gave a complete set of moves describing isotopy of framed links in  $\Sigma \times S^1$  :

**Proposition 2.1.** [14] *Two arrowed diagrams of framed links in  $\Sigma \times S^1$  correspond to isotopic links if and only if they are related by standard Reidemeister moves  $R'_1, R_2, R_3$  and the moves:*

$$(R_4) \begin{array}{c} \uparrow \\ | \\ \downarrow \end{array} \sim \begin{array}{c} | \\ | \\ \uparrow \end{array} \quad (R_5) \begin{array}{c} | \\ \sim \\ \uparrow \end{array} \sim \begin{array}{c} \uparrow \\ | \\ \sim \end{array}$$

The relations  $R_4$  and  $R_5$  imply the commutation relation: if  $\gamma \cdot \delta$  is the link obtained by stacking the diagram  $\gamma$  on top of  $\delta$ , then  $\gamma \cdot \delta = \delta \cdot \gamma$ . It is easy to see directly that those two links are isotopic in  $\Sigma \times S^1$ .

By relation  $R_4$ , we note if a strand of an arrowed diagram has some number of arrows (maybe with different directions) in succession, only the algebraic number of arrows matters. It will often be useful to write an arrow indexed by  $n \in \mathbb{Z}$  to denote  $n$  successive arrows all pointing to the direction of the arrow. If  $n$  is negative, it has to be understood as  $|n|$  arrows pointing in the opposite direction.

In all this article, by an *arrowed multicurve*, or *multicurve* for short, we mean an arrowed diagram without double points. In other words, this is a diagram whose projection on  $\Sigma$  is a 1-dimensional submanifold of  $\Sigma$ . Each of its components may be homotopically trivial, or essential, and we count these curves to form the degree and complexity of the diagram.

**Definition 2.2.** If  $\gamma$  is an arrowed multicurve and  $n$  (resp.  $m$ ) are its number of essential, non-separating (resp. non-trivial separating) simple closed curve components, then we define

$$\text{deg}(\gamma) = n + 2m.$$

We also define the complexity of  $\gamma$  as  $(\text{deg}(\gamma), n + m)$ . Complexities are ordered using the lexicographic order.

Notice that the above notion of degree does not depend on the number of arrows that decorate each component of  $\gamma$ .

Note that in the definition of the degree and complexity, we do not count the trivial components of  $\gamma$ . This is because we can essentially get rid of them, as follows.

**Proposition 2.3.** *For  $\Sigma$  a closed compact oriented surface, the skein module  $\mathcal{S}(\Sigma \times S^1)$  is spanned by arrowed multicurves containing no trivial component, and by the arrowed multicurves consisting of just the trivial curve with any number of arrows.*

*Moreover, every arrowed multicurve is a linear combination of arrowed multicurves as above and with same degree and complexity.*

*Proof.* It is clear that repeatedly applying Kauffman relations  $K_1$  to the crossings of an arrowed diagram will express it as a  $\mathbb{Q}(A)$ -linear combination of arrowed multicurves. Thus the main point of Proposition 2.3 is its second assertion: we can eliminate a trivial component (with arrows) if the multicurve has at least one other component, without changing its degree or complexity.

Let us consider an arrowed multicurve  $\gamma$  containing a trivial curve and another component. If the trivial curve contains no arrow then the Kauffman relation  $K_2$  gets rid of it. Otherwise, we use the relation:

$$\begin{array}{c}
 \begin{array}{c} \downarrow \\ \curvearrowright \\ \uparrow \end{array} \begin{array}{c} n \\ \rightarrow \end{array} = \begin{array}{c} \downarrow \\ \curvearrowright \\ \downarrow \end{array} \begin{array}{c} n - 1 \\ \rightarrow \end{array} \\
 \\
 A \begin{array}{c} \uparrow \\ n \\ \uparrow \end{array} + A^{-1} \left| \begin{array}{c} \circlearrowright \\ n \end{array} \right. = A^{-1} \begin{array}{c} \uparrow \\ n - 2 \\ \uparrow \end{array} + A \left| \begin{array}{c} \downarrow \\ \circlearrowright \\ n - 1 \end{array} \right.
 \end{array}$$

In the above, the left strand stands for a strand of another component of the starting multicurve (which is the second term of the left hand side of the second equality).

By using this relation, we can express  $\gamma$  as a linear combination of arrowed multicurves where the number of arrows on a trivial component has increased, or decreased: this number of arrows can therefore be pushed to 0, and then the Kauffman relation  $K_2$  gets rid of that trivial component. We then proceed inductively to erase the trivial components, until there is either no trivial component left, or just one trivial component and no other component.  $\square$

### 3. REDUCING THE DEGREE OF MULTICURVES

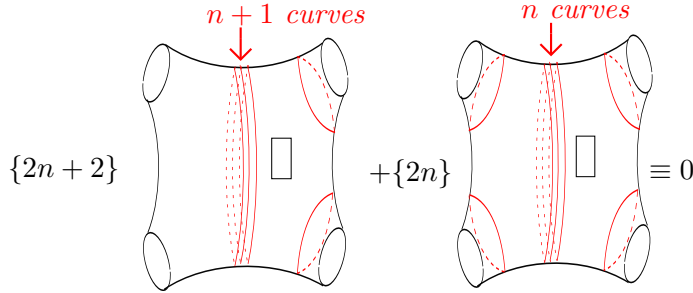
The main result of this long section will be the following.

**Proposition 3.1.** *The skein module  $\mathcal{S}(\Sigma \times S^1)$  is spanned by arrowed multicurves of degree  $\leq 1$ . That is  $\mathcal{S}(\Sigma \times S^1)$  is spanned by arrowed non-separating curves and by arrowed trivial curves.*

In the next subsection, we first introduce a few helpful relations that relate a few multicurves of the same degree, up to lower degree terms. We will use them to prove Proposition 3.1 inductively, showing that any arrowed multicurve of degree  $n \geq 2$  is a linear combination of arrowed multicurves with smaller degree.

**3.1. The sphere and the torus relation.** Our first relation relates multicurves that sit on a 5-holed sphere subsurface of  $\Sigma$  :

**Proposition 3.2.** *For any  $n \geq 1$ , we have the sphere relation  $(S_n)$  between multicurves of degree  $2n + 6$  :*



In the above figure, the two boundary components on the left are non-separating curves of the ambient surface  $\Sigma$ , while all the other red curves are essential, separating curves of  $\Sigma$ . The black square may be homotopically trivial in  $\Sigma$ , or not. Finally,  $\equiv$  is equality up to a linear combination of multicurves with degrees  $\leq 2n + 5$ .

We note that the multicurves above are indeed of degree  $2n+6$  : as the middle  $n$  components and the two rightmost ones are separating they contribute  $2n + 4$  to the degree, and for each side one adds either a separating curve of two non-separating curves, adding a total of 2 more to the degree. We will sometimes refer to the relations  $(S_n)$  as the *sphere relations*.

*Proof.* We prove Proposition 3.2 by considering the resolutions of the diagram shown in Figure 1. In this diagram, the 5-holed sphere is drawn as a four-holed disk, with the boundary of the disk and the leftmost hole corresponding to the two boundary components on the left of Relation  $(S_n)$ . We have to consider two slightly different patterns depending whether  $n$  is odd or even. The diagram shows two simple closed curves  $\gamma$  and  $\delta$ , with  $\gamma$  (in black) standing on top of  $\delta$  (in green). We will produce Relation  $(S_n)$  from the commutation relation  $\gamma \cdot \delta = \delta \cdot \gamma$ , after a careful study of the multicurves that appear after resolving the crossings using Kauffman relations. We recall that in  $\gamma \cdot \delta$  and  $\delta \cdot \gamma$  the exact same resolutions appear, with coefficients changed by replacing  $A$  by  $A^{-1}$ . With this in head, it is sufficient

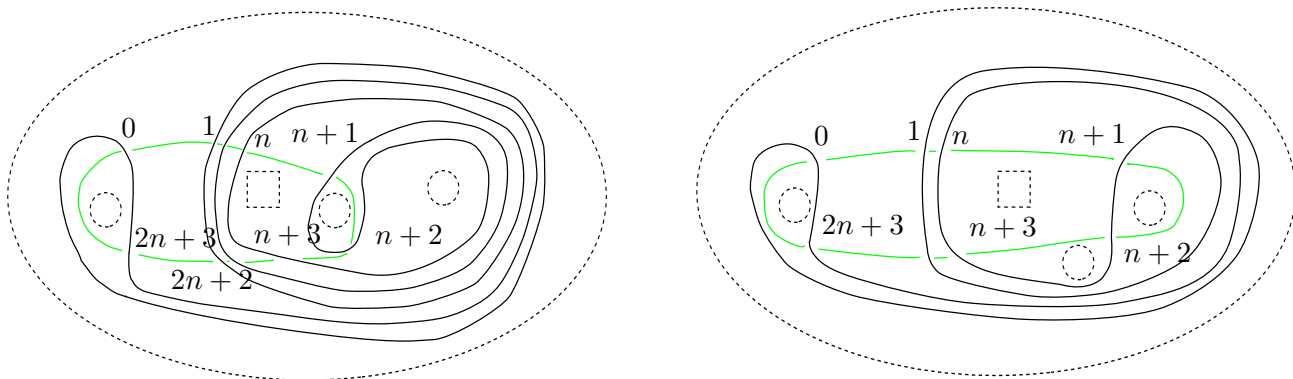


FIGURE 1. The black curve  $\gamma$  stands atop the green curve  $\delta$  inside a 4-holed disk. The two rightmost boundary components are the rightmost boundary components in Proposition 3.2. There are  $n \geq 1$  strands of  $\gamma$  going in between the two squares, and the left (resp. right) diagram correspond to whether  $n$  is odd or even. In each diagram, we labeled the crossings from 0 to  $2n + 3$ , following the green curve.

to study the resolutions of  $\gamma \cdot \delta$ . There are  $2n + 4$  crossings between  $\gamma$  and  $\delta$ , and thus  $2^{2n+4}$  resolutions. Our claim is that among all those resolutions, there are only 2 of maximal degree, and that this maximal degree is  $2n + 6$ .

Notice that the  $2n + 4$  crossings cut the black and green curves  $\gamma$  and  $\delta$  into  $2n + 4$  black arcs and  $2n + 4$  green arcs. Moreover, any component of any resolution consists of several green and black arcs, alternating between green and black: any such component is composed of an even number of arcs. We will focus mainly on the green arcs, and denote them by  $(i, i + 1)$  with  $0 \leq i \leq 2n + 3$ , with cyclic notation, as suggested in Figure 1.

Given a resolution  $\lambda$  of  $\gamma \cdot \delta$  and a green arc  $a$ , consider the connected component  $\lambda'$  of  $\lambda$  containing  $a$ . We will say that the *contribution of  $a$  to the degree of  $\lambda$*  equals  $\frac{p(a)}{q(a)}$ , where  $q(a)$  is the number of green arcs in  $\lambda'$ , and where  $p(a)$  equals 0 if  $\lambda'$  is a non essential curve of  $\Sigma$ , 1 if it is a non-separating simple closed curve, and 2 if it is a non-trivial separating curve. This way,  $\deg(\lambda)$  is the sum of the contributions of its green arcs.

Let us bound, individually, the contribution each green arc can have to the degree of a resolution of  $\gamma \cdot \delta$ . The arcs  $(2n + 3, 0)$ ,  $(0, 1)$ ,  $(n + 1, n + 2)$  and  $(n + 2, n + 3)$  are the only ones that can form a closed curve containing no other green arcs: for the two first ones that curve is non-separating, for the two others it is separating. Hence the maximal contributions they can give to the degree are respectively 1, 1, 2 and 2. For any other green arc  $a$ , we will have  $p(a) \leq 2$  and  $q(a) \geq 2$ , hence the contribution  $a$  cannot exceed 1. By summing up all these contributions, we deduce that for all resolution  $\lambda$  of  $\gamma \cdot \delta$  we have  $\deg(\lambda) \leq 2n + 6$ .

Finally, let us examine in which resolutions of  $\gamma \cdot \delta$  the degree  $2n + 6$  can indeed be reached. The contributions of both green arcs  $(n + 1, n + 2)$  and  $(n + 2, n + 3)$  need to equal 2. For this, these green arcs have to be matched to one black arc to make a separating curve in  $\Sigma$ . Thus, the crossings  $n + 1$ ,  $n + 2$  and  $n + 3$  need to receive the resolutions  $-$ ,  $+$  and  $-$  respectively. Now in order to reach maximal contribution to the degree, the green arc  $(n, n + 1)$  has to be paired with another green arc, and two black arcs, to form a separating curve of  $\Sigma$ . The only possibility for that is to be paired with the green arc  $(n + 3, n + 4)$ , and the crossings numbered  $n$  and  $n + 3$  have to receive both the resolution  $-$ . The same argument with the arc  $(n - 1, n)$  implies it is paired with the arc  $(n + 4, n + 5)$  and the crossings  $n - 1$  and  $n + 4$

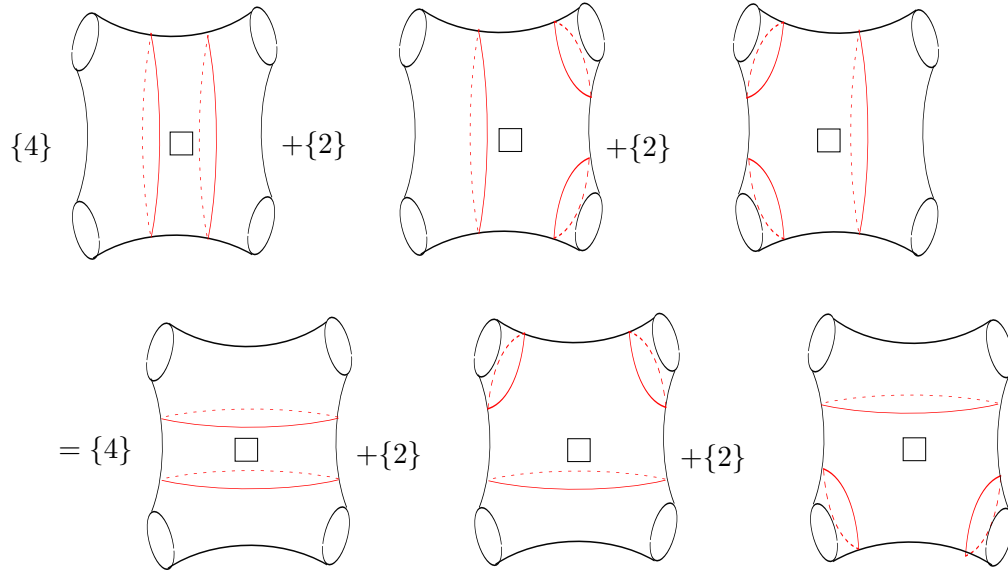


have the resolution  $-$ . We continue further left, until the arc  $(1, 2)$  which is paired with the arc  $(2n + 2, 2n + 3)$  (it can also be paired with the arc  $(2n + 3, 0)$  to form a closed curve, but that curve is trivial in  $\Sigma$ ). In conclusion, all crossings except  $n + 2$  and maybe  $0$ , have to receive the resolution  $-$ .

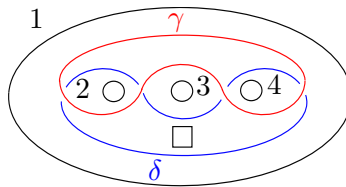
Depending on the resolution of the crossing labeled  $0$ , we have two possible diagrams of maximal degree. The one for which the resolution of  $0$  is  $-$  yields first diagram of the relation  $(S_n)$ . It has  $2g - 3$  signs  $-$  and one sign  $+$  hence it comes with coefficient  $A^{-2g+2}$  in  $\gamma \cdot \delta$ , and with coefficient  $A^{2g-2}$  in  $\delta \cdot \gamma$ . The positive resolution of  $0$  yields the second diagram, with coefficient  $A^{-2g}$  in  $\gamma \cdot \delta$  and  $A^{2g}$  in  $\delta \cdot \gamma$ . The equality  $\gamma \cdot \delta = \delta \cdot \gamma$  in the skein module thus proves the formula  $(S_n)$ .  $\square$

We defined the sphere relation  $(S_n)$  for any  $n \geq 1$ . We will need another relation which we will call  $(S_0)$ . It is again a relation between multicurves on a 5-holed sphere subsurface, although it has a slightly different form.

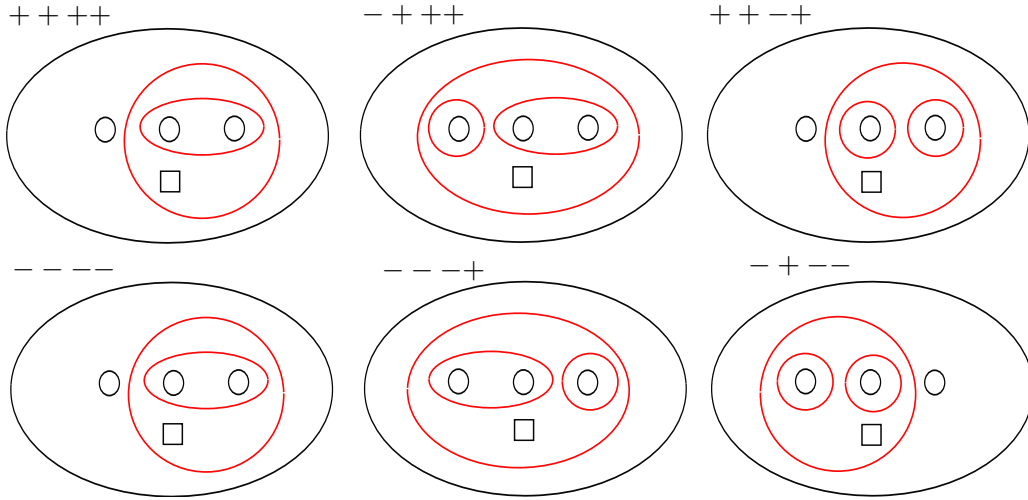
**Proposition 3.3.** *In  $\mathcal{S}(\Sigma \times S^1)$ , we have the following relation between multicurves that coincide except in a 5-holed sphere:*



*Proof.* We get the relation looking at resolutions of  $\gamma \cdot \delta$ , where  $\gamma$  and  $\delta$  are the following simple closed curves:



In the above, we numbered some boundary components from 1 to 4, where 1 and 2 correspond to the two leftmost boundary components in Proposition 3.3 and 3 and 4 to the two rightmost components. We get the sphere relation  $(S_0)$  from the equality  $\gamma \cdot \delta = \delta \cdot \gamma$ . Note that resolutions with an even number of positive resolutions at crossings will appear with the same coefficient 1 on both sides, so we need to analyse the other resolutions only. Let us order the crossings from left to right, the following sums up the different resolutions:



with the remaining odd resolutions  $+ - + +$ ,  $+ + + -$ ,  $- - + -$  and  $+ - - -$  all yielding trivial curves. Thus collecting all terms of the equality  $\gamma \cdot \delta = \delta \cdot \gamma$  we indeed get the relation  $(S_0)$ .  $\square$

We now establish another useful relation which we will call the Torus relation. This is a relation between multicurves that sit on a torus (with several holes) subsurface of  $\Sigma$ . We have:

**Proposition 3.4.** *Let  $\gamma$  be an arrowed multicurve. We suppose that there exists a simple curve  $\delta$  which intersects exactly once  $n \geq 3$  components of  $\gamma$  and which is disjoint from all other curves of  $\gamma$ . Then, up to multicurves of lower degree,  $\gamma$  is a linear combination of arrowed multicurves obtained from  $\gamma$  by replacing any number of pairs of consecutive components of  $\gamma$  along  $\delta$  by their connected sum along the arc of  $\delta$  connecting them, provided the resulting curves are non-trivial and separating.*

Note that in the torus relation all multicurves are of degree  $n$  : all components of  $\gamma$  are non-separating as they intersect  $\delta$  once, and the other multicurves have the same degree as we always replace two non-separating curves with one separating one. We also remark that any multicurve that satisfy the hypothesis of Proposition 3.4 is a linear combination of multicurves with smaller complexity. If moreover no pair of consecutive components bounds a subsurface  $\Sigma' \subset \Sigma$  with genus  $\geq 1$ , then  $\gamma$  is actually a linear combination of multicurves of smaller degree, as those connected sums are all either trivial curves or non-separating curves.

*Proof.* We prove Proposition 3.4 using a similar method as for Proposition 3.2. Let us introduce the curve  $\gamma'$  which is the  $1/n$  fractional Dehn twist of  $\gamma$  along  $\delta$ . The curves  $\gamma'$  and  $\delta$  are represented on Figure 2. We will deduce the relation from the equality  $\gamma' \cdot \delta = \delta \cdot \gamma'$ . This time, the diagram shown in Figure 2 has  $n$  crossings, thus  $n$  green arcs and  $n$  black arcs. The crossings are labeled 0 to  $n - 1$ , and considered as elements of  $\mathbb{Z}_n$ . Notice that the black arcs connect crossings  $i$  to crossing  $i + 1$ . We will find the bigons and squares of the diagram, and show that no bigon corresponds to a separating curve, which will imply that the maximal degree is at most  $n$ . As there are both green and black arcs connecting  $i$  to  $i + 1$ , we get exactly  $n$  bigons, and each green arc belongs to exactly 1 bigon. Notice that here it is important that  $n \geq 3$  : otherwise there are actually two black arcs that close up the green arc  $(0, 1)$ , as  $1 + 1 = 0 \pmod 2$ . Each of those bigons corresponds to a vertical curve in the square representing the one-holed torus: that is, the bigons correspond to parallel non-separating curves.

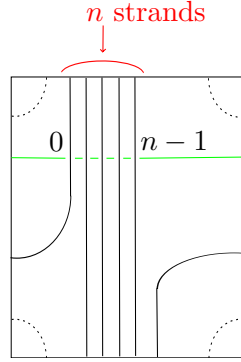


FIGURE 2. The black curve  $\gamma'$  stands atop the green curve  $\delta$ . The dashed quarter-circles correspond to the boundary component of the torus. We labeled the crossings from 0 to  $n-1$ , from left to right. The surface is represented as a torus using the square model, but regions of the diagram may actually contain some genus, or be connected by handles

Let us now search for the squares  $(i, i + 1, j, k)$ , where  $(i, i + 1)$  is a green arc. Given the form of black arcs, they must be of the form  $(i, i + 1, i + 2, i + 1)$ . We note that those squares all correspond the curves that are connected sums of consecutive components of  $\gamma$  along an arc of  $\delta$ . In a maximal degree resolution, all green arcs must belong to either a bigon (which has to be a non-separating curve as we saw) or a square that is a separating curve. So the different resolutions are exactly  $\gamma$  and the different possible multicurves described in the first part of Proposition 3.4. Thus the equality  $\gamma' \cdot \delta = \delta \cdot \gamma'$  gives exactly the equation of Proposition 3.4.  $\square$

Next we introduce a relation between arrowed multicurves in a two-holed torus subsurface of  $\Sigma$ . Those relations will involve multicurves with at most 2 components (in the subsurface), which are not covered by Proposition 3.4.

**Proposition 3.5.** *We have the relations:*

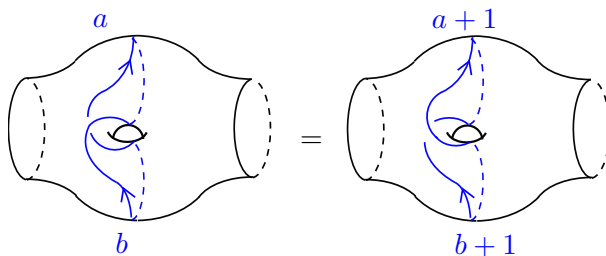
$$A^{-1} \left( \text{torus with arcs } a, b \right) - A \left( \text{torus with arcs } a+1, b+1 \right) = -A \left( \text{torus with arc } a+b \right) + A^{-1} \left( \text{torus with arc } a+b+2 \right)$$

and

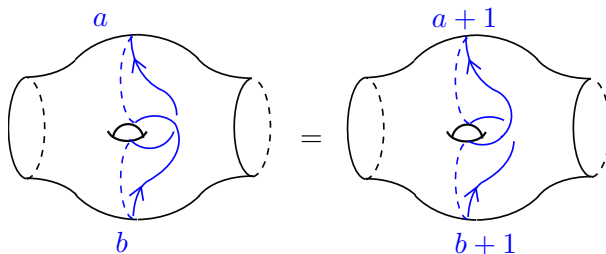
$$A \left( \text{torus with arcs } a, b \right) - A^{-1} \left( \text{torus with arcs } a+1, b+1 \right) = A \left( \text{torus with arc } a+b+2 \right) - A^{-1} \left( \text{torus with arc } a+b \right)$$

*In particular, the multicurves on the left hand side of two equations are linear combinations of multicurves of smaller complexity.*

*Proof.* From Reidemeister moves  $R_5$ , we have:



After resolving the crossings, this gives the first equation. Similarly, the second equation is a consequence of the relation:



Finally, let us call  $D_{a,b}$ ,  $D_{a+1,b+1}$  the two multicurves on the left hand of the two equations, let  $\equiv$  be equality modulo multicurves of smaller complexity. Note that  $D_{a,b}$  is composed of two non-separating curves, so it has degree 2. The multicurves on the right-hand side have only one component, thus have degree at most 2 and smaller complexity than  $D_{a,b}$ . Thus we have  $A^{-1}D_{a,b} - AD_{a+1,b+1} \equiv 0$  and  $AD_{a,b} - A^{-1}D_{a+1,b+1} \equiv 0$ , which implies that  $\{2\}D_{a,b} \equiv 0$  and  $\{2\}D_{a+1,b+1} \equiv 0$ . So the multicurves on the left hand side are linear combinations of multicurves of smaller complexity.  $\square$

**3.2. The dual graph of an arrowed multicurve.** Given an (arrowed) multicurve in  $\Sigma$  we define its dual tree as follows:

**Definition 3.6.** Let  $\gamma$  be a multicurve in  $\Sigma$ . Let  $c$  be the multicurve that consists of copy of each distinct homotopy classes of non-trivial separating curves among components of  $\gamma$ , and let  $V$  be the set of connected components of  $\Sigma \setminus c$ . Then the graph  $\Gamma$  dual to  $\gamma$  has one vertex for each element of  $V$  and one edge for each component of  $c$ , connecting the two connected components of  $\Sigma \setminus c$  that it bounds. For each vertex  $v \in V$ , we associate the corresponding connected component  $\Sigma(v) \subset \Sigma$ . We let  $g(v)$  denote its genus, and by  $\gamma \cap \Sigma(v)$  we mean the arrowed diagram of  $\Sigma(v)$  consisting of the non-separating connected components of  $\gamma$  lying inside  $\Sigma(v)$ .

Note that the graph  $\Gamma$  is actually a tree as any edge of  $\Gamma$  is disconnecting.

For  $\gamma \in \mathcal{S}(\Sigma \times S^1)$  a multicurve, we say that  $\gamma$  is *stable* if it is not a linear combination of multicurves of smaller complexity.

**Lemma 3.7.** *Let  $\gamma$  be a stable multicurve and let  $v$  be a vertex of the dual graph of  $\gamma$ . Then  $\gamma \cap \Sigma(v)$  consists of either 0 or 1 non-separating curve.*

*Proof.* Let  $\gamma'$  be the multicurve  $\gamma \cap \Sigma(v)$ , which consists only of non-separating curves. By the Torus relation, if we can find a curve  $\delta$  which intersects at least 3 components of  $\gamma'$  exactly once, then  $\gamma$  is a linear combination of multicurves of smaller complexity, hence  $\gamma$  is not stable. Also, if we can find a curve  $\delta$  which intersects exactly two components of  $\gamma'$  once,

then a neighborhood of the union of these three curves is a two-holed torus. Proposition 3.5 then asserts that  $\gamma$  is not stable.

Thus, we just need to prove that such a curve  $\delta$  exists provided  $\gamma'$  contains at least two non-separating curves. For this, consider the dual graph  $G(v)$  to  $\gamma'$  in the usual sense: its vertices are the connected components of  $\Sigma(v) \setminus \gamma'$  and each component of  $\gamma'$  yields one edge of  $G(v)$ . If  $G(v)$  has at least two vertices then we can find an embedded loop in  $G(v)$ , which can be followed to define a curve  $\delta$  as above. Similarly, if  $G(v)$  has only one vertex, but at least two loops, then the two corresponding components of  $\gamma'$  are non-separating, and mutually non-separating in  $\Sigma(v)$ , hence we can find a curve  $\delta$  intersecting just these two curves once: provided  $G(v)$  has at least two edges, such a curve  $\delta$  exists.  $\square$

**Proposition 3.8.** *Let  $\gamma$  be a stable multicurve on  $\Sigma$ . Then the dual graph of  $\gamma$  is linear.*

*Proof.* Let  $\gamma$  be a stable multicurve and  $\Gamma$  be its dual graph. Let us note that by definition of dual graphs, if a vertex  $v$  of  $\Gamma$  has valency  $\leq 2$  then its genus  $g(v)$  is at least 1. Hence, it is then sufficient to prove that if two vertices  $v$  and  $v'$  are connected by an edge and if  $g(v') \geq 1$ , then the valency of  $v$  is at most 2. Indeed, assuming this claim, starting at the leaves of the tree  $\Gamma$  which have genus  $\geq 1$ , then neighboring vertices also have valency  $\leq 2$  and thus genus  $\geq 1$ . One can proceed inductively to prove that  $\Gamma$  is linear.

Thus let us assume that  $\Gamma$  has two connected vertices  $v$  and  $v'$  such that  $g(v) \geq 1$  and suppose the valency of  $v'$  is at least 2. A neighborhood of the connected components of  $\gamma$  corresponding to the edge  $(v, v')$  looks like the second term of the sphere relation of Proposition 3.2, with  $n \geq 1$  curves in the middle.

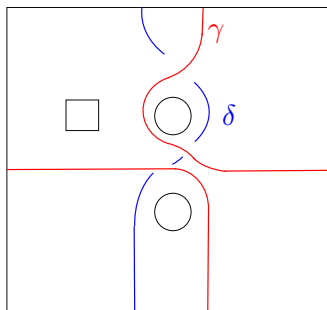
If  $n \geq 2$ , we have apply the sphere relation  $(S_{n-1})$ , reducing the value of  $n$  while adding copies of the two leftmost boundary components. Doing so we actually increase complexity, but we keep the same degree. We inductively reduce the value of  $n$  until we hit  $n = 1$ . Finally we apply the relation  $(S_0)$  of Proposition 3.3, which now expresses the multicurve we got in terms of multicurves of smaller degree. We deduce that the multicurve  $\gamma$  was not stable.  $\square$

We end this section with a lemma that refines Lemma 3.7 for vertices of valency 2 :

**Lemma 3.9.** *Let  $\gamma$  be a stable multicurve and  $v$  be a vertex of its dual graph  $\Gamma$  of valency 2. Then  $\gamma \cap \Sigma(v) = \emptyset$ .*

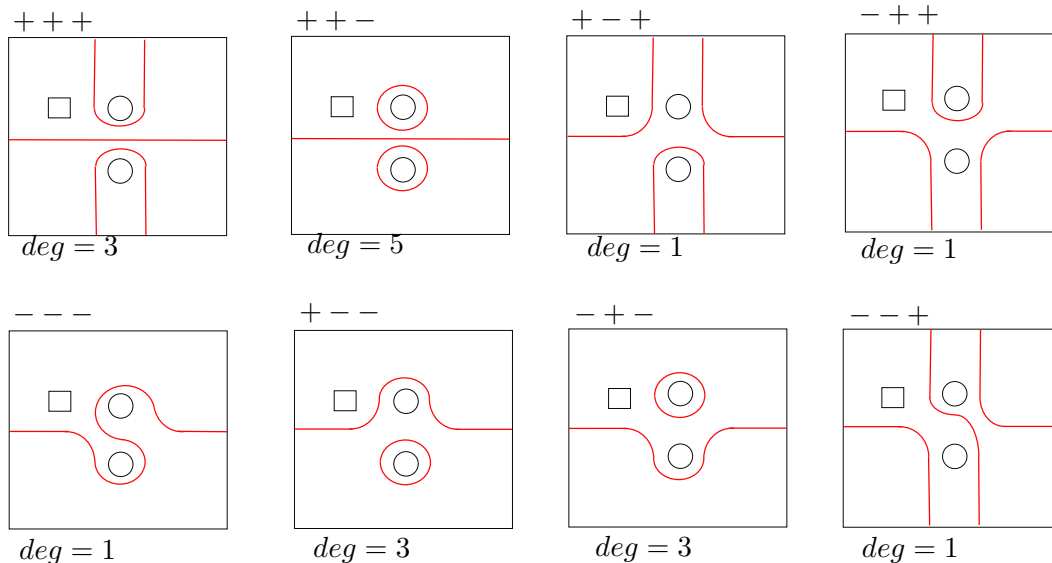
*Proof.* Because of Lemma 3.7, we know that  $\gamma \cap \Sigma(v)$  is either empty or a single non-separating curve. The subsurface  $\Sigma(v)$  has genus  $g \geq 1$  and 2 boundary components. We will provide a relation to show that a multicurve in  $\Sigma(v)$  consisting of the two boundary components and one non-separating curve is a linear combination of multicurves of smaller degree.

Consider two curves  $\gamma$  and  $\delta$  in a surface  $\Sigma_{g,2}$  as follows:



In the above, the two circles are the two boundary components of  $\Sigma_{g,2}$ , and the opposite sides of the big square are identified. Moreover, we attach  $g - 1$  handles to the little square, so that the surface indeed has genus  $g$ .

Let us order the crossings from bottom to top. We claim that the resolution  $++-$  consists of the two boundary components plus one non-separating curve, which gives total degree 5, and that all other resolutions have degree at most 3. The different resolutions are summed up in the following diagrams:



As the resolution  $++-$  appears in  $\gamma \cdot \delta - \delta \cdot \gamma$  with coefficient  $\{1\} \neq 0$ , that multicurve is a linear combination of multicurves with smaller degree, which is what we wanted.  $\square$

**3.3. Sausage decompositions of surfaces.** Thanks to Proposition 3.8 and Lemma 3.7, the skein module of  $\Sigma \times S^1$  is generated by arrowed multicurves which are put on  $\Sigma$  in a kind of standard form, that fits well with a special kind of pair of pants decomposition of  $\Sigma$  which we will call a *sausage-decomposition* of  $\Sigma$ . We define such a decomposition below:

**Definition 3.10.** Let  $\Sigma$  be an oriented compact closed surface of genus  $g$ . A sausage-decomposed subsurface  $\Sigma' \subset \Sigma$  is the data of a subsurface of  $\Sigma$  with 2 to 4 boundary components together with a pair of pants decomposition of the type described in Figure 3 with pair of pants being ordered from left to right. Moreover, a sausage decomposition of  $\Sigma$  is the data of a sausage subsurface of  $\Sigma$  composed of  $2g - 2$  pair of pants, and with 2 boundary components that each bound a disk in  $\Sigma$ .

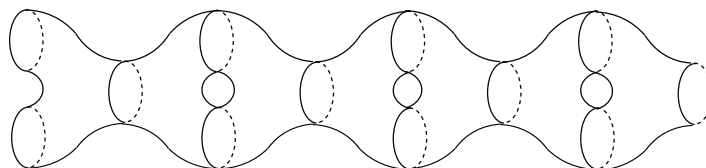
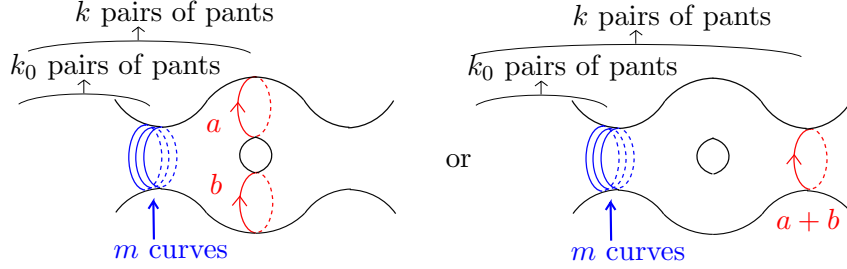


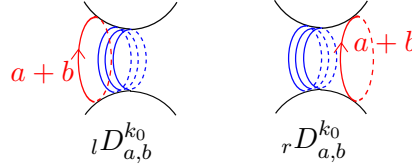
FIGURE 3. A sausage-decomposed subsurface of  $\Sigma$

Let us remark that in a sausage decomposition of  $\Sigma$  there is a well-defined left (and right) boundary component.

Let us fix a sausage-decomposed subsurface  $\Sigma' \subset \Sigma$ , containing  $N = |\chi(\Sigma')|$  pairs of pants. We fix an integer  $m \geq 0$  and  $k_0 \in \{1, \dots, N - 1\}$  so that the subsurface which is the union of the first  $k_0$  pairs of pants of  $\Sigma'$  has a single boundary component to its right. Let  $a, b \in \mathbb{Z}$  and  $k \in \{0, \dots, N\} \setminus \{k_0\}$ , we write  $D_{a,b}^k$  for the diagram (depending on the parity of  $k$ ):



When  $k = k_0$ , we would like to define a  $D_{a,b}^k$  similarly, except we need to specify the relative position of the blue and red curves. Thus we get two versions  ${}_l D_{a,b}^k$  and  ${}_r D_{a,b}^k$ , where the red curve is put respectively to the left and to the right of the  $m$  blue curves.



All those diagrams define elements of  $\mathcal{S}(\Sigma \times S^1)$ , which depend also on  $m$  and the sausage-decomposed subsurface  $\Sigma'$ , but for simplicity we omit those dependence from the notations.

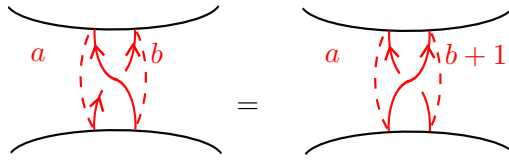
It is obvious that if  $m = 0$  then  ${}_l D_{a,b}^{k_0} = {}_r D_{a,b}^{k_0}$ . There is a more general relation between those two diagrams which we describe in the following lemma:

**Lemma 3.11.** *For any  $a, b \in \mathbb{Z}$  and  $m \geq 0$  we have*

$${}_l D_{a,b}^{k_0} \equiv A^{2m(a+b)} {}_r D_{a,b}^{k_0},$$

*modulo diagrams of smaller degree.*

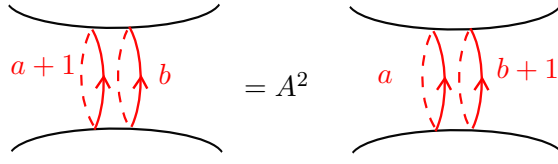
*Proof.* For any  $a, b \in \mathbb{Z}$  we have:



Thus:

$$A^{-1} \begin{array}{c} \text{---} \\ \uparrow \uparrow \\ a+1 \quad b \\ \downarrow \downarrow \\ \text{---} \end{array} + A \begin{array}{c} \text{---} \\ \circlearrowleft a-b+1 \\ \text{---} \end{array} = A \begin{array}{c} \text{---} \\ \uparrow \uparrow \\ a \quad b+1 \\ \downarrow \downarrow \\ \text{---} \end{array} + A^{-1} \begin{array}{c} \text{---} \\ \circlearrowleft a-b+1 \\ \text{---} \end{array}$$

so that:



Thus we can push any arrow on any curve to the curve immediately to its right, at the expense of multiplying by  $A^2$  each time. To push all arrows from the leftmost to the rightmost of the  $m+1$  curves, we multiply by  $A^{2m(a+b)}$ .  $\square$

The next proposition says that we can push multicurves  $D_{a,b}^k$  outward to the boundary of the subsurface  $\Sigma'$ .

**Proposition 3.12.** *Let  $V^{\partial\Sigma'}$  be the subspace of  $\mathcal{S}(\Sigma \times S^1)$  spanned by the elements  $D_{a,b}^0$  and  $D_{a,b}^N$ . Then, up to diagrams of smaller degree, for any  $a, b \in \mathbb{Z}$ , we have  ${}_l D_{a,b}^{k_0}, {}_r D_{a,b}^{k_0} \in V^{\partial\Sigma'}$ .*

This proposition rests upon the following relations between the  $D_{a,b}^k$ :

**Lemma 3.13.** *Let  $k \in \{0, \dots, N-1\}$  and  $a, b \in \mathbb{Z}$ .*

- *If  $k \notin \{k_0 - 1, k_0\}$  then*

$$AD_{a,b}^k - A^{-1}D_{a+1,b+1}^k = AD_{a+1,b+1}^{k+1} - A^{-1}D_{a,b}^{k+1}.$$

- *If  $k = k_0$  then*

$$A {}_r D_{a,b}^k - A^{-1} {}_r D_{a+1,b+1}^k = AD_{a+1,b+1}^{k+1} - A^{-1}D_{a,b}^{k+1}.$$

- *If  $k = k_0 - 1$  then*

$$AD_{a,b}^k - A^{-1}D_{a+1,b+1}^k = A {}_l D_{a+1,b+1}^{k+1} - A^{-1} {}_l D_{a,b}^{k+1}.$$

*Proof.* Those equations are direct applications of the first or second two-holed torus relation of Proposition 3.5. We apply them in the  $k+1$ -th pair of pants of the decomposition of  $\Sigma'$ , which may oriented to the right or to the left depending on the parity of  $k$ . The cases where  $k = k_0 - 1$  or  $k_0$  work the same as the others, as we can always keep the extra curves away.  $\square$

Let  $V$  be the  $\mathbb{Q}(A)$  vector space formally spanned by elements  $D_{a,b}^k$  (and elements  ${}_l D_{a,b}^{k_0}, {}_r D_{a,b}^{k_0}$ ) for  $a, b \in \mathbb{Z}$  and  $k \in \{0, \dots, N\}$ . We will by a slight abuse of notation, sometimes consider elements of  $V$  as elements of  $\mathcal{S}(\Sigma \times S^1)$  that might be thus subject to relations. We define on  $V$  a shift operator  $s: V \rightarrow V$  by  $s(D_{a,b}^k) = D_{a+1,b+1}^k$ . It should be kept in mind that this operator  $s$  acts on *the space of diagrams  $V$ , before quotienting by the skein relations*: it does not act on elements of the skein module. Let also  $A: V \rightarrow V$  be the multiplication operator by  $A$  and consider

$$\Delta_+ = As - A^{-1}, \quad \Delta_- = -A^{-1}s + A, \quad \text{and} \quad \Delta_{+,m} = A^{2m+1}s - A^{-1}.$$

We note that  $\Delta_{+,0} = \Delta_+$ .

**Lemma 3.14.** *For any  $a, b \in \mathbb{Z}$  we have  $\Delta_+^{k_0} {}_l D_{a,b}^{k_0} = 0$  and  $\Delta_-^{N-k_0} {}_r D_{a,b}^{N-k_0} = 0$ .*

*Proof.* By Lemma 3.13, we have  $\Delta_+ D_{a,b}^k = \Delta_- D_{a,b}^{k+1}$ , provided we do not run into the extra  $m$  curves. As the operators  $\Delta_+$  and  $\Delta_-$  commute, we get that  $\Delta_+^2 D_{a,b}^k = \Delta_- D_{a,b}^{k+2}$  and so on, as long as we do not collide with the  $m$  extra curves. In the end we get  $\Delta_+^{k_0} {}_l D_{a,b}^{k_0} = \Delta_-^{k_0} D_{a,b}^0 \in V^{\partial\Sigma'}$ , and  $\Delta_-^{N-k_0} {}_r D_{a,b}^{k_0} = \Delta_+ D_{a,b}^N \in V^{\partial\Sigma'}$ .  $\square$



We note after factoring in the relations of Lemma 3.11, up to smaller degree terms, the action of  $s$  on the elements  ${}_l D_{a,b}^{k_0}$  is like the action of  $A^{2m(a+b)} \circ s \circ A^{-2m(a+b)}$  on the  ${}_r D_{a,b}^{k_0}$ . So that, up to smaller degree terms,

$$s({}_l D_{a,b}^{k_0}) = {}_l D_{a+1,b+1}^{k_0} \equiv A^{2m(a+b+2)} {}_r D_{a+1,b+1}^{k_0} = A^{4m} s({}_r D_{a,b}^{k_0}).$$

Therefore,  $\Delta_+({}_l D_{a,b}^{k_0}) = \Delta_{+,m}({}_r D_{a,b}^{k_0})$  and  $\Delta_+^{k_0}({}_l D_{a,b}^{k_0}) = \Delta_{+,m}^{k_0}({}_r D_{a,b}^{k_0})$ . In the end, modulo diagrams of smaller degree, we have:

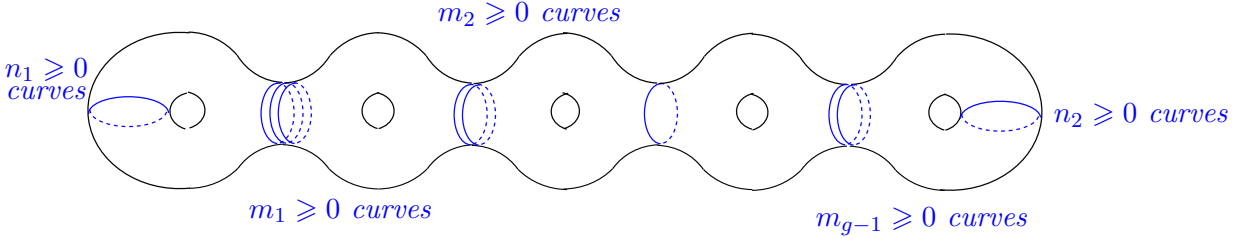
$$\begin{cases} \Delta_{+,m}^{k_0} {}_r D_{a,b}^{k_0} \in V^{\partial\Sigma'} \\ \Delta_-^{N-k_0} {}_r D_{a,b}^{k_0} \in V^{\partial\Sigma'}. \end{cases}$$

*Proof of Proposition 3.12.* We will write  $D \in V^{\partial\Sigma'}$  for a given diagram  $D$  to mean that  $D \in V^{\partial\Sigma'}$  up to smaller degree diagrams. By Lemma 3.11, we only want to show that  ${}_r D_{a,b}^{k_0} \in V^{\partial\Sigma'}$ . We have  $\Delta_{+,m}^{k_0}({}_r D_{a,b}^{k_0}) \in V^{\partial\Sigma'}$  and  $\Delta_-^{N-k_0}({}_r D_{a,b}^{k_0}) \in V^{\partial\Sigma'}$ . We note that  $\Delta_{+,m} = A^{4m+1}s - A^{-1}$  and  $\Delta_- = -A^{-1}s + A$  commute, and that

$$\text{id}_V = \frac{1}{A^{4m+2} - A^{-2}} (A^{-1}\Delta_{+,m} + A^{4m+1}\Delta_-).$$

We conclude that  ${}_r D_{a,b}^{k_0} = \text{id}_V^N({}_r D_{a,b}^{k_0}) = \frac{1}{(A^{4m+2} - A^{-2})^N} (A^{-1}\Delta_{+,m} + A^{4m+1}\Delta_-)^N ({}_r D_{a,b}^{k_0}) \in V^{\partial\Sigma'}$ , as, after expanding, any term will contain either  $\Delta_{+,m}^{k_0}$  or  $\Delta_-^{N-k_0}$ .  $\square$

**Corollary 3.15.** *Fix a sausage decomposition of  $\Sigma$ . Any (arrowed) multicurve of the form*



is actually a linear combination of curves of the same type with all  $m_i = 0$ , and multicurves of smaller degree. In the above, arrows may be added in an arbitrary way.

It should be noted that by applying Corollary 3.15 we may actually increase the complexity, as the separating curves are replaced by non-separating ones, twice in number.

*Proof.* Corollary 3.15 results of applying Proposition 3.12 many times, pushing the groups of separating curves out step by step.  $\square$

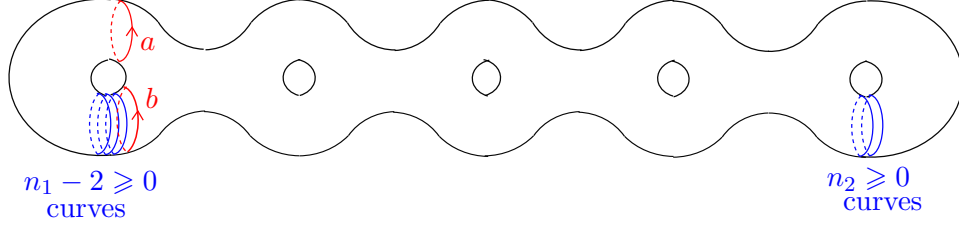
From the results of Section 3.2, the skein module  $\mathcal{S}(\Sigma \times S^1)$  is spanned by multicurves of the type described in Corollary 3.15. This may be used to strengthen Corollary 3.15 to the following proposition, which is the main result of the present section.

**Proposition 3.16.** *The skein module  $\mathcal{S}(\Sigma \times S^1)$  is spanned by arrowed trivial curves and arrowed non-separating simple closed curves.*

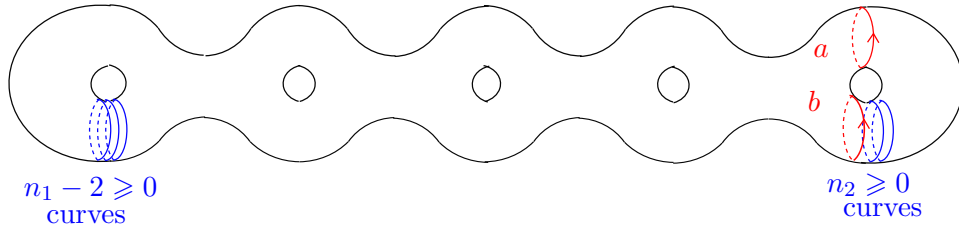
*Proof.* Thanks to Corollary 3.15, we will only need to show to multicurves of the type described in the corollary and with  $m_i = 0$  and  $n_1 + n_2 \geq 2$  are linear combinations of multicurves of smaller degree.

Let us first assume that  $n_1$  or  $n_2$  is at least 2. Without loss of generality, let us assume that  $n_1 \geq 2$ . Slightly adapting our previous notations, we define arrowed multicurves  $D_{a,b}^k$

compatible with the sausage decomposition of  $\Sigma$ , that have  $n_1 - 2$  extra curves on the left and  $n_2$  on the right. For example the multicurve we considered is of the form  $D_{a,b}^1$  and looks like:



By comparison, the multicurve  $D_{a,b}^{2g-1}$  would look like:



We will use the relations of Lemma 3.13 to relate the multicurves  $D_{a,b}^k$  for  $k \in \{0, \dots, N\}$ , while always leaving the “extra” blue curves unchanged. Note that we can always (up to multicurves of smaller degree) push all arrows from the blue curves to the red curves. Let  ${}_l D_{a,b}^1$  denote the arrowed multicurve where the red curve in  $D_{a,b}^1$  is put to the left of the blue curves. By Lemmas 3.13 and 3.11, we have  ${}_l D_{a,b}^1 \equiv A^{2(n_1-2)b} D_{a,b}^1$  and  $\Delta_+ {}_l D_{a,b}^1 \equiv 0$ . The same computation as before thus gives  $\Delta_{+,n_1-2} D_{a,b}^1 \equiv 0$ , where  $\Delta_{+,n_1-2} = A^{2n_1-3}s - A^{-1}$ .

Similarly, if  ${}_r D_{a,b}^{2g-1}$  is the diagram obtained from the diagram  $D_{a,b}^{2g-1}$  by putting the red arrowed curve to the right of the  $n_2$  blue curves, we would have  $\Delta_- {}_r D_{a,b}^{2g-1} \equiv 0$  and  ${}_r D_{a,b}^{2g-1} \equiv A^{-2n_2b} D_{a,b}^{2g-1}$ . Thus  $\Delta_{-,n_2} D_{a,b}^{2g-1} \equiv 0$ , where  $\Delta_{-,n_2} = -A^{1-2n_2}s + A$ .

Thanks to Lemma 3.13, we have:

$$\begin{cases} \Delta_{+,n_1-2} D_{a,b}^1 \equiv 0 \\ \Delta_{-,n_2} \Delta_-^{2g-2} D_{a,b}^1 \equiv \Delta_{-,n_2} \Delta_+^{2g-2} D_{a,b}^{2g-1} \equiv 0. \end{cases}$$

Let us note that  $\Delta_-$ ,  $\Delta_{+,n_1-2}$  and  $\Delta_{-,n_2}$  all commute. Moreover, let us note that

$$\text{id}_V = \frac{1}{A^{2n_1-2} - A^{-2-2n_2}} (A^{-1-2n_2} \Delta_{+,n_1-2} + A^{2n_1-3} \Delta_{-,n_2})$$

and

$$\text{id}_V = \frac{1}{A^{2n_1-2} - A^{-2}} (A^{-1} \Delta_{+,n_1-2} + A^{2n_1-3} \Delta_-).$$

When expanding the expression

$$\begin{aligned} \text{id}_V &= \frac{1}{(A^{2n_1-2} - A^{-2-2n_2})(A^{2n_1-2} - A^{-2})^{2g-2}} (A^{-1-2n_2} \Delta_{+,n_1-2} + A^{2n_1-3} \Delta_{-,n_2}) \\ &\quad \circ (A^{-1} \Delta_{+,n_1-2} + A^{2n_1-3} \Delta_-)^{2g-2}, \end{aligned}$$

any term will contain either a factor  $\Delta_{+,n_1-2}$  or a factor  $\Delta_{-,n_2}\Delta_-^{2g-2}$ . Applying  $\text{id}_V$  to  $D_{a,b}^1$ , we conclude that  $D_{a,b}^1 \equiv 0$ . This shows that as long as  $n_1 \geq 2$ , the multicurve above is a linear combination of arrowed multicurves of smaller degree.

Finally, in the remaining case where  $n_1 = n_2 = 1$ , we can fit the multicurve on a two-holed torus subsurface, so that the two boundary components of the two-holed torus are non-separating in  $\Sigma$ . Proposition 3.5 then shows that the multicurve with  $n_1 = n_2 = 1$  is a linear combination of non-separating simple closed curves (the two boundary components of the two-holed torus).  $\square$

#### 4. ELIMINATION OF ARROWS

By the previous section, the skein module  $\mathcal{S}(\Sigma \times S^1)$  is spanned by all arrowed multicurves whose underlying multicurve is either a non-separating curve or a trivial curve. We will now study the ‘‘vertical’’ part of those curves, that is, relate elements of  $\mathcal{S}(\Sigma \times S^1)$  that differ only by the number of arrows we put on them. We will treat the cases of non-separating curves and of the trivial curve separately.

##### 4.1. Arrows on non-separating curves. We have:

**Proposition 4.1.** *Let  $\gamma$  be a non-separating simple closed curve, with some choice of orientation, and for  $n \in \mathbb{Z}$  let  $\gamma_n$  be the arrowed curve  $\gamma$  with  $n$  arrows in the direction of  $\gamma$ . Then, for any  $n \in \mathbb{Z}$ , we have  $\gamma_n = \gamma_{n-2} \in \mathcal{S}(\Sigma \times S^1)$ .*

Based on the above proposition, to span  $\mathcal{S}(\Sigma \times S^1)$ , it is sufficient to consider non-separating curves with 0 or 1 arrow. Moreover, for non curves with 1 arrow the direction of the arrow can be chosen arbitrarily.

*Proof.* We begin by observing that by using relation  $R_5$  of Section 2 we have:

The diagram shows two curves separated by an equals sign. The left curve is a simple closed curve with  $n$  arrows pointing in the counter-clockwise direction. The right curve is a simple closed curve with  $n-1$  arrows pointing in the counter-clockwise direction.

Using Kauffman relations  $K_1$  and  $K_2$ , this gives the *arrow-shift relation*:

The diagram shows a vertical line with a circle attached to its right side. The vertical line has  $n$  arrows pointing upwards. This is equal to  $-A^2$  times a vertical line with  $n+1$  arrows pointing upwards, minus  $A^4$  times a vertical line with  $n-1$  arrows pointing upwards.

Hence, adding one trivial curve with one arrow in the direct orientation, to the diagram  $\gamma_n$ , yields the linear combination  $-A^2\gamma_{n+1} - A^4\gamma_{n-1}$ .

On the other hand, the equality

The diagram shows two curves separated by an equals sign. The left curve is a simple closed curve with  $n$  arrows pointing in the counter-clockwise direction. The right curve is a simple closed curve with  $n+1$  arrows pointing in the counter-clockwise direction.

gives, this time, that the same diagram obtained by adding one trivial curve with one arrow in the direct orientation to  $\gamma_n$  equals  $-A^2\gamma_{n-1} - A^4\gamma_{n+1}$ . The equality between these two expressions is equivalent to  $\{2\}(\gamma_{n+1} - \gamma_{n-1}) = 0$ . This proves the proposition.  $\square$

**4.2. Arrows on the trivial curve.** We now turn to the case of the trivial curve. To set things up, for  $n \in \mathbb{Z}$  let the arrowed curve  $S_n$  be the trivial curve with  $n$  arrows in the positive direction. If we fix a sausage decomposition of  $\Sigma$ , then  $S_n$  also corresponds to the curve  $D_{n,0}^0$  defined in Section 3.3. It also corresponds to the curve  $D_{-n,0}^{2g}$ . We introduce a last operator  $\theta$  on the vector space  $V$  formally spanned by the  $S_n$  by  $\theta(S_n) = S_{-n}$ . This operator will be treated similarly as we treated the shift operator  $s$  (which we recall is defined by  $s(S_n) = S_{n+1}$ ), and the operators  $\Delta_+ = As - A^{-1}$  and  $\Delta_- = -A^{-1}s + A$  in Section 3.3: they are only defined as linear operators on  $V$ , but we will use them to write relations in  $\mathcal{S}(\Sigma \times S^1)$ , as in Section 3.3.

In this context, as there are no ‘‘extra’’ curves that act as barriers here, the relations given by Lemma 3.13 simply read

$$\Delta_- D_{a,b}^k = \Delta_+ D_{a,b}^{k+1}$$

for any  $a, b \in \mathbb{Z}$  and any  $0 \leq k \leq 2g - 3$ . Thanks to these relations, we can show that the arrowed curves  $S_n$  satisfy the following system of relations:

**Proposition 4.2.** *The multicurves  $S_n \in \mathcal{S}(\Sigma \times S^1)$  satisfy:*

- (1)  $\forall n \geq 1, \quad A^{-n-2}S_n - A^{n+2}S_{-n} \in \text{Span}_{\mathbb{Q}(A)}(S_0, \dots, S_{n-1})$ .
- (2)  $\Delta_-^{2g}(S_n) = (\Delta_+^{2g} \circ \theta)(S_n)$ .

*Proof.* The second point of the proposition results from our remarks above. Indeed, from Lemma 3.13, we get for any  $n \in \mathbb{Z}$

$$\Delta_+^{2g} D_{n,0}^{2g} = \Delta_- D_{n,0}^0$$

which recalling that  $D_{n,0}^0 = S_n$  and  $D_{n,0}^{2g} = S_{-n}$  gives exactly point (2) of the proposition.

Thus we just need to prove point (1). Recall the arrow-shift relation obtained in the preceding section:

$$\begin{array}{c} \uparrow^n \\ \circlearrowleft \end{array} = -A^2 \begin{array}{c} \uparrow^{n+1} \end{array} - A^4 \begin{array}{c} \uparrow^{n-1} \end{array}$$

Applying this relation to the diagram  $\circlearrowleft \circlearrowleft$  will give us that  $A^{-2}S_1 = A^4S_{-1}$ , which is the  $n = 1$  case of (1). We proceed to prove (1) by induction on  $n$ .

Assume that (1) has been established for some  $n$ . Using the arrow-shift relation, we have:

$$\begin{aligned} A^{-n-2} \begin{array}{c} \uparrow^n \\ \circlearrowleft \end{array} \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} - A^{n+2} \begin{array}{c} \uparrow^{-n} \\ \circlearrowleft \end{array} \begin{array}{c} \uparrow \\ \circlearrowleft \end{array} \\ = -A^{-n}S_{n+1} - A^{-n+2}S_{n-1} + A^{n+4}S_{-n+1} + A^{n+6}S_{-n-1} \end{aligned}$$

By induction hypothesis and the arrow-shift relation, the left hand side is in  $\text{Span}_{\mathbb{Q}(A)}(S_0, \dots, S_n)$ . Thus  $A^{-n}S_{n+1} - A^{n+6}S_{-n-1} \in \text{Span}_{\mathbb{Q}(A)}(S_0, \dots, S_n)$ , and by induction, (1) holds for all  $n \geq 1$ .  $\square$

We will use this system to prove the following proposition, that shows that the subspace of  $\mathcal{S}(\Sigma \times S^1)$  spanned by arrowed trivial curves is finite dimensional:

**Proposition 4.3.** *The subspace of  $\mathcal{S}(\Sigma \times S^1)$  spanned by the arrowed curves  $(S_n)_{n \in \mathbb{Z}}$ , is actually spanned by the curves  $S_n$  for  $n = 0, \dots, 2g$ .*

*Proof.* Let us take a closer look at the equation

$$\Delta_-^{2g}(S_n) = (\Delta_+^{2g} \circ \theta)(S_n).$$

We expand both expressions, using that  $\Delta_- = -A^{-1}s + A$  and  $\Delta_+ = As - A^{-1}$ . We get:

$$\sum_{k=0}^{2g} \binom{n}{k} (-1)^k A^{2g-2k} S_{n+k} = \sum_{k=0}^{2g} \binom{n}{k} (-1)^k A^{2k-2g} S_{-n-k}.$$

Let us now assume  $n \geq 1$ . Extracting the terms in  $S_l$  with  $|l|$  maximal from both sides, we get  $A^{-2g}S_{n+2g} \equiv A^{2g}S_{-n-2g}$  modulo  $\text{Span}(S_{-n-2g+1}, \dots, S_{n+2g-1})$ . Now if  $\equiv$  is equality modulo  $\text{Span}(S_{-n-2g+1}, \dots, S_{n+2g-1})$ , using Equation (1) of Proposition 4.2, we get the invertible system

$$\begin{cases} A^{-2g}S_{n+2g} - A^{2g}S_{-n-2g} \equiv 0 \\ A^{n+2g+2}S_{n+2g} - A^{-n-2g-2}S_{-n-2g} \equiv 0 \end{cases}$$

which, using again Equation (1) of Proposition 4.2, implies that , for any  $n \geq 1$ ,

$$S_{n+2g}, S_{-n-2g} \in \text{Span}(S_0, \dots, S_{n+2g-1}).$$

We remark that using only Equation (1), by induction we can show that when  $|n| \leq 2g$  then  $S_n \in \text{Span}(S_0, \dots, S_{2g})$ . From there we can use the above result to inductively deduce that, for any  $n \in \mathbb{Z}$ , the element  $S_n$  is in  $\text{Span}(S_0, \dots, S_{2g})$ .  $\square$

## 5. RELATING NON-SEPARATING CURVES

In this last section, we conclude the proof of Theorem 1.2. Thanks to Section 3, we know that  $\mathcal{S}(\Sigma \times S^1)$  is spanned by arrowed non-separating curves and arrowed trivial curves and thanks to Section 4, we know that we need only non-separating curves with 0 or 1 arrow and trivial curves with at most  $2g$  arrows to span  $\mathcal{S}(\Sigma \times S^1)$ . Thus to prove Theorem 1.2, the only missing ingredient is to prove that two non-separating curves  $\gamma$  and  $\gamma'$  (both with 0 or 1 arrow) such that  $[\gamma] = [\gamma'] \in H_1(\Sigma, \mathbb{Z}/2)$  represent the same element in  $\mathcal{S}(\Sigma \times S^1)$ , which is what we prove in this section.

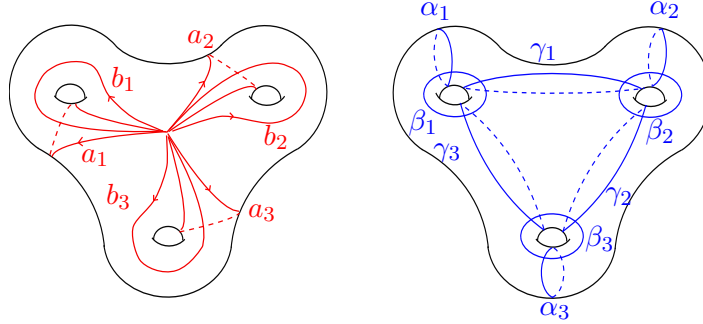


FIGURE 4. On the left, the standard generators of  $\pi_1(\Sigma)$ , on the right, the Lickorish generators of  $\text{Mod}(\Sigma)$ , drawn for  $\Sigma$  a genus 3 surface.

**5.1. Action of Dehn twists on the fundamental group.** In this section, we will set a few notations for the fundamental group and mapping class group of  $\Sigma$ , and perform some elementary computations, that we will need in Section 5.2.

Here, let  $\Sigma$  be a closed compact oriented surface of genus  $g$ . For elements  $a, b \in \pi_1(\Sigma)$ , we adopt the convention that  $a \cdot b$  is the path obtained by following first the oriented loop  $a$  then

following the loop  $b$ . Let  $a_1, b_1, \dots, a_g, b_g$  be the standard generators of  $\pi_1(\Sigma)$ , as shown on the left of Figure 4, so that  $\pi_1(\Sigma)$  is the group

$$\pi_1(\Sigma) = \langle a_1, b_1, \dots, a_g, b_g \rangle /_{[a_1, b_1][a_2, b_2] \dots [a_g, b_g] = 1}.$$

We also introduce three families of simple closed curves on  $\Sigma$  : the curves  $\alpha_i, \beta_i, \gamma_i$  represented on the right of Figure 4. One can easily check that the curve  $\alpha_i$  (resp.  $\beta_i$  and  $\gamma_i$ ) represents the free homotopy class  $[a_i]$  (resp.  $[b_i]$  and  $[a_{i+1}^{-1} b_i a_i b_i^{-1}]$ ). The Dehn twists along the curves  $\alpha_i, \beta_i, \gamma_i$  form the well-known Lickorish generators of the mapping class group  $\text{Mod}(\Sigma)$  :

**Theorem 5.1.** [11] *The  $3g - 1$  Dehn twists  $\tau_{\alpha_i}, \tau_{\beta_i}$  for  $1 \leq i \leq g$  and  $\tau_{\gamma_i}$  for  $1 \leq i \leq g - 1$  generate  $\text{Mod}(\Sigma)$ .*

For the use of the next section, let us collect here a few formulas expressing the action of the Dehn twists  $\tau_{\alpha_i}, \tau_{\beta_i}, \tau_{\gamma_i}$  on  $\pi_1(\Sigma)$ .

**Lemma 5.2.** *Let  $\varepsilon \in \{\pm 1\}$ .*

- *The map  $\tau_{\alpha_i}^\varepsilon$  sends  $b_i$  to  $b_i a_i^\varepsilon$ , and leaves all other generators of  $\pi_1(\Sigma)$  invariant.*
- *The map  $\tau_{\beta_i}^\varepsilon$  sends  $a_i$  to  $a_i b_i^{-\varepsilon}$ , and leaves all other generators invariant.*
- *The map  $\tau_{\gamma_i}^\varepsilon$  sends  $b_i$  to  $\gamma_i^\varepsilon b_i$ , sends  $a_{i+1}$  to  $\gamma_i^\varepsilon a_{i+1} \gamma_i^{-\varepsilon}$ , sends  $b_{i+1}$  to  $b_{i+1} \gamma_i^{-\varepsilon}$ , and leaves all other (including  $a_i$ ) generators of  $\pi_1(\Sigma)$  invariant.*

The proof of the lemma consists of homotopying the images of generators by the Dehn twists, and is left as exercise for the reader.

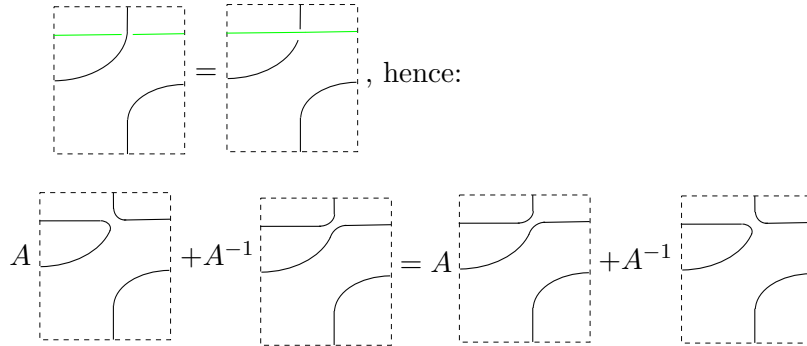
**5.2. An equivalence relation on the set of simple closed curves.** For  $\gamma, \delta$  two simple closed curves on  $\Sigma$ , let  $i(\gamma, \delta)$  be the geometric intersection number of  $\gamma$  and  $\delta$ . Let also  $[\gamma] \in H_1(\Sigma, \mathbb{Z}/2)$  be the  $\mathbb{Z}/2$ -homology class of  $\gamma$ . Finally, for any simple closed curve  $\gamma$  on  $\Sigma$ , let  $\tau_\gamma$  denote the Dehn twist along  $\gamma$ . We find some elementary equalities between different simple closed curves in  $\mathcal{S}(\Sigma \times S^1)$  :

**Proposition 5.3.** *Let  $\gamma$  and  $\delta$  be two simple closed curves on  $\Sigma$ , viewed as elements of  $\mathcal{S}(\Sigma \times S^1)$ . Then:*

- *If  $i(\gamma, \delta) = 1$ , then  $\gamma = \tau_\delta^2(\gamma)$ .*
- *If  $i(\gamma, \delta) = 2$ , then  $\gamma = \tau_\delta(\gamma)$ .*

*Moreover, those relations stay true after decorating  $\gamma$  and  $\tau_\delta^n(\gamma)$  with the same number of arrows.*

*Proof.* If  $i(\gamma, \delta) = 1$ , a neighborhood of  $a \cup b$  in  $\Sigma$  is a one-holed torus. Moreover,  $\tau_\delta(\gamma)$  and  $\delta$  also intersects once. Looking at the Kauffman resolution of the equation  $\tau_\delta(\gamma) \cdot \delta = \delta \cdot \tau_\delta(\gamma)$ , we have:



which gives us that  $(A - A^{-1})\gamma = (A - A^{-1})\tau_\delta^2(\gamma)$ , and thus  $\gamma = \tau_\delta^2(\gamma)$  as we work over  $\mathbb{Q}(A)$  coefficients.

If  $i(\gamma, \delta) = 2$ , if the algebraic intersection of  $a$  and  $b$  is 0, then a neighborhood of  $a \cup b$  in  $\Sigma$  is a 4-holed sphere. Otherwise, a neighborhood of  $a \cup b$  is a 2-holed torus. Let us assume the former. Let  $\gamma'$  be the  $1/2$  fractional Dehn twist of  $\gamma$  along  $\delta$ . The equation  $\gamma' \cdot \delta = \delta \cdot \gamma'$  reads:

, thus:

$$A^2 \left( \text{diagram 1} \right) + \left( \text{diagram 2} \right) + \left( \text{diagram 3} \right) + A^{-2} \left( \text{diagram 4} \right) \\ = A^{-2} \left( \text{diagram 5} \right) + \left( \text{diagram 6} \right) + \left( \text{diagram 7} \right) + A^2 \left( \text{diagram 8} \right)$$

which implies that  $(A^2 - A^{-2})\gamma = (A^2 - A^{-2})\tau_\delta(\gamma)$ , and thus  $\gamma = \tau_\delta(\gamma)$ .

The case where the algebraic intersection of  $a$  and  $b$  is  $\pm 2$  is fairly similar and left to the reader.  $\square$

This computation leads us to define an equivalence relation on non-separating simple closed curves on  $\Sigma$ .

**Definition 5.4.** On the set of non-separating simple closed curves on  $\Sigma$ , let  $\sim$  be the equivalence relation generated by:

- If  $\gamma, \delta$  are simple closed curves such that  $i(\gamma, \delta) = 1$ , then  $\gamma \sim \tau_\delta^2(\gamma)$ .
- If  $i(\gamma, \delta) = 2$ , then  $\gamma \sim \tau_\delta(\gamma)$ .

The proof of Theorem 1.2 now reduces to the following proposition.

**Proposition 5.5.** *Let  $\gamma$  and  $\gamma'$  be two simple closed non-separating curves on  $\Sigma$ . Then  $\gamma \sim \gamma'$  if and only if  $[\gamma] = [\gamma'] \in H_1(\Sigma, \mathbb{Z}/2)$ .*

Although we think it is likely that an appropriate use of the mapping class group literature could lead to a short proof of Proposition 5.5, we were unable to find a statement that would directly apply. Instead, we had to resort to a brute force proof.

*Proof of Proposition 5.5.* Notice that by definition, if  $\gamma \sim \gamma'$  then there is a mapping class group element  $\sigma \in \text{Mod}(\Sigma)$  such that  $\gamma' = \sigma(\gamma)$ . Moreover, if  $\gamma$  and  $\gamma'$  are related by a generating relation as in Definition 5.4, then clearly  $[\gamma] = [\gamma']$ . The direct implication follows.

Next we remark that if  $\gamma \sim \delta$ , then for any  $\sigma \in \text{Mod}(\Sigma)$ , we have that  $\sigma(\gamma) \sim \sigma(\delta)$ . Indeed, if for example  $\gamma' = \tau_\delta^2(\gamma)$  and  $i(\delta, \gamma) = 1$ , then  $\sigma(\gamma') = \sigma \circ \tau_\delta^2 \circ \sigma^{-1}(\sigma(\gamma)) = \tau_{\sigma(\delta)}(\sigma(\gamma))$ . Moreover  $i(\sigma(\delta), \sigma(\gamma)) = i(\delta, \gamma) = 1$ , so  $\sigma(\gamma') \sim \sigma(\gamma)$ . The same is true for the other generating relations, and the general case follows by transitivity.

Let us now introduce the finite  $\mathcal{F}$  of elements of  $\pi_1(\Sigma)$  of the type:

$$a_1^{\varepsilon_1} b_1^{-\delta_1} \dots a_g^{\varepsilon_g} b_g^{-\delta_g},$$

where the  $\varepsilon_i, \delta_i$  are elements of  $\{0, 1\}$ , non all-zero. Notice that  $\mathcal{F}$  contains exactly one element in each non-zero homology class of  $H_1(\Sigma, \mathbb{Z}/2)$ . Moreover, all of these loops actually represent simple closed curves on  $\Sigma$ , which are just connected sums of simple closed curves  $a_i^{\varepsilon_i} b_i^{-\delta_i}$ . Let also  $\mathcal{G}$  denote the set of Lickorish generators:  $\mathcal{G} = \{\tau_{\alpha_i}, \tau_{\beta_i}, \tau_{\gamma_i}\}$ . By the above discussion, Proposition 5.5 will follow once we prove:

**Lemma 5.6.** *For any simple closed curve  $c = a_1^{\varepsilon_1} b_1^{-\delta_1} a_2^{\varepsilon_2} b_2^{-\delta_2} \dots a_g^{\varepsilon_g} b_g^{-\delta_g} \in \mathcal{F}$ , for any Dehn twist  $\tau \in \mathcal{G} = \{\tau_{\alpha_i}, \tau_{\beta_i}, \tau_{\gamma_i}\}$ , we have that  $\tau(c)$  and  $\tau^{-1}(c)$  are equivalent to elements of  $\mathcal{F}$ .*

*Proof of Lemma 5.6.* Let us treat first the case of the generators  $\tau_{\alpha_i}$ . For  $c \in \mathcal{F}$ , let us write  $c = w a_i^{\varepsilon_i} b_i^{-\delta_i} z$  where  $w, z$  are expressed in generators of  $\pi_1(\Sigma)$  different than  $a_i, b_i$ . If  $\delta_i = 0$ , then we have  $\tau_{\alpha_i}^{\pm 1}(c) = c$ , so the  $\tau_{\alpha_i}^{\pm 1}(c)$  are equivalent to elements of  $\mathcal{F}$ . So let us assume  $\delta_i = 1$ . If  $\delta_i = 1$ , then  $i(c, \alpha_i) = 1$ , and thus  $\tau_{\alpha_i}(c) \sim \tau_{\alpha_i}^{-1}(c)$ . So it is sufficient to prove that one of the two is equivalent to an element of  $\mathcal{F}$ . Let  $\mu \in \{\pm 1\}$ . By the formulas in Lemma 5.2, we have that  $\tau_{\alpha_i}^{\mu}(c) = w a_i^{\varepsilon_i} a_i^{-\mu} b_i^{-1} z$ . Depending on the value of  $\varepsilon_i$ , we see that either  $\tau_{\alpha_i}(c)$  or  $\tau_{\alpha_i}^{-1}(c)$  is an element of  $\mathcal{F}$ , so both are equivalent to elements of  $\mathcal{F}$ .

Working with the generators  $\tau_{\beta_i}$  is similar: still writing  $c = w a_i^{\varepsilon_i} b_i^{-\delta_i} z$ , we have that if  $\varepsilon_i = 0$  then  $\tau_{\beta_i}^{\pm 1}(c) = c$ , which is an element of  $\mathcal{F}$  already. Else  $i(\beta_i, c) = 1$ , so that  $\tau_{\beta_i}(c) \sim \tau_{\beta_i}^{-1}(c)$  and moreover we find that one of the  $\tau_{\beta_i}^{\pm 1}(c)$  is an element of  $\mathcal{F}$ . We finally turn to the case of the Lickorish generators  $\tau_{\gamma_i}$ . This time, let us write  $c = w t z = w a_i^{\varepsilon_i} b_i^{-\delta_i} a_{i+1}^{\varepsilon_{i+1}} b_{i+1}^{-\delta_{i+1}} z$ . There are 16 possibilities for the middle word  $t$ , which we subdivide into 3 categories, depending on the geometric intersection number with  $\gamma_i$ :

- For  $t = 1, a_i, b_i^{-1} b_{i+1}^{-1}, a_i b_i^{-1} b_{i+1}^{-1}, b_i^{-1} a_{i+1} b_{i+1}^{-1}, a_i b_i^{-1} a_{i+1} b_{i+1}^{-1}$  we have that  $i([t], \gamma_i) = 0$ .
- For  $t = b_i^{-1}, a_i b_i^{-1}, b_i^{-1} a_{i+1}, a_i b_i^{-1} a_{i+1}, b_{i+1}^{-1}, a_i b_{i+1}^{-1}, a_{i+1} b_{i+1}^{-1}, a_i a_{i+1} b_{i+1}^{-1}$ , we have that  $i([t], \gamma_i) = 1$ .
- For  $t = a_{i+1}$  or  $a_i a_{i+1}$ , we have that  $i([t], \gamma_i) = 2$ .

Again, if  $i([t], \gamma_i) = 0$ , then  $\tau_{\gamma_i}^{\pm 1}(c) = c$  and we have nothing to prove. Moreover if  $i([t], \gamma_i) = 2$ , then by definition of the relation  $\sim$ , we have that  $\tau_{\gamma_i}^{\pm 1}(c) \sim c$ .

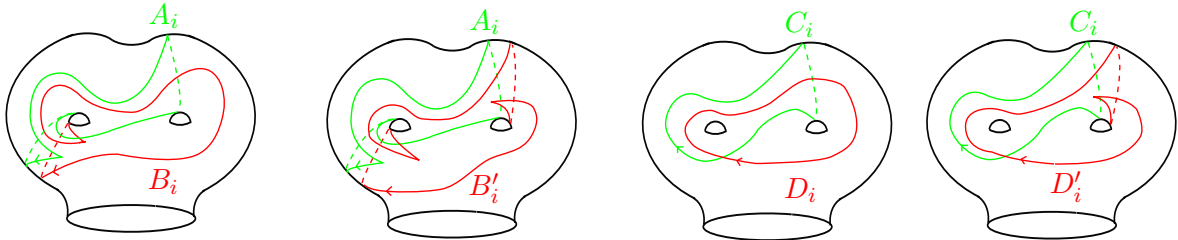


FIGURE 5. The curves  $A_i, B_i, B'_i, C_i, D_i, D'_i$  in the genus 2 subsurface with boundary the curve corresponding to the free homotopy class of  $[a_i, b_i][a_{i+1}, b_{i+1}] \in \pi_1(\Sigma)$ .

It remains to show that if  $i([t], \gamma_i) = 1$  then the elements  $\tau_{\gamma_i}^{\pm 1}(c)$  are equivalent to elements of  $\mathcal{F}$ . As before, those two elements are equivalent, so we only have to find that one of them is equivalent to an element of  $\mathcal{F}$  in each case. We proceed to show this in the remaining 8 cases:



1) If  $t = b_i^{-1}$  then  $\tau_{\gamma_i}(c) = wb_i^{-1}\gamma_i^{-1}z = wa_i^{-1}b_i^{-1}a_{i+1}z$ . But then we have that  $i(\tau_{\gamma_i}(c), \alpha_i) = 1$ , so that  $\tau_{\gamma_i}(c) \sim \tau_{\alpha_i}^{-2} \circ \tau_{\gamma_i}(c)$ . But we compute that:

$$\tau_{\alpha_i}^{-2} \circ \tau_{\gamma_i}(c) = wa_i b_i^{-1} a_{i+1} z$$

which is an element of  $\mathcal{F}$ .

2) If  $t = a_i b_i^{-1}$  then  $\tau_{\gamma_i}(c) = wa_i b_i^{-1} \gamma_i^{-1} z = wb_i^{-1} a_{i+1} z \in \mathcal{F}$ .

3) If  $t = b_i^{-1} a_{i+1}$  then  $\tau_{\gamma_i}^{-1}(c) = wb_i^{-1} a_{i+1} \gamma_i z = wa_i b_i^{-1} z$  which is an element of  $\mathcal{F}$ .

4) If  $t = a_i b_i^{-1} a_{i+1}$  then  $\tau_{\gamma_i}^{-1}(c) = wa_i b_i^{-1} a_{i+1} \gamma_i z = wa_i^2 b_i^{-1} z$ . Thus  $i(\alpha_i, \tau_{\gamma_i}^{-1}(c)) = 1$  so that  $\tau_{\gamma_i}^{-1}(c) \sim \tau_{\alpha_i}^2 \circ \tau_{\gamma_i}^{-1}(c)$ . We compute that

$$\tau_{\alpha_i}^2 \circ \tau_{\gamma_i}^{-1}(c) = wb_i^{-1} z$$

is an element of  $\mathcal{F}$ .

5) If  $t = b_{i+1}^{-1}$  then  $\tau_{\gamma_i}^{-1}(c) = w\gamma_i^{-1} b_{i+1}^{-1} z = wb_i a_i^{-1} b_i^{-1} a_{i+1} b_{i+1}^{-1} z$ . We find that  $i(\tau_{\gamma_i}^{-1}(c), \beta_i) = 1$ , so that  $\tau_{\gamma_i}^{-1}(c) \sim \tau_{\beta_i}^{-2} \circ \tau_{\gamma_i}^{-1}(c)$ . We compute that:

$$c' = \tau_{\beta_i}^{-2} \circ \tau_{\gamma_i}^{-1}(c) = wb_i^{-1} a_i^{-1} b_i^{-1} a_{i+1} b_{i+1}^{-1} z.$$

But now  $i(c', \alpha_i) = 2$ , so that  $c \sim c' \sim \tau_{\alpha_i}^{-1}(c')$ . We compute that

$$\tau_{\alpha_i}^{-1}(c') = wa_i b_i^{-1} a_{i+1} b_{i+1}^{-1} z$$

is an element of  $\mathcal{F}$ .

6) If  $t = a_{i+1} b_{i+1}^{-1}$  then  $\tau_{\gamma_i}^{-1}(c) = w\gamma_i^{-1} a_{i+1} b_{i+1}^{-1} z = wb_i^{-1} a_i^{-1} b_i b_{i+1}^{-1} z$ . Similarly to the previous case, we find that

$$c \sim \tau_{\beta_i}^{-2} \circ \tau_{\gamma_i}^{-1}(c) = c' \sim \tau_{\alpha_i}^{-1}(c') = wa_i b_i^{-1} b_{i+1}^{-1} z$$

which is an element of  $\mathcal{F}$ .

7) If  $t = a_i b_{i+1}^{-1}$  then  $\tau_{\gamma_i}(c) = wa_i \gamma_i b_{i+1}^{-1} z = wa_i a_{i+1}^{-1} b_i a_i b_i^{-1} b_{i+1}^{-1} z = c'$ . Let us introduce  $A_i = a_i a_{i+1}^{-1} b_i$  and  $B_i = a_i b_i^{-1} b_{i+1}^{-1}$ . By abuse of notation, we also write  $A_i$  and  $B_i$  for the simple closed curves corresponding to the free homotopy classes  $[A_i], [B_i]$ . Those simple closed curves are represented on Figure 5. We see on the Figure that  $i(A_i, B_i) = 1$ , and also that  $\tau_{A_i}(B_i) = A_i^2 B_i$ . It is moreover clear that  $\tau_{A_i}$  leaves all the generators  $a_j, b_j$  with  $j$  not  $i$  or  $i+1$  invariant. Thus we have that

$$\tau_{\gamma_i}(c) = c' \sim \tau_{A_i}^{-2}(c') = wA_i^{-1} B_i z = w(b_i^{-1} a_{i+1})(b_i^{-1} b_{i+1}^{-1}) z = c''.$$

Now calling  $C_i$  and  $D_i$  the simple closed curves corresponding to the free homotopy classes  $[b_i^{-1} a_{i+1}]$  and  $[b_i^{-1} b_{i+1}^{-1}]$ . Again we have that  $i(C_i, D_i) = 1$ , and that  $\tau_{C_i}(D_i) = C_i^2 D_i$ , with  $\tau_{C_i}$  leaving the  $a_j, b_j$  with  $j$  not  $i$  or  $i+1$  invariant. So,

$$\tau_{\gamma_i}(c) \sim c'' \sim \tau_{C_i}^{-2}(c'') = wC_i^{-1} D_i z = wa_{i+1}^{-1} b_{i+1}^{-1} z.$$

Finally, this last element is equivalent to  $wa_{i+1} b_{i+1}^{-1} z$ , an element of  $\mathcal{F}$ , using the square of the Dehn twist along  $\alpha_{i+1}$ .

8) Finally, if  $t = a_i a_{i+1} b_{i+1}^{-1}$  then  $\tau_{\gamma_i}(c) = wa_i \gamma_i a_{i+1} b_{i+1}^{-1} z = wa_i a_{i+1}^{-1} b_i a_i b_i^{-1} a_{i+1} b_{i+1}^{-1} z = c'$ . Let this time  $A_i = a_i a_{i+1}^{-1} b_i$  and  $B'_i = a_i b_i^{-1} a_{i+1} b_{i+1}^{-1}$ . We still have that  $i(A_i, B'_i) = 1$ , and that  $\tau_{A_i}(B'_i) = A_i^2 B'_i$ , so that similarly to the previous case we get:

$$c' \sim wA_i^{-1} B'_i z = wb_i^{-1} a_{i+1} b_i^{-1} a_{i+1} b_{i+1}^{-1} z.$$

Setting again  $C_i = b_i^{-1} a_{i+1}$  and  $D'_i = b_i^{-1} a_{i+1} b_{i+1}^{-1}$ , we still check that  $i(C_i, D'_i) = 1$  and thus that

$$c' \sim \tau_{C_i}^{-2}(c') = c'' = wC_i^{-1} D'_i z = wb_{i+1}^{-1} z$$

which is an element of  $\mathcal{F}$ . □

Lemma 5.6 now being established, Proposition 5.5 follows: by induction, for any word in the Lickorish generators  $w$  and any element  $s$  of  $\mathcal{F}$ , there is  $s' \in \mathcal{F}$  so that  $w(s) \sim s'$ . As any non-separating simple closed curve is of the form  $w([a_1])$  for some  $w \in \text{Mod}(\Sigma)$ , any non-separating simple closed curve is equivalent to an element of  $\mathcal{F}$ . □

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