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► **To cite this version:**

Pierre Julg. How to prove the Baum–Connes conjecture for the groups $S p (n , 1)$?. Journal of Geometry and Physics, 2019, 141, pp.105 - 119. 10.1016/j.geomphys.2019.02.009 . hal-03484995

HAL Id: hal-03484995

<https://hal.science/hal-03484995v1>

Submitted on 20 Dec 2021

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How to prove the Baum-Connes conjecture for the groups $Sp(n, 1)$?

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Abstract. *We present a programme of proof of the Baum-Connes conjecture with coefficients for $G=Sp(n,1)$ or $F_4(-20)$, i.e. simple Lie groups of real rank one having Kazhdan's property (T). We use the geometry of the boundary sphere to produce a G -Fredholm module, together with a homotopy to the trivial representation through uniformly bounded representations. The strip of uniformly bounded representations of $M. Cowling$ plays here the role of the complementary series. We explain how, modulo some conjectural estimates, this construction would prove the conjecture.*

In his famous work on the Novikov conjecture [K1][K3], G. Kasparov introduced the equivariant bivariant K -theory groups. For any locally compact group G and A, B two $G - C^*$ -algebras (i.e. C^* -algebras equipped with a strong continuous action by automorphisms of the group G), he defines an abelian group $KK_G(A, B)$. The main tool in the theory is the cup product

$$KK_G(A, B) \times KK_G(B, C) \rightarrow KK_G(A, C).$$

In particular, if \mathbf{C} is the field of complex numbers equipped with the trivial G -action, $KK_G(\mathbf{C}, \mathbf{C})$ is a commutative ring, and all the $KK_G(A, B)$'s are $KK_G(\mathbf{C}, \mathbf{C})$ modules. There are also maps $j_{G,\text{red}}$ and $j_{G,\text{max}}$:

$$KK_G(A, B) \rightarrow KK(C_{\text{red}}^*(G, A), C_{\text{red}}^*(G, B))$$

$$KK_G(A, B) \rightarrow KK(C_{\text{max}}^*(G, A), C_{\text{max}}^*(G, A))$$

where $C_{\text{red}}^*(G, A)$ and $C_{\text{max}}^*(G, A)$ denote respectively the reduced and the full crossed product.

Let us recall how KK_G theory gives a formulation of the Baum-Connes conjecture with coefficients. We follow [BCH]:

For any proper G -space Z , the space $C_0(Z)$ is a module of finite type over the algebra $C^*(G, C_0(Z))$ (which is both the full and the reduced one) whose class in $K_0(C^*(G, C_0(Z))) = KK(\mathbf{C}, C^*(G, C_0(Z)))$ will be denoted e_Z . Then for any $G - C^*$ -algebra A , the map $j_{G,\text{red}}$:

$$KK_G(C_0(Z), A) \rightarrow KK(C^*(G, C_0(Z)), C_{\text{red}}^*(G, A))$$

can be composed with the left multiplication by e_Z :

$$KK(C^*(G, C_0(Z)), C_{\text{red}}^*(G, A)) \rightarrow KK(\mathbf{C}, C_{\text{red}}^*(G, A))$$

to define a map

$$KK_G(C_0(Z), A) \rightarrow K_*(C_{\text{red}}^*(G, A)).$$

Baum, Connes and Higson consider $\mathbf{E}G$, the classifying space of proper actions of G , and the inductive limit

$$K_*^G(\mathbf{E}G; A) = \varinjlim KK_G(C_0(Z), A)$$

over all closed subsets Z of $\mathbf{E}G$ which are G -invariant and G -compact. The above maps form an inductive system which defines the Baum-Connes assembly map

$$\mu_{A, \text{red}} : K_*^G(\mathbf{E}G, A) \rightarrow K_*(C_{\text{red}}^*(G, A)).$$

Baum-Connes conjecture for G , with coefficients in A : *The map $\mu_{A, \text{red}}$ is an isomorphism.*

From the point of view of C^* -algebras, it would have been more natural, in some sense, to work with full crossed products $C_{\text{max}}^*(G, A)$ instead of reduced, and to consider the corresponding assembly map

$$\mu_{A, \text{max}} : K_*^G(\mathbf{E}G, A) \rightarrow K_*(C_{\text{max}}^*(G, A)).$$

defined in the same way. Note that the two maps are related by $\mu_{A, \text{red}} = \lambda_A^* \circ \mu_{A, \text{max}}$ where

$$\lambda_A^* : K_*(C_{\text{max}}^*(G, A)) \rightarrow K_*(C_{\text{red}}^*(G, A))$$

is the group homomorphism obtained from the regular representation by the K -functor.

However, we shall see that the map $\mu_{A, \text{max}}$ cannot in general be surjective, whereas $\mu_{A, \text{red}}$ can reasonably be conjectured to be bijective.

1 The γ element of Kasparov

Let us consider the case where G is a connected group and K a maximal compact subgroup. Kasparov shows that the ring $KK_G(\mathbf{C}, \mathbf{C})$ is a direct sum of two subrings

$$KK_G(\mathbf{C}, \mathbf{C}) = \gamma KK_G(\mathbf{C}, \mathbf{C}) \oplus (1 - \gamma) KK_G(\mathbf{C}, \mathbf{C})$$

given by some idempotent element γ of $KK_G(\mathbf{C}, \mathbf{C})$, such that the restriction map $KK_G(\mathbf{C}, \mathbf{C}) \rightarrow KK_K(\mathbf{C}, \mathbf{C}) = R(K)$ is an isomorphism from $\gamma KK_G(\mathbf{C}, \mathbf{C})$ to $R(K)$, and vanishes on the complement $(1 - \gamma) KK_G(\mathbf{C}, \mathbf{C})$.

More generally for any A, B as above,

$$KK_G(A, B) = \gamma KK_G(A, B) \oplus (1 - \gamma) KK_G(A, B)$$

and the restriction map is an isomorphism from $\gamma KK_G(A, B)$ to $KK_K(A, B)$ and vanishes on $(1 - \gamma) KK_G(A, B)$.

The element γ plays a key rôle in the Baum-Connes conjecture.

In the case of a connected group, the geometric group (with coefficients in A) is $KK_G(C_0(Z), A)$ where $Z = G/K$. One has $(1 - \gamma) KK_G(C_0(Z), A) = 0$ so that the γ element acts trivially on the geometric group.

On the other hand the element γ acts on the K -theory of $C^*(G, A)$ (where C^* denotes either the full or the reduced crossed product) by an idempotent map which can be described as follows. Consider the composition of ring homomorphisms

$$\begin{aligned} KK_G(\mathbf{C}, \mathbf{C}) &\rightarrow KK_G(A, A) \rightarrow KK(C^*(G, A), C^*(G, A)) \\ &\rightarrow \text{End}(K_*(C^*(G, A))) \end{aligned}$$

and take the images of the idempotent γ by the above maps:

$$\tilde{\gamma}_{A, \max} \in \text{End}(K_*(C_{\max}^*(G, A)))$$

$$\tilde{\gamma}_{A, \text{red}} \in \text{End}(K_*(C_{\text{red}}^*(G, A)))$$

respectively. The results of Kasparov [K1][K3] can then be summarized as follows:

Theorem 1 *The maps $\mu_{A, \text{red}}$ and $\mu_{A, \max}$ are both injective. Their respective images in $K_*(C_{\text{red}}^*(G, A))$ and $K_*(C_{\max}^*(G, A))$ are equal to the images of the idempotent maps $\tilde{\gamma}_{A, \text{red}}$ and $\tilde{\gamma}_{A, \max}$.*

Corollary 2 *The Baum-Connes conjecture (i.e. the statement that $\mu_{A, \text{red}}$ is an isomorphism) is equivalent to the equality $\tilde{\gamma}_{A, \text{red}} = \text{Id}$.*

Corollary 3 *If $\gamma = 1$ in $KK_G(\mathbf{C}, \mathbf{C})$, then both maps $\mu_{A, \text{red}}$ and $\mu_{A, \max}$ are isomorphisms, and so is $\lambda_A^* : K_*(C_{\max}^*(G, A)) \rightarrow K_*(C_{\text{red}}^*(G, A))$.*

The property for a group G , that λ_A^* is an isomorphism for any A , is essentially J. Cuntz' K -amenability [Cu], see [JV]. Alain Connes pointed out, in the early 1980's, that if G is non compact and has Kazhdan's property (T) , the map $\lambda^* : K_*(C_{\max}^*(G)) \rightarrow K_*(C_{\text{red}}^*(G))$ is not injective. Indeed, the trivial representation of G defines an idempotent of $C_{\max}^*(G)$ which vanishes in $C_{\text{red}}^*(G)$. Therefore in such cases $\gamma \neq 1$ in $KK_G(\mathbf{C}, \mathbf{C})$.

This happens in particular when G is a connected simple Lie group of real rank greater or equal to 2, e.g. $G = SL(n, \mathbf{R})$ for $n \geq 3$. Then G has property (T) . The challenge about the Baum-Connes conjecture with coefficients is to

show that $\tilde{\gamma}_{A,\text{red}} = \text{Id}$ despite the fact that $\gamma \neq 1$. In other words we expect that full and reduced crossed products behave in a different manner: $\tilde{\gamma}_{A,\text{red}} = \text{Id}$ on the one hand, $\tilde{\gamma}_{A,\text{max}} \neq \text{Id}$ on the other.

The Baum-Connes conjecture with coefficients is completely open for G of higher rank, and so is its corollary: the Baum-Connes conjecture (even without coefficients) for discrete subgroups of such a group.

The only positive result was obtained by V. Lafforgue in his 1998 thesis [L2][L3]: let $G = SL(3, \mathbf{R})$ or $SL(3, \mathbf{C})$, then the Baum-Connes conjecture *without coefficients* is true for *discrete cocompact subgroups* of G . This follows from the fact that such subgroups do have the Jolissaint (RD) property. The Baum-Connes conjecture then follows from the Banach analogue (a variant of Bost conjecture) whose proof is the central point of Lafforgue's thesis. We should also mention that the same argument applies to $SL(3, \mathbf{H})$ and $E_{6(-26)}$ which also have Jolissaint's (RD) property [Cha]. In fact, A. Valette has conjectured that discrete cocompact subgroups of semi-simple Lie groups have Jolissaint's property. However, no progress has been made in the last 20 years on that question. On the other hand, Lafforgue's strong property (T) for higher rank Lie groups [L4][Pu] appears to be a fundamental obstacle to a proof of the conjecture.

2 The rank one case.

Let us now consider the case of a connected *real rank one* simple Lie group G . There is a dichotomy between two classes of groups.

2.1 The groups $SO_0(n, 1)$ and $SU(n, 1)$

Assume G is locally isomorphic to $SO_0(n, 1)$ or $SU(n, 1)$ ($n \geq 2$). Then G has the Haagerup approximation property (also known, following M. Gromov, as a-T-menability), see [J2]. Then a theorem proved by Higson and Kasparov in 1997 [HK][J2] shows that $\gamma = 1$, so that the three maps $\mu_{A,\text{red}}$, $\mu_{A,\text{max}}$ and λ_A^* are isomorphisms. In particular the Baum-Connes conjecture with coefficients is true for G . However the cases of $SO_0(n, 1)$ and $SU(n, 1)$ had already been treated by Kasparov, Kasparov and Julg, and Chen in the 1980's and early 1990's. We shall here briefly review their approaches.

In 1983 Kasparov [K2] considered the group $SO_0(2n+1, 1)$. His construction of a Fredholm module representing the element γ is based on the de Rham complex on the boundary of the hyperbolic space of dimension $2n+1$, which is a sphere S^{2n} carrying a G -invariant *conformal structure*. Let us explain briefly this point. To obtain a Fredholm module, we must modify both the operator d and the action of G on differential forms:

1) The natural action is not unitary since the metric is K -invariant, but not G -invariant. However, one can modify the representation thanks to the conformal structure. An element g of G multiplies the metric on the sphere by some function λ_g^2 . It follows that the metric on k -forms is multiplied by λ_g^{-2k}

whereas the volume form is multiplied by λ_g^{2n} . One can then twist the natural representation of G by a cocycle:

$$\pi(g)\alpha = \lambda_g^{n-k} g^{-1*} \alpha.$$

The representation π is unitary on the Hilbert space of L^2 forms of degree k .

2) The operator d can be made bounded by the usual trick: $F = d(1+\Delta)^{-1/2}$ is bounded, if $\Delta = dd^* + d^*d$ is the Laplace-Beltrami operator. But F is no more G -invariant under the natural action of G : the action of $g \in G$ takes Δ to $\lambda_g^{-2}\Delta$ plus a first order operator, as easily seen at the principal symbol level. Therefore F is a zero order operator, and its conjugate by the natural action is $\lambda_g F$ plus a negative order pseudodifferential operator.

3) Combining the two preceding idems (and the fact that F maps k -forms to $(k+1)$ -forms) we easily see that the conjugate $\pi(g)F\pi(g)^{-1}$ equals F plus a negative order pseudodifferential operator, hence the compactness of the commutator $[F, \pi(g)]$.

Finally, Kasparov notes that the Fredholm module thus obtained is not quite the good one, since its index is 2 (the Euler characteristic of S^{2n}). The following trick, again using conformal structure, solves the problem: the space $\Omega^n(S^{2n})$ of forms of the middle degree n (on which the representation π coincides with the natural one) splits under the (G -invariant) Hodge star operator into two eigenspaces. A truncated module with index 1 is obtained considering forms up to degree $\leq n-1$, and ending with forms of degree n in the i^n -eigenvalue of the Hodge operator. This gives a G -Fredholm module representing the element γ . To prove that $\gamma = 1$, Kasparov made use of the theory of unitary representations of $SO_0(2n+1, 1)$, more precisely the existence of complementary series.

In Kasparov's 1983 work, the case of $SO_0(2n, 1)$ was settled as a mere corollary of the case of $SO_0(2n+1, 1)$. Indeed $SO_0(2n, 1)$ is a subgroup of $SO_0(2n+1, 1)$ and the element γ restricts to closed subgroups. However it was most interesting to treat the case of $SO_0(2n, 1)$ in itself before passing to the other rank one groups. Indeed, $SO_0(2n, 1)$ shares with $SU(n, 1)$, $Sp(n, 1)$ or $F_{4(-20)}$ the property that the boundary sphere is odd dimensional. The direct proof for $SO_0(2n, 1)$ has been written by my student Z.Q. Chen in his thesis [Che] as an interesting toy model for the $SU(n, 1)$ case that I was then working on with Kasparov. Here again, the G -equivariant de Rham complex on S^{2n-1} is again turned, thanks to the conformal structure, into a G -Fredholm module with the bounded operator $F = d(1+\Delta)^{-1/2}$ and the unitary representation $\pi(g)\alpha = \lambda_g^{n-1/2-k} g^{-1*} \alpha$. This time the index is 0 (the Euler characteristic of S^{2n-1}). To get an Fredholm module of index 1, something new is needed, which had no analogue in the $SO_0(2n+1, 1)$ case. One must use the L^2 -cohomology of the hyperbolic space of dimension $2n$, i.e. the Hilbert space \mathcal{H}^n of square integrable harmonic forms (which are of degree n). The truncated module (with index 1) is obtained by considering only forms of degree $\leq n-1$, and completing by a map from $\Omega^{n-1}(S^{2n-1})$ to \mathcal{H}^n . For $n=1$, the map $\Omega^0(S^1) \rightarrow \mathcal{H}^1$ is just the composition of the classical Poisson transform with the de Rham differential. In general one must use P.-Y. Gaillard's Poisson transform for forms [G].

The complementary series of $SO_0(2n, 1)$ is finally used by Chen to prove that $\gamma = 1$.

In the 1990's Kasparov and I [JK] gave a proof of $\gamma = 1$ for $SU(n, 1)$. The boundary of the symmetric space is a sphere S^{2n-1} equipped with a G -invariant *contact structure*. Our construction combines the Rumin complex (which replaces the de Rham complex) with the L^2 -cohomology of the symmetric space. We do not explain it here since it will be a special case of the BGG complex and Poisson map constructions of sections 4 and 6. The homotopy is again provided by the complementary series of $SU(n, 1)$ which had already been used by Fox and Haskell in their proof of K -amenability [FH].

2.2 The groups $Sp(n, 1)$ and $F_{4(-20)}$

Assume G is locally isomorphic to $Sp(n, 1)$ ($n \geq 2$) or $F_{4(-20)}$. Then G has Kazhdan's property (T) and, as explained above, $\gamma \neq 1$. However we have suspected for a long time [J1] that it should be possible to prove the Baum-Connes conjecture with any coefficients (i.e. $\tilde{\gamma}_{A, \text{red}} = \text{Id}$) using M. Cowling's strip of uniformly bounded representations [Co].

Evidence in favour of the Baum-Connes conjecture with coefficients for groups of rank one is given by the results of V. Lafforgue. Already in his 1998 thesis [L1][L2], using Banach KK -theory and the Jolissaint rapid decrease property, he had proved the following result: if Γ is a cocompact discrete subgroup of such a group G , and more generally if Γ is any Gromov hyperbolic group, then Γ satisfies the Baum-connes conjecture *without coefficients*. But more recently, in the 2010's, he proved that such groups Γ do satisfy the Baum-Connes conjecture *with any coefficients* [L5][Pu].

We would like to present here our programme of proof of the Baum-Connes conjecture with coefficients for $G = Sp(n, 1)$ ($n \geq 2$) or $F_{4(-20)}$:

1) The first step is a geometric and analytic construction (sections 3 to 6) generalizing the case of the Kasparov-Julg-Chen constructions for $SO_0(n, 1)$ or $SU(n, 1)$. We produce in section 7 a G -Fredholm module whose class in $KK_G(\mathbf{C}, \mathbf{C})$ should be γ . Some gap remains and we state the result as conjecture 1 in section 7.4.

2) The second is a generalization of the homotopy argument. We construct in section 8 a homotopy with uniformly bounded representations. The strip of uniformly bounded representations of M. Cowling plays here the rôle of the complementary series for $SO_0(n, 1)$ and $SU(n, 1)$. Some of the estimates for the homotopy are still missing, and the result is stated in conjecture 2 in 8.4.

3 Geometry on the sphere at infinity

3.1 Lie algebra structure on the cotangent bundle

We now assume that G is one of the groups in the series $SO_0(n, 1)$, $SU(n, 1)$, $Sp(n, 1)$ ($n \geq 2$) or the exceptional group $F_{4(-20)}$. Each of the four cases

correspond to a division algebra \mathbf{K} , respectively the field \mathbf{R} of real numbers, the field \mathbf{C} of complex numbers, the (noncommutative) division algebra \mathbf{H} of quaternions and the (nonassociative) division algebra \mathbf{O} of octonions, which are respectively of real dimension $\kappa = 1, 2, 4$ and 8 . The associated symmetric space Z has dimension κn , and M the boundary at infinity is a sphere of dimension $\kappa n - 1$. Note that $n = 2$ in the exceptional case.

The group G acts transitively on M . For any point x of the sphere M , its stabilizer in G is a parabolic subgroup P_x , which admits a maximal nilpotent normal subgroup N_x . The Lie algebras \mathfrak{n}_x form a bundle of Lie algebras, which we denote by \mathfrak{n} .

Lemma 4 *There is a G -equivariant isomorphism of vector bundles between the cotangent bundle T^*M and the bundle \mathfrak{n} . We can therefore equip T^*M with a structure of G -equivariant Lie algebra bundle.*

Proof: The tangent space at x is the quotient of Lie algebras $\mathfrak{g}/\mathfrak{p}_x$. The cotangent space is the orthogonal of \mathfrak{p}_x in the dual \mathfrak{g}^* . Let us now identify \mathfrak{g}^* with \mathfrak{g} by the Killing form, which is a nondegenerate quadratic form on \mathfrak{g} . The orthogonal of \mathfrak{p}_x in \mathfrak{g}^* is identified with the orthogonal of \mathfrak{p}_x for the Killing form, which is precisely the maximal nilpotent ideal \mathfrak{n}_x of \mathfrak{p}_x . This defines a P_x -invariant isomorphism between T_x^*M and the vector space \mathfrak{n}_x . Which proves the lemma.

In the case of $SO_0(n, 1)$ Lie algebra $\mathfrak{n} = \mathfrak{n}_x$ is abelian of dimension $n - 1$. In the other cases, it is a 2-step nilpotent Lie algebra of dimension $\kappa n - 1$, with a center \mathfrak{z} of dimension $\kappa - 1$ and $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{z}$. More precisely the Lie algebra \mathfrak{n} is (non canonically) isomorphic to its associated graded Lie algebra $\text{gr}(\mathfrak{n}) = \mathfrak{n}/\mathfrak{z} \oplus \mathfrak{z}$ which is itself a generalized Heisenberg Lie algebra of the form $\mathbf{K}^{n-1} \oplus \text{Im}\mathbf{K}$ (for $\mathbf{K} = \mathbf{C}, \mathbf{H}$ or \mathbf{O}) where the bracket of two vectors $\xi, \eta \in \mathbf{K}^{n-1}$ is defined as $[\xi, \eta] = \text{Im}(\langle \xi, \eta \rangle)$ where $\langle \xi, \eta \rangle = \sum \xi_i \eta_i$.

3.2 Lie algebra structure on the graded tangent space

It is important to note that the above Lie algebra structure is defined on the cotangent space, not on the tangent space. There is no G -invariant riemannian metric on M , and it is not possible to transport the Lie bracket from T^*M to TM . However there is another Lie algebra structure, not on the bundle TM but on the graded bundle associated to the filtration given by the subbundle E defined below.

Notations. Let F be the G -equivariant subbundle of T^*M whose fiber at x corresponds to the center \mathfrak{z}_x of \mathfrak{n}_x via the isomorphism between \mathfrak{n}_x and T_x^*M . The fibers of F have dimension respectively 1, 3 or 7 (for $G = SU(n, 1), Sp(n, 1)$ or $F_{4(-20)}$). Let E be the G -equivariant subbundle of TM which is the orthogonal of F . Then E has codimension 1, 3 or 7 respectively. We consider $E \subset TM$ as a

filtration of vector bundles, and define $\text{gr}(TM) = E \oplus TM/E$ as the associated graded bundle.

Lemma 5 *Consider the Lie algebra $\Gamma(TM)$ of vector fields as filtered by the subspace $\Gamma(E)$. Then the associated graded Lie algebra is identified to $\Gamma(E \oplus TM/E)$ and its Lie bracket is bilinear with respect to multiplication by smooth functions on M . The graded tangent bundle $\text{gr}(TM) = E \oplus TM/E$ therefore carries a G -equivariant structure of graded Lie algebra bundle.*

The proof of the lemma is standard. Let us rather explain the link between the to Lie algebra bundles T^*M and $\text{gr}(TM) = E \oplus TM/E$. Given x and x' two distinct points of the sphere M , the Killing form defines a nondegenerate pairing between \mathfrak{n}_x and $\mathfrak{n}_{x'}$, in other words an isomorphism between \mathfrak{n}_x^* and $\mathfrak{n}_{x'}$. Transporting the Lie algebra structure from $\mathfrak{n}_{x'}$ to \mathfrak{n}_x^* yields a Lie bracket on \mathfrak{n}_x^* which depends on the choice of x' . However, if we equip \mathfrak{n}_x and $\mathfrak{n}_{x'}$ with the grading given by the one parameter group fixing x and x' (the group of translations on the geodesic from x to x'), the map $\text{gr}(\mathfrak{n}_{x'}) \rightarrow \text{gr}(\mathfrak{n}_x^*)$ defines on the associated graded space $\text{gr}(\mathfrak{n}_x^*) = \text{gr}(T_x M) = E_x \oplus T_x M/E_x$ a Lie algebra structure which is independent of the choice of x' . This follows from two facts: N_x acts transitively on the complement $M - \{x\}$ (so that two $\mathfrak{n}_{x'}$ for different x' 's are conjugate under N_x); and any inner automorphism of N_x acts trivially on the graded group $\text{gr}(\mathfrak{n}_x)$. One can check that the Lie algebra structure thus obtained on $\text{gr}(T_x M) = E_x \oplus T_x M/E_x$ does coincide with the structure defined in lemma 5.

Remark 1. At a given point x of M are therefore attached two Lie algebras $\mathfrak{n}_x = T_x^*M$ and

$$\tilde{\mathfrak{n}}_x = \text{gr}(\mathfrak{n}_x^*) = \text{gr}(T_x M) = E_x \oplus T_x M/E_x$$

which are isomorphic, but not canonically (the isomorphism depends on the choice of a point $z \in Z$.) The second carries a canonical grading, whereas the first does not. The Lie algebra $T_x^*M = \mathfrak{n}_x$ has a filtration, but no preferred grading. In other words, the subbundle E of TM has no canonical supplementary subbundle.

Remark 2. Consider the graded Lie group \tilde{N}_x associated to the Lie algebra $\tilde{\mathfrak{n}}_x$. The cotangent bundle $T^*\tilde{N}_x$ has a left invariant trivialisation as $\tilde{N}_x \times \tilde{\mathfrak{n}}_x^* = \tilde{N}_x \times \text{gr}(\mathfrak{n}_x)$. As a consequence the cotangent bundle $T^*\tilde{N}_x$ is equipped with a left invariant Lie algebra structure. One can consider the group \tilde{N}_x with that left invariant Lie algebra structure on $T^*\tilde{N}_x$ as a local model for the manifold M equipped with the G -invariant Lie algebra structure on T^*M .

3.3 Metrics

Let z be a point of the symmetric space Z , K_z its stabilizer (a maximal compact subgroup of G), and θ_z the associated Cartan involution. The Killing form

defines a positive definite, K_z -invariant quadratic form on the tangent space $T_z Z = \mathfrak{g}/\mathfrak{k}_z$ (identified to the orthogonal of \mathfrak{k}_z in \mathfrak{g}), thus defining on TZ a G -invariant metric, i.e. a G -invariant Riemannian metric on Z .

The situation is however very different on the boundary M : the restriction of the Killing form to any \mathfrak{n}_x is zero. To any point x in the sphere M we associate its opposite x' with respect to z . The Cartan involution θ_z maps the Lie subalgebra \mathfrak{n}_x to $\mathfrak{n}_{x'}$ which is itself mapped to \mathfrak{n}_x^* by the Killing form. This defines a map $\mathfrak{n}_x \rightarrow \mathfrak{n}_x^*$, i.e. a quadratic form on $T_x M$, which is positive definite and $K_z \cap P_x$ -invariant. One gets a K_z -invariant Riemannian metric on M .

The action of $g \in G$ on the above metrics behaves as follows. There is a cocycle λ on G , with values in the group of positive smooth functions on M , such that the metric restricted to the subbundle E , under the action of $g \in G$, is multiplied by λ_g^2 , whereas the induced metric on the quotient bundle TM/E is multiplied by λ_g^4 . We shall say that the action of G on the K_z -invariant Riemannian structure on M is quasi-conformal. In the special case of $SO_0(n, 1)$ the action of G on M is conformal in the classical sense.

Let us consider the induced metric on the graded space $\text{gr}(T_x M) = E_x \oplus T_x M/E_x$. Then a group element g acts by composing the metric with the graded Lie algebra automorphism given by multiplication by $\lambda_g(x)$ on E_x , by $\lambda_g(x)^2$ on $T_x M/E_x$. Note the difference of the action of g on the metrics on TM and $E \oplus TM/E$: on the second, the action is conformal on each of the two components, whereas on the first, the action is only quasi-conformal, with an off-diagonal component.

4 The BGG complex on the sphere at infinity.

The results and constructions below are due to A. Čap, J. Slovák, and V. Souček [CSS]. M. Rumin has also introduced [R] a non G -invariant version in the context of subriemannian geometry.

Let $\delta : \bigwedge^k T^* M \rightarrow \bigwedge^{k-1} T^* M$ be the bundle map which is at each point x the boundary map defining the *homology* of the Lie algebra $T_x^* M$. The formula defining δ is

$$\delta(\xi_1 \wedge \dots \wedge \xi_k) = \sum_{i < j} (-1)^{i+j} [\xi_i, \xi_j] \wedge \xi_1 \wedge \dots \wedge \hat{\xi}_i \wedge \dots \wedge \hat{\xi}_j \wedge \dots \wedge \xi_k.$$

In particular $\delta(\xi \wedge \eta) = [\xi, \eta]$ for $\xi, \eta \in T_x^* M$.

Let $\Omega = \Omega(M)$ be the graded algebra of differential forms on M . We consider on Ω the two operators d and δ , respectively of degree 1 and -1 . Recall that $d^2 = 0$ and $\delta^2 = 0$. It follows that the degree zero map $d\delta + \delta d$ commutes both with d and δ .

Let \mathcal{E} be the space of differential forms α on M such that $\delta\alpha = 0$ and $\delta d\alpha = 0$. It is graded by $\mathcal{E}^k = \mathcal{E} \cap \Omega^k$ and is stable by d . We thus have a subcomplex (\mathcal{E}, d) of the complex (Ω, d) .

Consider the map ι from \mathcal{E} to the sections of the quotient bundle $\ker\delta/\text{im}\delta$ obtained by composing the canonical injection $\mathcal{E} = \ker\delta \cap \ker\delta d \rightarrow \ker\delta$ with the canonical surjection $\ker\delta \rightarrow \ker\delta/\text{im}\delta$.

Theorem 6 1) $\mathcal{E} = \ker(d\delta + \delta d)$ 2) The canonical injection $\mathcal{E} \rightarrow \Omega$ induces an isomorphism in cohomology. 3) The map ι is an isomorphism from \mathcal{E} to the space of sections of the quotient bundle $\ker\delta/\text{im}\delta$.

Note that $(\ker\delta/\text{im}\delta)^k$ is the space of sections of the bundle whose fiber at $x \in M$ is the homology group $H_k(\wedge T_x^*M, \delta)$ of the Lie algebras T_x^*M at each point $x \in M$.

All the assertions of the theorem follow from the lemma:

Lemma 7 The map $d\delta + \delta d$ induces on the sections of the vector bundle $\text{im}\delta$ a differential operator which is invertible in the algebra of differential operators.

The lemma implies theorem 6:

Let us consider q the differential operator on Ω defined by $q\alpha = (d\delta + \delta d)^{-1}\delta\alpha$ where $(d\delta + \delta d)^{-1}$ is the inverse of $d\delta + \delta d$ on $\text{im}\delta$. Note that the kernel of q (resp. the image of q) is the space of sections of the vector bundle $\ker\delta$ (resp. $\text{im}\delta$).

One easily checks that $q^2 = 0$ and $qdq = q$. The operator $\pi = dq + qd$ therefore satisfies $\pi^2 = \pi$, $\pi d = d\pi = dqd$ and $\pi q = q\pi = q$. It follows immediately that $\ker\pi = \ker q \cap \ker qd = \mathcal{E}$.

Let us consider the decomposition $\Omega = \ker\pi \oplus \text{im}\pi$. Note that $\ker\pi$ and $\text{im}\pi$ are subcomplexes and that $H^*(\text{im}\pi, d) = 0$ since the map $dq + qd$ vanishes in cohomology. Therefore $H^*(\ker\pi, d) = H^*(\Omega, d)$.

It remains to show that the map from $\ker\pi = \ker q \cap \ker qd$ to $\ker q/\text{im}q$ is an isomorphism. We construct explicitly its inverse: it is given by the map $1 - qd$ which leaves $\ker q$ stable and vanishes on $\text{im}q$.

Proof of lemma 7:

The key tool in the proof of the lemma is the filtration of the bundle of exterior algebras $\wedge T^*M$ by weight: the space $\wedge_w T^*M$ of forms of weight $\geq w$ is the span of forms $\alpha \wedge \beta$ where $\alpha \in \wedge^i T^*M$, $\beta \in \wedge^F$ and $i + 2j \geq w$. Recall that F_x is the center of the Lie algebra T_x^*M . One has $\wedge_{w+1} T^*M \subset \wedge_w T^*M$ and $\wedge_w T^*M \wedge \wedge_{w'} T^*M \subset \wedge_{w+w'} T^*M$. Note that δ preserves the filtration: $\delta(\wedge_w T^*M) \subset \wedge_w T^*M$.

Passing to quotients, the bundle of graded algebras $\text{gr}(\wedge T^*M)$ i.e. the direct sum of all the quotients $\wedge_w T^*M / \wedge_{w+1} T^*M$ is canonically identified to the exterior bundle $\wedge \text{gr}(T^*M)$, where $\text{gr}(T^*M)$ is the nilpotent graded algebra bundle associated to the filtered Lie algebra bundle T^*M . Note that the boundary map δ passes to the quotient, and the induced map is the boundary map δ_0 of the Lie algebra bundle $\text{gr}(T^*M)$.

Let Ω_w be the algebra of sections of $\bigwedge_w T^*M$. The associated graded algebra

$$\text{gr}(\Omega) = \bigoplus_w \Omega_w / \Omega_{w+1}$$

is the set of sections of $\bigwedge(\text{gr}(T^*M))$. The important fact about the filtration of Ω is the following:

Proposition 8 *The de Rham operator d preserves the filtration: $d(\Omega_w) \subset \Omega_w$. Moreover, it induces on the quotient the bundle map δ_0^* which is the adjoint of the above defined δ_0 for the metric associated to any point $z \in Z$.*

The first statement is straightforward since for a section τ of F , $d\tau$ is a 2-form, hence an element of Ω_2 . When passing to the quotient, d induces (point-wise) the coboundary operator d_0 for the Lie algebra $E \oplus TM/E = \text{gr}(TM)$. The adjoint operator d_0^* on $\bigwedge(\text{gr}(T^*M))$ is given by $d_0^* = jb_0j^{-1}$ where b_0 is the boundary operator on $\bigwedge(\text{gr}(TM))$ and j the isomorphism (depending on z) : $\text{gr}(TM) \rightarrow \text{gr}(T^*M)$. But since j is a Lie algebra homomorphism, one has $jb_0 = \delta_0j$ so that $d_0^* = \delta_0$.

Let us now prove the lemma. The map $d\delta + \delta d$ preserves the filtration of Ω . Passing to the quotient yields the bundle map $d_0\delta_0 + \delta_0d_0 = d_0d_0^* + d_0^*d_0$, which is clearly invertible on $\text{im}d_0^* = \text{im}\delta_0$. The conclusion follows from the elementary fact that if a linear map preserving a filtration induces an invertible map at the graded level, then it is invertible.

Theorem 6 has the following corollary:

Corollary 9 *The operator $D = \iota dt^{-1}$ defines a differential operator on the space of sections of the bundle $\ker\delta/\text{im}\delta$ which satisfies $D^2 = 0$ and has degree one for the grading of $\ker\delta/\text{im}\delta$.*

The complex $(\Gamma(\ker\delta/\text{im}\delta), D)$ is called the *Bernstein-Gel'fand-Gel'fand complex*, or *BGG complex* [CSS]. Its cohomology is the de Rham cohomology of the manifold M . Note that D is G -invariant by construction: let $\rho(g)$ be the natural action of G on Ω and $\Gamma(C)$: for $\alpha \in \Gamma(C)$, $\rho(g)\alpha = g^{*-1}\alpha$. Then $D\rho(g) = \rho(g)D$. Let us finally mention the following fact:

Proposition 10 *The filtration $\Omega_w(M)$ of Ω restricts to a filtration of the space \mathcal{E} . It also induces a filtration of the bundle $\ker\delta/\text{im}\delta$, and the map ι is filtration preserving. Moreover the filtration on $\ker\delta/\text{im}\delta$ is associated to a canonical grading.*

The vector bundle $\ker\delta/\text{im}\delta$ is therefore given with a bigrading (degree and weight). To prove the proposition, note that the existence of the grading by weight on $\ker\delta/\text{im}\delta$ follows from the observation that two (non canonical) gradings of \mathfrak{n}_x are conjugate under N_x . But it is a standard fact that the inner automorphisms of a Lie algebra act trivially on the homology, as follows from the formula $\text{ad}X = e_X\delta + \delta e_X$.

4.1 Differential operators.

The algebra $\text{Diff}(M)$ of differential operators on a manifold M is the subalgebra of operators on $C^\infty(M)$ generated by multiplications by functions and derivations by vector fields. It has a natural filtration (order of differential operators) such that functions have order 0 and vector fields order 1. But in our situation, we rather equip $\text{Diff}(M)$ with the filtration by *weighted order* associated to the filtration of TM by the subbundle E . Only vector fields in $\Gamma(E)$ are required to be of order 1. As a result, a vector field not in $\Gamma(E)$ has weighted order 2. The symbol of an operator of weighted order l is its image in the algebra $\text{Diff}_l(M)/\text{Diff}_{l-1}(M)$ which is the space of sections of the vector bundle with fibers $U_l(E_x \oplus T_x M/E_x)$, the degree l component of the graded universal algebra of the graded Lie algebra $\tilde{\mathfrak{n}}_x = E_x \oplus T_x M/E_x$. Note that in the trivial case ($E = 0$) the universal algebra coincides with the symmetric algebra $ST_x M$ of polynomial functions on the cotangent space $T_x^* M$.

More generally one can define the algebra $\text{Diff}(M, V)$ of differential operators on the space of sections of a vector bundle V . Choose a connection ∇ on V and consider the algebra generated by bundle homomorphisms and by derivations ∇_X for vector fields X in $\Gamma(E)$. The filtration by weighted order is defined by requiring that bundle homomorphisms have order 0 and operators ∇_X for $X \in \Gamma(E)$ have order 1. The algebra and the filtration do not depend on the choice of the connection. The space of sections $\text{Diff}_l(M, V)/\text{Diff}_{l-1}(M, V)$ is the space of sections of the bundle with fibers $U_l(E_x \oplus T_x M/E_x) \otimes \text{End}(V_x)$.

Let us now turn to the case of differential operators on $\Omega(M)$, i.e. on the space of sections of the bundle $\bigwedge T^* M$ equipped as above with a decreasing filtration. There is an increasing \mathbf{Z} -filtration on $\text{End}(\bigwedge T^* M)$ defined as follows: we say $a \in \text{End}_l(\bigwedge T^* M)$ if and only if for any w ,

$$a\left(\bigwedge_w T^* M\right) \subset \bigwedge_{w-l} T^* M.$$

Choose a connection on $\Gamma(TM)$ preserving the Lie algebra structure on $T^* M$. In particular $\nabla_X(\Gamma(E)) \subset \Gamma(E)$. Such connections do exist [JvE]. We define a filtration on $\text{Diff}(M, \bigwedge T^* M)$ extending the above filtration on $\text{End}(\bigwedge T^* M)$ by requiring that ∇_X has order 1 for $X \in \Gamma(E)$. The new filtration thus obtained will be called filtration by *superweighted order*. It is independent of the choice of the connection ∇ . The space $\text{Diff}_l(M, \bigwedge T^* M)/\text{Diff}_{l-1}(M, \bigwedge T^* M)$ of symbols of superweighted order l is the space of sections of the degree l component of the graded algebra bundle $U(E_x \oplus T_x M/E_x) \otimes \text{End}(\bigwedge T_x^* M)$.

Similarly one defines a filtration on $\text{Diff}(M, C)$ where the bundle C is as above the subquotient $\ker \delta / \text{im} \delta$ of $\bigwedge T^* M$. Recall that the filtration of C comes with a grading $C = \bigoplus_w C_w$. An operator of superweighted degree l is a differential operator on $\Gamma(C)$ which is a sum of operators of weighted order j mapping sections of each C_w to C_{w-l+j} .

Proposition 11 *The operator d has superweighted order 0 in $\text{Diff}(M, \bigwedge T^* M)$. Similarly, the operator D has superweighted order 0 in $\text{Diff}(M, C)$. The operator*

D has the form $D = D_1 + D_2 + \dots$ where each operator D_l ($l \geq 1$) is a differential operator of weighted order l mapping sections of C_w^k to sections of C_{w+l}^{k+1} .

The symbol of D at a point $x \in M$ is the sum of the symbols of D_l ,

$$\sigma_x(D_l) \in U_l(\tilde{\mathfrak{n}}_x) \otimes \text{Hom}(C_{w,x}, C_{w+l,x})$$

where $\tilde{\mathfrak{n}}_x = \text{gr}(\mathfrak{n}_x^*) = E_x \oplus T_x M / E_x$.

Note that the symbol of D can be interpreted as a left invariant differential operator on the graded Lie group \tilde{N}_x associated to the Lie algebra $\tilde{\mathfrak{n}}_x$. Recall that the cotangent bundle $T^*\tilde{N}_x$ carries a left invariant Lie algebra structure. The construction of the BGG complex on M can therefore be made similarly on each group \tilde{N}_x , and one can define an operator D_x which is a left invariant operator acting on the space $\Omega(\tilde{N}_x) = C^\infty(\tilde{N}_x) \otimes C_x$ where $C_x = \ker \delta_x / \text{im} \delta_x$. Then the symbol of D is $\sigma_x(D) = D_x$.

The cohomology of the complex $(\Gamma(C), D)$ is finite dimensional since it is equal to the de Rham cohomology of a compact manifold. The finite dimensional character is related to the existence of a parametrix Q , which lives in a suitable *pseudodifferential calculus*, namely the pseudodifferential calculus modeled on convolution algebras of *graded nilpotent groups*. Unfortunately such a calculus is not very documented in the literature. We can refer to [BG] for the contact case, or [CGGP] on a graded nilpotent group. The general case is treated in [Me] but it is an unpublished preprint very difficult to find. The best approach is certainly the one based on groupoids: see [Po], [vEY][DS], or [Mo].

If N is a 2-step graded nilpotent groups, $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ its Lie algebra, with $[\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2$ and \mathfrak{n}_2 central. Equip \mathfrak{n} with a euclidian metric such that \mathfrak{n}_1 and \mathfrak{n}_2 are orthogonal. A classical fact is that the left invariant sublaplacian (or Kohn laplacian) $\sum_i X_i^2$ (where (X_i) is an orthonormal basis of \mathfrak{n}_1) has an inverse which is the right convolution by a homogeneous distribution on N .

The filtrations of the algebras of differential operators on M by weighted order (or by superweighted order) extends to similar filtration of the algebra of pseudodifferential operators. We shall consider below pseudodifferential operators acting on sections of the graded bundle C . As in the case of differential operators, a pseudodifferential operator of superweighted j is an operator whose component mapping C_w to C_{w+l} is a pseudodifferential operator of weighted order $l + j$.

Theorem 12 *There exists a pseudodifferential operator Q in the above calculus, of superweighted order 0, such that the operators $DQ + QD - 1$, Q^2 and $\pi(g)Q\pi(g)^{-1} - Q$ for $g \in G$ have superweighted order -1 .*

Sketch of proof : By the above pseudodifferential calculus, it is enough to work at the symbol level. At a given point $x \in M$, consider the graded nilpotent group \tilde{N}_x associated to the Lie algebra $\tilde{\mathfrak{n}}_x = \text{gr}(T_x M) = E_x \oplus T_x M / E_x$, and the left invariant Lie algebra structure on the bundle $T^*\tilde{N}_x$ as above. For the operator D_x Rumin [R] constructs Q_x such that $D_x Q_x + Q_x D_x = 1$ and $Q_x^2 = 0$;

Q_x is a sum of convolution operators by homogeneous distributions of weight l with values in linear maps from C_w to C_{w+l} . After lifting from symbols to pseudodifferential operators, one obtains that $DQ + QD - 1$ and Q^2 are of superweighted order -1 .

The construction of Q_x only depends on a metric on $\tilde{\mathfrak{n}}_x$. Let us choose the K_z -invariant metric for some $z \in Z$. Now if one changes the choice of z , the new metric differ from the old one by some dilation automorphism (λ, λ^2) of the Lie algebra $\tilde{\mathfrak{n}}_x = E_x \oplus T_x M/E_x$. It follows easily that the operator Q_x does not change. In other words, one has for $g \in G$, $Q_{gx} = g^{*-1} Q_x g^*$ where g^* is the natural transport of functions on \tilde{N}_{gx} with values in C_{gx} to functions on \tilde{N}_x with values in C_x . Therefore the family Q_x is a G -equivariant symbol, hence its lifting Q is such that $\pi(g)Q\pi(g)^{-1} - Q$ has superweighted order -1 .

5 The bounded version of the complex.

Let us fix a point z of the symmetric space Z and consider on the boundary M the K_z -invariant metric as above. Let λ_g be the cocycle describing its quasiconformal behaviour as in 3.3. Let us complete the space of sections $\Gamma(C)$ by the corresponding L^2 -norm. An element $g \in G$ multiplies the metric on E by λ_g^2 , and on TM/E by λ_g^4 . Therefore the metric on $\bigwedge^i E^* \otimes \bigwedge^j (TM/E)^*$ is multiplied by $\lambda_g^{-2(i+2j)}$ and that on C_w by λ_g^{-2w} . On the other hand, the volume form on M is multiplied by λ_g^ν where $\nu = \dim E + 2 \dim(TM/E)$.

It follows that the cocycle λ allows to twist the action of G on the Hilbert space $L^2\Gamma(C)$ into a unitary one. Let us consider indeed the representation $\pi(g) = \lambda_g^{\nu/2-w} \rho(g)$ where the operator $W = w$ on the subbundle C_w . In other words if $\alpha \in \Gamma(C_w)$ is a section of weight w , $\pi(g)\alpha = \lambda_g^{\nu/2-w} \rho(g)\alpha$.

Proposition 13 *The representation π of G in the Hilbert space $L^2\Gamma(C)$ is unitary.*

The complex $(L^2\Gamma(C), D)$ of unbounded operators can be turned into a complex of bounded operators by the usual trick of passing to Sobolev spaces. The Sobolev spaces adapted to our situation are not the classical ones, but those defined by a K_z -invariant *sublaplacian* on M , i.e. the operator $\Delta_E = \nabla_E^* \nabla_E$ defined as follows. We consider a connection ∇ compatible with the subbundle E as in section 4, and $\nabla_E : \Gamma(M, C) \rightarrow \Gamma(M, C \otimes E^*)$ the composition of $\nabla : \Gamma(M, C) \rightarrow \Gamma(M, C \otimes T^*M)$ with the restriction $T^*M \rightarrow E^*$ of 1-forms to the subbundle E , and ∇_E^* its formal adjoint with respect to the K_z -invariant metric on M .

The sublaplacian operator $\Delta_E = \nabla_E^* \nabla_E$ is not elliptic (except of course in the case $G = SO_0(n, 1)$) but subelliptic with $(1 + \Delta_E)^{-1}$ a pseudodifferential operator of weighted order -2 in the above calculus. More generally for any real number s , the operator $(1 + \Delta_E)^{s/2}$ has weighted order s .

Instead of considering Sobolev spaces, we prefer to remain on the space $L^2\Gamma(C)$ and conjugate the operator D by the operator $(1 + \Delta_E)^{(\nu/2-W)/2}$ on $\Gamma(C)$. The latter is by definition equal to $(1 + \Delta_E)^{(\nu/2-w)/2}$ on $\Gamma(C_w)$. Cf. [R].

Proposition 14 *The operator $D_0 = (1 + \Delta_E)^{(\nu/2-W)/2}D(1 + \Delta_E)^{(-\nu/2+W)/2}$ is of weighted order zero.*

Proof: the component of D_0 which sends C_w to C_{w+l} is

$$(1 + \Delta_E)^{(\nu/2-w-l)/2}D_l(1 + \Delta_E)^{(-\nu/2+w)/2}$$

which has weighted order $(\nu/2 - w - l) + l + (-\nu/2 + w) = 0$.

Corollary 15 *The operator D_0 is bounded on the Hilbert space $L^2\Gamma(C)$. It satisfies $D_0^2 = 0$, and the commutator $[D_0, f]$ is compact for any continuous function f on M .*

Indeed, for f smooth, the operator $[D_0, f]$ is an operator of weighted order -1 .

Let us now consider the behaviour of the above operators under the action of G . First note that $\rho(g)\Delta_E\rho(g)^{-1} - \lambda_g^{-2}\Delta_E$ is a differential operator of weighted order -1 . More generally:

Lemma 16 *For any real s ,*

$$\rho(g)(1 + \Delta_E)^{s/2}\rho(g)^{-1} - \lambda_g^{-s}(1 + \Delta_E)^{s/2}$$

is a pseudodifferential operator of weighted order $s - 1$.

Let us compare the following two representations of the group G in the space $L^2\Gamma(C_w)$

$$\pi(g) = \lambda_g^{\nu/2-W}\rho(g)$$

and

$$\rho_0(g) = (1 + \Delta_E)^{(\nu/2-w)/2}\rho(g)(1 + \Delta_E)^{-(\nu/2-w)/2}$$

Lemma 17 *The representation ρ_0 is a representation by bounded operators, and for any $g \in G$, the operator $\rho_0(g) - \pi(g)$ is compact.*

Let us denote $s = -\nu/2+w$. The operator $(\rho_0(g) - \pi(g))\rho(g)^{-1}$ is equal to the product of $(1 + \Delta_E)^{-s/2}$ (of order $-s$) by $\rho(g)(1 + \Delta_E)^{s/2}\rho(g)^{-1} - (1 + \Delta_E)^{s/2}\lambda_g^{-s}$ which is of weighted order $s - 1$ according to the lemma.

Corollary 18 *For any $g \in G$, $[D_0, \rho_0(g)] = 0$ and $[D_0, \pi(g)]$ is compact. In other words, $\pi(g)D_0\pi(g)^{-1} - D_0$ is compact.*

Proof: The equality $[D, \rho(g)] = 0$ implies $[D_0, \rho_0(g)] = 0$ after conjugaison by $(1 + \Delta_E)^{(\nu/2-W)/2}$. Hence $[D_0, \pi(g)] = [D_0, \pi(g) - \rho_0(g)]$ is compact.

Corollary 19 *The operator $Q_0 = (1 + \Delta_E)^{(\nu/2-W)/2}Q(1 + \Delta_E)^{(-\nu/2+W)/2}$ is bounded and the operators $Q_0D_0 + D_0Q_0 - 1$, Q_0^2 and $\pi(g)Q_0\pi(g)^{-1} - Q_0$ for $g \in G$ are compact.*

6 Poisson transformation for differential forms.

We first construct a G -invariant kernel on the space $M \times Z$. Let $z \in Z$ and $x \in M$. The obvious map $\mathfrak{n}_x \rightarrow \mathfrak{g}/\mathfrak{k}_z$ yields an injective map $\Phi_{z,x} : T_x^*M \rightarrow T_zZ = T_x^*Z$. Its transpose $T_zZ \rightarrow T_xM$ is the visual map in the sense of Haefliger and Gaillard [G]. We shall still denote $\Phi_{z,x}$ the functorially associated map $\Phi_{z,x} : \bigwedge^k T_x^*M \rightarrow \bigwedge^k T_x^*Z$ for any integer k .

Note that $\Phi_{z,x} \in \text{Hom}(\bigwedge^k T_x^*M, \bigwedge^k T_x^*Z)$ is $K_z \cap P_x$ -invariant. Consider the volume form $d\mu_z$ on M associated to the K_z invariant metric. Note that the form $d\mu_z(x)$ is a $K_z \cap P_x$ -invariant element of $\bigwedge^{\text{top}} T_x^*M$.

It follows that the formula

$$P(\alpha)(z) = \int_M \Phi_{z,x}(\alpha_x) d\mu_z(x)$$

defines a G -equivariant map

$$P : \Omega^k(M) \rightarrow \Omega^k(Z).$$

Let us consider the Casimir operator of the semisimple Lie algebra \mathfrak{g} . It acts on $\Omega^k(Z)$ by $-\Delta = -(dd^* + d^*d)$ (Kuga's theorem, see [BW], [Pe]) and by $-2(d\delta + \delta d)$ on $\Omega^k(M)$ (see the proof by Čap and Souček in [CS]).

From the commutation of the Casimir operator with P , one deduces the formula:

$$\Delta P(\alpha) = 2P((d\delta + \delta d)(\alpha))$$

for any $\alpha \in \Omega^k(M)$. In particular, the map P sends the subcomplex $\mathcal{E} = \ker(d\delta + \delta d)$ to the space of harmonic forms.

Recall the following classical fact (see [Pe], appendix A) about L^2 harmonic forms.

Theorem 20 *Let \mathcal{H}^k be the Hilbert space of L^2 harmonic forms of degree k on the rank one symmetric space Z . Then $\mathcal{H}^k \neq \{0\}$ if and only if $k = \dim Z/2$. In particular the L^2 cohomology of Z vanishes for $G = SO_0(2n+1, 1)$. In the other cases, it is infinite dimensional, equal to \mathcal{H}^n for $G = SO_0(2n, 1)$ or $SU(n, 1)$, to \mathcal{H}^{2n} for $G = Sp(n, 1)$ and to \mathcal{H}^8 for $G = F_{4(-20)}$.*

We now assume that $\dim Z = 2m$ is even.

Lemma 21 *The Poisson transform P induces a G -equivariant bijection from the space $d(\mathcal{E}^{m-1}) \subset \mathcal{E}^m$ to a dense subspace of \mathcal{H}^m .*

Let us recall that one has a bijective linear map $\iota : \mathcal{E}^k \rightarrow \Gamma(C^k)$ such that $D\iota = \iota d$. Let us consider the composition $S = P d \iota^{-1} = P \iota^{-1} D$ as a map $\Gamma(C^{m-1}) \rightarrow \mathcal{H}^m$. One has $SD = 0$. The operator S is a G -equivariant linear map from $\Gamma(C^{m-1}) \rightarrow \mathcal{H}^m$ with dense image and kernel $\ker D$. Let σ denote the unitary representation of G in \mathcal{H}^m , one has $\sigma(g)S = S\rho(g)$.

Let us define a parametrix as follows: take $Q' = Q\iota R$ where R is the inverse of P from \mathcal{H}^m to $d(\mathcal{E}^{m-1}) \subset \mathcal{E}^m$. The operator $DQ + Q'S - 1$ on $\Gamma(C^{m-1})$ is

pseudodifferential of superweighted order -1 , and $SQ' = 1$ on \mathcal{H}^m . Moreover QQ' and $\rho(g)Q'\sigma(g) - Q'$ are composition with the map $\iota R : \mathcal{H}^m \rightarrow \Gamma(C^m)$ of pseudodifferential operators of superweighted order -1 .

One can make S bounded as follows:

Proposition 22 *The map $S_0 = S(1 + \Delta_E)^{(\nu/2-W)/2}$ is bounded. One has $S_0D_0 = 0$.*

Proposition 23 *For any $g \in G$, the operator $\sigma(g)S_0\pi(g)^{-1} - S_0$ is compact.*

Indeed the relation $\sigma(g)S = S\rho(g)$ implies $\sigma(g)S_0 = S_0\rho_0(g)$. The proposition follows from the fact that $\pi(g) - \rho_0(g)$ is compact.

Take $Q'_0 = (1 + \Delta_E)^{(-\nu/2+W)/2}Q'$.

One has $S_0Q'_0 = 1$ on \mathcal{H}^m , and the operator $D_0Q_0 + Q'_0S_0 - 1$ on $L^2\Gamma(C^{m-1})$ is compact. Moreover $Q_0Q'_0$ and $\rho(g)Q'_0\sigma(g) - Q'_0$ are compact on \mathcal{H}^m .

7 Representing the γ element.

7.1 KK_G -element associated to a G -Fredholm complex.

Let H be a Hilbert space equipped with:

- (i) a \mathbf{Z} -grading $H = \bigoplus H_k$
- (ii) a degree 0 unitary representation π
- (iii) a degree 1 bounded operator F satisfying the following three conditions: $F^2 = 0$; $[F, \pi(g)]$ is compact for any $g \in G$; there exists a degree -1 bounded operator Q such that Q^2 , $FQ + QF - 1$ and $[Q, \pi(g)]$ (for $g \in G$) are compact operators.

To such a G -Fredholm complex we associate a class in $KK_G(\mathbf{C}, \mathbf{C})$ as follows: let us equip the Hilbert space H with the $\mathbf{Z}/2$ -grading by the parity of the \mathbf{Z} -degree, the representation π of G and the operator $T = F + Q$. We obtain a G -Fredholm module whose class in $KK_G(\mathbf{C}, \mathbf{C})$ is independent of the choice of the parametrix Q (see [L2]).

7.2 The original G -complex.

Let us consider the following complex of Hilbert spaces:

$$L^2\Gamma(C^0) \xrightarrow{D_0} L^2\Gamma(C^1) \xrightarrow{D_0} \dots \xrightarrow{D_0} L^2\Gamma(C^{2m-1})$$

The \mathbf{Z} -graded Hilbert space

$$L^2\Gamma(C) = \bigoplus_{k=0}^{2m-1} L^2\Gamma(C^k)$$

together with the unitary representation π defined by

$$\pi(g) = \lambda_g^{\nu/2-w} \rho(g)$$

on each $L^2\Gamma(C_w^k)$, and the bounded operator D_0 do satisfy the above conditions.

Note that the class of the complex $(L^2\Gamma(C), \pi, D_0)$ in $KK_G(\mathbf{C}, \mathbf{C})$ cannot be γ since its restriction to a maximal compact subgroup K of G is 0 (indeed, the cohomology of D_0 is the same as the de Rham cohomology of the sphere M , which vanishes except in dimensions 0 and $2m-1$).

However one can show that the class of the complex $(L^2\Gamma(C), \pi, D_0)$ is in the subgroup $\gamma KK_G(\mathbf{C}, \mathbf{C})$. This follows from the observation that it belongs to the image of $KK_G(C(M), \mathbf{C})$ with $M = G/P$ and P amenable. See [K2].

7.3 The truncated complex.

Let us truncate the complex $(L^2\Gamma(C), D_0)$ to make it of index 1. We keep the left half of the complex and complete in degree m with the L^2 -cohomology \mathcal{H}^m of $Z = G/K$.

In other words we consider the complex of Hilbert spaces:

$$L^2\Gamma(C^0) \xrightarrow{D_0} L^2\Gamma(C^1) \xrightarrow{D_0} \dots \xrightarrow{D_0} L^2\Gamma(C^{m-1}) \xrightarrow{S_0} \mathcal{H}^m$$

The Hilbert space

$$H = \bigoplus_{k=0}^{m-1} L^2\Gamma(C^k) \oplus \bigoplus \mathcal{H}^m$$

is equipped with:

- (i) the \mathbf{Z} -grading by degree.
- (ii) the degree 0 unitary representation π defined by

$$\pi(g) = \lambda_g^{\nu/2-w} \rho(g)$$

on each $L^2\Gamma(C_w^k)$ for $k \leq m-1$ and $\pi(g) = \sigma(g)$ on \mathcal{H}^m .

(iii) the degree 1 bounded operator F equal to D_0 on $L^2\Gamma(C^k)$ ($k \leq m-2$) and to S_0 on $L^2\Gamma(C^{m-1})$

The complex (H, π, F) defines an element of $KK_G(\mathbf{C}, \mathbf{C})$ whose image in $R(K)$ is equal to the unit 1_K of $R(K)$.

7.4 Link with γ .

Conjecture 1 *The class of the complex (H, π, F) is equal to $\gamma \in KK_G(\mathbf{C}, \mathbf{C})$.*

To show conjecture 1 we consider the compactification $\bar{Z} = Z \cup M$ of $Z = G/K$ by the boundary sphere $M = G/P$. Let us use the lemma of [JK]:

Lemma 24 *An element of $KK_G(\mathbf{C}, \mathbf{C})$ which is in the image of $KK_G(C(\bar{Z}), \mathbf{C}) \rightarrow KK_G(\mathbf{C}, \mathbf{C})$ and maps to 1 in $R(K)$ is equal to γ .*

The conjecture would therefore follow from the fact that the class of (H, π, F) in $KK_G(\mathbf{C}, \mathbf{C})$ is in the image of $KK_G(C(\bar{Z}), \mathbf{C})$.

The algebra $C(\bar{Z})$ acts on $L^2\Gamma(C^k)$ via $C(M)$. It also has an action via $C_b(Z)$ on $L^2\Omega^m(Z)$, but the subspace \mathcal{H}^m is not stable.

To fix the latter difficulty, we enlarge the complex (H, π, F) to allow an action of $C(\bar{Z})$. Let

$$\tilde{H} = \bigoplus_{k=0}^{m-1} L^2\Gamma(C^k) \oplus \bigoplus_{k=0}^{2m} L^2\Omega^k(Z)$$

be graded by form degree. We equip \tilde{H} with the unitary representations $\tilde{\pi}$ equal to π on $L^2\Gamma(C)$ and to σ (the natural action) on $L^2\Omega(Z)$. The operator is modified as follows: \tilde{F} is equal to D_0 on $L^2\Gamma(C^k)$ ($0 \leq k \leq m-2$), to S_0 on $L^2\Gamma(C^{m-1})$ (with values in $\mathcal{H}^m \subset L^2\Omega^m(Z)$) and to $d_0 = d(1 + \Delta)^{-1/2}$ on $L^2\Omega^k(Z)$ ($0 \leq k \leq 2m$).

The complex $(\tilde{H}, \tilde{\pi}, \tilde{F})$ clearly has the same image as (H, π, F) . To show that it belongs to the image of $KK_G(C(\bar{Z}), \mathbf{C})$, one has to estimate the commutators of the operator \tilde{F} with the action of $f \in C(\bar{Z})$.

The case of the D_0 -component has already been checked. The d_0 -component is dealt with as in [JK]. It remains to show that $[S_0, f]$ is compact as an operator from $L^2\Gamma(C^{m-1})$ to $L^2\Omega^m(Z)$. A proof has been given in the case of $SO_0(2n, 1)$ by Chen and in the case of $SU(n, 1)$ by Kasparov and myself. We do not yet have the estimates for the case of $Sp(n, 1)$ and $F_{4(-20)}$. In principle it should be possible to generalize the method of [JK] with explicit computations, but unfortunately the combinatorics of the representations of $K = Sp(n)Sp(1)$ are much more complicated than for $U(n)$. We look for some new ideas involving explicit formulas for the Poisson kernels and estimates near the boundary.

8 How to prove that $\tilde{\gamma}_{A,\text{red}} = \text{Id}$?

8.1 From unitary to uniformly bounded representations.

We shall produce a homotopy of Fredholm complexes from (H, π, F) to a trivial complex of index 1. The Hilbert space H and the operator F will be constant, but the representations on each $L^2\Gamma(C_w^k)$ will be deformations from

$$\pi_0(g) = \pi(g) = \lambda_g^{\nu/2-w} \rho(g)$$

to

$$\pi_1(g) = \rho_0(g) = (1 + \Delta_E)^{(\nu/2-w)/2} \rho(g) (1 + \Delta_E)^{-(\nu/2-w)/2}.$$

Consider to that effect the family of representations

$$\pi_s(g) = (1 + \Delta_E)^{(\nu/2-w)s/2} \lambda_g^{(\nu/2-w)(1-s)} \rho(g) (1 + \Delta_E)^{-(\nu/2-w)s/2}$$

Lemma 25 *For any $s \in \mathbf{C}$ and $g \in G$ the operator $\pi_s(g)$ is bounded and $\pi_s(g) - \pi_0(g)$ is compact.*

The proof is the same as in lemma 16.

Let us now state a special case of a theorem due to Michael Cowling [Co][ACD]. We say that a representation π is uniformly bounded if

$$\sup_{g \in G} \|\pi(g)\| < \infty.$$

Theorem 26 *The representation π_s on $L^2\Gamma(C_w)$ is uniformly bounded for any $s \in \mathbf{C}$ such that*

$$|\operatorname{Res}| < \frac{\nu/2}{|\nu/2 - w|}$$

Recall that $\nu = \dim E + 2 \dim(TM/E)$ and note that the values of w range between 0 and ν , so that $\nu/2 \leq |\nu/2 - w|$ with equality only for $w = 0$ or ν , i.e. $k = 0$ or $2m - 1$.

Corollary 27 *The representations π_s are uniformly bounded if $s \in [0, 1[$. The representation π_1 is uniformly bounded on $L^2\Gamma(C^k)$ for $k \neq 0$ or $2m - 1$.*

We consider the homotopy from $s = 0$ to $s = 1$. We start at $s = 0$ with the unitary representations π_0 and the Fredholm complex which, according to conjecture 1, represents the element γ . At the other end, for $s = 1$ the operator F exactly commutes with the representation π_1 . *Formally* the family (H, π_s, F) from $s = 0$ to $s = 1$ provides the homotopy from γ to 1, but only in a weak sense, since the representations involved are no more unitary.

8.2 Uniformly bounded representations and K -theory

The fact which makes the uniformly bounded representations work for our problem is the following theorem. We define a uniformly bounded G -Fredholm module as the data (H, π, F) where H is a Hilbert space, π a *uniformly bounded representation* and F a Fredholm operator. We define a *homotopy* of uniformly bounded G -Fredholm module in the same way, replacing Hilbert spaces by $C[0, 1]$ -Hilbert modules (i.e. continuous fields of Hilbert spaces).

Let us denote after Kasparov [K1] $R(G) = KK_G(\mathbf{C}, \mathbf{C})$. Define as in [J1] $R_{\text{ub}}(G)$ the group of homotopy classes of uniformly bounded G -Fredholm modules.

Theorem 28 *For any $G - C^*$ -algebra A , the Kasparov map*

$$R(G) \rightarrow \operatorname{End}K_*(C_{\text{red}}^*(G, A))$$

factors through the map $R(G) \rightarrow R_{\text{ub}}(G)$.

This follows from the following easy lemma and its corollary:

Lemma 29 1) Let π be a uniformly bounded representation of G in a Hilbert space H . Let λ be the left regular representation of G on $L^2(G)$. There exists an operator U on $H \otimes L^2(G)$, which is bounded and has a bounded inverse, such that

$$\pi(g) \otimes \lambda(g) = U(1 \otimes \lambda(g))U^{-1}$$

2) If moreover π is a unitary representation, U is a unitary operator.

To any Hilbert space H equipped with a uniformly bounded representation π , let us associate as in [K1] the Hilbert module $E = H \otimes C_{\text{red}}^*(G, A)$ and the covariant representation of (G, A) with values in $\mathcal{L}_{C_{\text{red}}^*(G, A)}(E)$ defined by:

$$a \mapsto 1 \otimes a, \quad g \mapsto \pi(g) \otimes \lambda(g).$$

Corollary 30 The representation $\pi_A : C_c(G, A) \rightarrow \mathcal{L}_{C_{\text{red}}^*(G, A)}(E)$ extending the above covariant representation factors through the reduced crossed product $C_{\text{red}}^*(G, A)$.

Proof. The reduced crossed product $C_{\text{red}}^*(G, A)$ is by definition a sub- C^* -algebra of $\mathcal{L}_A(L^2(G) \otimes A)$. The C^* -algebra $\mathcal{L}_{C_{\text{red}}^*(G, A)}(E)$ is a sub- C^* -algebra of $\mathcal{L}_A(H \otimes L^2(G) \otimes A)$. It follows easily from lemma 1 that for any $a \in C_c(G, A)$, one has

$$\pi_A(a) = U(1 \otimes \lambda_A(a))U^{-1}$$

where

$$\lambda_A : C_c(G, A) \rightarrow C_{\text{red}}^*(G, A) \rightarrow \mathcal{L}_A(L^2(G) \otimes A)$$

The map π_A therefore extends to a continuous (but in general not $*$) homomorphism

$$\pi_A : C_{\text{red}}^*(G, A) \rightarrow \mathcal{L}_{C_{\text{red}}^*(G, A)}(E).$$

Proof of the theorem. Let us construct the map $R_{\text{ub}}(G) \rightarrow \text{End}K_*(C_r^*(G, A))$. To a G -Fredholm module (H, π, T) we associate the triple $(H \otimes C_{\text{red}}^*(G, A), \pi_A, T_A)$ where $\pi_A : C_{\text{red}}^*(G, A) \rightarrow \mathcal{L}_{C_{\text{red}}^*(G, A)}(E)$ is the Banach algebra homomorphism defined above, and $T_A = T \otimes \mathbb{1} \in \mathcal{L}_{C_{\text{red}}^*(G, A)}(E)$.

The Banach G -Fredholm module thus obtained defines a map from the group $K_*(C_{\text{red}}^*(G, A))$ to itself. Note that such a construction has no analogue for $C_{\text{max}}^*(G, A)$ since it relies upon a specific feature of the regular representation.

8.3 Slow exponential growth

Unfortunately, it is not quite enough to work with uniformly bounded representations. Indeed, the representation π_1 on $L^2(M)$ is not uniformly bounded, and when $s \rightarrow 1$, the uniform bound of π_s on $L^2(M)$ is not bounded. Therefore the family π_s cannot define a homotopy on the closed segment $[0, 1]$. To overcome the difficulty we introduce a wider class of representations.

Fix $\varepsilon > 0$. We say that a representation π of G is of ε -exponential type if there is a constant C such that for any $g \in G$,

$$\|\pi(g)\| \leq Ce^{\varepsilon l(g)}$$

where $l(g) = d(gz, z)$ is the length of g (z is a chosen origin in Z as above). We define as above a G -Fredholm module of ε -exponential type, and similarly a homotopy of such modules. Let $R_\varepsilon(G)$ be the abelian group of homotopy classes. The obvious maps $R_\varepsilon(G) \rightarrow R_{\varepsilon'}(G)$ for $\varepsilon < \varepsilon'$ form a projective system and we consider the projective limit $\varprojlim R_\varepsilon(G)$ when $\varepsilon \rightarrow 0$.

We would like to have an analogue of theorem 28 with the group $\varprojlim R_\varepsilon(G)$ instead of $R_{\text{ub}}(G)$. In fact there is a slightly weaker result, due to N. Higson and V. Lafforgue (cf [L5], théorème 2.3) which is enough for our purpose:

Theorem 31 *The kernel of the map*

$$R(G) \rightarrow \varprojlim R_\varepsilon(G)$$

is included in the kernel of the map

$$R(G) \rightarrow \text{End}K_*(C_r^*(G, A)).$$

Let us sketch the proof following [L5]. As above, to any representation π of G is associated a $*$ -homomorphism

$$\pi_A : C_c(G, A) \rightarrow \mathcal{L}_{C_r^*(G, A)}(E)$$

where $E = H \otimes C_{\text{red}}^*(G, A)$.

For all $\varepsilon > 0$ there is a Banach algebra C_ε which is a completion of $C_c(G, A)$ such that for any representation π of ε -exponential type, the above map π_A extends to a bounded map $C_\varepsilon \rightarrow \mathcal{L}_{C_r^*(G, A)}(E)$. The Banach Fredholm module thus obtained defines a map

$$R_\varepsilon(G) \rightarrow \text{Hom}(K_*(C_\varepsilon), K_*(C_r^*(G, A))).$$

This being done for each ε , we have a system of maps compatible with the maps $C_\varepsilon \rightarrow C_{\varepsilon'}$ for $\varepsilon' < \varepsilon$, so that there is a commutative diagramme (cf [L5] prop 2.5)

$$\begin{array}{ccc} R(G) & \rightarrow & \text{End}K_*(C_r^*(G, A)) \\ \downarrow & & \downarrow \\ \varprojlim R_\varepsilon(G) & \rightarrow & \varprojlim \text{Hom}(K_*(C_\varepsilon), K_*(C_r^*(G, A))). \end{array}$$

The theorem of Higson-Lafforgue then follows immediately, thanks to the following lemma:

Lemma 32 *The group $K_*(C_r^*(G, A))$ is the union of the images of the maps $K_*(C_\varepsilon) \rightarrow K_*(C_r^*(G, A))$.*

To prove the lemma, Higson and Lafforgue use the fact that the symmetric space Z has finite asymptotic dimension. They give an estimate of the form (prop 2.6 in [L6])

$$\|f\|_{C_\varepsilon} \leq k_\varepsilon e^{\varepsilon(ar+b)} \|f\|_{C_r^*(G,A)}$$

for $f \in C_c(G, A)$ with support in a ball of radius r (for the length l).

The spectral radius formula in Banach algebras then implies for such an f ,

$$\rho_{C_\varepsilon}(f) \leq e^{\varepsilon ar} \rho_{C_r^*(G,A)}(f),$$

so that $\rho_{C_r^*(G,A)}(f) = \inf \rho_{C_\varepsilon}(f)$. A fact which, by standard holomorphic calculus techniques, implies the lemma.

8.4 Towards the end of the proof.

Conjecture 2 For any $\varepsilon > 0$, the class of the module (H, π, F) defined in 7.3. maps to 1 under the map $R(G) = KK_G(\mathbf{C}, \mathbf{C}) \rightarrow R_\varepsilon(G)$.

Conjecture 2, together with theorem 31, implies that for any $G - C^*$ -algebra A , the class of (H, π, F) in $KK_G(\mathbf{C}, \mathbf{C})$ maps to the identity under the map

$$R(G) = KK_G(\mathbf{C}, \mathbf{C}) \rightarrow \text{End}K_*(C_r^*(G, A)).$$

Therefore conjecture 1 and conjecture 2 combined together imply that the element γ maps to 1 in $\text{End}K_*(C_r^*(G, A))$. In other words, with the notations of section 1, one has $\tilde{\gamma}_{A, \text{red}} = \text{Id}$. This proves the Baum-Connes conjecture for G with coefficients in A .

The main step in the proof of conjecture 2 should be the solution of the following. To simplify notations, put $L = (1 + \Delta_E)^{1/2}$. Let $\varepsilon > 0$ and $w < \nu/2$.

Problem. Show that there exists constants C and a such that for any $s \in [1/2, 1]$ the formula

$$\pi_s^\varepsilon(g) = L^{-\varepsilon} \pi_s(g) L^\varepsilon$$

defines a representation of G on $L^2\Gamma(C_w)$, satisfying

$$\|\pi_s^\varepsilon(g)\| \leq C e^{\varepsilon a l(g)}.$$

Note that

$$\pi_s^\varepsilon(g) = \Lambda_g \pi_{s'}(g)$$

where $s' = s - \frac{\varepsilon}{\nu/2-w}$ and where the cocycle operator Λ_g is defined by

$$\Lambda_g = L^{(\nu/2-w)s'} \lambda_g^{-\varepsilon} L^{-(\nu/2-w)s'}$$

The representations $\pi_{s'}$ are uniformly bounded with a bound independant of s because (for ε small enough, $s' = s - \frac{\varepsilon}{\nu/2-w}$ belongs to a compact interval

strictly contained in $[0, \frac{\nu/2}{|\nu/2-w|}]$. To solve our problem, it remains to show that $\|\Lambda_g\| \leq e^{\varepsilon a l(g)}$ for some $a > 0$.

Let us sketch how conjecture 2 would then follow. Fix any $\varepsilon > 0$. We want to show that the image of the element γ in $R(G)$ in $R_\varepsilon(G)$ is equal to 1.

Consider conjugation by $L^{-\varepsilon}$ and define $F_\varepsilon = L^{-\varepsilon} F L^\varepsilon$. There is a parametrix $Q_\varepsilon = L^{-\varepsilon} Q L^\varepsilon$. The commutator $[D_\varepsilon, \pi_s^\varepsilon(g)]$ is compact for any $g \in G$. For any $s \in [0, 1]$ the complex $(H, \rho_s^\varepsilon, F_\varepsilon)$ has a class in $R_\varepsilon(G)$ independent of s . Then observe that:

At $s = 0$ this class is the image of γ : one can indeed consider the homotopy $(H, \rho_s^{t\varepsilon}, F_{t\varepsilon})$ for $t \in [0, 1]$.

At $s = 1$ the class is equal to 1 since $[D^\varepsilon, \rho_1^\varepsilon(g)] = 0$. Indeed the complex tD_ε has parametrix $t^{-1}Q_\varepsilon$ and the commutator $[tD_\varepsilon + t^{-1}Q_\varepsilon, \rho_1^\varepsilon(g)]$ tends to 0 as t goes to infinity. Cf. [L2].

This would end the proof.

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