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► **To cite this version:**

Pierre Collet. A short historical account of period doublings in the pre-renormalization era. *Comptes Rendus. Mécanique*, 2019, 347, pp.287 - 293. 10.1016/j.crme.2019.03.007 . hal-03485050

HAL Id: hal-03485050

<https://hal.science/hal-03485050v1>

Submitted on 20 Dec 2021

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A Short Historical Account of Period Doublings in the Pre-Renormalization Era.

P. Collet

*Centre de Physique Théorique, CNRS UMR 7644, Ecole Polytechnique, F-91128
Palaiseau Cedex (France)*

Abstract

I will shortly review the history of experimental and theoretical findings on period doubling until the discovery of the quantitative universal properties of the infinite period doubling cascade.

Résumé

Cet article décrit brièvement l'histoire des expériences et des développements théoriques du doublage de période jusqu'à la découverte des propriétés quantitatives universelles de la cascade infinie.

Version française abrégée

Avant d'être renormalisé à Nice et à Los Alamos et de devenir une route robuste (ouverte) vers le chaos, le phénomène de doublage de période a vécu une longue histoire. Cet article présente quelques étapes de ces développements aussi bien théoriques qu'expérimentaux. Les notions de familles à un paramètre et de bifurcation (bien que cette dernière fut introduite par Poincaré) ont mises un certain temps avant d'émerger comme les concepts naturels. La lecture des articles "anciens" frappe par la capacité de calcul (manuelle) et l'inventivité des dispositifs expérimentaux. De nombreuses observations et développements théoriques ont suivi la découverte des propriétés d'universalité dans la suite d'accumulation de doublage de période et ce flux de résultats n'est certainement pas près de s'arrêter.

Email address: pierre.collet@cpht.polytechnique.fr (P. Collet).

1 Introduction.

More than one hundred years separate the first observations of the doubling period phenomenon and the discovery of the quantitative universal properties of accumulation of period doublings. As we will see, there were many experimental and theoretical works on the subject from which the contemporary concepts emerged. It is interesting to try to understand in the light of our present knowledge what was observed or computed, and how some concepts emerged quite early (like perturbation techniques) while some remained without use for a long time (like the Poincaré map). Of course we should keep in mind that Science is in constant evolution and we are somewhat biased by our actual knowledge. The major concepts and ideas of tomorrow may largely encompass those of today and shed a different light on the subject.

It is sometimes difficult to ascertain what was really observed in experiments (notwithstanding the fact that some experimental reports seem to be lost). Very clever experimental settings had to compensate for the lack of sophisticated equipments and measurements techniques. For example in the famous frequency demultiplication experiment [78], the system was observed through a telephone receiver and a human ear was used as a frequency analyzer. The amazing precision and details of observations had probably a lot to do with the fact that Balthazar Van der Pol had absolute pitch [9].

Quite probably there were many more manifestations of period doubling which were missed or discarded. It is of course much easier to search experimentally for a phenomenon that has been already predicted theoretically or previously observed in other situations. This probably explains why many experiments concentrated on the frequency demultiplication and in particular on period 3. Note also that frequency demultiplication found early applications [9] for example in building a television system [75]. Quite probably higher period doubling bifurcations were observed without being reported.

This paper is organized as follows, based more or less on successive epochs of more and more sophisticated theory and experiments. We will first discuss the history of the period doubling phenomenon in itself and the development of the theoretical analysis until the occurrence of the notion of period doubling bifurcation. We then describe some early observations of secondary period doubling bifurcations and of the beginning of the cascade. This is followed by the qualitative observation of the infinite cascade and some Mathematical related results. Then came the quantitative results on the infinite cascade including universality and openness. The bibliography does not pretend to be exhaustive but tries to mention the earliest articles about experimental and theoretical discoveries about the phenomenon of period doubling.

In the sequel we will say period n to mean period nT when the reference period T is obvious from the context.

2 Double period, subharmonics, undertone, frequency demultiplication, fractional harmonics.

One of the first observation of a double period is due to Faraday [22] on surfaces of vibrating liquid layers. Few years later, Savart [62] observed a “son rauque” in cane vibrations, see also [72] and [36]. Von Melde observed a similar phenomenon in the vibration of strings, see Raman [58]. The related phenomenon of parametric resonance had been observed many times even much earlier (the swing was known in very ancient times like for example in the Aiora festival of ancient Greece or in a statue of Minoean time). See also [51] for the history of electrical circuits.

In all the previously mentioned experiments, there is a periodic forcing at a frequency which is roughly twice the “natural” frequency of the system. Note that this is very different from the harmonic generation by nonlinearities. For a nice review about these early experiments and theory, see Von Kármán’s Gibbs lecture [79].

Surprisingly enough some scientists have raised doubts on the existence of subharmonic solutions, we refer to [55] section 2 for a discussion and historical references. The landmark paper of Van der Pol and Van der Mark [78] on frequency demultiplication also played a major role in the further developments. It is also one of the first papers analyzing the evolution of the dynamics along a one parameter family.

The first theoretical approaches were made by Lord Rayleigh [59] and [60] §65b and Stephenson [69] using perturbation theory. A more mathematical approach was described by H.Poincaré [57] chapter XXVIII “solutions du deuxième genre” using the time T map.

In order to briefly describe the findings at this stage of history, we will use the example of an electrical system discussed by Pedersen [55] (formula 3.31), namely (with slightly different notations and some rescalings)

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \left(\frac{1}{4} + \xi + g x \right) x = \sin(t) , \quad (2.1)$$

where $\gamma \geq 0$ is a friction coefficient, ξ measures the detuning with respect to the period 4π , and g measures the intensity of the nonlinearity. Considering the equation for the function $-x(t + \pi)$, one can see that one can assume

$g \geq 0$, and it is enough to consider the parameter space $\mathcal{P} = \{(\gamma, \xi, g)\}$ (with $\gamma \geq 0$ and $g \geq 0$).

Note that if $\gamma = \xi = g = 0$, there is a unique periodic solution of period 2π given by

$$x(t) = -\frac{4}{3} \sin(t).$$

One can prove that if γ, ξ and g are small enough, equation (2.1) still has a unique solution of period 2π denoted by $x_0(t)$. It is natural to ask for the stability of this solution. One first transforms the equation by the translation $x(t) = x_0(t) + q(t)$ which leads to the equation

$$\frac{d^2 q}{dt^2} + \gamma \frac{dq}{dt} + \left(\frac{1}{4} + \xi + 2g x_0(t) + g q\right) q = 0, \quad (2.2)$$

or in the more convenient system form

$$\frac{d}{dt} \begin{pmatrix} q \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\gamma v - \left(\frac{1}{4} + \xi + 2g x_0(t) + g q\right) q \end{pmatrix}, \quad (2.3)$$

and we are interested in the stability of the trivial solution $q(t) = v(t) = 0$. The Poincaré time 2π map denoted by \mathcal{P} is particularly convenient for this purpose. The 2×2 matrix $D\mathcal{P}(0, 0)$ is obtained as the value at $t = 2\pi$ of the solution of the matrix differential equation

$$\frac{d}{dt} W(t) = \begin{pmatrix} 0 & 1 \\ -\left(\frac{1}{4} + \xi + 2g x_0(t)\right) & -\gamma \end{pmatrix} W(t)$$

with initial condition $W(0) = \text{Id}$. It follows from Theorem 7.3 chapter 1 in [15] that

$$\det(D\mathcal{P}(0, 0)) = e^{-2\pi\gamma}.$$

Therefore if $\gamma > 0$, a change of stability in the matrix $D\mathcal{P}(0, 0)$ (as a function of (γ, ξ, g)) can only occur when an eigenvalue is equal to ± 1 (the other eigenvalue being of modulus strictly smaller than one). In other words, a Neimark-Sacker bifurcation is impossible.

One can prove that in a neighborhood of the origin in \mathcal{P} there is a surface (essentially a cone) defined as the zero set of a regular function $H(\gamma, \xi, g)$ such that

$$H = \frac{16g^2}{9} - \xi^2 - \frac{\gamma^2}{4} + \mathcal{O}(|\gamma|^3 + |\xi|^3 + g^3).$$

The equation $H = 0$ can also be solved in g and we get

$$g = F(\gamma, \xi) = \frac{3}{8} \sqrt{\gamma^2 + 4\xi^2} + \mathcal{O}(\gamma^2 + \xi^2), \quad (2.4)$$

where $F \geq 0$ is a regular function except at the origin such that for $0 \leq g < F(\gamma, \xi)$ ($H < 0$), equation (2.1) has a stable periodic solution of period 2π . For g larger but near $F(\gamma, \xi)$ ($H > 0$), this solution is unstable and the instability develops a solution of period 4π . In other words, if one considers a one parameter family $(\gamma(s), \xi(s), g(s))$ in \mathcal{P} , an instability occurs when crossing upward the surface $g = F$ ($H = 0$).

In [55], one has to combine equations 3.36 and the equation following 3.39 to get the above function $F(H)$ to the leading order (with different notations).

Equation (2.2) is a particular case of the equation

$$\frac{d^2q}{dt^2} + \gamma\left(q, \frac{dq}{dt}, t\right) \frac{dq}{dt} + \left[\frac{1}{4} + A\left(q, \frac{dq}{dt}, t\right)\right] q = 0, \quad (2.5)$$

where the functions $\gamma(u, v, t)$ and $A(u, v, t)$ are regular and periodic of period 2π in t . This equation includes also many models of parametric resonance, and we are interested in the stability of the trivial solution $q = dq/dt = 0$ which can be analyzed using Floquet theory (see for example [4]). Let

$$\gamma(0, 0, t) = \gamma_0 + \gamma_1 \cos(t + \varphi) + \tilde{\gamma}(t),$$

and

$$A(0, 0, t) = A_0 + A_1 \cos(t + \psi) + \tilde{A}(t),$$

where $\gamma_0, \gamma_1, A_0, A_1, \varphi$ and ψ are constants and the Fourier spectrum of $\tilde{\gamma}$ and \tilde{A} do not contain 0 neither ± 1 . Then if $\gamma_0 < 0$ an instability occurs only at an eigenvalue ± 1 of the differential of the Poincaré map at the origin (the determinant of this matrix is $\exp(-2\pi\gamma_0)$), and there is a real valued functional $\mathcal{F}(\gamma, A)$ such that

$$\mathcal{F}(\gamma, A) = 16A_0^2 + 4\gamma_0^2 - \gamma_1^2 - 4A_1^2 + 4A_1\gamma_1 \sin(\varphi - \psi) + h.o.t.$$

where the higher order terms are at least cubic (also in $\tilde{\gamma}$ and \tilde{A}), and such that if $\mathcal{F} > 0$ the periodic solution is stable while if $\mathcal{F} < 0$ and small enough, it is unstable and the instability develops a solution of period 4π .

In the previously mentioned theoretical papers (see also [55]), the function F was computed to the lowest nontrivial order in its Taylor expansion. As often stated one expects that the nonlinearities will saturate the instability. This of course brings the question of computing the amplitude of the new solution (of double period) to the lowest nontrivial order in perturbation theory. This question was discussed first by Van der Pol [76], and then by Russian groups around Maldelstam, Papalexi, Andronov, Chaikin and Vitt, see [40] and references therein and Bogoljubov, Krylov, see [37] and references therein. See also [77] and references therein. We refer to [25] for a nice and detailed description of these works, their relations and more references. See also [20] for the history of the Andronov school at Gorky.

In the 30's and later many experimental papers reported on the observation of double period, while several others discussed the theoretical aspects. It is impossible to list all these works and I will just refer the reader to some of them. On the theoretical aspects [61] is one of the few papers using the Poincaré map to analyze the problem of existence of subharmonics. The stability of subharmonics is discussed in [30]. The paper [39] contains a figure of the domain of double period in a section of the parameter space from an earlier article of Andronov and Leontovich. See also [40] and references therein. The perturbation method of analysis of period doubling was exposed in several reference books like [7], [53], [8], [3] [48], [70].

Many experimental examples of subharmonics are described in the book [44]. The paper by Okumura [54] contains many early references about subharmonic oscillations in power circuits (mostly in Japanese). For other experimental observations one can see for example [73] for an iron-core system, [66] for a pendulum with vertically moving suspension, [19] for an experiment with cochlea, and [51] for the history of transducers.

Of course one would like to go beyond perturbation theory and for example in the case of the system (2.1), one wants to prove the existence of the periodic orbit of period 4π , namely the occurrence of a period doubling bifurcation of the cycle. Three (at least) general approaches were developed later on.

- Pursue the idea of perturbation theory and then control the equation for the remainder term (by contraction mapping principle or related techniques). This was done in Gambill Hale [24], see also [29].
- Use the time T map (time 2π map in our example (2.1)) following Poincaré, and prove there is a period doubling bifurcation. This is done by considering a center manifold (of dimension one). See [10] Theorem 3 page 576, [11] Theorem 3 page 781, and [67] page 575. The proof also follows by considering the time $2T$ map and applying the bifurcation theorem from simple eigenvalue [18] which was proved around the same time.
- One can also observe that in the case of the dynamical system (2.1) expressed as a system of two coupled one dimensional odes, for $\gamma = \xi = g = 0$, the time 2π map is equal to $-\text{Id}$. In other words we have a double eigenvalue -1 . One can unfold the three parameters family of vector fields around this point and then study the bifurcations along a one parameter family in the parameter space \mathcal{P} . See [71], [4], and [35] for the unfolding. This method can also be used to understand frequency demultiplication.

For example, consider a one parameter family of equations (2.1), namely a curve $(\gamma(s), \xi(s), g(s)) \in \mathcal{P}$ crossing the surface (2.4) at a parameter value s_0 . In order to conclude from the bifurcation theorem from simple eigenvalue

applied to the Poincaré time 4π map, one should check first the hypothesis of Theorem 1.7 in [18]. They are all trivially satisfied in our case except hypotheses (d). With our notations, this hypothesis can be expressed as

$$\left. \frac{d}{ds} H(\gamma(s), \xi(s), g(s)) \right|_{s=s_0} \neq 0.$$

This means that the one parameter family crosses the surface (2.4) transversally and with non zero velocity in the parameter s . From now on we will assume that

$$\left. \frac{d}{ds} H(\gamma(s), \xi(s), g(s)) \right|_{s=s_0} > 0.$$

The final step is to check that a certain quantity Q (essentially a curvature term) is not zero (see [18] Theorem 1.18 with $n = 2$ for the details). This quantity relates the amplitude A of the bifurcated solution in the marginal (dominant) direction to the parameter s by

$$s - s_0 = Q A^2 + h.o.t.$$

In order to express this coefficient in a compact form, it is convenient to use polar coordinates for the parameters γ and ξ , namely

$$\gamma = 2r \sin(\theta), \quad \xi = r \cos(\theta)$$

with $r \geq 0$ and $\theta \in]0, \pi[$. In these coordinates the function H is given by

$$H = \frac{16g^2}{9} - r^2 + h.o.t.$$

After some computations one gets

$$Q = -240\pi^2 r^5 \cos\theta \frac{(1 + \sin\theta)}{\left. \frac{d}{ds} H(\gamma(s), \xi(s), g(s)) \right|_{s=s_0}} + h.o.t.$$

We see that this coefficient can vanish only if $\theta = \pi/2$, namely zero detuning. Moreover if $\theta \in]-\pi/2, 0[$ the period doubling bifurcation is supercritical ($Q > 0$) and the doubled cycle is stable, while for $\theta \in]0, \pi/2[$ the period doubling bifurcation is subcritical ($Q < 0$) and the doubled cycle is unstable.

3 The beginning of the cascade.

To go beyond the first period doubling, there are several difficulties in observing subsequent ones. As we now know, one loses almost one digit of precision

at each bifurcation requiring a high precision in the experimental setting or in numerical computations using analogic or digital computers.

Moreover most experiments until the seventies were done with systems having periodic forcings (recall that the period doubling of a cycle is impossible in a plane system). These systems exhibit in general frequency demultiplication [78]. It is then easy to misinterpret a period $4T$ arising by a second period doubling and the period $4T$ of the frequency demultiplication which has a very different origin. Many experimental efforts also concentrated on the period $3T$. The sequence 1, 2, 3, ... being of course more intuitive than the sequence 1, 2, 4 ...

An experiment with loudspeakers was performed by Pedersen [55] and [56] in 1933. Besides containing a discussion on the occurrence of the double period (using perturbation techniques) he mentions the experimental observation of an oscillation with a period four times the forcing period (see figure 1 in [55]) inside the domain of existence of the double period. Although the figure is presented as a “sketch”, this may be the first example of an experiment where a quadruple cycle occurred.

We will mention few other “early” papers dealing with higher subharmonics. Figure V-11 page 57 in [13] studying nonlinear control systems seem to describe an overlapping domain of periods 2 and 4. Numerical simulations of voltage regulators in [2] seem to show transitions 2 to 4 tables 2-1, 5-1 and 5-3 with some period 8 harder to interpret. Numerical computations in [33] using the Poincaré map shows clearly up to period 4, see figure 6 page 248. The book [31] shows period 4 page 167, 173 and describes experiments page 176 figure 7.26 and page 177 figure 7.27. See also sections 7.6 and 7.7 and references therein. For a review of subharmonic observations in acoustic systems we refer to [14]

The above list of references is certainly incomplete since higher period doublings must have been observed in many experiments and perhaps also in numerical simulations. As mentioned at the beginning of this section, it is sometimes difficult to discriminate from the content of the publication if there is a secondary period doubling or a period four coming from frequency demultiplication.

4 The infinite qualitative cascade.

We already mentioned that in the early days the experimental setup had to be cleverly designed to overcome the lack of precise measurement techniques. From a theoretical point of view, hand made calculations of perturbation series

were rather limited. The advent of computer simulation of dynamical systems [68] and later of computer algebra systems (CAS) opened the road to a much deeper understanding and allowed to easily perform “computer experiments”.

In the 60’s and 70’s the infinite cascade of period doubling was proved in theoretical works and also observed in numerical computations.

Myrberg in the late 50’s [52] proved the existence of infinitely many doubling bifurcations in the quadratic family by using an interspersed argument between bifurcations and superstable periods.

Around the same time Sharkovsky proved his famous result on the ordering.

Theorem 4.1 (Sharkovsky) *Define an order on the integers by*

$$\begin{array}{cccccc}
 1 & \prec & 2 & \prec & 4 & \prec & 8 & \prec & \dots \\
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 \dots & \prec & 4 * 9 & \prec & 4 * 7 & \prec & 4 * 5 & \prec & 2 * 3 \\
 \dots & \prec & 2 * 9 & \prec & 2 * 7 & \prec & 2 * 5 & \prec & 2 * 3 \\
 \dots & \prec & 9 & \prec & 7 & \prec & 5 & \prec & 3
 \end{array}$$

Then if $q \prec p$ and a continuous map of the interval has a periodic orbit of period p , it has a periodic orbit of period q .

See [65], [27], and [12].

The infinite cascade and combinatorial structure of the orbits of dimensional maps was studied by Metropolis Stein and Stein in [46] and later in [45]. This developed later on in the kneading theory of Milnor and Thurston [47].

Hayashi, Ueda and Akamatsu [32] based on Levinson Massera relations argued for the existence of 2^n unstable periodic points for the Poincaré return map of a periodically forced nonlinear second order ode.

According to [38] and [5], Shapiro [64] observed the infinite period doubling cascade in an ecological Ricker model. Around the same time, May [41] [42] and May and Oster [43] saw clearly in numerical experiments the infinite cascade and discovered that it was present in many one parameter families of maps of the interval modeling in particular the evolution of ecological systems.

Mira and Gumowski developed a technique similar to renormalization to understand the successive bifurcations, see [28], [49] and references therein.

5 The quantitative universal cascade.

The quantitative properties of the infinite cascade were first reported in the papers [17], [74] and [23], see also [21]. The universal properties, stability (openness) of the “road to chaos”, other universality classes, experimental observations have been studied in many publications and I will not review them here.

One can refer more generally to papers on the history of dynamical systems like [1], [6], [34] [16], [50], [26], among many others. See also [63] for an occurrence in literature.

An important property of the period doubling “road to chaos” is its openness. In the space of all (regular) one parameter families of dynamical systems (of fixed dimension) there is an open set such that any one parameter family inside this set will present the qualitative and quantitative universal properties of the accumulation of period doubling. In particular any not too large perturbation of a one parameter family in the set will still be in the set and will have the same universal properties. Of course this set is far from including all the one parameter families of dynamical systems. It is easy to construct (by regular surgery) one parameter families having n period doublings and then a Hopf bifurcation (or other “roads to chaos”).

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