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▶ To cite this version:

Anna Kamińska, Yves Raynaud. Abstract Lorentz spaces and Köthe duality. Indagationes Mathematicae, 2019, 30, pp.553 - 595. 10.1016/j.indag.2019.02.002 . hal-03484973

HAL Id: hal-03484973

https://hal.science/hal-03484973

Submitted on 20 Dec 2021

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Version of Record: https://www.sciencedirect.com/science/article/pii/S0019357719300175
Manuscript_e458a78f5a0f33982df6c9099a283480

ABSTRACT LORENTZ SPACES AND KÖTHE DUALITY

ANNA KAMIŃSKA AND YVES RAYNAUD

Abstract. Given a symmetric Banach function space E and a decreasing positive weight w on $I = (0, a)$, $0 < a \leq \infty$, the generalized Lorentz space $\Lambda_{E,w}$ is defined as the symmetrization of the canonical copy E_w of E on the measure space associated with the weight. A class of functions $M_{E,w}$ is similarly defined in the spirit of Marcinkiewicz spaces as the symmetrization of the space $w\,E_w$. Differently from the Lorentz space, which is a Banach function space, the class $M_{E,w}$ does not need to be even a linear space; but we show that if the weight w is regular then this class is normable. Let also $Q_{E,w}$ be the smallest fully symmetric Banach function space containing $M_{E,w}$. The Köthe duality of these classes is developed here. The Köthe dual of the class $M_{E,w}$ is identified as the Lorentz space $\Lambda_{E',w}$, while the Köthe dual of $\Lambda_{E,w}$ is $Q_{E',w}$. Several characterizations of $Q_{E,w}$ are obtained, one of them states that a function belongs to $Q_{E,w}$ if and only if its level function in Halperin's sense with respect to w, belongs to $M_{E,w}$. The other characterizations are by optimization with respect to the Hardy-Littlewood submajorization order. These results are applied to a number of concrete Banach function spaces. In particular a new description of the Köthe dual space is provided for the classical Lorentz space $\Lambda_{p,w}$ and for the Orlicz-Lorentz space $\Lambda_{\varphi,w}$, which correspond respectively to the cases $E = L_p$ and $E = L_\varphi$.

Given a positive locally integrable weight w on an interval $I = (0, a)$, $0 < a \leq \infty$, and $p \in [1, \infty)$, the classical Lorentz space $\Lambda_{p,w}$ is the set of measurable functions f having a non-increasing rearrangement f^* such that $\int_I (f^*)^p w\, dm < \infty$, where m denotes the Lebesgue measure. This class is a symmetric Banach function space and the formula $\|f\|_{p,w} = (\int_I (f^*)^p w\, dm < \infty)^{1/p}$ defines a norm if and only if the weight w is non-increasing [1, 13]. Orlicz-Lorentz spaces may be defined in a similar way. Following [9], given an Orlicz function φ, consider the modular Φ defined on the set of Lebesgue measurable functions $L^0(I)$ by

$$\Phi(f) = \int_I \varphi(f^*) w\, dm.$$

Then the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ is the set of $f \in L^0(I)$ such that $\{c : \Phi(f/c) < \infty\} \neq \emptyset$. If w is non-increasing then Φ is convex, $\Lambda_{\varphi,w}$ is a linear subset and an ideal in $L^0(I)$, and the formula $\|f\|_{\varphi,w} := \inf\{c : \Phi(f/c) \leq 1\}$ defines a norm, called the Luxemburg or second Nakano norm, for which $\Lambda_{\varphi,w}$ is complete and symmetric. Clearly if $\varphi(t) = t^p$ we recover the classical Lorentz space $\Lambda_{p,w}$, so that Orlicz-Lorentz spaces are a generalization of ordinary Lorentz spaces. We refer to [12, 10] for a study of Köthe duality of Orlicz-Lorentz spaces.

Our goal in this paper is to generalize further the class of Orlicz-Lorentz spaces by replacing Orlicz spaces by general symmetric Köthe function spaces. We will use the fact that the classical Lorentz spaces $\Lambda_{p,w}$ and the Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ respectively, are symmetrizations [11] with respect to the Lebesgue measure of the spaces L_p and Orlicz

Date: January 31, 2019.
2010 *Mathematics Subject Classification.* 46B20, 46E30, 47B38.
Key words and phrases. Lorentz space, Köthe duality, Banach function space, symmetric space, rearrangement inequalities, modular space.

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spaces L_φ on the measure space (I, wdm), respectively. Clearly the latter spaces are weighted $L_p(I, wdm)$ and $L_\varphi(I, wdm)$ spaces respectively, and thus they are symmetric with respect to the measure wdm on I. Thus it is natural to consider symmetrizations, with respect to m, of arbitrary Köthe function spaces $\mathcal{E}$ which are symmetric with respect to the measure wdm. However, since we do not want the space parameter, which will play the role of the exponent p or the Orlicz function φ, to depend on the weight w, we choose to take as space $\mathcal{E}$ the natural copy E_w on the measure space (I, wdm) of a symmetric Köthe function space E defined on the measure space (J, m) where J is an interval $(0, b)$, $b \in (0, \infty]$. The symmetrization of E_w will thus be denoted by $\Lambda_{E,w}$ and called a *generalized Lorentz space*.

In this paper w will always denote a non-increasing weight. We assume also that E is *fully symmetric* that is E is hereditary by Hardy-Littlewood submajorization and the norm is monotone with respect to this submajorization. It follows from these hypotheses that the set $\Lambda_{E,w}$ is a linear space and that the formula $\|f\|_{\Lambda_{E,w}} = \|f^*\|_{E_w}$ defines a norm on it. In fact $\Lambda_{E,w}$ is a fully symmetric Banach function space.

The goal is to provide a description of the Köthe dual space of the Lorentz space $\Lambda_{E,w}$ in this abstract form, following the pattern of our previous article [12] on Orlicz-Lorentz spaces. It was proved in [7] that when the weight w is *regular* the Köthe dual of $\Lambda_{\varphi,w}$ is equal to the symmetrization of the Köthe dual of the weighted Orlicz space $(L_\varphi)_w$. Since $(E_w)' = w \cdot (E')_w$ with equal norms, where E' means the Köthe dual of E, it is natural to introduce for a general symmetric space E, not only for Köthe duals, the "class" $M_{E,w}$ defined by

$$M_{E,w} = \{f \in L^0(I) : f^* \in w \cdot E_w\} = \{f \in L^0(I) : f^*/w \in E_w\}.$$

This class is closed under scalar multiplication but not necessarily by sums, hence it may even not be a linear subspace of $L^0(I)$. It may be equipped with a gauge $\|f\|_{M_{E,w}} = \|f^*/w\|_{E_w}$, which is positively homogeneous, faithful, monotone and symmetric. In section 4 we prove that if the weight w is regular then the class $M_{E,w}$ is a linear subspace of $L^0(I)$ and its gauge is equivalent to a norm. The proof of this latter result is based on an optimization formula for the gauge which is of interest by itself, namely

$$\|f\|_{M_{E,w}} = \inf\{\|f/v\|_{E_v} : v \geq 0, v^* = w, \operatorname{supp} v \supset \operatorname{supp} f\}.$$

A similar formula was proved in the setting of Orlicz spaces and modulars, in our article [12]. The proof depended on a certain inequality for rearrangements and weights [12, Proposition 2.1], that cannot have any equivalent form in the present setting. Here this argument is replaced by a completely new one, namely a submajorization formula for rearrangements and weights, which is stated and proved in section 3 (see Theorem 3.1).

Although the class $M_{E,w}$ may not be a vector space and its gauge may not be convex, its Köthe dual space can be defined as the domain of finiteness of the dual function norm $L^0(I) \to [0, +\infty]$,

$$\|f\|_{(M_{E,w})'} = \sup\left\{\int_I |fg|\, dm : g \in M_{E,w}, \|g\|_{M_{E,w}} \leq 1\right\},$$

which is an ideal in $L^0(I)$ normed by the above function norm, and a Banach function space with Fatou property.

The next step, performed in section 5, is to show that the Köthe dual of $M_{E,w}$ coincides isometrically with the Lorentz space $\Lambda_{E',w}$, where E' is the Köthe dual space of E. The proof of this fact is very similar to that given in the setting of Orlicz-Lorentz spaces in [12]. As a corollary we obtain that if the weight w is regular, and E has Fatou property, then the Köthe dual to $\Lambda_{E,w}$ is equal as a set to $M_{E',w}$, with equivalence of their respective norm and gauge.

In section 6 we introduce the class $Q_{E,w}$ consisting of all elements of $L^0(I)$ which are submajorized by some element of $M_{E,w}$. It is easy to verify that $Q_{E,w}$ is an ideal in $L^0(I)$, which is clearly hereditary by submajorization and contains $M_{E,w}$. The formula

$$\|f\|_{Q_{E,w}} = \inf\{\|g\|_{M_W} : f \prec g\}, \tag{0.1}$$

where the symbol $\prec$ denotes the Hardy-Littlewood submajorization, defines a very natural gauge on $Q_{E,w}$ which turns out to be a norm. Equipped with this norm, $Q_{E,w}$ is a fully symmetric Banach function space, the smallest one containing $M_{E,w}$. Moreover its Köthe dual space coincides isometrically with $\Lambda_{E',w}$.

If E has Fatou property one may exchange the roles of E and E', thus $(Q_{E',w})' = \Lambda_{E,w}$, and $(\Lambda_{E,w})' = (Q_{E',w})''$. For deriving our final duality result that $(\Lambda_{E,w})' = Q_{E',w}$, we need to know that $Q_{E',w}$ has Fatou property, and thus equals to its second Köthe dual. This is shown in section 6. A general proof of the latter fact does not seem easy without knowledge of a minimizer g for the right hand side of the equation (0.1) defining the $Q_{E,w}$ norm of an element f. In fact Halperin's level function f^0 of the decreasing rearrangement f^* is such a minimizer, in other words we prove that $\|f\|_{Q_{E,w}} = \|f^0\|_{M_W}$.

At this point we should remark that the path followed here differs from that in [12], where the spaces $Q_{\varphi,w}$ were not introduced. Instead we initiated there another scale of spaces $P_{\varphi,w}$, the analogue of which we define and discuss now.

In section 7 we define the class $P_{E,w}$ consisting of the union of all classes $M_{E,v}$, for all positive decreasing weights v submajorized by w. This class is equipped with the gauge

$$\|f\|_{P_{E,w}} = \inf\{\|f\|_{M_{E,v}} : v, 0 < v \prec w\}.$$

Contrary to the case of Orlicz-Lorentz spaces, we did not find direct evidence that these classes are linear and these gauges are norms. In the present paper this fact is proven indirectly, at least if E has Fatou property, by showing that $P_{E,w} = Q_{E,w}$, with equality of gauges.

Finally we obtain three different formulas of the norm in the dual Köthe space to Lorentz space $\Lambda_{E,w}$. In fact we have that for $f \in (\Lambda_{E,w})'$,

$$\|f\|_{(\Lambda_{E,w})'} = \inf\{\|g\|_{M_{E',w}} : f \prec g\} = \inf\{\|f\|_{M_{E',v}} : v \prec w, v > 0, v \downarrow\} = \|f^0\|_{M_{E',w}}. \tag{0.2}$$

Let us mention that the expression of the dual norm on $(\Lambda_{E,w})'$ given in terms of the level function by equation (0.2) is implicit in Sinnamon's work [22] (see Theorem 2.2 and Corollary 2.4 there), as it appears clearly once the relationship between Sinnamon's level functions and Halperin's ones has been elucidated like in [4, p. 64]. Our methods however are different and the two infimal expressions in (0.2) seem to be new.

If $E = L_\varphi$ is an Orlicz space then $\Lambda_{E,w}$ is an Orlicz-Lorentz space $\Lambda_{\varphi,w}$ [11, 12], and we obtain that the norm in its dual space is expressed in three different ways following from equalities (0.2). But in section 8.2 we consider $L_\varphi(J)$ as a modular function space [16], equipped with its natural convex modular

$$I_\varphi(f) = \int_J \varphi(|f|)\, dm.$$

Then the Orlicz-Lorentz space $\Lambda_{\varphi,w}$ inherits of a modular structure defined by the convex modular $\Phi_w(f) = \int_I \varphi(f^*) w\, dm$, while the class $M_{\varphi,w}$ is equipped with the (non-convex) modular $\mathrm{M}_w(f) := \int_I \varphi\left(\frac{f^*}{w}\right) w\, dm$. Set

$$\mathrm{P}(f) = \inf\{M_v(f) : v \prec w, v > 0, v \downarrow\} \text{ and } \mathrm{Q}(f) = \inf\{M_w(g) : f \prec g\}.$$

These formulas define convex modulars on L^0, the associated modular spaces of which coincide with the space $P_{\varphi,w} = Q_{\varphi,w}$. The modular P was introduced in [12] and further studied in [10], where it was proved that $\mathrm{P}(f) = \mathrm{M}_w(f^0)$ under the additional hypothesis

that φ is an N-function. In section 8.2 we show that $Q(f) = M_w(f^0)$ (without any hypothesis on φ). Combined with the preceding result of [10] we obtain that $Q(f) = P(f) = M_w(f^0)$ if φ is an N-function.

At the end of section 2 for generalized Lorentz spaces, as well as in the final section 9 for dual spaces, we discuss a number of examples where the space E is more specified. In particular if E is itself a classical Lorentz space it turns out that $\Lambda_{E,w}$ is another Lorentz space.

1. Preliminaries

Let μ be a measure defined on a σ-algebra $\mathcal{A}$ of subsets of Ω and $L^0(\Omega, \mathcal{A}, \mu)$ be the set of all classes of μ-measurable real valued functions on Ω, modulo equality almost everywhere, and let $L^0_+(\Omega, \mathcal{A}, \mu)$ be the cone of all non-negative functions from $L^0(\Omega, \mathcal{A}, \mu)$. Since in this article Ω will typically be an interval of the real line and μ a measure equivalent to the Lebesgue measure m, there will be no ambiguity in the shorter notation $L^0(\Omega)$, where $\mathcal{A}$ will be implicitly the algebra of Lebesgue-measurable sets. Similarly the space of bounded measurable functions will be unambiguously denoted by $L_\infty(\Omega)$. As for the spaces $L_p(\Omega, \mathcal{A}, \mu)$, $1 \le p < \infty$, they depend on μ and should be denoted by $L_p(\Omega, \mu)$, or simply $L_p(I)$ when Ω is an interval I and μ is the Lebesgue measure. When there is no ambiguity, the usual symbols L_p and L_∞ stand for the spaces of p-integrable functions and essentially bounded functions on Ω, respectively. Their norms are denoted by $\|f\|_p$ for $f \in L_p$ and $\|f\|_\infty$ for $f \in L_\infty$.

A subset S of $L^0(\Omega)$ is called *solid* if for any $f \in L^0(\Omega)$ and $g \in S$ with $|f| \le |g|$ a.e. we have $f \in S$. An *ideal* in $L^0(\Omega)$ is a solid linear subspace. *A Banach function space* E over $(\Omega, \mathcal{A}, \mu)$, is an ideal in $L^0(\Omega)$, equipped with a monotone norm $\|\cdot\|_E$, that is $\|f\|_E \le \|g\|_E$ whenever $|f| \le |g|$ a.e., $f, g \in E$, complete with respect to this norm, and with full support (no element in L_0, except 0, is disjoint from all elements in E). The Banach function space E satisfies the *Fatou property* whenever for any $f \in L^0(\Omega)$, $f_n \in E$ such that $f_n \uparrow f$ a.e. and $\sup_n \|f_n\|_E < \infty$ it follows that $f \in E$ and $\|f_n\|_E \uparrow \|f\|_E$. We say that E is *order continuous* whenever for every sequence $(f_n) \subset E$ with $f_n \downarrow 0$ a.e. we have $\|f_n\|_E \downarrow 0$.

For any $f \in L^0(\Omega)$, we will use the notation $\{f > s\}$ for the set $\{t \in \Omega : f(t) > s\}$, where the symbol ">" can be replaced by $<$, $\le$ or $\ge$. Throughout the whole paper the terms increasing or decreasing are reserved for non-decreasing or non-increasing, respectively. Given $f \in L^0(\Omega)$, the *distribution* of f with respect to μ is the function $d^\mu_f(s) = \mu\{|f| > s\}$, $s \ge 0$, and its *decreasing rearrangement* $f^{*,\mu}(t) = \inf\{s > 0 : d^\mu_f(s) \le t\}$, $t \in [0, \mu(\Omega))$. Given two measure spaces $(\Omega_i, \mathcal{A}_i, \mu_i)$, $i = 1, 2$, we say that $f_i \in L^0(\Omega_i)$ are *equimeasurable* if $d^{\mu_1}_{f_1}(s) = d^{\mu_2}_{f_2}(s)$, $s \ge 0$, which equivalently means that $f_1^{*,\mu_1} = f_2^{*,\mu_2}$.

A Banach function space E over $(\Omega, \mathcal{A}, \mu)$ is called a *symmetric* space whenever $\|f\|_E = \|g\|_E$ for every equimeasurable functions $f, g \in E$. Recall that the *fundamental function* of a symmetric space E is $\phi_E(t) = \|\chi_A\|_E$, $\mu(A) = t$, $t \in [0, \mu(\Omega))$. We say that the *support* of the symmetric space E is the entire set Ω whenever $\chi_A \in E$ for any $A \in \mathcal{A}$ with $\mu(A) < \infty$.

The Hardy-Littlewood order $f \prec_\mu g$ for locally integrable $f, g \in L^0(\Omega)$ is defined by the inequality $\int_0^x f^{*,\mu}\, dm \le \int_0^x g^{*,\mu}\, dm$ for every $x \in (0, \mu(\Omega))$. If $\Omega = (0, a)$, $a \le \infty$, and $\mu = m$ one writes simply $f \prec g$. Clearly $f \prec_\mu g$ if and only if $f^{*,\mu} \prec g^{*,\mu}$. Recall that $(f + g)^{*,\mu} \prec f^{*,\mu} + g^{*,\mu}$. We call E a *fully symmetric* space if E is symmetric and if for any $f \in L^0(\Omega)$ and $g \in E$ with $f \prec_\mu g$ we have that $f \in E$ and $\|f\|_E \le \|g\|_E$.

The *Köthe dual* space E' of a Banach function space E is the collection of all measurable functions $f \in L^0(\Omega)$ such that

$$\|f\|_{E'} = \sup\left\{\int_\Omega |fg|\,d\mu : \|g\|_E \le 1\right\} < \infty.$$

The space E' equipped with the norm $\|\cdot\|_{E'}$ is a complete Banach function space satisfying the Fatou property. If E is order continuous then the dual space E^* equals the Köthe dual E', in the sense that the only functionals in E^* are the maps $f \mapsto \int_\Omega fg\,d\mu$, $g \in E'$. If in addition E is a symmetric space then E' is fully symmetric, and

$$\|f\|_{E'} = \sup\left\{\int_0^{\mu(\Omega)} f^{*,\mu} g^{*,\mu}\,dm : \|g\|_E \le 1\right\}.$$

For the theory of Banach function and symmetric spaces we refer to the excellent books [1, 13, 23].

Given $f, g \in L^0(\Omega)$ denote $f \wedge g = \min\{f, g\}$ a.e., $f \vee g = \max\{f, g\}$ a.e., $f_+ = f \vee 0$ a.e. and $f_- = -f \vee 0$ a.e.. By m denote always the Lebesgue measure on subsets of real numbers $\mathbb{R}$. Recall that for $f \in L_1 + L_\infty(\Omega)$, $x \in (0, \mu(\Omega))$,

$$\begin{aligned}\int_0^x f^{*,\mu}\,dm &= \inf\{\|g\|_1 + x\|h\|_\infty : g \in L_1, h \in L_\infty, f = g + h\} \\ &= \inf_{\lambda>0}\left[\int_\Omega (|f| - \lambda)_+\,d\mu + \lambda x\right]\end{aligned} \tag{1.1}$$

(see e.g. Theorem 6.2 in [1, Ch. 2] and its proof; Exercise 1 on p. 87). It is well known (cf. Proposition 1.8, p.43, [1]) that for any $0 < p < \infty$,

$$\int_\Omega |f|^p\,d\mu = \int_0^{\mu(\Omega)} (f^{*,\mu})^p\,dm,$$

in which formula we can replace $|f|^p$ by $\varphi(|f|)$ where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is any increasing continuous function.

Let $I = (0, a)$, where $0 < a \le \infty$, and $L^0 = L^0(I)$ be the space of all real valued Lebesgue measurable functions on I. If $\Omega = I$ and $\mu = m$ then the distribution and decreasing rearrangement of a measurable function f are denoted by d_f and f^*, respectively. The support of f is denoted by $\operatorname{supp} f$.

Let us recall a useful connection between a measurable function and its decreasing rearrangement. Let f be a measurable function on I and f^* be its decreasing rearrangement.

Proposition 1.1.

(i) [1, Ryff's Theorem 7.5] *If $a < \infty$, or if* $\operatorname{supp} f$ *has finite measure, there exists an onto and measure preserving transformation $\tau : I \to I$, that is τ is measurable and $m(\tau^{-1}(A)) = m(A)$ for each measurable subset of I, such that $|f| = f^* \circ \tau$.*

(ii) [1, Corollary 7.6] *If* $\operatorname{supp} f$ *has infinite measure, and* $\lim_{t\to\infty} f^*(t) = 0$*, then such a measure preserving transformation τ exists but only from* $\operatorname{supp} f$ *onto the support of f^*. The equation $|f|(t) = f^* \circ \tau(t)$ is valid for $t \in \operatorname{supp} f$.*

We shall need to consider a third case that we settle as follows.

Lemma 1.2. *Let $I = (0, \infty)$ and f be a measurable function in I such that* $\lim_{t\to\infty} f^*(t) = \alpha > 0$*. Then for each $\varepsilon > 0$ there exists an onto and measure preserving transformation $\tau : I \to I$ such that $|f| \le (1 + \varepsilon)\, f^* \circ \tau$.*

Proof. Set $\tilde{f} = |f| \vee (1 + \varepsilon)\alpha$. Note that $(\tilde{f})^* = f^* \vee (1 + \varepsilon)\alpha \le (1 + \varepsilon)\, f^*$. Since $m\{|f| \ge (1 + \varepsilon)\alpha\} < \infty$, by Ryff's theorem we may find an onto measure preserving

transformation $\tau : I \to I$ such that $(|f| - (1+\varepsilon)\alpha)_+ = (f^* - (1+\varepsilon)\alpha)_+ \circ \tau$. Adding the constant $(1+\varepsilon)\alpha$ to both sides yields $\tilde{f} = (\tilde{f})^* \circ \tau$. Then

$$|f| \le \tilde{f} = (\tilde{f})^* \circ \tau \le (1+\varepsilon)\, f^* \circ \tau. \qquad \square$$

We will assume throughout the paper that $w : I \to (0,\infty)$ is a decreasing positive *weight* function. Then $d\omega = wdm$ is a measure on I such that $\omega(A) = \int_A w\, dm$ for Lebesgue measurable subsets $A \subset I$. The symbols d_f^w and $f^{*,w}$ will be reserved for the distribution and decreasing rearrangement of f respectively, with respect to the measure ω. Define

$$W(t) = \int_0^t w\, dm, \quad t \in I, \quad W(\infty) = \int_0^\infty w\, dm \quad \text{if} \quad I = (0,\infty).$$

Let further $b = \omega(I) = W(a) \in (0, +\infty]$ and $J = (0, b)$. The interval J will be always equipped with the Lebesgue measure m. It may happen that $a = \infty$ and $b < \infty$ if w is integrable on I, or that $a < \infty$ and $b = \infty$ if w is not integrable near 0. If the weight w is integrable near 0, it is integrable on any finite interval, and then clearly $W(t) < \infty$ for all $t \in I$. We say that the weight w is *regular* if $W(t) \le Ctw(t)$ for some $C \ge 1$ and all $t \in I$.

Throughout the paper the symbol E will always stand for a fully symmetric Banach function space contained in $L^0(J)$ with its support equal to J.

2. Lorentz spaces $\Lambda_{E,w}$

2.1. **Spaces E_w.** Given a fully symmetric space $E \subset L^0(J)$, let E_w be the subset of $L^0 = L^0(I)$ and $\|\cdot\|_{E_w}$ the functional on E_w such that

$$E_w = \{f \in L^0 : f^{*,w} \in E\}, \quad \|f\|_{E_w} = \|f^{*,w}\|_E, \quad f \in E_w.$$

The space E_w is a fully symmetric space on I for the measure ω. Note that if $f \in L^0(I)$ then $f^{*,w} \in L^0(J)$. If $E = L_p(J)$, $1 \le p < \infty$, then $E_w = (L_p)_w$ is traditionally called a weighted L_p space on I, which is not symmetric with respect to the measure m. However this is an ordinary L_p-space on (I,ω) in the sense that for $f \in E_w = (L_p)_w$ we have [1, Proposition 1.8, p.43]

$$\int_J (f^{*,w})^p dm = \int_J (|f|^p)^{*,w} dm = \int_I |f|^p d\omega = \int_I |f|^p w dm,$$

so that $\|f\|_{(L_p)_w} = (\int_I |f|^p w dm)^{1/p}$. Clearly it is symmetric with respect to the measure ω.

Let $\varphi : [0,\infty) \to [0,\infty)$ be an Orlicz function, that is $\varphi(0) = 0$, φ is convex and positive on $(0,\infty)$. Then for $f \in L^0(J)$ define the Orlicz modular as

$$I_\varphi(f) = \int_J \varphi(|f|)\, dm,$$

and the Orlicz space $L_\varphi(J)$ [1] as a collection of $f \in L^0(J)$ such that for some $\lambda > 0$, $I_\varphi(|f|/\lambda) < \infty$. It is a Banach fully symmetric space equipped with either of the norms, the Luxemburg norm $\|f\|_\varphi = \inf\{\lambda > 0 : I_\varphi(|f|/\lambda) \le 1\}$ or the Orlicz norm $\|f\|_\varphi^0 = \inf_{t>0} t(1 + I_\varphi(f/t))$. Analogously as for L_p-spaces, if $E = L_\varphi(J)$ then $E_w = (L_\varphi)_w$ is a weighted Orlicz space symmetric with respect to the measure ω, associated with the Orlicz modular

$$\int_J \varphi(f^{*,w})\, dm = \int_I \varphi(|f|)\, w dm.$$

Remark 2.1. The space E_w over (I,ω) where $d\omega = wdm$ can be called a generalized weighted space induced by the space E over (J,m) and the weight w on I. *In general, E_w is a Banach function space in $L^0(I)$ which is non symmetric with respect to the Lebesgue measure but isometrically order isomorphic to E on (J,m).*

This is a simple consequence of a general theorem of Caratheodory on isomorphisms of separable atomless measure algebras [20, Chap. 15, Theorem 4], but a far more elementary proof may be given in the present case.

Indeed, there exists a bijective, bimeasurable map $S : I \to J$ which is measure preserving i.e. $m(S(A)) = \omega(A)$ for all measurable $A \subset I$. This result follows from general facts in measure theory, but such a map will be explicitly exhibited below. Since for every $f \in L^0(I)$ and $t > 0$ we have $\{|f \circ S^{-1}| > t\} = S\{|f| > t\}$ we see that

$$d_{f\circ S^{-1}}(t) = m(|f \circ S^{-1}| > t) = m(S\{|f| > t\}) = \omega(|f| > t) = d_f^w(t). \tag{2.1}$$

Hence $(f \circ S^{-1})^* = f^{*,\omega}$. Thus $f \in E_w$ if and only if $f \circ S^{-1} \in E$ and $\|f\|_{E_w} = \|f^{*w}\|_E = \|(f \circ S^{-1})^*\|_E = \|f \circ S^{-1}\|_E$. The map $T : L^0(I) \to L^0(J) : f \mapsto f \circ S^{-1}$ is a linear order isomorphism, so that $E_w = T^{-1}(E)$ must be an ideal of $L^0(I)$. The restriction of T to E_w is the wished Banach lattice isometry.

Now for the sake of constructing a map S as requested in the preceding paragraph, we consider two cases.

a) If $W < \infty$ on I, then W is a bijective, bimeasurable, measure preserving map from (I, ω) onto (J, m), so that we may set $S = W$.

Indeed, since $w > 0$ is integrable on every finite segment $(0, x) \subset I$, the map W is a homeomorphism from I onto $W(I) = J$. The pushforward measure of ω by W is $\omega \circ W^{-1} = m$, the Lebesgue measure, as can be seen easily on intervals $[x, y] \subset I$,

$$\omega(W^{-1}([x, y])) = \omega([W^{-1}(x), W^{-1}(y)]) = \int_{W^{-1}(x)}^{W^{-1}(y)} w\, dm = y - x = m([x, y]),$$

it follows that $m(W(A)) = \omega(A)$ for all measurable $A \subset I$.

b) If $W(t) = \infty$ for $t > 0$ we choose $\alpha \in I = (0, a)$, and set $W_\alpha(t) = \int_\alpha^t w\, dm$ for $t \in I$. Letting $c = \int_\alpha^a w\, dm$, and $K = (-\infty, c)$, W_α is a bijective, bimeasurable, measure preserving map from (I, ω) onto (K, m). It is then a standard exercise to exhibit a bijective, bimeasurable, measure preserving map U from (K, m) onto (J, m), and we set $S = U \circ W_\alpha$.

Since the case $W < \infty$ is the main one considered in this article, except in sections 3 and 4, we collect the preceding information relative to this case in the following proposition.

Proposition 2.2. *Assume that $W < \infty$ on I. Then*

(i) Every $f \in L^0(I)$ is equimeasurable with respect to ω to $f \circ W^{-1} \in L^0(J)$ with respect to m. Consequently,

$$(f \circ W^{-1})^* = f^{*,w}.$$

(ii) $f \in L^0(I)$ belongs to E_w if and only if $f \circ W^{-1}$ belongs to E, and then

$$\|f\|_{E_w} = \|f \circ W^{-1}\|_E.$$

Consequently, E_w is an ideal in $L^0(I)$, it is fully symmetric for the measure $d\omega = wdm$, and the map $f \mapsto f \circ W^{-1}$ induces an order isometry from E_w onto E.

2.2. **Generalized Lorentz spaces.** Define now the *Lorentz space* $\Lambda_{E,w}$ as the symmetrization of E_w [12], that is

$$\Lambda_{E,w} = \{f \in L^0(I) : f^* \in E_w\}, \qquad \|f\|_{\Lambda_{E,w}} = \|f^*\|_{E_w}.$$

If $W(t) = \infty$ for $t > 0$, then $J = (0, \infty)$ and if f is a decreasing nonnegative function in $L^0(I)$, then $d_f^w = \infty \cdot \chi_{[0,f(0_+)]}$ and $f^{*,w} = f(0_+) \cdot \chi_J$. It follows that $\Lambda_{E,w} = \{0\}$ except if E contains the function 1, in which case $\Lambda_{E,w} = L_\infty(I)$.

For the rest of this section we disregard the above degenerate case and assume that $W < \infty$ on I.

For the Orlicz space $E = L_\varphi(J)$, $\Lambda_{E,w}$ is the Orlicz-Lorentz space $\Lambda_{\varphi,w}$, defined in [12], that is $\|f\|_{\Lambda_{\varphi,w}} = \|f^*\|_{(L_\varphi)_w}$. If $\varphi(t) = t^p$, $1 \le p < \infty$, then $\Lambda_{E,w} = \Lambda_{p,w}$ [3, 6].

If $E = L_\infty(J)$ then $E_w = L_\infty(I) = \Lambda_{E,w}$.

Other examples are given at the end of the present section.

Proposition 2.3. *Let $W < \infty$ on I.*

(i) *The support of $\Lambda_{E,w}$ is I.*

(ii) *For all $f \in \Lambda_{E,w}$,*
$$\|f\|_{\Lambda_{E,w}} = \|f^* \circ W^{-1}\|_E.$$

(iii) *The functional $\|\cdot\|_{\Lambda_{E,w}}$ is a norm, and the Lorentz space $\Lambda_{E,w}$ is a fully symmetric Banach space. If E has Fatou property then $\Lambda_{E,w}$ has also this property. If E is order continuous then $\Lambda_{E,w}$ is also order continuous.*

Proof. (i) Let $A \subset I$ with $m(A) < \infty$. Then $W(m(A)) < \infty$ and
$$\|\chi_A\|_{\Lambda_{E,w}} = \|\chi_{(0,m(A))}\|_{E_w} = \|\chi^{*,w}_{(0,m(A))}\|_E = \|\chi_{(0,W(m(A)))}\|_E < \infty$$
since by assumption the support of E is J.

(ii) In view of $w > 0$ on I, the function $W : I \to J$ is a strictly increasing homeomorphism. By Proposition 2.2 the functions f for ω and $f \circ W^{-1}$ for m are equimeasurable, that is $d^w_f = d_{f\circ W^{-1}}$. So $f^{*,w} = (f \circ W^{-1})^*$ and hence
$$\|f\|_{\Lambda_{E,w}} = \|f^*\|_{E_w} = \|(f^*)^{*,w}\|_E = \|f^* \circ W^{-1}\|_E.$$

(iii) For $f \in L_1 + L_\infty$ and $g \in \Lambda_{E,w}$ with $f \prec g$, and $x \in J$ we have
$$\int_0^x f^* \circ W^{-1}\, dm = \int_0^{W^{-1}(x)} f^* w\, dm \le \int_0^{W^{-1}(x)} g^* w\, dm = \int_0^x g^* \circ W^{-1}\, dm$$
by Hardy's inequality [1, Proposition 3.6, Ch.2]. Hence $f^* \circ W^{-1} \prec g^* \circ W^{-1} \in E$ and so by the assumption of full symmetry of E and by (ii) we get $f^* \circ W^{-1} \in E$, hence $f \in \Lambda_{E,w}$, and
$$\|f\|_{\Lambda_{E,w}} = \|f^* \circ W^{-1}\|_E \le \|g^* \circ W^{-1}\|_E = \|g\|_{\Lambda_{E,w}}.$$
Now if $f, g \in \Lambda_{E,w}$ we have $f^*, g^* \in E_w$, hence $f^* + g^* \in E_w$, which means that $f^* + g^* \in \Lambda_{E_w}$. Moreover $\|f^* + g^*\|_{\Lambda_{E_w}} = \|f^* + g^*\|_{E_w} \le \|f^*\|_{E_w} + \|g^*\|_{E_w} = \|f^*\|_{\Lambda_{E_w}} + \|g^*\|_{\Lambda_{E_w}}$. Then by the well known submajorization $(f + g)^* \prec f^* + g^*$ [1, Theorem 3.4], it follows from the preceding observation that $f + g \in \Lambda_{E,w}$ and
$$\|f + g\|_{\Lambda_{E,w}} \le \|f^* + g^*\|_{\Lambda_{E,w}} \le \|f^*\|_{\Lambda_{E,w}} + \|g^*\|_{\Lambda_{E,w}} = \|f\|_{\Lambda_{E,w}} + \|g\|_{\Lambda_{E,w}}$$
Therefore $\|\cdot\|_{\Lambda_{E,w}}$ is a fully symmetric norm.

The normed function space $\Lambda_{E,w}$ is complete since it is a symmetrization of the complete space E_w [12, Lemma 1.4].

Suppose now that E has the Fatou property. Take $f_n, f \in L^0(I)$, $f_n \uparrow f$ a.e., and $\sup \|f_n\|_{\Lambda_{E,w}} < \infty$. Then $f_n^* \circ W^{-1} \uparrow f^* \circ W^{-1}$ a.e., and by (ii) $\sup \|f_n^* \circ W^{-1}\|_E = \sup \|f_n\|_{\Lambda_{E,w}} < \infty$. Now by the Fatou property of E, $f^* \circ W^{-1} \in E$ so $f \in \Lambda_{E,w}$, and $\|f_n\|_{\Lambda_{E,w}} = \|f_n^* \circ W^{-1}\|_E \uparrow \|f^* \circ W^{-1}\|_E = \|f\|_{\Lambda_{E,w}}$. The statement on order continuity of $\Lambda_{E,w}$ can be proved analogously. □

Applications. Proposition 2.3(ii) allows to compute some Lorentz spaces.

Example 2.4 (Reiteration). *Let w_1, w_2 be two locally integrable decreasing positive weights on $I_1 = (0, a_1)$, resp. $I_2 = (0, W_1(a_1))$, where $W_1(x) = \int_0^x w_1\, dm$ for $x \in I_1$, and $W_2(x) = \int_0^x w_2\, dm$ for $x \in I_2$. For every symmetric space E on $J = (0, b)$, $b = W_2(W_1(a_1))$, it holds that $\Lambda_{\Lambda_{E,w_2},w_1} = \Lambda_{E,w}$ with equal norms, where $w = (w_2 \circ W_1)w_1$.*

Proof. For $f \in L^0(I_1)$ we have $f \in \Lambda_{\Lambda_{E,w_2},w_1}$ if and only if $f^* \circ W_1^{-1} \in \Lambda_{E,w_2}(I_2)$, that is if $(f^* \circ W_1^{-1}) \circ W_2^{-1}$ belongs to E. Setting $W = W_2 \circ W_1$, W is an increasing concave function with a derivative defined almost everywhere by $W' = (w_2 \circ W_1)w_1 =: w$, which is a decreasing weight on I_1. Then $f \in \Lambda_{\Lambda_{E,w_2},w_1}$ if and only if $f^* \circ W^{-1} \in E$, that is $f \in \Lambda_{E,w}$. The fact that both norms coincide is straightforward. □

For definition of the Marcinkiewicz space M_W see section 6.

Example 2.5 (Marcinkiewicz-Lorentz spaces). *Let I_1, I_2, w_1, w_2 be as in Example 2.4 and $M_{W_2}(I_2)$ be the Marcinkiewicz space associated with the weight w_2. Then the Lorentz space $\Lambda_{M_{W_2},w_1}$ consists of $f \in L^0(I_1)$ such that*

$$\|f\| := \sup_{x \in I_1} \frac{1}{W_2 \circ W_1(x)} \int_0^x f^* w_1 \, dm < \infty.$$

Proof. Clearly $f \in \Lambda_{M_{W_2},w_1}$ if and only if $f^* \circ W_1^{-1} \in M_{W_2}(I_2)$, that is

$$\|f^* \circ W_1^{-1}\|_{M_{W_2}} = \sup_{t \in I_2} \frac{1}{W_2(t)} \int_0^t f^* \circ W_1^{-1}(s) \, ds < \infty.$$

The result follows by performing first the substitution for $W_1^{-1}(s)$ in the integral, then the change $t = W_1(x)$ in the supremum. □

Recall if $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ are two fully symmetric Banach function spaces over the same interval J, then the Banach function spaces $E \cap F$ and $E + F$ equipped with the standard norms $\|f\|_{E\cap F} = \max\{\|f\|_E, \|f\|_F\}$ and $\|f\|_{E+F} = \inf\{\|f_1\|_E + \|f_2\|_F : f = f_1 + f_2, f_1 \in E, f_2 \in F\}$ respectively, are also fully symmetric. This is evident for the intersection space $E \cap F$, while for the sum space $E + F$ it is an immediate consequence of the following decomposition property for the submajorization.

Fact 2.6. *If $f, g_1, g_2 \in L^0_+$ are locally integrable with $f \prec g_1 + g_2$ then there is a decomposition $f = f_1 + f_2$ into non-negative functions such that $f_1 \prec g_1$ and $f_2 \prec g_2$.*

This fact is an easy consequence of the well known characterization of submajorization by Calderón, namely that $f \prec g$ if and only if there exists a substochastic linear operator T such that $|f| = T|g|$ ([13, Theorem II-3.4], or [1, Chap.3, Proposition 2.4 and Theorem 2.10]).

In the following example we shall use a monotone version of Fact 2.6, that is based on a monotone refinement of Calderón's theorem by Bennett and Sharpley [2, Theorem 5], [1, Remark 7.6, Theorem 7.7] (see also [14, §3] for a different proof), i.e. if f, g are non-negative locally integrable and decreasing functions, such that $f \prec g$ then $f = Tg$ for some positive substochastic operator T *which preserves the cone of decreasing non-negative functions.* Thus we obtain.

Fact 2.7. *If f, g_1, g_2 are non-negative decreasing locally integrable functions with $f \prec g_1 + g_2$ then there is a decomposition $f = f_1 + f_2$ into non-negative decreasing functions such that $f_1 \prec g_1$, $f_2 \prec g_2$.*

Example 2.8 (Intersections and sums). *Let E, F be fully symmetric Banach function spaces defined on the same interval J, and w a locally integrable decreasing positive weight on I with $W(I) = J$. Then $\Lambda_{E\cap F,w} = \Lambda_{E,w} \cap \Lambda_{F,w}$ and $\Lambda_{E+F,w} = \Lambda_{E,w} + \Lambda_{F,w}$ with equality of norms.*

Proof. The formula for the Lorentz space of an intersection is straightforward, so we treat only the sum case.

From $E \subset E+F$, with norm-decreasing inclusion it follows immediately that $\Lambda_{E,w} \subset \Lambda_{E+F,w}$, with norm-decreasing inclusion. Similarly $\Lambda_{F,w} \subset \Lambda_{E+F,w}$, and thus $\Lambda_{E,w} + \Lambda_{F,w} \subset \Lambda_{E+F,w}$. Moreover this inclusion is norm-decreasing.

As for the converse inclusion, let $f \in \Lambda_{E+F,w}$. We have $f^* \circ W^{-1} \in E+F$, hence for any $\varepsilon > 0$ there are $g \in E, h \in F$ such that $f^* \circ W^{-1} = g + h$ and $\|g\|_E + \|h\|_F \leq (1+\varepsilon)\|f^* \circ W^{-1}\|_{E+F}$. Then $f^* \circ W^{-1} \prec g^* + h^*$, and by of Fact 2.7 there exist decreasing non-negative functions g_1, h_1 such that

$$g_1 \prec g^*, h_1 \prec h^* \text{ and } f^* \circ W^{-1} = g_1 + h_1.$$

We have then $g_1 \in E$ and $h_1 \in F$. Setting $k = g_1 \circ W$, $l = h_1 \circ W$, we have $f^* = k + l$. Since k, l are non-negative decreasing and $k \circ W^{-1} \in E$, $l \circ W^{-1} \in F$, we have $k \in \Lambda_{E,w}$, $l \in \Lambda_{F,w}$ with $\|k\|_{\Lambda_{E,w}} = \|g_1\|_E \leq \|g\|_E$, $\|l\|_{\Lambda_{F,w}} = \|h_1\|_F \leq \|h\|_F$. It follows $f \in \Lambda_{E,w} + \Lambda_{F,w}$ with

$$\|f\|_{\Lambda_{E,w}+\Lambda_{F,w}} \leq \|g\|_E + \|h\|_F \leq (1+\varepsilon)\|f^* \circ W^{-1}\|_{E+F} = (1+\varepsilon)\|f\|_{\Lambda_{E+F,w}}$$

□

3. An inequality for rearrangements of functions and weights

Let $v \in L^0_+ = L^0_+(I)$, $I = (0,a)$. It defines a measure $d\nu = v dm$ on I in the usual way by setting $\nu(A) = \int_A v\, dm$, where $A \subset I$ is Lebesgue measurable. If $f \in L^0$ then by $f^{*,v}$ we denote the decreasing rearrangement of f with respect to the measure ν. This is a decreasing function on the interval $J_v := (0, \nu(I))$. Clearly $f = \chi_{\{v>0\}} f$ ν-a.e., so $f^{*,v} = (\chi_{\{v>0\}} f)^{*,v}$. If v has a rearrangement v^* such that $v^* = w$, then we have

$$\nu(I) = \int_I v = \int_I v^* = \int_I w = \omega(I) = b, \tag{3.1}$$

and so $J_v = (0,b) = J$ does not depend on v in that case. If E is a symmetric space on J then E_v is defined as in the case of a decreasing weight by $f \in E_v \iff f^{*,v} \in E$, where $f^{*,v}$ is the decreasing rearrangement of $|f|$ relative to the measure ν. Then again, E_v is a symmetric Banach function space on I equipped with the measure ν, which is order-isometric to E.

If $\operatorname{supp} f \subset \operatorname{supp} v$ then we agree that $(f/v)(t) = 0$ for $t \notin \operatorname{supp} f$.

Theorem 3.1. *Let $v \in L^0_+$ be such that $v^* = w$. Assume $f \in L_1 + L_\infty(I)$ with $\operatorname{supp} f \subset \operatorname{supp} v$. Then*

$$\left(\frac{f^*}{w}\right)^{*,w} \prec \left(\frac{f}{v}\right)^{*,v}.$$

In particular if $f/v \in E_v$ then $f^*/w \in E_w$ and $\|f^*/w\|_{E_w} \leq \|f/v\|_{E_v}$.

We prove first two lemmas.

Lemma 3.2. *For any $f, g \in L^0_+$ we have $(f \wedge g)^* \leq f^* \wedge g^*$.*

Proof. First notice that $m(\{f^* > s\} \cap \{g^* > s\}) = m\{f^* > s\} \wedge m\{g^* > s\}$, $s \geq 0$, since the sets $\{f^* > s\}$ and $\{g^* > s\}$ are two intervals with the same lower bound 0. Thus we have

$$\begin{aligned} d_{f\wedge g}(s) &= m\{f \wedge g > s\} = m(\{f > s\} \cap \{g > s\}) \\ &\leq m\{f > s\} \wedge m\{g > s\} = m\{f^* > s\} \wedge m\{g^* > s\} \\ &= m(\{f^* > s\} \cap \{g^* > s\}) = m\{f^* \wedge g^* > s\} = d_{f^*\wedge g^*}(s), \end{aligned}$$

which implies $(f \wedge g)^* \leq f^* \wedge g^*$. □

Lemma 3.3. *For every $f, g \in L^0_+$ such that $f^*, g^* < \infty$, it holds*

$$\int_I (f^* - g^*)_+ dm \le \int_I (f-g)_+ dm.$$

Proof. We assume first that $0 \le f$ is bounded. Note that

$$(f-g)_+ = f - f \wedge g \qquad (f^* - g^*)_+ = f^* - f^* \wedge g^*.$$

Then by Lemma 3.2 we have for every $t \in I$,

$$\int_0^t (f^* - g^*)_+ dm = \int_0^t (f^* - f^* \wedge g^*) dm \le \int_0^t (f^* - (f \wedge g)^*)) dm.$$

But since $f^* \prec (f - f \wedge g)^* + (f \wedge g)^*$ and $\int_0^t (f \wedge g)^* < \infty$ by boundedness of f,

$$\int_0^t (f^* - (f \wedge g)^*)) dm = \int_0^t f^* dm - \int_0^t (f \wedge g)^* dm \le \int_0^t (f - f \wedge g)^* dm = \int_0^t [(f-g)_+]^* dm.$$

Therefore for every $t \in I = (0, a)$,

$$\int_0^t (f^* - g^*)_+ \le \int_0^t [(f-g)_+]^*.$$

Letting $t \uparrow a$ we obtain

$$\int_0^a (f^* - g^*)_+ \, dm \le \int_0^a [(f-g)_+]^* dm = \int_0^a (f-g)_+ \, dm.$$

If $0 \le f$ is not bounded, letting $f_n = f \wedge n$, $n \in \mathbb{N}$, we get $f_n^* \uparrow f^*$ a.e. and thus $(f_n^* - g^*)_+ \uparrow (f^* - g^*)_+$ a.e. as well as $(f_n - g)_+ \uparrow (f-g)_+$ a.e.. Now by the monotone convergence theorem,

$$\int_I (f^* - g^*)_+ \, dm = \lim_{n \to \infty} \int_0^a (f_n^* - g^*)_+ \, dm \le \lim_{n \to \infty} \int_0^a (f_n - g)_+ \, dm = \int_I (f-g)_+ \, dm.$$

□

Remark 3.4. Using Lemma 3.2 and Lorentz-Shimogaki inequality [1, Chapter 3, Theorem 7.4] for rearrangements, we obtain in fact the more powerful result

$$(f^* - g^*)_+ \prec (f-g)_+.$$

Indeed since $f \ge f \wedge g$, Lorentz-Shimogaki's theorem gives $f^* - (f \wedge g)^* \prec f - f \wedge g$ and

$$(f^* - g^*)_+ = f^* - f^* \wedge g^* \le f^* - (f \wedge g)^* \prec f - f \wedge g = (f-g)_+.$$

However Lemma 3.3, which requires only quite elementary ingredients in its proof, will suffice for our purpose.

Proof of Theorem 3.1. By Lemma 3.3, for every $\lambda > 0$ we have

$$\begin{aligned} \int_I \left(\frac{f^*}{w} - \lambda \right)_+ w \, dm &= \int_I (f^* - \lambda w)_+ \, dm = \int_I (f^* - (\lambda v)^*)_+ \, dm \\ &\le \int_I (|f| - \lambda v)_+ \, dm = \int_I \left(\frac{|f|}{v} - \lambda \right)_+ v \, dm. \end{aligned}$$

Now in view of the equality (1.1), for any $x \in J$,

$$\begin{aligned} \int_0^x \left(\frac{f^*}{w} \right)^{*,w} dm &= \inf_{\lambda > 0} \left[\int_I \left(\frac{f^*}{w} - \lambda \right)_+ w dm + \lambda x \right] \\ &\le \inf_{\lambda > 0} \left[\int_I \left(\frac{|f|}{v} - \lambda \right)_+ v \, dm + \lambda x \right] = \int_0^x \left(\frac{f}{v} \right)^{*,v} dm, \end{aligned}$$

and the proof is completed. □

Proposition 3.5. *Let $f \in L^0$ have a finite decreasing rearrangement f^*. If I is a finite interval $(0,a)$, or $I=(0,\infty)$ with $\lim_{t\to\infty} f^*(t)=0$, then there exists $v \in L^0_+$ such that*

$$v^* = w, \quad \operatorname{supp} v \supset \operatorname{supp} f \quad \textit{and} \quad \left(\frac{f^*}{w}\right)^{*,w} = \left(\frac{f}{v}\right)^{*,v}.$$

If $I=(0,\infty)$ and $\lim\limits_{t\to\infty} f^*(t) > 0$ then for every $\varepsilon > 0$ there exists $0 < v \in L^0$ such that

$$v^* = w \quad \textit{and} \quad \left(\frac{f}{v}\right)^{*,v} \le (1+\varepsilon)\left(\frac{f^*}{w}\right)^{*,w}.$$

Proof. The proof will make use of the following fact.

(a) If $\tau : I \to I$ is a measure preserving transformation, w is a weight on I and $v = w\circ\tau$, then clearly $v^* = w$. Moreover for every $h \in L^0_+$ we have $h^{*,w} = (h\circ\tau)^{*,v}$.

Indeed for every $\lambda > 0$, and $g \in L^0_+$ we have $\{g\circ\tau > \lambda\} = (g\circ\tau)^{-1}(\lambda,\infty) = (\tau^{-1}\circ g^{-1})(\lambda,\infty) = \tau^{-1}(\{g>\lambda\})$. Thus $g\circ\tau$ and g are equimeasurable for the measure m, and it follows that $\int_I g\circ\tau\, dm = \int_I g\, dm$. Setting now $g = \chi_{\{h>\lambda\}} w$, we get

$$\begin{aligned}\omega(\{h>\lambda\}) &= \int_I \chi_{\{h>\lambda\}}\, w dm = \int_I (\chi_{\{h>\lambda\}}\circ\tau)\,(w\circ\tau)\, dm \\ &= \int_I \chi_{\{h\circ\tau>\lambda\}} v\, dm = \nu(\{h\circ\tau>\lambda\}),\end{aligned}$$

hence h for ω and $h\circ\tau$ for ν are equimeasurable, and so $h^{*,w} = (h\circ\tau)^{*,v}$.

Let us come back now to the proof of Proposition 3.5 itself. We consider several cases.

If the support of f has finite measure, then by Proposition 1.1 (i), there exists a measure preserving onto transformation $\tau : I \to I$ such that $|f|(t) = f^*\circ\tau(t)$, $t \in I$. Then setting $v = w\circ\tau$, we have $v > 0$, $v^* = w$, and by (a), f^*/w for $d\omega = wdm$ and $(f^*/w)\circ\tau$ for $d\nu = vdm$ are equimeasurable. But $(f^*/w)\circ\tau = (f^*\circ\tau)/(w\circ\tau) = |f|/v$, and the desired equality of rearrangements follows.

If now the support of f has infinite measure and $\lim_{t\to\infty} f^*(t) = 0$, by Proposition 1.1 (ii) there exists a measure preserving transformation τ from the support of f onto the support of f^*, such that $|f|(t) = f^*\circ\tau(t)$ for $t \in \operatorname{supp} f$. Define $v(t) = w\circ\tau(t)$ for t in the support of f and $v(t) = 0$ otherwise. By the assuption that the support f has infinite measure we have that $\operatorname{supp} f^* = (0,\infty)$. Then we have $\operatorname{supp} v = \operatorname{supp} f$ and again $v^* = w$. In fact the conclusions of (a) remain valid when defining $h\circ\tau(t) = 0$ for any $t \notin \operatorname{supp} f = \operatorname{supp} v$. Thus the conclusion $(f^*/w)^{*,w} = (f/v)^{*,v}$ remains valid provided we define $(f/v)(t) = 0$ for $t \notin \operatorname{supp} f$.

Finally if $I = (0,\infty)$ and $\lim_{t\to\infty} f^*(t) > 0$, then by Lemma 1.2 for every $\varepsilon > 0$ there exists a measure preserving onto transformation $\tau : I \to I$ such that $|f| \le (1+\varepsilon)\, f^*\circ\tau$. Defining the weight $v = w\circ\tau$, we have $v > 0$ on I. By (a), $v^* = w$ and

$$\left(\frac{f}{v}\right)^{*,v} \le \left(\frac{(1+\varepsilon)f^*\circ\tau}{v}\right)^{*,v} = (1+\varepsilon)\left(\frac{f^*\circ\tau}{w\circ\tau}\right)^{*,v} = (1+\varepsilon)\left(\frac{f^*}{w}\right)^{*,w}.$$

□

Given $0 \le v \in L^0(I)$, let us introduce some notation. Set

$$V(t) = \int_0^t v\, dm \quad \text{and assume} \quad v^* = w, \quad V(t) < \infty, \quad t \in I.$$

Then V is an increasing, not necessarily strictly increasing, and continuous function from I onto $J = (0,b)$ since $V(a) = \int_0^a v^*\, dm = \int_0^a w\, dm = W(a) = b$. For $t \in J$, the set $V^{-1}\{t\}$ is a closed subinterval of I. Let

$$N_v = \{t \in J : m(V^{-1}\{t\}) > 0\}.$$

Clearly the set N_v is finite or countable. If $t \in N_v$ then v vanishes a.e. on $V^{-1}\{t\}$. If $t \notin N_v$ then $V^{-1}(t)$ is defined unambiguously as the unique element in $V^{-1}\{t\}$.

For $f \in L^0$ with $\operatorname{supp} f \subset \operatorname{supp} v$ define $f \circ V^{-1}$ by

$$f \circ V^{-1}(t) = \begin{cases} 0 & \text{if } t \in N_v, \\ f(V^{-1}(t)) & \text{if } t \in J \setminus N_v. \end{cases} \tag{3.2}$$

With the convention (3.2) above the submajorization result of Theorem 3.1 as well as Proposition 3.5 may be restated in a more transparent way when the weight w is such that $W(t) < \infty$ for all $t \in I$.

Corollary 3.6. *If $W < \infty$ on I, then for any $v \in L^0_+(I)$ with $v^* = w$, and every $f \in L_1 + L_\infty(I)$ with $\operatorname{supp} f \subset \operatorname{supp} v$ we have*

$$\frac{f^*}{w} \circ W^{-1} \prec \frac{f}{v} \circ V^{-1}.$$

Moreover if $I = (0, a)$ with $a < \infty$ or if $I = (0, \infty)$ and $\lim_{t\to\infty} f^*(t) = 0$, then there exists $v \in L^0_+$ with $\operatorname{supp} f \subset \operatorname{supp} v$ such that $v^* = w$ and

$$\left(\frac{f^*}{w} \circ W^{-1}\right)^* = \left(\frac{f}{v} \circ V^{-1}\right)^*.$$

If $I = (0, \infty)$ and $\lim\limits_{t\to\infty} f^*(t) > 0$ then for every $\varepsilon > 0$ there exists $v > 0$ on I such that $v^* = w$ and

$$\left(\frac{f}{v} \circ V^{-1}\right)^* \le (1 + \varepsilon) \left(\frac{f^*}{w} \circ W^{-1}\right)^*.$$

Proof. Let $N_v = \{t_n\}$ be an enumeration of N_v and set $A = \bigcup_n V^{-1}\{t_n\}$. Then $A \subset I$ and $\nu(A) = \int_A v\, dm = 0$. If $t \notin A$ then $(f \circ V^{-1}) \circ V(t) = f(t)$, and so $(f \circ V^{-1}) \circ V = f$ ν-a.e. on I. Moreover for any $h \in L^0_+$ and $t \ge 0$ by the change of variable formula it holds

$$m\{h > t\} = \int_I \chi_{(t,\infty)} \circ h\, dm = \int_I \chi_{(t,\infty)} \circ h \circ V d\nu = \nu\{h \circ V > t\}. \tag{3.3}$$

It follows that h for m and $h \circ V$ for ν are equimeasurable. In particular

$$m\{|f| \circ V^{-1} > t\} = \nu\{(|f| \circ V^{-1}) \circ V > t\} = \nu\{|f| > t\},$$

and so $f \circ V^{-1}$ for m and f for ν are equimeasurable. Hence $\frac{f}{v} \circ V^{-1}$ for m and $\frac{f}{v}$ for ν are equimeasurable, and so

$$\left(\frac{f}{v} \circ V^{-1}\right)^* = \left(\frac{f}{v}\right)^{*,v}.$$

By a similar argument $\frac{f^*}{w} \circ W^{-1}$ for m and $\frac{f^*}{w}$ for ω are equimeasurable as well, and hence

$$\left(\frac{f^*}{w} \circ W^{-1}\right)^* = \left(\frac{f^*}{w}\right)^{*,w}.$$

Now the conclusion follows directly from Theorem 3.1 and Proposition 3.5. □

Remark 3.7. *Let $0 \le v \in L^0(I)$ with $V(t) < \infty$ for all $t \in I$, ν be the measure $v\, dm$ and $J_v = (0, \nu(I))$. Let E be a symmetric space on J_v. Then for every $h \in E$, $h \circ V \in E_v$ and the map $T : h \mapsto h \circ V$ is a surjective order isometry from E onto E_v.*

Proof. Indeed by (3.3), h for m and $h \circ V$ for ν are equimeasurable, thus T embeds isometrically E into E_v. Moreover for every $f \in E_v$ we have $f = T(f \circ V^{-1})$, thus T is surjective. Here $f \circ V^{-1}$ is defined as (3.2) where J is replaced by J_v. □

4. Spaces $M_{E,w}$

In this section we define a class $M_{E,w}$ of functions contained in $L^0 = L^0(I)$ which will be used later for investigating the Köthe dual of the Lorentz space $\Lambda_{E,w}$.

4.1. **Definition and properties.** Let the class $M_{E,w}$ and the *gauge* on $M_{E,w}$ be defined by

$$M_{E,w} = \left\{ f \in L^0 : \frac{f^*}{w} \in E_w \right\} \quad \text{and} \quad \|f\|_{M_{E,w}} = \left\| \frac{f^*}{w} \right\|_{E_w} = \left\| \left(\frac{f^*}{w} \right)^{*,w} \right\|_E .$$

Although the class $M_{E,w}$ does not need to be even linear it has several properties analogous to those in symmetric spaces, so a similar terminology is used here as may be seen below.

Proposition 4.1. (i) *The class $M_{E,w}$ is a solid symmetric subset of L^0, that is $\|f\|_{M_{E,w}} = \|f^*\|_{M_{E,w}}$ and if $f \in L^0$, $g \in M_{E,w}$ and $|f| \le |g|$ a.e. then $f \in M_{E,w}$ and $\|f\|_{M_{E,w}} \le \|g\|_{M_{E,w}}$.*

(ii) *For all $x \in I$, $\chi_{(0,x)} \in M_{E,w}$. Consequently the support of $M_{E,w}$ is equal to the entire interval I.*

(iii) *The fundamental function $\phi_{M_{E,w}}(x) = \|\chi_{(0,x)}\|_{M_{E,w}}$, $x \in I$, verifies*

$$\phi_{M_{E,w}}(x) \le 2\phi_E(1 \wedge b) \left(x + \frac{1}{w(x)} \right).$$

(iv) *If $W < \infty$ on I, then*

$$f \in M_{E,w} \iff \frac{f^*}{w} \circ W^{-1} \in E \quad \text{and} \quad \|f\|_{M_{E,w}} = \left\| \frac{f^*}{w} \circ W^{-1} \right\|_E .$$

(v) *If E has the Fatou property then the class $M_{E,w}$ has this property, that is for every $f \in L^0$, $0 \le f_n \in M_{E,w}$ with $f_n \uparrow f$ a.e. and $\sup_n \|f_n\|_{M_{E,w}} = K < \infty$ we have $f \in M_{E,w}$ and $\|f\|_{M_{E,w}} = K$.*

Proof. (i) It is clear by symmetry and ideal properties of E_w.

(ii) For every $x \in I$ we have

$$\int_0^x \frac{1}{w} d\omega = \int_0^x \frac{1}{w} w dm = x,$$

thus the function $h_x = \frac{1}{w}\chi_{(0,x)} \in L_1(I,\omega)$. On the other hand $h_x \le 1/w(x)$ a.e. equivalently ω-a.e. on I, and so it is bounded ω-a.e. on I. Hence $h_x \in L_\infty(I,\omega)$. Consequently $h_x \in L_1 \cap L_\infty(I,\omega)$. Therefore $h_x^{*,w} \in L_1 \cap L_\infty(J,m)$. Indeed, it is clear that

$$\|h_x^{*,w}\|_\infty = 1/w(x). \tag{4.1}$$

We also have that $m\{h_x^{*,w} > t\} = \omega\{h_x > t\}$, $t \ge 0$, in view of equimeasurability of $h_x^{*,w}$ with respect to m on J and h_x with respect to ω on I. Hence

$$\begin{aligned} \|h_x^{*,w}\|_1 &= \int_J h_x^{*,w}\, dm = \int_0^\infty m\{h_x^{*,w} > t\}\, dm(t) \\ &= \int_0^\infty \omega\{h_x > t\}\, dm(t) = \int_I h_x w\, dm = x. \end{aligned} \tag{4.2}$$

It is well known [1, 13] that $L_1 \cap L_\infty(J,m) \subset E$, and so $h_x^{*,w} = (\chi_{(0,x)}/w)^{*,w} \in E$. The latter means that $\chi_{(0,x)} \in M_{E,w}$ for every $x \in I$. Thus the support of the space $M_{E,w}$ is the entire interval I.

(iii) Since E is a symmetric Banach function space it is well known that $\|f\|_E \leq C\|f\|_{L_1\cap L_\infty}$, $f \in E$, where $C = 2\varphi_E(1\wedge b)$ (see [13], Ch. II, Theorem 4.1 and its proof). From (4.1) and (4.2), $\|h_x^{*,w}\|_{L_1\cap L_\infty} \leq x + 1/w(x)$. Thus

$$\phi_{M_{E,w}}(x) = \|h_x\|_{E_w} = \|h_x^{*,w}\|_E \leq 2\varphi_E(1\wedge b)\left(x + \frac{1}{w(x)}\right).$$

(iv) This condition follows directly from Proposition 2.2.

(v) It is immediate by the definition of the space $M_{E,w}$ and the properties of the rearrangements. □

From Theorem 3.1, Proposition 3.5 and Corollary 3.6 we obtain directly the next result.

Proposition 4.2. *For any $f \in M_{E,w}$ we have*

$$\|f\|_{M_{E,w}} = \inf\left\{\left\|\frac{f}{v}\right\|_{E_v} : v \geq 0, v^* = w, \operatorname{supp} v \supset \operatorname{supp} f\right\}$$

with the convention that $\|g\|_E = \infty$ for every $g \notin E$, and $f(t)/v(t) = 0$ whenever $f(t) = 0$.

Moreover if $W < \infty$ on I, then for $f \in L^0$ we have that $f \in M_{E,w}$ if and only if $\frac{f}{v} \circ V^{-1} \in E$ for some $v \geq 0$ with $v^* = w$ and $\operatorname{supp} v \supset \operatorname{supp} f$.

Remark 4.3. The class $M_{E,w}$ does not need to be either linear or normable. Let E be an Orlicz space L_φ, then the class $M_{E,w}$ is the class $M_{\varphi,w}$ considered in [12]. In view of [12, Proposition 3.4] the class $M_{\varphi,w}$ may not be linear, while by [12, Proposition 4.14 and Example 4.15] it may be linear but not normable.

4.2. **Normability.** Before we prove the main result on normability of the class $M_{E,w}$ we need the following lemma.

Lemma 4.4. *Let w_1, w_2 be two decreasing positive weights on I such that for some constant $C \geq 1$ it holds that $w_1 \leq Cw_2$ a.e.. Then for every function $f \in L^0$ we have*

$$\left(\frac{f}{w_2}\right)^{*,w_2} \prec C\left(\frac{f}{w_1}\right)^{*,w_1}.$$

Consequently, if $\int_I w_1 dm = \int_I w_2 dm = b$ and E is a fully symmetric space on $J = (0,b)$ then $M_{E,w_1} \subset M_{E,w_2}$ with $\|f\|_{M_{E,w_2}} \leq C\|f\|_{M_{E,w_1}}$ for $f \in M_{E,w_1}$.

Proof. Setting $\omega_2 = w_2\, dm$, by the well known formula ([1], Ch. 2, Proposition 3.3, [13], p.64, (2.14)) we get for $x \in I$,

$$\int_0^x \left(\frac{f}{w_2}\right)^{*,w_2} dm = \sup_{\omega_2(A)\leq x} \int_A \frac{|f|}{w_2} d\omega_2 = \sup_{\omega_2(A)\leq x} \int_A |f|\, dm,$$

and a similar equation holds true for w_1. Clearly $w_1 \leq Cw_2$ a.e. implies that $\sup\limits_{\omega_2(A)\leq x} \int_A |f| dm \leq \sup\limits_{\omega_1(A)\leq Cx} \int_A |f| dm$. Thus

$$\int_0^x \left(\frac{f}{w_2}\right)^{*,w_2} dm \leq \int_0^{Cx} \left(\frac{f}{w_1}\right)^{*,w_1} dm.$$

But for $C \geq 1$, $Cx \in (0,a)$ and a non-negative decreasing function h on $(0,a)$ we have

$$\int_0^{Cx} h\, dm \leq \int_0^x h\, dm + h(x)x(C-1) \leq \int_0^x h\, dm + (C-1)\int_0^x h\, dm = C\int_0^x h\, dm,$$

and the conclusion follows. □

Proposition 4.5. *Assume that the weight w is regular that is $W(t) \le Ctw(t)$ for some $C \ge 1$ and all $t \in I$. Then $M_{E,w}$ is a vector space and the formula*

$$|||f||| := \inf\left\{\sum_{i=1}^n \|f_i\|_{M_{E,w}} : \sum_{i=1}^n |f_i| \ge |f|\right\} \tag{4.3}$$

defines a lattice norm $|||\cdot|||$ on $M_{E,w}$ such that

$$|||f||| \le \|f\|_{M_{E,w}} \le C\,|||f|||\,. \tag{4.4}$$

Consequently the class $M_{E,w}$ is a normable vector lattice.

Proof. We will prove that for any finite family $f_1, \dots, f_n$ in $M_{E,w}$ we have

$$\left\|\sum_{i=1}^n f_i\right\|_{M_{E,w}} \le C\sum_{i=1}^n \|f_i\|_{M_{E,w}}, \tag{4.5}$$

where C is the constant of regularity of w. Then $|||\cdot|||$ defined by (4.3) is a vector lattice norm on $M_{E,w}$ equivalent to the gauge $\|f\|_{M_{E,w}}$. In fact we will verify (4.4).

We claim that

$$\left(\frac{1}{w}\left(\sum_{i=1}^n f_i\right)^*\right)\circ W^{-1} \prec C\sum_{i=1}^n\left(\frac{f_i}{v_i}\circ V_i^{-1}\right)^* \tag{4.6}$$

for every non-negative functions $v_1, \dots, v_n$ with $\operatorname{supp} f_i \subset \operatorname{supp} v_i$, $v_i^* = w$, $i = 1, \dots, n$, where V_i^{-1} are defined as in the proof of Corollary 3.6, since $V_i(t) = \int_0^t v_i\,dm \le \int_0^t v_i^*\,dm = \int_0^t w\,dm = W(t) < \infty$ for all $t \in I$. The statement of the claim then implies the following

$$\left\|\left(\frac{1}{w}\left(\sum_{i=1}^n f_i\right)^*\right)\circ W^{-1}\right\|_E \le C\sum_{i=1}^n\left\|\frac{f_i}{v_i}\circ V_i^{-1}\right\|_E.$$

Taking the infimum of every right term with respect to v_i with $v_i^* = w$ and $\operatorname{supp} f_i \subset \operatorname{supp} v_i$ for $i = 1, \dots, n$, we get by Proposition 4.2,

$$\left\|\left(\frac{1}{w}\left(\sum_{i=1}^n f_i\right)^*\right)\circ W^{-1}\right\|_E \le C\sum_{i=1}^n\left\|\frac{f_i^*}{w}\circ W^{-1}\right\|_E,$$

and consequently in view of Proposition 4.1(iv) we obtain the desired inequality (4.5).

Now in order to finish it is enough to prove claim (4.6), which is equivalent to the following inequality

$$\int_0^x\left(\frac{(\sum_{i=1}^n f_i)^*}{w}\circ W^{-1}\right)^* dm \le C\sum_{i=1}^n\int_0^x\left(\frac{|f_i|}{v_i}\circ V_i^{-1}\right)^* dm, \quad x \in J. \tag{4.7}$$

For any measurable $v \ge 0$ with $V(t) = \int_0^t v\,dm < \infty$, $t \in I$, and $f \in L^0$ such that $\operatorname{supp} f \subset \operatorname{supp} v$, by equimeasurability of f/v for $d\nu = v dm$ and $(f/v)\circ V^{-1}$ for m we have that $(f/v)^{*,v} = ((f/v)\circ V^{-1})^*$. Hence by (1.1) for any $x \in J$,

$$\begin{aligned}\int_0^x\left(\frac{f}{v}\circ V^{-1}\right)^* dm &= \int_0^x\left(\frac{f}{v}\right)^{*,v} dm = \inf_{\lambda>0}\left\{\int_I\left(\frac{|f|}{v}-\lambda\right)_+ d\nu + \lambda x\right\}\\ &= \inf_{\lambda>0}\left\{\int_I(|f|-\lambda v)_+ dm + \lambda x\right\}.\end{aligned}$$

Thus the righthand side of (4.7) has the following form

$$(4.8)\quad R(x) := \sum_{i=1}^n \int_0^x \left(\frac{|f_i|}{v_i} \circ V_i^{-1}\right)^* dm = \inf_{\substack{\lambda_i>0\\ i=1,\dots,n}} \left\{ \int_I \sum_{i=1}^n (|f_i| - \lambda_i v_i)_+ dm + \sum_{i=1}^n \lambda_i x \right\}.$$

The function $s \mapsto s_+$ is subadditive and non-decreasing on $\mathbb{R}$. Hence a.e. on I,

$$\left(\left|\sum_{i=1}^n f_i\right| - \sum_{i=1}^n \lambda_i v_i\right)_+ \le \left(\sum_{i=1}^n |f_i| - \sum_{i=1}^n \lambda_i v_i\right)_+ \le \sum_{i=1}^n (|f_i| - \lambda_i v_i)_+.$$

Thus by (4.8), in view of (1.1) we get for $x \in J$,

$$\begin{aligned}
R(x) &\ge \inf_{\lambda_1,\dots,\lambda_n>0} \left[\int_I \left(\left|\sum_{i=1}^n f_i\right| - \sum_{i=1}^n \lambda_i v_i\right)_+ dm + x\sum_{i=1}^n \lambda_i\right] \\
&= \inf_{\substack{\alpha_1,\dots,\alpha_n>0\\ \sum \alpha_i=1}} \inf_{\lambda>0} \left[\int_I \left(\left|\sum_{i=1}^n f_i\right| - \lambda \sum_{i=1}^n \alpha_i v_i\right)_+ dm + \lambda x\right] \\
&= \inf_{\substack{\alpha_1,\dots,\alpha_n>0\\ \sum \alpha_i=1; v=\sum \alpha_i v_i}} \inf_{\lambda>0} \left[\int_I \left(\frac{\left|\sum_{i=1}^n f_i\right|}{v} - \lambda\right)_+ v\, dm + \lambda x\right] \\
&= \inf_{v\in \mathrm{conv}(v_1,\dots,v_n)} \int_0^x \left(\frac{\sum_{i=1}^n f_i}{v}\right)^{*,v} dm \\
&= \inf_{v\in \mathrm{conv}(v_1,\dots,v_n)} \int_0^x \left(\frac{\left|\sum_{i=1}^n f_i\right|}{v} \circ V^{-1}\right)^* dm.
\end{aligned}$$

If $v \in \mathrm{conv}(v_1,\dots,v_n)$ we have $v = \sum_{i=1}^n \alpha_i v_i$ for some $\alpha_i \ge 0$ with $\sum_{i=1}^n \alpha_i = 1$. Since by $v_i^* = w$ we have $V_i(t) \le W(t)$ for every $0 \le t < a$, with equality $V_i(a) = \lim_{t\to a^-} V_i(t) = W(a) = \lim_{t\to a^-} W(t)$, we obtain $V(t) = \sum_{i=1}^n \alpha_i V_i(t) \le \sum_{i=1}^n \alpha_i W(t) = W(t)$ for $t \in I$ with $V(a) = W(a)$, so that the continuous function V maps I onto J, and we may define V^{-1} as in the proof of Corollary 3.6. We also have $v^* \prec \sum_{i=1}^n \alpha_i v_i^* = w$, hence

$$tv^*(t) \le \int_0^t v^* \le W(t) \le Ctw(t), \qquad t \in I,$$

by regularity of w. But then for every $v \in \mathrm{conv}(v_1,\dots,v_n)$, letting $V_*(t) := \int_0^t v^*$, we get for $x \in J$,

$$\begin{aligned}
\int_0^x \left(\frac{\left|\sum_{i=1}^n f_i\right|}{v} \circ V^{-1}\right)^* dm &\ge \int_0^x \left(\frac{\left(\sum_{i=1}^n f_i\right)^*}{v^*} \circ V_*^{-1}\right)^* dm \\
&\ge \frac{1}{C}\int_0^x \left(\frac{\left(\sum_{i=1}^n f_i\right)^*}{w} \circ W^{-1}\right)^* dm =: L(x),
\end{aligned}$$

where the first inequality results from Corollary 3.6 with v^*, V_* playing the role of w, W respectively, and the second one by Lemma 4.4 applied to the weights v^* and w. Thus $CR(x) \succ L(x)$, and this proves the claim and completes the proof.

□

5. Köthe duality of $M_{E,w}$.

The Köthe dual of the class $M_{E,w}$ is defined as for a Banach function space, as the set of elements $f \in L^0 = L^0(I)$ such that

$$\|f\|_{(M_{E,w})'} := \sup\left\{ \int_I |fg|\, dm : g \in M_{E,w}, \|g\|_{M_{E,w}} \le 1 \right\} < \infty.$$

The set $(M_{E,w})'$ is an ideal in L^0 on which $f \mapsto \|f\|_{(M_{E,w})'}$ defines a vector lattice norm. Equipped with this norm, the space $(M_{E,w})'$ becomes a symmetric Banach function space, as it may be shown directly; but this will be also a consequence of the next theorem.

Theorem 5.1. *If $W < \infty$ on I, then the Köthe dual of $M_{E,w}$ equals $\Lambda_{E',w}$ isometrically, that is $\|f\|_{(M_{E,w})'} = \|f\|_{\Lambda_{E',w}}$.*

Proof. The proof will be done in several steps.

a) $\Lambda_{E',w} \subset (M_{E,w})'$ and the inclusion is norm-decreasing i.e. $\|f\|_{(M_{E,w})'} \le \|f\|_{\Lambda_{E',w}}$. Indeed if $f \in \Lambda_{E',w}$ and $g \in M_{E,w}$ then in view of the assumption $W < \infty$ and Proposition 4.1 (iv) we get

$$\begin{aligned}(5.1)\quad \int_I |fg|\, dm &\le \int_I f^* g^*\, dm = \int_I f^* \frac{g^*}{w} w dm = \int_J (f^* \circ W^{-1}) \left(\frac{g^*}{w} \circ W^{-1} \right) dm \\ &\le \|f^* \circ W^{-1}\|_{E'} \left\| \frac{g^*}{w} \circ W^{-1} \right\|_E = \|f^*\|_{(E')_w} \left\| \frac{g^*}{w} \right\|_{E_w} = \|f\|_{\Lambda_{E',w}} \|g\|_{M_{E,w}},\end{aligned}$$

which shows that $\|f\|_{(M_{E,w})'} \le \|f\|_{\Lambda_{E',w}}$.

b) Now we will show that for every $f \in \Lambda_{E',w}$ we get the equality of the norms $\|f\|_{(M_{E,w})'} = \|f\|_{\Lambda_{E',w}}$. Assume first that $0 \le f \in \Lambda_{E',w}$ is decreasing, and so $f \circ W^{-1}$ is also decreasing. Then for any $\varepsilon > 0$ we can find a decreasing non-negative function $h \in E$ with $\|h\|_E = 1$ and satisfying

$$\|f\|_{\Lambda_{E',w}} - \varepsilon = \|f \circ W^{-1}\|_{E'} - \varepsilon \le \int_J (f \circ W^{-1}) h\, dm = \int_I f\, (h \circ W) w\, dm.$$

Setting $g = (h \circ W)\, w$, we have

$$(5.2)\qquad \int_I fg\, dm \ge \|f\|_{\Lambda_{E',w}} - \varepsilon,$$

while $g/w = h \circ W \in E_w$ with $\|g/w\|_{E_w} = \|h\|_E = 1$ by Proposition 2.2. Now since g is decreasing we have $g \in M_{E,w}$ and $\|g\|_{M_{E,w}} = \|g/w\|_{E_w} = 1$. Then by (5.1) and (5.2) we get $\|f\|_{(M_{E,w})'} = \|f\|_{\Lambda_{E',w}}$.

Let us reduce now the general case when f is not decreasing to the preceding one.

First assume that $m(\operatorname{supp} f) < \infty$. Then by Proposition 1.1(i) there exists a measure preserving and onto transformation τ on I such that $|f| = f^* \circ \tau$. Let g be chosen to satisfy (5.2) for f^* in place of f. Then

$$(5.3)\quad \int_I |f|(g \circ \tau)\, dm = \int_I (f^* \circ \tau)(g \circ \tau)\, dm = \int_I f^* g\, dm \ge \|f^*\|_{\Lambda_{E',w}} - \varepsilon = \|f\|_{\Lambda_{E',w}} - \varepsilon,$$

and $\|g \circ \tau\|_{M_{E,w}} = \|g\|_{M_{E,w}} = 1$.

Now let $m(\operatorname{supp} f) = \infty$. There exists a sequence of functions f_n with $m(\operatorname{supp}(f_n)) < \infty$ and such that $|f_n| \uparrow |f|$ a.e.. Hence $f_n^* \uparrow f^*$ a.e., and by the Fatou property of $\Lambda_{E',w}$ (see Proposition 2.3) we get $\|f_n\|_{\Lambda_{E',w}} \uparrow \|f\|_{\Lambda_{E',w}}$.

Now by (5.3) for each f_n we can find $g_n \ge 0$ with $\|g_n\|_{M_{E_w}} = 1$ and such that

$$\int_I |f_n| g_n\, dm \ge \|f_n\|_{\Lambda_{E',w}} - \frac{1}{n}, \qquad n \in \mathbb{N}.$$

Then

$$\|f\|_{(M_{E,w})'} \geq \limsup_{n\to\infty} \int_I |f_n| g_n \, dm \geq \lim_{n\to\infty} \left(\|f_n\|_{\Lambda_{E',w}} - \frac{1}{n} \right) = \|f\|_{\Lambda_{E',w}}.$$

c) By a) and b) we have that $\Lambda_{E',w} \subset (M_{E,w})'$ and this inclusion is isometric, so $\Lambda_{E',w}$ is a closed ideal in $(M_{E,w})'$. This ideal is order dense, since it contains the bounded functions with finite measure supports, and moreover it has the Fatou property. It follows that $\Lambda_{E',w}$ is equal to $(M_{E,w})'$. In fact if $0 \leq f \in (M_{E,w})'$ there exists a sequence $(f_n) \subset \Lambda_{E',w}$ with $0 \leq f_n \uparrow f$ a.e.. Moreover $\|f_n\|_{\Lambda_{E',w}} = \|f_n\|_{(M_{E,w})'} \leq \|f\|_{(M_{E,w})'}$. Then by the Fatou property of $\Lambda_{E',w}$, $f \in \Lambda_{E',w}$. □

The next result is a generalization of [7, Theorem 2(i)].

Corollary 5.2. *Let $W < \infty$ on I. If E has the Fatou property and w is regular, then $(\Lambda_{E',w})' = M_{E,w}$ as sets with the gauge $\|\cdot\|_{M_{E,w}}$ equivalent to the norm $\|\cdot\|_{(\Lambda_{E',w})'}$.*

Proof. It is well known that a Banach function lattice F has the Fatou property if and only if $F = F''$ isometrically [15, 23]. The gauge $\|\cdot\|_{M_{E,w}}$ is not a norm, but it is equivalent to a lattice norm on $M_{E,w}$ by Proposition 4.5. Moreover by Proposition 4.1 the class $(M_{E,w}, \|\cdot\|_{M_{E,w}})$ has the Fatou property. Now analogously as in the proof of Theorem 1, page 470 in [23], or page 30 in [15] one can show that

$$(M_{E,w})'' = M_{E,w} \quad \text{as sets, and} \quad \|\cdot\|_{(M_{E,w})''} \quad \text{is equivalent to} \quad \|\cdot\|_{M_{E,w}}.$$

Then by Theorem 5.1 we get the equality of sets $M_{E,w} = (M_{E,w})'' = (\Lambda_{E',w})'$ with equivalence of $\|\cdot\|_{M_{E,w}}$ and $\|\cdot\|_{(\Lambda_{E',w})'}$.

□

6. Spaces $Q_{E,w}$

In this chapter we introduce a new space related to the class $M_{E,w}$.

6.1. Definition and properties.

Definition 6.1. We denote by $Q_{E,w}$ the set of elements of $L^0 = L^0(I)$ which are submajorized by elements of $M_{E,w}$. For $f \in Q_{E,w}$ we set

$$\|f\|_{Q_{E,w}} = \inf\{\|g\|_{M_{E,w}} : f \prec g\}.$$

Given a positive and decreasing weight w on I and assuming that $W < \infty$, recall that the *Marcinkiewicz function space* M_W is defined as

$$M_W = \left\{ f \in L^0 : \|f\|_{M_W} = \sup_{x\in I} \frac{\int_0^x f^*}{W(x)} < \infty \right\},$$

and the space $L_1 + M_W$ is the set of all functions $f \in L^0$ such that

$$\|f\|_{L_1+M_W} = \inf\{\|h\|_1 + \|g\|_{M_W} : \ f = h + g, \ h \in L_1, \ g \in M_W\} < \infty.$$

The spaces $(M_W, \|\cdot\|_{M_W})$ and $(L_1 + M_W, \|\cdot\|_{L_1+M_W})$ are fully symmetric spaces [1, 13].

Theorem 6.2. *Let w be a weight function such that $W < \infty$ on I.*

(i) *The class $Q_{E,w}$ is a solid linear subspace of $L_1 + M_W$ such that*

$$\|f\|_{L_1+M_W} \leq C\|f\|_{Q_{E,w}} \quad \text{with} \quad C \leq (1 \wedge b)/\phi_E(1 \wedge b). \tag{6.1}$$

(ii) *The functional $\|\cdot\|_{Q_{E,w}}$ is a norm on $Q_{E,w}$.*

(iii) *$Q_{E,w}$ equipped with the norm $\|\cdot\|_{Q_{E,w}}$, is the smallest fully symmetric Banach function space containing the class $M_{E,w}$.*

(iv) *We have $(Q_{E,w})' = \Lambda_{E',w}$ with equality of norms.*

Proof. (i) If $f \in M_{E,w}$ then $f^*/w \in E_w$. The space E_w is fully symmetric with respect to the measure $d\omega = wdm$ on I by Proposition 2.2, so $E_w \hookrightarrow (L_1 + L_\infty)(I,\omega)$ with the embedding constant $C \le \frac{1\wedge b}{\phi_E(1\wedge b)}$ by [13, Ch. II, Theorem 4.1] and the fact that E and E_w have the same fundamental function. Since w is positive, the norms in $L_\infty(I,\omega)$ and $L_\infty(I)$ are equal. Thus for any $\epsilon > 0$ there exist $g \in L_1(I,\omega), h \in L_\infty(I,\omega)$ such that

$$f^*/w = g + h \text{ and } \|g\|_{L_1(I,\omega)} + \|h\|_\infty \le C\|f^*/w\|_{E_w} + \epsilon = C\|f\|_{M_{E,w}} + \epsilon.$$

Then $f^* = gw + hw$, $\|gw\|_1 = \|g\|_{L_1(I,\omega)}$ and $\|hw\|_{M_W} \le \|h\|_\infty \|w\|_{M_W} = \|h\|_\infty$. Hence

$$\|f^*\|_{L_1+M_W} \le \|gw\|_1 + \|hw\|_{M_W} \le \|g\|_{L_1(I,\omega)} + \|h\|_\infty \le C\|f\|_{M_{E,w}} + \epsilon,$$

which gives $\|f\|_{L_1+M_W} \le C\|f\|_{M_{E,w}}$ for any $f \in M_{E,w}$.

Assume now that $f \in Q_{E,w}$ and choose $g \in M_{E,w}$ such that $f \prec g$ and $\|g\|_{M_{E,w}} \le (1+\varepsilon)\|f\|_{Q_{E,w}}$. Since L_1+M_W is fully symmetric and by the previous paragraph $g \in L_1+M_W$, we have $f \in L_1 + M_W$ and

$$\|f\|_{L_1+M_W} \le \|g\|_{L_1+M_W} \le C\|g\|_{M_{E,w}} \le C(1+\varepsilon)\|f\|_{Q_{E,w}}.$$

Letting then $\varepsilon \to 0$, we obtain (6.1). It is also clear that $Q_{E,w}$ is a solid subset in L_1+M_W.

(ii) By (6.1) we have that $\|\cdot\|_{Q_{E,w}}$ is faithful, that is $\|f\|_{Q_{E,w}} = 0$ implies $f = 0$ a.e.. Since the homogeneous property of $\|\cdot\|_{Q_{E,w}}$ is clear, we need only to show the triangle inequality. For any $\epsilon > 0$ and $f_1, f_2 \in Q_{E,w}$, choose $g_1, g_2 \in M_{E,w}$ with

$$f_i \prec g_i \text{ and } \|g_i\|_{M_{E,w}} \le (1+\varepsilon)\|f_i\|_{Q_{E,w}}, \quad i = 1,2.$$

Then

$$(f_1 + f_2) \prec f_1^* + f_2^* \prec g_1^* + g_2^*.$$

Since $g_i^*/w \in E_w$, $i = 1,2$, and E_w is a linear space, we have $(g_1^* + g_2^*)/w \in E_w$ and so $g_1^* + g_2^* \in M_{E,w}$. Thus $f_1 + f_2 \in Q_{E,w}$. Moreover, since E_w is a normed space we get

$$\|g_1^* + g_2^*\|_{M_{E,w}} = \|(g_1^* + g_2^*)/w\|_{E_w} \le \|g_1^*/w\|_{E_w} + \|g_2^*/w\|_{E_w} = \|g_1\|_{M_{E,w}} + \|g_2\|_{M_{E,w}}.$$

Thus

$$\|f_1 + f_2\|_{Q_{E,w}} \le \|g_1^* + g_2^*\|_{M_{E,w}} \le \|g_1\|_{M_{E,w}} + \|g_2\|_{M_{E,w}} \le (1+\varepsilon)(\|f_1\|_{Q_{E,w}} + \|f_2\|_{Q_{E,w}}).$$

Letting $\varepsilon \to 0$ we obtain that the homogeneous functional $\|\cdot\|_{Q_{E,w}}$ is subadditive, and thus it is a norm on $Q_{E,w}$.

(iii) By definition of $\|\cdot\|_{Q_{E,w}}$, if $f \prec g$, $f \in L^0$ and $g \in Q_{E,w}$ then $f \in Q_{E,w}$ and $\|f\|_{Q_{E,w}} \le \|g\|_{Q_{E,w}}$. Clearly $\|f^*\|_{Q_{E,w}} = \|f\|_{Q_{E,w}}$. Hence $Q_{E,w}$ is fully symmetric. To prove that $Q_{E,w}$ is complete, by the Riesz criterion it is sufficient to show that if (f_n) is a non-negative sequence in $Q_{E,w}$ with $\sum_{n=1}^{\infty} \|f_n\|_{Q_{E,w}} < \infty$ then the series $\sum_{n=1}^{\infty} f_n$ converges in $Q_{E,w}$. In view of completeness of $L_1 + M_W$ and (6.1), $\sum_{n=1}^{\infty} f_n$ converges in $L_1 + M_W$.

For every n choose $g_n \in M_{E,w}$ with $\|g_n\|_{M_{E,w}} \le (1+\varepsilon)\|f_n\|_{Q_{E,w}}$ and $f_n \prec g_n$. Then $\sum_{n=1}^{\infty} \|g_n\|_{M_{E,w}} \le (1+\varepsilon) \sum_{n=1}^{\infty} \|f_n\|_{Q_{E,w}} < \infty$, and since $\|g_n\|_{M_{E,w}} = \|g_n^*/w\|_{E_w}$ it follows that $\frac{1}{w}\sum_{n=1}^{\infty} g_n^*$ converges in the Banach function space E_w. Therefore

$$\left\|\sum_{n=1}^{\infty} g_n^*\right\|_{M_{E,w}} = \left\|\frac{1}{w}\sum_{n=1}^{\infty} g_n^*\right\|_{E_w} \le \sum_{n=1}^{\infty} \left\|\frac{g_n^*}{w}\right\|_{E_w} \le (1+\varepsilon)\sum_{n=1}^{\infty} \|f_n\|_{Q_{E,w}}.$$

On the other hand $\sum_{n=1}^{\infty} f_n \prec \sum_{n=1}^{\infty} g_n^*$, thus $\sum_{n=1}^{\infty} f_n \in Q_{E,w}$ and by the above $\left\|\sum_{n=1}^{\infty} f_n\right\|_{Q_{E,w}} \leq \left\|\sum_{n=1}^{\infty} g_n^*\right\|_{M_{E,w}} \leq (1+\varepsilon) \sum_{n=1}^{\infty} \|f_n\|_{Q_{E,w}}$. Letting $\varepsilon \to 0$ we obtain $\left\|\sum_{n=1}^{\infty} f_n\right\|_{Q_{E,w}} \leq \sum_{n=1}^{\infty} \|f_n\|_{Q_{E,w}}$.

Similarly for every $m \in \mathbb{N}$ we have $\left\|\sum_{n=m}^{\infty} f_n\right\|_{Q_{E,w}} \leq \sum_{n=m}^{\infty} \|f_n\|_{Q_{E,w}} \to 0$ when $m \to \infty$ and thus $\sum_{n=1}^{\infty} f_n$ converges in $Q_{E,w}$, which achieves the proof of the completness of $Q_{E,w}$.

Finally if F is a fully symmetric Banach function space containing $M_{E,w}$, it contains also any function that is submajorized by a function of $M_{E,w}$, that is, it contains $Q_{E,w}$, which shows that $Q_{E,w}$ is the smallest fully symmetric Banach function space containing the class $M_{E,w}$.

(iv) In view of the assumption $W < \infty$, by Theorem 5.1 it is enough to show that the Köthe dual spaces $(Q_{E,w})'$ and $(M_{E,w})'$ are equal as sets with equal norms. Since $M_{E,w} \subset Q_{E,w}$, and the norm in $Q_{E,w}$ is clearly smaller than the gauge in $M_{E,w}$, the reverse inclusion $(Q_{E,w})' \subset (M_{E,w})'$ holds for their Köthe duals and $\|h\|_{(M_{E,w})'} \leq \|h\|_{(Q_{E,w})'}$.

Conversely if $h \in (M_{E,w})'$, $f \in Q_{E,w}$ and $\varepsilon > 0$, let us choose $g \in M_{E,w}$ with $f \prec g$ and $\|g\|_{M_{E,w}} \leq (1+\varepsilon)\|f\|_{Q_{E,w}}$. Then

$$\begin{aligned}
\int_I |fh|\, dm &\leq \int_I f^* h^*\, dm \quad \text{(Hardy-Littlewood inequality [1, Theorem 2.2])} \\
&\leq \int_I g^* h^*\, dm \quad \text{(Hardy's lemma [1, Proposition 3.6], } f^* \prec g^*,\ h^* \text{ is decreasing)} \\
&\leq \|g^*\|_{M_{E,w}} \|h^*\|_{(M_{E,w})'} \leq (1+\varepsilon)\|f\|_{Q_{E,w}} \|h\|_{(M_{E,w})'}.
\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain that $h \in (Q_{E,w})'$ with $\|h\|_{(Q_{E,w})'} \leq \|h\|_{(M_{E,w})'}$, and so $\|h\|_{(M_{E,w})'} = \|h\|_{(Q_{E,w})'}$. □

6.2. **Link with Halperin's level functions.** In this section let w be a positive decreasing weight function on I such that $W < \infty$ on I. For $f = f^*$ locally integrable on I, define after Halperin [6] for $0 \leq \alpha < \beta < \infty$, $\alpha, \beta \in I = (0, a)$, $a \leq \infty$,

$$W(\alpha,\beta) = \int_\alpha^\beta w dm, \quad F(\alpha,\beta) = \int_\alpha^\beta f\, dm, \quad R(\alpha,\beta) = \frac{F(\alpha,\beta)}{W(\alpha,\beta)},$$

and for $\beta = \infty$,

$$R(\alpha,\beta) = R(\alpha,\infty) = \limsup_{t\to\infty} R(\alpha,t).$$

Then $(\alpha,\beta) \subset I$ is called a *level interval (resp. degenerate level interval) of f with respect to w* if $\beta < \infty$ (resp. $\beta = \infty$) and for each $t \in (\alpha,\beta)$,

$$R(\alpha,t) \leq R(\alpha,\beta) \text{ and } 0 < R(\alpha,\beta).$$

Level intervals can be equivalently assumed to be open, closed or half-closed. If a level interval is not contained in any larger level interval, then it is called *maximal level interval of f with respect to w*, or just maximal level interval and in short m.l.i.. In [6], Halperin proved that maximal level intervals of f with respect to w are pairwise disjoint and unique and therefore there is at most countable number of maximal level intervals.

Definition 6.3. [6] Let $f \in L^0$ be non-negative, decreasing and locally integrable on I. Then the *level function* f^0 of f with respect to w is defined as

$$f^0(t) = \begin{cases} R(\alpha,\beta)\, w(t) & \text{if } t \text{ belongs to some maximal level interval } (\alpha,\beta), \\ f(t) & \text{otherwise.} \end{cases}$$

For a general $f \in L^0$, $0 \le \alpha < \beta < \infty$, $\alpha, \beta \in I$, we define

$$f^0 = (f^*)^0, \quad F(\alpha,\beta) = \int_\alpha^\beta f^*\,dm, \quad \text{and} \quad F(t) = \int_0^t f^*\,dm, \quad t \in I.$$

Fact 6.4 (Properties of level functions). *Let $f \in L_1 + L_\infty$ and w be a decreasing locally integrable weight function on I.*

(i) [6, Theorem 3.6] *f^0/w is decreasing. Consequently in view of w being decreasing, f^0 is decreasing as well.*
(ii) [6, Theorem 3.2] *$f \prec f^0$. Moreover if x does not belong to a m.l.i., $\int_0^x f^0\,dm = \int_0^x f^*\,dm$, and so if I is finite, $\int_I f^0\,dm = \int_I f^*\,dm$.*
(iii) [6, Theorem 3.7] *If $f \prec g$ then $f^0 \prec g^0$.*

Remark 6.5. (1) If $I = (0,a)$ with $a < \infty$ then for every $f \in L_1$, $\|f\|_1 = \|f^0\|_1$ by (ii) in Fact 6.4. Therefore $f^0(t) < \infty$ for $t \in (0,a)$.

(2) If $I = (0,\infty)$ there exist functions $f \in L_1 + L_\infty$ with a degenerate level function, that is $f^0 \equiv \infty$ on I. Indeed, consider $f \equiv 1$ on I, then $R(0,t) = t/W(t)$ is increasing. Hence $(0,\infty)$ is a m.l.i. of f, and if $\lim_{t\to\infty} t/W(t) = \infty$ then $R(0,\infty) = \infty$, and so $f^0 = R(0,\infty)\cdot w \equiv \infty$.

Note that if an interval (a,∞) with $a > 0$ is a m.l.i. of a function f then $R(a,\infty) < \infty$ since $f^0(a) < \infty$ and f^0 is decreasing. Thus the only possible degenerate level function is identically equal to ∞ on $I = (0,\infty)$. For f^0 to be degenerate it is necessary and sufficient that $\limsup_{t\to\infty} F(t)/W(t) = \infty$.

(3) When $I = (0,\infty)$ there are two simple cases where f^0 is non-degenerate.

(3a) Let $f \in L_1$. Then $\lim_{t\to\infty} F(t)/W(t) = \lim_{t\to\infty} (\int_0^t f^*)/W(t) = \|f\|_1/W(\infty) < \infty$.

If $W(\infty) = \infty$ and (a,∞), $a > 0$, is a m.l.i. of f, then $R(a,\infty) = 0$ and so $R(a,t) \le R(a,\infty) = 0$ for all $t > a$. Hence $f^*(t) = 0$ for $t > a$, and so $\|f\|_1 = \|f^0\|_1$ by (ii) in Fact 6.4, and consequently $f^0 < \infty$ on $(0,\infty)$ and so f^0 is non-degenerate.

If $W(\infty) < \infty$ and if f has an infinite m.l.i. say (a,∞) with $a > 0$, then for $t > a$ we have $f^0(t) = R(a,\infty)w(t) = \frac{F(a,\infty)}{W(a,\infty)} w(t) < \infty$. Clearly $f^0(t) < \infty$ for $t \in (0,a)$, and so f^0 is non-degenerate. Moreover $\|f\|_1 = \|f^0\|_1$.

(3b) Let $f \in M_W$. Then by definition we have $f \prec Cw$ where $C = \|f\|_{M_W}$. Hence $f^0 \prec Cw^0$ by (iii) of Fact 6.4. But $w^0 = w$, and so $\int_0^t f^0 dm \le CW(t)$ for $t \in I$. Thus $f^0 \in M_W$ with $\|f^0\|_{M_W} \le \|f\|_{M_W}$. Therefore f^0 is non-degenerate. In addition by $f \prec f^0$ we have $\|f\|_{M_W} \le \|f^0\|_{M_W}$, and it follows the equality of norms $\|f\|_{M_W} = \|f^0\|_{M_W}$.

The above two simple cases (3a) and (3b) may be combined as follows.

Lemma 6.6. *$f \in L_1 + M_W$ if and only if $f^0 \in L_1 + M_W$, and $\|f\|_{L_1+M_W} = \|f^0\|_{L_1+M_W}$.*

Proof. Assume $\|f\|_{L_1+M_W} < 1$. We have $f = g + h$ with some $g \in L_1$, $h \in M_W$ such that $\|g\|_1 + \|h\|_{M_W} < 1$. Then $f^* \prec g^* + h^* \prec g^* + \|h\|_{M_W} w$. It follows that $f^0 \prec (g^* + \|h\|_{M_W} w)^0$. It is easy to see that $g^* + Cw$ and g^* have the same m.l.i. and that $(g^* + Cw)^0 = g^0 + Cw$, $C = \|h\|_{M_W}$. Then

$$\|f^0\|_{L_1+M_W} \le \|g^0 + Cw\|_{L_1+M_W} \le \|g^0\|_1 + \|h\|_{M_W}\|w\|_{M_W} = \|g\|_1 + \|h\|_{M_W} < 1.$$

This shows that $\|f^0\|_{L_1+M_W} \le \|f\|_{L_1+M_W}$ for every $f \in L_1 + M_W$. The converse inclusion and inequality follow from $f \prec f^0$. □

Notation & Remark 6.7. If $g, h \in L^0$ then we write $g \prec_w h$ if $g^{*,w} \prec h^{*,w}$. Clearly if $h \in E_w$ and $g \prec_w h$ then $g \in E_w$ and $\|g\|_{E_w} \le \|h\|_{E_w}$.

Lemma 6.8. *For $f \in M_{L_1+L_\infty,w}$ we have $\dfrac{f^0}{w} \prec_w \dfrac{f^*}{w}$.*

Proof. Note that the hypothesis $f \in M_{L_1+L_\infty,w}$ is the right one for ensuring that f^*/w is locally integrable in measure ω, that is integrable on every set of finite measure ω. It implies also that $f^* \in L_1 + M_{L_\infty,w} \subset L_1 + M_W$ (see Example 9.2), thus by Lemma 6.6, the level function $f^0 < \infty$ belongs to $L_1 + M_W$ too.

By (1.1) we have to prove that for each $x \in J$,

$$\inf_{\lambda>0}\left[\int_I \left(f^0-\lambda w\right)_+ dm + \lambda x\right] \le \inf_{\lambda>0}\left[\int_I \left(f^*-\lambda w\right)_+ dm + \lambda x\right]. \tag{6.2}$$

If $(\alpha,\beta) \subset I$ is a non-degenerate m.l.i. of f^* we have for any $\lambda > 0$,

$$\begin{aligned}\int_\alpha^\beta \left(f^*-\lambda w\right)_+ dm &\ge \left(\int_\alpha^\beta \left(f^*-\lambda w\right) dm\right)_+ \\ &= \left(F(\alpha,\beta)-\lambda W(\alpha,\beta)\right)_+ = \left(R(\alpha,\beta)-\lambda\right)_+ W(\alpha,\beta) \\ &= \int_\alpha^\beta \left(R(\alpha,\beta)-\lambda\right)_+ w dm = \int_\alpha^\beta \left(f^0-\lambda w\right)_+ dm.\end{aligned}$$

Consider now a degenerate m.l.i. (α,∞) of f^*. Since $R(\alpha,t) \le R(\alpha,\infty)$ for all $t \ge \alpha$ and $R(\alpha,\infty) = \limsup_{t\to\infty} R(\alpha,t)$, there exists a sequence (t_n) such that $t_n \uparrow \infty$ with $R(\alpha,t_n) \uparrow R(\alpha,\infty)$. Then as above we have for each $n \in \mathbb{N}$,

$$\int_\alpha^{t_n} \left(f^*-\lambda w\right)_+ dm \ge \int_\alpha^{t_n} \left(R(\alpha,t_n)-\lambda\right)_+ w dm.$$

Passing to the limit $n \to \infty$ we obtain

$$\begin{aligned}\int_\alpha^\infty \left(f^*-\lambda w\right)_+ dm &\ge \int_\alpha^\infty \left(R(\alpha,\infty)-\lambda\right)_+ w dm \\ &= \int_\alpha^\infty \left(f^0-\lambda w\right)_+ dm.\end{aligned}$$

On the complementary set C of the union of all the m.l.i. we have $f^0 = f^*$, and thus

$$\int_C \left(f^0-\lambda w\right)_+ dm = \int_C \left(f^*-\lambda w\right)_+ dm.$$

Adding this equality with all the inequalities we obtained on each m.l.i. we get

$$\int_I \left(f^0-\lambda w\right)_+ dm \le \int_I \left(f^*-\lambda w\right)_+ dm,$$

which implies (6.2). □

If $f \in M_{E,w}$ then $f \in M_{L_1+L_\infty,w}$ and so by Lemma 6.8, $\frac{f^0}{w} \prec_w \frac{f^*}{w}$, and so by Notation & Remark 6.7, $\|f^0\|_{M_{E,w}} = \|f^0/w\|_{E_w} \le \|f^*/w\|_{E_w} = \|f\|_{M_{E,w}}$. Thus we get the next result.

Lemma 6.9. *If $f \in M_{E,w}$ then $f^0 \in M_{E,w}$ and $\|f^0\|_{M_{E,w}} \le \|f\|_{M_{E,w}}$.*

Now we prove that submajorization for level functions implies an inequality for their gauges in $M_{E,w}$.

Lemma 6.10. *For $f, g \in L_1 + L_\infty$, $f^0 \prec g^0$ if and only if $\dfrac{f^0}{w} \prec_w \dfrac{g^0}{w}$. Consequently, if $f \prec g$ and $g^0 \in M_{E,w}$ then $f^0 \in M_{E,w}$, and moreover $\|f^0\|_{M_{E,w}} \le \|g^0\|_{M_{E,w}}$.*

Proof. Let $f^0 \prec g^0$. By Fact 6.4(i) the functions f^0/w, g^0/w are both decreasing. Therefore by Proposition 2.2 (i), for $x \in J$,

$$\int_0^x \left(\frac{f^0}{w}\right)^{*,w} dm = \int_0^x \left(\frac{f^0}{w} \circ W^{-1}\right)^* dm = \int_0^x \frac{f^0}{w} \circ W^{-1} dm = \int_0^{W^{-1}(x)} \frac{f^0}{w} w dm$$
$$= \int_0^{W^{-1}(x)} f^0\, dm \le \int_0^{W^{-1}(x)} g^0\, dm = \int_0^x \left(\frac{g^0}{w}\right)^{*,w} dm,$$

and so $\dfrac{f^0}{w} \prec_w \dfrac{g^0}{w}$. The proof of the opposite implication is similar.

Now by Fact 6.4(iii) if $f \prec g$ then $f^0 \prec g^0$, and by the preceding $\dfrac{f^0}{w} \prec_w \dfrac{g^0}{w}$, which implies that $\|f^0\|_{M_{E,w}} = \|f^0/w\|_{E_w} \le \|g^0/w\|_{E_w} = \|g^0\|_{M_{E,w}}$.

□

Theorem 6.11. *A function $f \in L_1 + M_W$ belongs to $Q_{E,w}$ if and only if its level function f^0 relative to w belongs to $M_{E,w}$, and then $\|f\|_{Q_{E,w}} = \|f^0\|_{M_{E,w}}$.*

Proof. If $f \in L_1 + M_W$ then $f^*, f^0 < \infty$ on I and $f \prec f^0$ by Fact 6.4 (ii). Thus if $f^0 \in M_{E,w}$, then $f \in Q_{E,w}$, and $\|f\|_{Q_{E,w}} \le \|f^0\|_{M_{E,w}}$. Conversely if $f \in Q_{E,w}$, there is $g \in M_{E,w}$ with $f \prec g$, and for any such g we have by Lemmas 6.9 and 6.10 that $g^0 \in M_{E,w}$ and $f^0 \in M_{E,w}$ and moreover

$$\|f^0\|_{M_{E,w}} \le \|g^0\|_{M_{E,w}} \le \|g\|_{M_{E,w}}.$$

It follows that $\|f^0\|_{M_{E,w}} \le \|f\|_{Q_{E,w}}$. □

Proposition 6.12. *If E has the Fatou property then so has $Q_{E,w}$, and moreover $(\Lambda_{E',w})' = Q_{E,w}$ with equal norms.*

Proof. If $Q_{E,w}$ has the Fatou property, then since by Theorem 6.2 we have $\Lambda_{E',w} = (Q_{E,w})'$ with equal norms, it follows that $(\Lambda_{E',w})' = (Q_{E,w})'' = Q_{E,w}$ with equal norms.

It remains to prove that $Q_{E,w}$ has the Fatou property when E has the property. Let $f_n \uparrow f$ a.e. with $f_n \in Q_{E,w}$ and $\sup_n \|f_n\|_{Q_{E,w}} = K < \infty$. Since by Theorem 6.2 (i), $\|f_n\|_{L_1+M_W} \le C\|f_n\|_{Q_{E,w}}$ and L_1+M_W has the Fatou property, we have that $f \in L_1+M_W$, and $f_n^* \uparrow f^*$ a.e..

Letting $g_n = f_n^0$, by Theorem 6.11 we have $g_n \in M_{E,w}$ with $\|g_n\|_{M_{E,w}} = \|f_n\|_{Q_{E,w}}$. Moreover g_n and f^0 are decreasing and $f_n \prec g_n \prec f^0$ by Fact 6.4. Now by Helly's Selection Theorem [19, Chapter 8, Section 4] we may find a subsequence (g_{n_k}) which converges a.e. to some g. By Proposition 4.1, $M_{E,w}$ has the Fatou property and so $g \in M_{E,w}$ with $\|g\|_{M_{E,w}} \le \liminf \|g_{n_k}\|_{M_{E,w}} = \liminf \|f_{n_k}\|_{Q_{E,w}} \le K$. If we show that $f \prec g$ then $f \in Q_{E,w}$, and

$$\limsup \|f_{n_k}\|_{Q_{E,w}} \le \|f\|_{Q_{E,w}} \le \|g\|_{M_{E,w}} \le \liminf \|f_{n_k}\|_{Q_{E,w}},$$

which shows that $\|f_n\|_{Q_{E,w}} \uparrow \|f\|_{Q_{E,w}}$, the Fatou property of $Q_{E,w}$.

Assume further without loss of generality that $g_n \to g$ a.e.. By the monotone convergence theorem we get for $t \in I$,

$$\lim_n \int_0^t f_n^*\, dm = \int_0^t f^*\, dm.$$

For every $\alpha \in I$ there is N such that for $n > N$ we have $g_n(\alpha) \le g(\alpha^-) + 1 < \infty$, and so for all $t \ge \alpha$, $n > N$, $g_n(t) \le g(\alpha^-) + 1$. Now by the Lebesgue Dominated Convergence

Theorem, for $t \in I$,

$$\lim_n \int_\alpha^t g_n \, dm = \int_\alpha^t g \, dm.$$

On the other hand by $g_n \prec f^0$,

$$\int_0^\alpha g_n \, dm \le \int_0^\alpha f^0 \, dm =: \varepsilon(\alpha),$$

where the function $\varepsilon(\cdot)$ depends only on f and w, and it is continuous because $f^0 \in L_1 + M_W$ by Lemma 6.6 and thus is not degenerate. Then since $f_n \prec g_n$, for every $t \in I$,

$$\int_0^t f^* \, dm = \lim_n \int_0^t f_n^* \, dm \le \limsup_n \int_0^t g_n \, dm \le \varepsilon(\alpha) + \int_\alpha^t g \, dm \le \varepsilon(\alpha) + \int_0^t g \, dm.$$

Since $\varepsilon(\alpha) \to 0$ when $\alpha \to 0^+$, we get for all $t \in I$,

$$\int_0^t f^* \, dm \le \int_0^t g \, dm,$$

and we obtain that $f \prec g$ as desired. □

Remark 6.13. A shorter proof of Proposition 6.12 can be given using the fact that the level functions of an increasing sequence of functions form themselves an increasing sequence, and if f_n converge to f a.e. then f_n^0 converge to f^0 a.e.. This result was given by G. Sinnamon for his version of level functions, in the special case of a uniformly bounded sequence on a right-finite interval [21]. It may be transferred to Halperin's level functions using the results of [4], in the corresponding special case of functions in $M_{L^\infty,w}$ on a finite interval while the correct frame for our study is that of functions having a W-concave majorant [4]. The proof presented above avoids this problem and moreover it uses only Halperin's reference paper [6] for the sake of bibliographical simplicity.

Despite that the space E_w is considered over I with the measure $d\omega = wdm$, the space $(E_w)'$ will always denote its Köthe dual computed with respect to the Lebesgue measure m on I as it is done below.

Lemma 6.14. *For any $f \in (E_w)'$ we have $\|f\|_{(E_w)'} = \left\|\frac{f}{w}\right\|_{(E')_w}$. Moreover $(E_w)'' = (E'')_w$ with equality of norms.*

Proof. In view of Proposition 2.2 we get

$$\begin{aligned}
\|f\|_{(E_w)'} &= \sup\left\{\int_I |f| g \, dm : \|g\|_{E_w} \le 1\right\} = \sup\left\{\int_I \frac{|f|}{w} g \, wdm : \|g\|_{E_w} \le 1\right\} \\
&= \sup\left\{\int_J \left(\frac{|f|}{w} \circ W^{-1}\right) \cdot (g \circ W^{-1}) \, dm : \|g \circ W^{-1}\|_E \le 1\right\} \\
&= \sup\left\{\int_J \left(\frac{|f|}{w} \circ W^{-1}\right) \cdot h \, dm : \|h\|_E \le 1\right\} \\
&= \left\|\frac{|f|}{w} \circ W^{-1}\right\|_{E'} = \left\|\frac{f}{w}\right\|_{(E')_w}.
\end{aligned}$$

This proves the first part. Using this once for E, then for E' we get

$$\begin{aligned}
\|f\|_{(E_w)''} &= \sup\left\{\int_I |f| h \, dm : \|h\|_{(E_w)'} \le 1,\right\} = \sup\left\{\int_I |f| \frac{h}{w} \, wdm : \left\|\frac{h}{w}\right\|_{(E')_w} \le 1\right\} \\
&= \sup\left\{\int_I (|f|w) g \, dm : \|g\|_{(E')_w} \le 1\right\} = \left\|\frac{|f|w}{w}\right\|_{(E'')_w} = \|f\|_{(E'')_w},
\end{aligned}$$

which proves the second part. □

Lemma 6.15. *The equality* $(\Lambda_{E,w})'' = \Lambda_{E'',w}$ *holds with equal norms.*

Proof. We use the fact that if F is a Banach function space, then $f \geq 0$ belongs to F'' with $\|f\|_{F''} \leq 1$ if and only if there exists a sequence $0 \leq f_n \uparrow f$ a.e., with $f_n \in F$, $\|f_n\|_F \leq 1$ for all $n \in \mathbb{N}$ [23, Ch. 15, §66, Theorem 1].

Assume first that $f \in (\Lambda_{E,w})''$ with norm ≤ 1, and let $0 \leq f_n \uparrow f$ a.e. with $\|f_n\|_{\Lambda_{E,w}} \leq 1$. Then $f_n^* \uparrow f^*$ a.e., and $f_n^* \in E_w$, $\|f_n^*\|_{E_w} \leq 1$. Hence $f^* \in (E_w)''$ with $\|f^*\|_{(E_w)''} \leq 1$. However by Lemma 6.14, $(E_w)'' = (E'')_w$ and so $f \in \Lambda_{E'',w}$ with $\|f\|_{\Lambda_{E'',w}} \leq 1$.

Conversely, let $f \in \Lambda_{E'',w}$ with $\|f\|_{\Lambda_{E'',w}} \leq 1$, then $f^* \in (E'')_w$ with $\|f^*\|_{(E'')_w} \leq 1$. Since $(E'')_w = (E_w)''$, there exists $0 \leq g_n \uparrow f^* \in E_w$, with $\|g_n\|_{E_w} \leq 1$. Then

$$g_n^{*,w} = (g_n \circ W^{-1})^* \uparrow f^* \circ W^{-1}.$$

Setting $h_n = g_n^{*,w} \circ W$, we have h_n are non-negative and decreasing on I. Clearly $h_n \uparrow f^*$ and $h_n^{*,w} = g_n^{*,w}$, so $\|h_n\|_{\Lambda_{E,w}} = \|h_n\|_{E_w} = \|h_n^{*,w}\|_E = \|g_n^{*,w}\|_E = \|g_n\|_{E_w} \leq 1$. Therefore $f^* \in (\Lambda_{E,w})''$ with $\|f\|_{(\Lambda_{E,w})''} = \|f^*\|_{(\Lambda_{E,w})''} \leq 1$, which shows the desired equality of spaces and norms. □

The next corollary states an important result on Köthe duality of generalized Lorentz spaces $\Lambda_{E,w}$. As a corollary we obtain a new description of the Köthe dual space of the Orlicz-Lorentz space (see section 8.2 for details).

Corollary 6.16. *Let w be a decreasing positive weight on I and $W < \infty$. We have $(\Lambda_{E,w})' = Q_{E',w}$ with equal norms.*

Proof. By general theory of Banach function lattices [23, Theorem 2, p.457], $\Lambda_{E,w}$ and its Köthe bidual $(\Lambda_{E,w})''$ have the same Köthe duals. The result follows then by applying Proposition 6.12 to E' since E' has the Fatou property, and then Lemma 6.15. □

As an immediate corollary of Theorem 6.11 and Corollary 6.16 we obtain a generalization of the Hölder-Halperin inequality [6, Theorem 4.2].

Corollary 6.17. *Let w be a decreasing positive weight on I and $W < \infty$. For $f \in L^0$ we have*

$$\sup\left\{\int_I |fg| : g \in \Lambda_{E,w}, \|g\|_{\Lambda_{E,w}} \leq 1\right\} = \begin{cases} \|f^0\|_{M_{E',w}} & \text{if } f^0 \in M_{E',w}, \\ \infty & \text{otherwise.} \end{cases}$$

Consequently $\|f\|_{(\Lambda_{E,w})'} = \|f^0\|_{M_{E',w}} = \|f\|_{Q_{E',w}}$ *for every* $f \in (\Lambda_{E,w})'$.

Proof. The left member is finite if and only if $f \in (\Lambda_{E,w})' = Q_{E',w}$. In this case $\|f\|_{(\Lambda_{E,w})'} = \|f\|_{Q_{E',w}} = \|f^0\|_{M_{E',w}}$. Conversely if the right side is finite then f^0 is non-degenerate and belongs to $M_{E',w}$. Thus $f \prec f^0$ implies that $f \in Q_{E',w} = (\Lambda_{E,w})'$.

□

7. Spaces $P_{E,w}$

We assume in this chapter that $W < \infty$ on I.

Definition 7.1. We denote by $P_{E,w}$ the union of the classes $M_{E,v}$, where v is a positive decreasing weight submajorized by w on I. The symbol $v \downarrow$ means that v is decreasing. This set is equipped with the gauge

$$\|f\|_{P_{E,w}} = \inf\left\{\|f\|_{M_{E,v}} : v > 0, v \downarrow, v \prec w\right\}.$$

Our goal is to show that $\|\cdot\|_{P_{E,w}}$ is a symmetric norm, and in fact $P_{E,w} = Q_{E,w}$ as sets and $\|\cdot\|_{P_{E,w}} = \|\cdot\|_{Q_{E,w}}$. From the next lemma it follows that the gauge on $P_{E,w}$ is faithful.

Lemma 7.2. *We have $M_{E,w} \subset P_{E,w} \subset M_{E,\tilde{w}}$, where $\tilde{w}(t) := \frac{W(t)}{t}$, $t \in I$, and these inclusions are gauge-decreasing.*

Proof. The first inclusion and the corresponding gauge inequality are clear. Conversely for each $v \prec w$ we have $tv(t) \le V(t) \le W(t)$, where $V(t) = \int_0^t v\,dm$, $t \in I$. Hence $v(t) \le \tilde{w}(t)$, $t \in I$, and in view of Lemma 4.4, $M_{E,v} \subset M_{E,\tilde{w}}$, with $\|f\|_{M_{E,\tilde{w}}} \le \|f\|_{M_{E,v}}$. Taking the infimum with respect to $v \prec w$ we obtain $P_{E,w} \subset M_{E,\tilde{w}}$ with $\|f\|_{M_{E,\tilde{w}}} \le \|f\|_{P_{E,w}}$ for $f \in P_{E,w}$. □

Lemma 7.3. *If v is a positive decreasing weight such that $v \prec w$ and $h \in E_w$ is decreasing then $h \in E_v$ and $\|h\|_{E_v} \le \|h\|_{E_w}$.*

Proof. By Hardy's Lemma [1, Proposition 3.6, p. 56] since $(h-\lambda)_+$ is decreasing and $v \prec w$, for every $\lambda > 0$ we have

$$\int_I (h-\lambda)_+\, v dm \le \int_I (h-\lambda)_+\, w dm.$$

Then in view of identity (1.1) for any $x \in J$,

$$\begin{aligned}\int_0^x h^{*,v}\,dm &= \inf_{\lambda>0}\left[\int_I (h-\lambda)_+\, v dm + \lambda x\right]\\ &\le \inf_{\lambda>0}\left[\int_I (h-\lambda)_+\, w dm + \lambda x\right] = \int_0^x h^{*,w}\,dm,\end{aligned}$$

and so $h^{*,v} \prec h^{*,w}$. Thus since E is fully symmetric and $h^{*,w} \in E$ we have that $h^{*,v} \in E$ and so $h \in E_v$. Moreover $\|h\|_{E_v} = \|h^{*,v}\|_E \le \|h^{*,w}\|_E = \|h\|_{E_w}$. □

Proposition 7.4. *We have $(P_{E,w})' = \Lambda_{E',w}$ with equal norms.*

Proof. Since $M_{E,w} \subset P_{E,w}$ with gauge decreasing inclusion, we have $(P_{E,w})' \subset (M_{E,w})' = \Lambda_{E',w}$ by Theorem 5.1, and the inclusion is norm decreasing.

Conversely if $g \in P_{E,w}$ and $\varepsilon > 0$, there is $v \prec w$ such that

$$g \in M_{E,v} \quad \text{and} \quad \|g\|_{M_{E,v}} \le (1+\varepsilon)\|g\|_{P_{E,w}}.$$

Let $f \in \Lambda_{E',w}$. Then $f^* \in (E')_w$ and by Lemma 7.3, $f^* \in (E')_v$, hence $f \in \Lambda_{E',v}$ with $\|f\|_{\Lambda_{E',v}} = \|f^*\|_{(E')_v} \le \|f^*\|_{(E')_w} = \|f\|_{\Lambda_{E',w}}$. Then by Theorem 5.1, $fg \in L_1$ with

$$\int_I |fg|\,dm \le \|f\|_{\Lambda_{E',v}}\|g\|_{M_{E,v}} \le (1+\varepsilon)\|f\|_{\Lambda_{E',w}}\|g\|_{P_{E,w}}.$$

Thus $f \in (P_{E,w})'$ with $\|f\|_{(P_{E,w})'} \le (1+\varepsilon)\|f\|_{\Lambda_{E',w}}$. Since $\varepsilon > 0$ is arbitrary we obtain that the inclusion $\Lambda_{E',w} \subset (P_{E,w})'$ is norm decreasing. □

Consider the inverse level function w^f of w with respect to a non-negative decreasing and locally integrable function f, that was introduced in [10, Remark 4.4]. It is defined as

$$w^f(t) = \begin{cases} \frac{f(t)}{R(\alpha,\beta)} & \text{if } t \text{ belongs to some maximal level interval } (\alpha,\beta),\\ w(t) & \text{otherwise.}\end{cases}$$

Comparing this with Definition 6.3 of f^0 we have that $f^0(t) = R(\alpha,\beta)w(t)$ if $t \in (\alpha,\beta)$, and thus

$$w^f(t) = \begin{cases} \dfrac{f(t)}{f^0(t)} w(t) & \text{if } t \in (\alpha,\beta), \\ w(t) & \text{otherwise.} \end{cases} \tag{7.1}$$

By definition of the level function we can show directly that $f(t) > 0$ for $t \in (\alpha,\beta)$. Hence $f^0 > 0$ on (α,β) and since w is positive on I, so w^f is also positive on I. Moreover w^f is decreasing and $w^f \prec w$ [10, Remark 4.4]. For arbitrary $f \in L_1 + M_W$ we define $w^f = w^{f^*}$. Now we are ready to compare the classes $Q_{E,w}$ with $P_{E,w}$.

Proposition 7.5. *$Q_{E,w} \subset P_{E,w}$ and the inclusion is gauge decreasing.*

Proof. By Theorem 6.11 we have $\|f\|_{Q_{E,w}} = \|f^0\|_{M_{E,w}}$. Clearly $\dfrac{f^*}{w^f} = \dfrac{f^0}{w}$. By Fact 6.4(i) the latter function is decreasing. Hence by Lemma 7.3 and Theorem 6.11 we get

$$\|f\|_{M_{E,w^f}} = \left\| \frac{f^*}{w^f} \right\|_{E_{w^f}} \le \left\| \frac{f^*}{w^f} \right\|_{E_w} = \left\| \frac{f^0}{w} \right\|_{E_w} = \|f^0\|_{M_{E,w}} = \|f\|_{Q_{E,w}},$$

and a fortiori $\|f\|_{P_{E,w}} \le \|f\|_{Q_{E,w}}$. □

Remark 7.6. By Lemma 7.2 and Proposition 7.5 we have $M_{E,w} \subset Q_{E,w} \subset P_{E,w} \subset M_{E,\tilde{w}}$, with gauge-decreasing inclusions. In particular if w is regular the four classes coincide as sets, and the gauges are equivalent, and we recover that in this case the class $M_{E,w}$ is normable (Proposition 4.5).

Corollary 7.7. *If E has the Fatou property then $P_{E,w} = Q_{E,w}$ isometrically, that is $\|f\|_{P_{E,w}} = \|f\|_{Q_{E,w}}$ for every $f \in P_{E,w}$. Consequently the class $P_{E,w}$ is a fully symmetric Banach function space having all properties discussed in Section 6.*

Proof. By the Fatou property $E'' = E$, and Propositions 7.4, 7.5 and Theorem 6.16 we have $Q_{E,w} \subset P_{E,w} \subset (P_{E,w})'' = (\Lambda_{E',w})' = Q_{E'',w} = Q_{E,w}$, and these inclusions are gauge decreasing. Hence $P_{E,w} = Q_{E,w}$ with equality of norms. □

Since E' has the Fatou property we have $P_{E',w} = Q_{E',w}$ by Corollary 7.7, and $Q_{E',w} = (\Lambda_{E,w})'$ by Corollary 6.16, thus we get the following result which generalizes [10, Theorem 2.2], [12, Corollary 4.12] from Orlicz-Lorentz to abstract Lorentz spaces:

Corollary 7.8. *For any fully symmetric Banach function space E, we have $(\Lambda_{E,w})' = P_{E',w}$ isometrically.*

Now we investigate the order continuity of spaces $M_{E,w}$ and $P_{E,w}$.

Proposition 7.9. *If E is an order continuous symmetric space then $M_{E,w}$ and $P_{E,w}$ are order continuous.*

Proof. By Proposition 4.2 and the definition of $P_{E,w}$, for each $f \in M_{E,w}$, resp. $f \in P_{E,w}$, we have

$$\|f\|_{M_{E,w}} = \inf\{\|f/v\|_{E_v} : v \in \mathcal{V}_M\}, \text{ resp. } \|f\|_{P_{E,w}} = \inf\{\|f/v\|_{E_v} : v \in \mathcal{V}_P\}, \tag{7.2}$$

where

$$\mathcal{V}_M = \{v \in L^0_+, v^* = w, \operatorname{supp} v \supset \operatorname{supp} f\},$$
$$\mathcal{V}_P = \{v \in L^0_+, 0 < v^* \prec w, \operatorname{supp} v \supset \operatorname{supp} f\}.$$

Since for each $v \in \mathcal{V}_M$, $v^* = w$, we have $V(a) = \int_I v\,dm = \int_I w\,dm = b$. Thus $J_v = J$, and by Remark 3.7 each E_v, $v \in \mathcal{V}_M$, is order isometric with E, so by order continuity of E, E_v is also order continuous.

As for E_v, $v \in \mathcal{V}_P$, we note that $V(a) \le W(a) = b$, so that $J_v = (0, V(a)) \subset J = (0, b)$. By Remark 3.7, the space E_v is order isometric to the space $\chi_{J_v} E$, a band in E, and thus it is also order continuous.

If (f_n) is a non-negative decreasing sequence in $M_{E,w}$, respectively $P_{E,w}$, with $f_n \downarrow 0$ a.e., choose v in $\mathcal{V}_M$, respectively $\mathcal{V}_P$, such that $f_1/v \in E_v$. Then $f_n/v \downarrow 0$ a.e. and thus $\|f_n\|_{E_v} \downarrow 0$. Since $\|f_n\|_{M_{E,w}} \le \|f_n/v\|_{E_v}$, respectively $\|f_n\|_{P_{E,w}} \le \|f_n/v\|_{E_v}$, we get $f_n \to 0$ in $M_{E,w}$, respectively $P_{E,w}$. □

Recall that the norm of E is p-concave for some $1 \le p < \infty$, if for some $C > 0$, for every $f_i \in E$, $i = 1, \dots, n$, $n \in \mathbb{N}$, it holds

$$\left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_E \ge C \left(\sum_{i=1}^n \|f_i\|_E^p \right)^{1/p}.$$

The largest such constant C is called concavity constant of E.

Applying the approach as in the proof of Proposition 7.9, we can show the following statement about the p-concavity of $M_{E,w}$ or $P_{E,w}$, which generalizes [12, Corollary 3.6].

Proposition 7.10. *If E is p-concave, $1 \le p < \infty$, then so are the gauge of $M_{E,w}$ and the norm of $P_{E,w}$, with p-concavity constants not exceeding that of E.*

Proof. Let $f_i \in M_{E,w}$, $i = 1, \dots, n$, and $\epsilon > 0$. Then by (7.2) there exists $v \in \mathcal{V}_M$ such that

$$L := \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_{M_{E,w}} + \epsilon \ge \left\| \left(\frac{1}{v} \sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|_{E_v}.$$

Since E_v is order isometric to E, so the norm $\|\cdot\|_{E_v}$ is also p-concave with the same constant, and thus

$$L \ge C \left(\sum_{i=1}^n \left\| \frac{f_i}{v} \right\|_{E_v}^p \right)^{1/p} \ge C \left(\sum_{i=1}^n \|f_i\|_{M_{E,w}}^p \right)^{1/p}.$$

The part on the space $P_{E,w}$ we do analogously applying that E_v is order isometric to the space $\chi_{J_v} E$.

□

Recall the definition of the Banach envelope of a quasi-normed linear space $(X, \|\cdot\|_X)$ [8, pp. 27-28]. Denote by $(X^*, \|\cdot\|_{X^*})$ the dual space to X, that is the space of linear functionals which are bounded with respect to the quasinorm $\|\cdot\|_X$. It is a Banach space equipped with the usual norm $\|\cdot\|_{X^*}$. Let us define a functional on X by

$$\|x\|_{\widehat{X}} = \sup\{|f(x)| : f \in X^*, \text{ and } \|f\|_{X^*} \le 1\}.$$

If X^* separates the points of X then $\|\cdot\|_{\widehat{X}}$ is a norm on X. Then the *Banach envelope* $\widehat{X}$ of X is simply the completion of the normed linear space $(X, \|\cdot\|_{\widehat{X}})$. One can show that the Banach envelop of X is the smallest Banach space $(\widehat{X}, \|\cdot\|_{\widehat{X}})$ such that $\|x\|_{\widehat{X}} \le \|x\|_X$ for $x \in X$ and $(\widehat{X})^* = X^*$.

The following result is a generalization of [12, Corollary 4.13]. We refer to section 8.2 where the spaces $M_{E,w}$ and $P_{E,w}$ are interpreted in the case of E being an Orlicz space L_φ.

Corollary 7.11. *Let E be a fully symmetric order continuous Banach function space with Fatou property. If moreover $M_{E,w}$ is a linear space and its gauge is a quasinorm, and if E is order continuous, then $P_{E,w}$ is the Banach envelope of $M_{E,w}$. Consequently if w is regular and E is order continuous then $P_{E,w} = M_{E,w}$ with equivalent norm and gauge.*

Proof. Clearly $M_{E,w} \subset P_{E,w}$. Since by Proposition 7.9, $M_{E,w}$ and $P_{E,w}$ are both order continuous their topological dual spaces coincide isometrically with their Köthe duals. By Theorem 5.1 and Proposition 7.4 we have that $(M_{E,w})' = \Lambda_{E',w} = (P_{E,w})'$. Since moreover $P_{E,w}$ is a Banach space by Corollary 7.7, it must be the Banach envelope of $M_{E,w}$. The second part results from Proposition 4.5. □

8. Applications to modular and Orlicz-Lorentz spaces

Here we apply the results obtained in the previous sections to Orlicz spaces $E = L_\varphi$. A special feature of these spaces, as well as of Orlicz-Lorentz spaces, is that their Banach space structure is induced by a modular space structure. In the present section we introduce modular structures on the spaces $P_{L_\varphi,w}$ and $Q_{L_\varphi,w}$ by defining two convex modulars P, Q, which have the same domain $\mathcal{M}_{L_\varphi,w} : P_{L_\varphi,w} = Q_{L_\varphi,w}$. These modulars have the same Luxemburg, resp. Orlicz norms, which are also the norms on $P_{L_\varphi,w}$ and $Q_{L_\varphi,w}$ when L_φ is equipped with its Luxemburg, resp. Orlicz norms. The modular P has been already defined in [12, 10]. This allows to compare the present work for L_φ spaces with the results in those papers. The introduction of the modular Q seems however to be new.

8.1. **Modular spaces.** We start with an introduction to modular spaces [16, 18].

Definition 8.1. Let X be a real vector space. For an extended real valued functional $\rho : X \to [0, \infty]$ consider the following conditions.

(i) $\rho(0) = 0$ and $\rho(-x) = \rho(x)$ for every $x \in X$.
(ii) If $x \in X$ and $\rho(tx) = 0$ for every $t \geq 0$ then $x = 0$.
(iii) ρ is convex.
(iii') For every $x \in X$, the extended real valued function $t \to \rho(tx)$ is convex.

If ρ satisfies conditions (i), (iii) then ρ is called a *pseudo-modular*, and a *modular* if it satisfies also (ii). If ρ fulfills (i), (ii), (iii′) then ρ will be called a *convex along rays-modular* (in short, CAR-modular). There is also a notion of CAR-pseudo-modular for which (ii) has not to be satisfied. In all preceding cases, the *modular domain* X_ρ consists of all $x \in X$ such that $\rho(tx) < \infty$ for some $t > 0$.

Note that in Musielak's classical terminology [16], our 'modular' functionals would be called 'convex semi-modular'.

It is easy to check that for ρ a (pseudo-) modular, X_ρ is a vector space, and for ρ a CAR-modular it may be only shown to be a symmetric cone.

If ρ is a modular (resp. a pseudo-modular) then two norms (resp. semi-norms) on X_ρ are classically associated with ρ, which are defined as follows.

– the *Luxemburg* (or second Nakano [18]) norm is the Minkowski functional of the convex set $U = \{x \in E : \rho(x) \leq 1\}$, thus

$$\|x\|_\rho = \inf\{\lambda > 0 : \rho(x/\lambda) \leq 1\}, \tag{8.1}$$

– the *Orlicz* (or first Nakano [18]) norm is given by Amemiya's formula [16]

$$\|x\|_\rho^0 = \inf_{\lambda>0} \frac{1 + \rho(\lambda x)}{\lambda} = \inf_{t>0} \left(t + t\rho\left(\frac{x}{t}\right)\right). \tag{8.2}$$

There is another expression of the Luxemburg norm, similar to Amemiya's formula. In fact we have

$$\|x\|_\rho = \inf_{\lambda>0} \frac{1 \vee \rho(\lambda x)}{\lambda} = \inf_{t>0} \left(t \vee t\rho\left(\frac{x}{t}\right)\right). \tag{8.3}$$

Indeed,

$$\inf_{t>0}(t \vee t\rho(t^{-1}x)) \geq \inf_{t\geq\|x\|_\rho} t \wedge \inf_{t<\|x\|_\rho} t\rho(t^{-1}x) = \|x\|_\rho \wedge \lim_{t\uparrow\|x\|_\rho} t\rho(t^{-1}x) = \|x\|_\rho,$$

since by convexity of ρ, the map $t \mapsto t\rho(t^{-1}x)$ is decreasing on $(0,\infty)$. On the other hand

$$\inf_{t>0}(t \vee t\rho(t^{-1}x)) \leq \inf_{t>\|x\|_\rho}(t \vee t\rho(t^{-1}x)) = \|x\|_\rho$$

since $\rho(t^{-1}x) \leq 1$ for $t > \|x\|_\rho$.

It is clear that a pseudo-modular is a modular if and only if the associated Luxemburg or Orlicz semi-norms are norms.

If we replace the modular ρ by a CAR-modular then all formulas (8.1), (8.2) and (8.3) remain valid although the functionals $\|\cdot\|_\rho$ and $\|\cdot\|_\rho^0$ are not norms on X_ρ since the triangle inequality may be not satisfied. They are however *gauges* that is positively homogeneous functionals.

By (8.2) and (8.3), the equivalence of $\|\cdot\|_\rho$ and $\|\cdot\|_\rho^0$ is immediate.

Lemma 8.2. *Let X be a vector space and $\rho : X \to [0,\infty]$ be a (pseudo-, CAR-) modular on X. Let $\rho_v : X \to [0,\infty]$, $v \in \mathcal{V}$, be a family of CAR-modulars on X. If*

$$\rho(x) = \inf_{v\in\mathcal{V}} \rho_v(x),$$

then the modular domain of ρ is

$$X_\rho = \bigcup_{v\in\mathcal{V}} X_{\rho_v}$$

and its associated norms are

$$\|x\|_\rho = \inf\{\|x\|_{\rho_v} : \ v \in \mathcal{V}\} \quad \textit{and} \quad \|x\|_\rho^0 = \inf\{\|x\|_{\rho_v}^0 : \ v \in \mathcal{V}\}.$$

Proof. For $x \in X$ we have

$$\|x\|_\rho = \inf_{t>0}(t \vee t\rho(t^{-1})) = \inf_{t>0}(t \vee t \inf_{v\in\mathcal{V}} \rho_v(t^{-1}x)) = \inf_{v\in\mathcal{V}} \inf_{t>0} t(1 \vee \rho_v(t^{-1}x)) = \inf_{v\in\mathcal{V}} \|x\|_{\rho_v}.$$

The formula for Amemiya norm follows analogously. □

Lemma 8.3. *Let $X \subset L^0(\Omega)$ be a vector space which is closed under rearrangements, i.e. $f^* \in X$ whenever $f \in X$. Assume $\rho : X \to [0,\infty]$ satisfies conditions* (i), (ii) *of Definition 8.1, ρ is convex on the cone of decreasing non-negative functions in X, ρ is symmetric that is $\rho(f^*) = \rho(f)$, and ρ is monotone that is $\rho(f) \leq \rho(g)$ if $|f| \leq |g|$, $f, g \in X$. Then for $f \in X$,*

$$\bar{\rho}(f) = \inf\{\rho(g^*) : f \prec g, g \in X\}$$

is a symmetric pseudo-modular on X, monotone with respect to the relation $\prec$, with associated Luxemburg and Amemiya semi-norms given respectively by

$$\|f\|_{\bar{\rho}} = \inf\{\|g\|_\rho : f \prec g, g \in X\} \quad \textit{and} \quad \|f\|_{\bar{\rho}}^0 = \inf\{\|g\|_\rho^0 : f \prec g, g \in X\}.$$

Proof. It is clear that the functional $\bar{\rho}$ satisfies (i) of Definition 8.1, and is symmetric and monotone with respect to $\prec$. Now let $f_1, f_2 \in X_{\bar{\rho}}$ with $\bar{\rho}(f_i) < \infty$, $i = 1, 2$, and $t_1, t_2 \geq 0$ with $t_1 + t_2 = 1$. Given $\varepsilon > 0$ choose $g_1, g_2 \in X$ such that $f_i \prec g_i$ and $\rho(g_i) \leq \bar{\rho}(f_i) + \varepsilon$, $i = 1, 2$. Then in view of

$$t_1 f_1 + t_2 f_2 \prec t_1 f_1^* + t_2 f_2^* \prec t_1 g_1^* + t_2 g_2^*,$$

we have, by symmetry and convexity of ρ on the cone of decreasing functions

$$\bar{\rho}(t_1 f_1 + t_2 f_2) \leq \rho(t_1 g_1^* + t_2 g_2^*) \leq t_1\rho(g_1) + t_2\rho(g_2) \leq t_1\bar{\rho}(f_1) + t_2\bar{\rho}(f_2) + \varepsilon,$$

which shows that $\bar{\rho}$ is convex. Since ρ is a CAR-modular, formulas (8.2) and (8.3) are satisfied. Moreover,

$$\|f\|_{\bar{\rho}} = \inf_{t>0} t(1 \vee \bar{\rho}(t^{-1}f)) = \inf_{t>0} t(1 \vee \inf_{f\prec g} \rho(t^{-1}g^*)) = \inf_{f\prec g} \inf_{t>0} t(1 \vee \rho(t^{-1}g) = \inf_{f\prec g} \|g\|_\rho,$$

where $g \in X$. Similarly we get the second formula associated with Amemiya functional. □

8.2. **Orlicz-Lorentz spaces and their Köthe duals.** Assume in this section that $W < \infty$ on I. Let $E = L_\varphi$ be an Orlicz space on J. As was mentioned in section 2.1, L_φ is a modular space generated by the modular $I_\varphi(f) = \int_J \varphi(|f|)\,dm$. Then $E_w = (L_\varphi)_w$ is the set of $f \in L^0(J)$ such that for some $\lambda > 0$, $\int_J \varphi(\lambda|f|)w\,dm < \infty$, so it is a modular space defined by the modular $\int_J \varphi(|f|)w\,dm$. Hence the generalized Lorentz space $\Lambda_{L_\varphi,w}$ consists of all $f \in L^0(I)$ such that $f^* \in (L_\varphi)_w$, so it is a modular space corresponding to the modular

$$\Phi(f) := \int_I \varphi(f^*)w\,dm. \tag{8.4}$$

This space is usually called an Orlicz-Lorentz space and is denoted by $\Lambda_{\varphi,w}$ [9, 11, 12]. Setting now for $f \in L^0 = L^0(I)$,

$$\mathrm{M}(f) := \int_I \varphi\left(\frac{f^*}{w}\right) w\,dm,$$

then the functional M is a CAR-modular on L^0. By definition, the space $M_{L_\varphi,w}$ consists of all $f \in L^0$ such that $f^*/w \in (L_\varphi)_w$. It follows that this space is the modular space induced by the CAR-modular M. Moreover the Luxemburg and Amemiya gauges associated with the modular M on $M_{L_\varphi,w}$ coincide with those defined in section 4.1 on $M_{E,w}$ when $E = L_\varphi$ is equipped with its Luxemburg and Amemiya norms respectively.

Now we will characterize the spaces $Q_{L_\varphi,w}$ and $P_{L_\varphi,w}$.

Lemma 8.4. *Let for $f \in L^0$,*

$$\mathrm{P}(f) := \inf\{\mathrm{M}_v(f) : v \prec w,\ v > 0,\ v \downarrow\} \quad \textit{where} \quad \mathrm{M}_v(f) = \int_I \varphi\left(\frac{f^*}{v}\right) v\,dm. \tag{8.5}$$

Then P *is a convex modular with domain $P_{L_\varphi,w}$ and the Luxemburg and Orlicz norms associated with this modular coincide with the norms on $P_{L_\varphi,w}$ given by Definition 7.1, associated with the Luxemburg and Orlicz norms respectively on L_φ.*

Proof. The modular P is convex by [12, Theorem 4.7] and its proof. By convexity of φ it is clear that the function $t \mapsto \mathrm{M}_v(tf)$ is convex for every $f \in L^0$. Therefore M_v is a CAR-modular for every $v > 0$. The last part of the lemma is a consequence of Lemma 8.2 by letting $\rho(f) = \mathrm{P}(f)$, $\mathcal{V} = \{v \prec w,\ v > 0,\ v \downarrow\}$ and $\rho_v(f) = \mathrm{M}_v(f)$. □

Lemma 8.5. *Let for $f \in L^0$*

$$\mathrm{Q}(f) := \inf\left\{\mathrm{M}(g) : f \prec g,\ g \in M_{L_\varphi,w}\right\}. \tag{8.6}$$

Then Q *is a convex modular with modular domain $Q_{L_\varphi,w}$ and the Luxemburg and Orlicz norms associated with this modular coincide with the norms on $Q_{L_\varphi,w}$ given by Definition 6.1, associated with the Luxemburg and Orlicz norms respectively on L_φ.*

Proof. Applying Lemma 8.3, with $\rho(f) = \mathrm{M}(f)$ and $\bar{\rho}(f) = \mathrm{Q}(f)$ gives that Q is a symmetric pseudo-modular, and by Lemma 8.2 its Luxemburg and Orlicz semi-norms coincide with the norms on $Q_{L_\varphi,w}$ given by Definition 6.1, when L_φ is equipped with its Luxemburg and Orlicz norms, respectively. In particular those semi-norms are in fact norms and Q is a modular. □

The next fact is well known and can be easily deduced from [5, Theorem 7.4.1]. We provide in Appendix a completely different and self-contained proof of it for the convenience of the reader.

Fact 8.6. *Let $\psi : [0,\infty) \to [0,\infty)$ be a convex increasing function. If $f, g \in L_1 + L_\infty \subset L^0(\Omega, \mathcal{A}, \mu)$ with $f \prec_\mu g$ then $\psi(f) \prec_\mu \psi(g)$.*

Proposition 8.7. *The modular* $\mathrm{Q}(f)$ *for* $f \in Q_{L_\varphi,w}$ *is expressed in terms of the level function* f^0 *by*

$$\mathrm{Q}(f) = \mathrm{M}(f^0) = \int_I \varphi\left(\frac{f^0}{w}\right) w\, dm.$$

Proof. Let $f \in Q_{L_\varphi,w}$, then for each $g \in M_{L_\varphi,w}$ such that $f \prec g$ we have $f^0 \prec g^0$ by Fact 6.4(iii). Since $(f^0/w) \circ W^{-1}$ is decreasing, it follows that $(f^0/w) \circ W^{-1} \prec (g^0/w) \circ W^{-1}$, which by Proposition 2.2 (i) is equivalent to $f^0/w \prec_w g^0/w$. We also have $g^0/w \prec_w g^*/w$ by Lemma 6.8, whence by Fact 8.6 above

$$\varphi(f^0/w) \prec_w \varphi(g^0/w) \prec_w \varphi(g^*/w).$$

It follows that $\mathrm{M}(f^0) \le \mathrm{M}(g)$, and so $\mathrm{M}(f^0) \le \mathrm{Q}(f)$. Since $f \prec f^0$ by Fact 6.4(ii), $\mathrm{Q}(f) \le \mathrm{M}(f^0)$, and the proof is finished. □

In view of Corollary 7.7, $Q_{L_\varphi,w} = P_{L_\varphi,w}$, with equal norms, and we will further use the notation (introduced in [12] for the domain of the modular P)

$$\mathcal{M}_{\varphi,w} := Q_{L_\varphi,w} = P_{L_\varphi,w}$$

According to whether L_φ is equipped with its Luxemburg or Orlicz norm, the space $\mathcal{M}_{\varphi,w}$ is equipped with two different norms that we denote by $\|\cdot\|_{\mathcal{M}_{\varphi,w}}$, resp. $\|\cdot\|^0_{\mathcal{M}_{\varphi,w}}$. Each of these norms has two different expressions corresponding to the respective definitions of the norms in $Q_{L_\varphi,w}$ and $P_{L_\varphi,w}$. Moreover by Theorem 6.11 the norm of a function in $Q_{L_\varphi,w}$ is the gauge of the corresponding level function in $M_{L_\varphi,w}$. We have thus:

Theorem 8.8. *Let* φ *be an Orlicz function and* w *be a decreasing positive weight function on* $I = (0,a)$, $a \le \infty$, *such that* $W < \infty$ *on* I. *Then for* $f \in \mathcal{M}_{\varphi,w}$ *we have*

$$\|f\|_{\mathcal{M}_{\varphi,w}} = \inf\{\|f\|_{\mathrm{M}_v} :\ v \prec w, v > 0,\ v \downarrow\} = \inf\{\|g\|_{\mathrm{M}};\ f \prec g\} = \|f^0\|_{\mathrm{M}}, \tag{8.7}$$

$$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \inf\{\|f\|^0_{\mathrm{M}_v} :\ v \prec w, v > 0,\ v \downarrow\} = \inf\{\|g\|^0_{\mathrm{M}};\ f \prec g\} = \|f^0\|^0_{\mathrm{M}}, \tag{8.8}$$

where $\|\cdot\|_{\mathrm{M}}$, $\|\cdot\|_{\mathrm{M}_v}$ *are Luxemburg, and* $\|\cdot\|^0_M, \|\cdot\|^0_{M_v}$ *are Amemiya gauges.*

On the other hand, $\mathcal{M}_{\varphi,w}$ is the modular space induced by both modular Q and P. To each of the modulars Q, P are associated its Luxemburg and Orlicz norms. It appears that both the Luxemburg norms of Q, P coincide with $\|\cdot\|_{\mathcal{M}_{\varphi,w}}$, and the Orlicz norms with $\|\cdot\|^0_{\mathcal{M}_{\varphi,w}}$.

Applying the results developed so far we obtain additional insight on these modular structures.

Theorem 8.9. *Let* φ *be an Orlicz function and* w *be a decreasing positive weight function on* $I = (0,a)$, $a \le \infty$, *such that* $W < \infty$ *on* I. *Then*

$$\mathrm{Q}(f) = \mathrm{M}(f^0) = \mathrm{M}_{w^f}(f) \ge \mathrm{P}(f). \tag{8.9}$$

For $f \in \mathcal{M}_{\varphi,w}$ *we have*

$$\|f\|_{\mathcal{M}_{\varphi,w}} = \|f\|_{\mathrm{P}} = \|f\|_{\mathrm{Q}}, \tag{8.10}$$

$$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \|f\|^0_{\mathrm{P}} = \|f\|^0_{\mathrm{Q}} \tag{8.11}$$

If in addition φ *is a* N*-function that is* $\lim_{s\to 0} \varphi(s)/s = 0$ *and* $\lim_{s\to\infty} \varphi(s)/s = \infty$, *and either* I *is finite or* $W(\infty) = \int_0^\infty w\, dm = \infty$, *then*

$$\mathrm{P}(f) = \mathrm{M}(f^0) = \mathrm{Q}(f). \tag{8.12}$$

Proof. The first part of (8.9) follows from Proposition 8.7 and the second one from equality (7.1). Since $w^f \prec w$, $\mathrm{M}_{w^f}(f) \geq \mathrm{P}(f)$.

Equations (8.10), (8.11) follow from Lemmas 8.5 and 8.4.

Under the additional assumptions when φ is N-function and $W(\infty) = \infty$, the first equation in (8.12) has been presented in Theorem 4.8 in [10]. □

Now let us summarize all known results describing the Köthe dual of the Orlicz-Lorentz space $\Lambda_{\varphi,w}$. For the space $\Lambda_{\varphi,w}$ by $\|\cdot\|_{\Lambda_{\varphi,w}}$ and $\|\cdot\|^0_{\Lambda_{\varphi,w}}$ denote the Luxemburg and Orlicz norm respectively. Recall that $\varphi_*(t) = \sup_{s\geq 0}\{st - \varphi(s)\}$, $t \geq 0$, is the complementary function to the Orlicz function φ.

In the next theorem we state complete descriptions of the dual spaces of the Orlicz-Lorentz space equipped with two standard Luxemburg and Orlicz norms. Recall indeed that the Orlicz-Lorentz space $\Lambda_{L_\varphi,w} = \Lambda_{\varphi,w}$ has a natural modular space structure given by the modular Φ defined in the equation (8.4), with respect to which $\Lambda_{\varphi,w}$ is equipped with both a Luxemburg norm $\|\cdot\|_{\Lambda_{\varphi,w}}$ and an Orlicz norm $\|\cdot\|^0_{\Lambda_{\varphi,w}}$. It is easy to see that these norms are identical to the norms of $\Lambda_{L_\varphi,w}$ when the Orlicz space L_φ is equipped respectively with its own Luxemburg or Orlicz norm.

Theorem 8.10. *Let φ be an Orlicz function and w be a decreasing positive weight function on $I = (0, a)$, $a \leq \infty$, such that $W < \infty$ on I. Then the Köthe dual spaces to the Orlicz-Lorentz spaces $(\Lambda_{\varphi,w}, \|\cdot\|_{\Lambda_{\varphi,w}})$ and $(\Lambda_{\varphi,w}, \|\cdot\|^0_{\Lambda_{\varphi,w}})$ are as follows*

$$(\Lambda_{\varphi,w}, \|\cdot\|_{\Lambda_{\varphi,w}})' = (\mathcal{M}_{\varphi_*,w}, \|\cdot\|^0_{\mathcal{M}_{\varphi_*,w}}) \quad \textit{and} \quad (\Lambda_{\varphi,w}, \|\cdot\|^0_{\Lambda_{\varphi,w}})' = (\mathcal{M}_{\varphi_*,w}, \|\cdot\|_{\mathcal{M}_{\varphi_*,w}}),$$

where the norms $\|\cdot\|_{\mathcal{M}_{\varphi_*,w}}$ and $\|\cdot\|^0_{\mathcal{M}_{\varphi_*,w}}$ are given by (8.7) and (8.8), respectively, where φ is replaced by φ_*.

Proof. This is is a consequence of Corollary 6.16, and the fact that when $E = L_\varphi$ is an Orlicz space equipped with its Luxemburg (resp. Orlicz) norm then its Köthe dual E' is L_{φ_*} equipped with its Orlicz (resp. Luxemburg) norm. □

Comparing to Theorem 4.8 in [10], the above theorem is more general since it is proved here without additional assumptions that φ is N-function and $W(\infty) = \infty$. It is also more informative since it provides three different formulas for the norms in the dual space $\mathcal{M}_{\varphi_*,w}$. In fact each Luxemburg and Orlicz norm have three formulas expressed by (8.7) and (8.8), corresponding either to modular Q or P or to level functions. The ones related to the modular Q are new here.

Finally we obtain a corollary on representation of the dual space for the classical Lorentz space $\Lambda_{p,w}$. If $\varphi(t) = t^p$, $1 \leq p < \infty$, then we use the following notations

$$\Lambda_{p,w} := \Lambda_{\varphi,w} \quad \text{and} \quad \mathcal{M}_{p,w} := \mathcal{M}_{\varphi,w}.$$

In this case $\varphi_*(t) = \frac{p^{1-q}}{q} t^q$ and the Orlicz norms on L_{φ_*} and $\mathcal{M}_{\varphi_*,w}$ coincide with the classical norms on L_q and $\mathcal{M}_{q,w}$ respectively. We provide below three different formulas of the norm in the dual space $(\Lambda_{p,w})^*$. The formula (8.14) has been presented as Corollary 4.9 in [10], and (8.15) has been proved by Halperin in [6, Theorem 6.1, Corollary, p. 288]. The first expression however, (8.13), is new and it results from the introduction of the space $Q_{E,w}$ and $(\Lambda_{E,w})' = Q_{E',w}$.

Theorem 8.11. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and w be a decreasing positive weight function on $I = (0, a)$, $a \leq \infty$, such that $W < \infty$ on I. Then*

$$(\Lambda_{p,w})' = \mathcal{M}_{q,w}.$$

If in addition $W(\infty)=\infty$ when $I=(0,\infty)$, then the dual space $(\Lambda_{p,w})^*$ is isometric to $\mathcal{M}_{q,w}$. In fact for every $F\in(\Lambda_{p,w})^*$ there exists $f\in\mathcal{M}_{q,w}$ such that

$$F(g)=\int_I fg\,dm,\quad g\in\Lambda_{p,w},$$

and

$$\|F\|=\|g\|_{\mathcal{M}_{q,w}}=\inf\left\{\left(\int_I (g^*)^q w^{1-q}\right)^{1/q}: f\prec g\right\} \tag{8.13}$$

$$=\inf\left\{\left(\int_I (f^*)^q v^{1-q}\right)^{1/q}: v\prec w, v>0, v\downarrow\right\} \tag{8.14}$$

$$=\left(\int_I [(f^*)^0]^q w^{1-q}\right)^{1/q}. \tag{8.15}$$

Proof. The Köthe duality follows from Theorem 8.10. It is also well known and easy to show that $\Lambda_{p,w}$ is order continuous when $W(\infty)=\infty$ in the case of $I=(0,\infty)$. Therefore the Köthe dual space is isometric to the dual space via integral functionals [1, Theorem 4.1].

□

Remark 8.12. For $f\in L^0$ define

$$\mathrm{Q}_{\varphi,w}(f)=\inf\left\{\int_I \varphi(h)w\,dm : h\downarrow \text{ and } f\prec hw\right\}.$$

This formula was introduced once by K. Nakamura [17], who determined the modular dual to the natural modular in $\Lambda_{\varphi,w}$ as being $Q_{\varphi_*,w}$, when φ is a N-function satisfying a Δ_2-condition and φ_* a complementary function to φ. Note that

$$Q_{\varphi,w}(f)=\inf\left\{\mathrm{M}(g): f\prec g \text{ and } g/w\downarrow\right\}.$$

Hence clearly $\mathrm{Q}(f)\le Q_{\varphi,w}(f)$. On the other hand since f^0/w is non-increasing, we have $Q_{\varphi,w}(f)\le M(f^0)=\mathrm{Q}(f)$ and finally $Q_{\varphi,w}(f)=\mathrm{Q}(f)$.

9. Examples of $M_{E,w}$ and $Q_{E,w}$ spaces

Let w be a positive decreasing weight on $I=(0,a)$ such that $W<\infty$ on I, and E be a Banach function space defined on the interval $J=(0,b)$, $b=W(a)$, equipped with the Lebesgue measure m. In this section we will identify the spaces $M_{E,w}$ and $Q_{E,w}$ for some classical spaces E. Note that $M_{E,w}$, $Q_{E,w}\subset L^0(I)$.

Example 9.1. *If $E=L_1(J)$, then $(M_{L_1,w},\|\cdot\|_{M_{L_1,w}})=(Q_{L_1,w},\|\cdot\|_{Q_{L_1,w}})=(L_1(I),\|\cdot\|_1)$.*

Proof. Clearly $E_w=(L_1)_w=L_1(I,\omega)$ is a weighted L_1 space. We also have

$$f\in M_{L_1,w}\iff \frac{f^*}{w}\in(L_1)_w\iff \int_I \frac{f^*}{w}w\,dm<\infty\iff \int_I f^*\,dm<\infty\iff f\in L_1(I).$$

Hence $M_{L_1,w}=L_1(I)$ with the same norms. It follows that $Q_{L_1,w}=L_1(I)$, also with the same norms. □

Example 9.2. *If $E=L_\infty(J)$ then*

$$M_{L_\infty,w}=\{f:\|f\|_{M_{L_\infty,w}}<\infty\}\quad \textit{with}\quad \|f\|_{M_{L_\infty,w}}=\inf\{C: f^*\le Cw\}=\|f^*/w\|_{L_\infty(I)}$$

$$(Q_{L_\infty,w},\|\cdot\|_{Q_{L_\infty,w}})=(M_W,\|\cdot\|_{M_W}).$$

Proof. The weighted space $(L_\infty)_w$ consists of all essentially bounded functions on I with respect to the measure $d\omega = wdm$. Since w is positive both spaces $(L_\infty)_w$ and $L_\infty(I)$ coincide with equality of norms. Thus $f \in M_{L_\infty,w}$ if and only if $\frac{f^*}{w} \in (L_\infty)_w = L_\infty(I)$, and

$$\|f\|_{M_{L_\infty,w}} = \left\|\frac{f^*}{w}\right\|_{(L_\infty)_w} = \left\|\frac{f^*}{w}\right\|_\infty = \inf\{C : f^* \le Cw\}.$$

Note that the gauge $\|\cdot\|_{M_{L_\infty,w}}$ is not a norm. As for the space $Q_{L_\infty,w}$, by its definition $f \in Q_{L_\infty,w}$ if and only if there exists $g \in M_{L_\infty,w}$ with $f \prec g$. This is equivalent to

$$\exists g \in L^0_+,\ g\downarrow, C > 0 \text{ with } g \le Cw \text{ and } \forall x \in I, \int_0^x f^*\,dm \le \int_0^x g\,dm. \tag{9.1}$$

The above statement is equivalent to $\int_0^x f^*\,dm \le C\int_0^x w\,dm$ for all $x \in I$ with the same constant C as in (9.1). It follows that $f \in M_W$ and

$$\begin{aligned}\|f\|_{Q_{L_\infty,w}} &= \inf\{\|g\|_{M_{L_\infty,w}} : f \prec g,\ g \in M_{L_\infty,w}\} = \inf\{C : f \prec g,\ g \le Cw\}\\ &= \inf\{C :\ f \prec Cw\} = \sup_{x\in I}\frac{1}{W(x)}\int_0^x f^* dm = \|f\|_{M_W}.\end{aligned}$$

Thus $Q_{L_\infty,w}$ coincides with the Marcinkiewicz space M_W with the same norms. □

Example 9.3. *If $E = L_1 \cap L_\infty(J)$ then*

$$(M_{L_1\cap L_\infty,w}, \|\cdot\|_{M_{L_1\cap L_\infty,w}}) = (L_1 \cap M_{L_\infty,w}, \|\cdot\|_{L_1\cap M_{L_\infty,w}})$$
$$(Q_{L_1\cap L_\infty,w}, \|\cdot\|_{Q_{L_1\cap L_\infty,w}}) = (L_1 \cap M_W, \|\cdot\|_{L_1\cap M_W}).$$

Proof. Let $f \in M_{L_1\cap L_\infty,w}$. Then by Proposition 4.1

$$\begin{aligned}\|f\|_{M_{L_1\cap L_\infty,w}} &= \left\|\frac{f^*}{w}\right\|_{(L_1\cap L_\infty)_w} = \left\|\frac{f^*}{w}\circ W^{-1}\right\|_{L_1\cap L_\infty(J)}\\ &= \left\|\frac{f^*}{w}\circ W^{-1}\right\|_{L_1(J)} \vee \left\|\frac{f^*}{w}\circ W^{-1}\right\|_{L_\infty(J)}\\ &= \|f\|_{L_1(I)} \vee \left\|\frac{f^*}{w}\right\|_{(L_\infty)_w} = \|f\|_{L_1(I)} \vee \|f\|_{M_{L_\infty,w}} = \|f\|_{L_1\cap M_{L_\infty,w}}.\end{aligned}$$

Thus $M_{L_1\cap L_\infty,w} = L_1 \cap M_{L_\infty,w}$ with identical gauges.

For every $g \in L_1 \cap L_{\infty,w}$ and $f \prec g$ we have $\|f\|_1 \le \|g\|_1$, and

$$\|f\|_{M_W} = \inf\{C : f \prec Cw\} \le \inf\{C : g^* \le Cw\} = \|g\|_{M_{L_\infty,w}}.$$

Thus

$$\|f\|_{L_1\cap M_W} = \|f\|_1 \vee \|f\|_{M_W} \le \|g\|_1 \vee \|g\|_{M_{L_\infty,w}} = \|g\|_{L_1\cap M_{L_\infty,w}}.$$

It follows that $Q_{L_1\cap L_\infty,w} \subset L_1 \cap M_W$, and that for every $f \in Q_{L_1\cap L_\infty,w}$,

$$\|f\|_{L_1\cap M_W} \le \|f\|_{Q_{L_1\cap L_\infty,w}}.$$

Conversely if $f \in L_1 \cap M_W$ then $f^* \in L_1$ and $f \prec Cw$ where $C = \|f\|_{M_W}$. Then for every $x \in I$ we have

$$\int_0^x f^* dm \le CW(x) \wedge \|f\|_1.$$

We have $b = W(a) = \sup_{x\in I} W(x)$. If $Cb \le \|f\|_1$ then the preceding inequalities mean that $f \prec Cw$. Setting $g = Cw$ we have $\|g\|_1 \le \|f\|_1$ and $\|g\|_{M_{L_\infty,w}} = C = \|f\|_{M_W}$, hence $g \in L_1 \cap M_{L_\infty,w} = M_{L_1\cap L_\infty,w}$ and by $f \prec g$ it follows that $f \in Q_{L_1\cap L_\infty,w}$ with

$$\|f\|_{Q_{L_1\cap L_\infty,w}} \le \|g\|_{M_{L_1\cap L_\infty,w}} \le \|f\|_1 \vee \|f\|_{M_W}.$$

On the other hand if $Cb > \|f\|_1$, there exists $x_f \in I$ such that $CW(x_f) = \|f\|_1$. Setting now $g = Cw\chi_{(0,x_f)}$, observe that $f \prec g$, $\|g\|_1 = \|f\|_1$ and $\|g\|_{M_{L_\infty,w}} = C = \|f\|_{M_W}$, and conclude as above. □

Example 9.4. *If* $E = L_1 + L_\infty(J)$ *then*

$$(M_{L_1+L_\infty,w}, \|\cdot\|_{M_{L_1+L_\infty,w}}) = (L_1 + M_{L_\infty,w}, \|\cdot\|_{L_1+M_{L_\infty,w}}),$$
$$(Q_{L_1+L_\infty,w}, \|\cdot\|_{Q_{L_1+L_\infty,w}}) = (L_1 + M_W, \|\cdot\|_{L_1+M_W}).$$

Proof. We can show directly that $(L_1 + L_\infty)_w = (L_1)_w + (L_\infty)_w = (L_1)_w + L_\infty(I)$ with equality of norms. By Example 9.2, a function v belongs to $M_{L_\infty,w}$ if there exists $C > 0$ such that $v^* \le Cw$. Thus

$$\begin{aligned}
&f \in M_{L_1+L_\infty,w} \text{ with } \|f\|_{M_{L_1+L_\infty,w}} < 1\\
&\iff \exists\, g \in (L_1)_w, h \in L_\infty(I) : \frac{f^*}{w} = g + h, \|g\|_{(L_1)_w} + \|h\|_\infty < 1\\
&\iff \exists u \in L_1(I), v \in L^0, C \ge 0 : f^* = u + v, |v| \le Cw, \|u\|_1 + C < 1 \qquad (9.2)\\
&\implies \exists u \in L_1(I), v \in L^0, C \ge 0 : f^* = u + v, v^* \le Cw, \|u\|_1 + C < 1\\
&\iff f^* \in L_1 + M_{L_\infty,w} \text{ with } \|f^*\|_{L_1+M_{L_\infty,w}} < 1.
\end{aligned}$$

We want to prove that $\|f^*\|_{L_1+M_{L_\infty,w}} < 1$ implies $\|f\|_{L_1+M_{L_\infty,w}} < 1$. Let $f^* = u + v$ with $v^* \le Cw$ and $\|u\|_1 + C < 1$. Let us consider two cases.

Assume first that either the interval I is finite or $\lim_{t\to\infty} f^*(t) = 0$. Then by Proposition 1.1 there exists a measure preserving transformation σ, either from I onto I if the support of f has finite measure or from the support of f onto I if the support of f has infinite measure, such that $f = f^* \circ \sigma$. Then $f = u \circ \sigma + v \circ \sigma$ with $u \circ \sigma, v \circ \sigma$ equimeasurable with u, v respectively. In particular $\|u \circ \sigma\|_1 = \|u\|_1$ and $(v \circ \sigma)^* = v^* \le Cw$. Hence $\|f\|_{L_1+M_{L_\infty,w}} < 1$.

The proof is similar if I is infinite and $\lim_{t\to\infty} f^*(t) > 0$, by using now Lemma 1.2.

Therefore we have shown that if $\|f\|_{M_{L_1+L_\infty,w}} < 1$ then $\|f\|_{L_1+M_{L_\infty,w}} < 1$, which implies $M_{L_1+L_\infty,w} \subset L_1 + M_{L_\infty,w}$ and that this inclusion is gauge-decreasing.

Let us prove now the converse inclusion. Let $f \in L_1 + M_{L_\infty,w}$ with $\|f\|_{L_1+M_{L_\infty,w}} < 1$ and $f = u + v$ be a decomposition with $u \in L_1(I), v^* \le Cw$ and $\|u\|_1 + C < 1$. By the Lorentz-Shimogaki inequality [1, Chapter 3, Theorem 7.4],

$$f^* - v^* \prec (f - v)^* = u^*$$

it follows that $u_1 := f^* - v^* \in L_1$, with $\|u_1\|_1 \le \|u\|_1$. Thus

$$f^* = u_1 + v^*, \text{ with } v^* \le Cw \text{ and } \|u_1\|_1 + C < 1.$$

By (9.2) it means that $\|f\|_{M_{L_1+L_\infty,w}} < 1$. Hence $L_1 + M_{L_\infty,w} \subset M_{L_1+L_\infty,w}$, and finally $L_1 + M_{L_\infty,w} = M_{L_1+L_\infty,w}$ with equal norms.

Now we will show that $Q_{L_1+L_\infty,w} = L_1 + M_W$ with equality of norms. First,

$$\begin{aligned}
f \in Q_{L_1+L_\infty,w} &\iff \exists\, g \in M_{L_1+L_\infty,w},\ f \prec g\\
&\iff \exists\, g \in L_1 + M_{L_\infty,w},\ f \prec g\\
&\iff \exists u \in L_1, \exists v \in M_{L_\infty,w} : f \prec u + v\\
&\implies \exists u \in L_1, \exists v \in M_{L_\infty,w} : f^* \prec u^* + v^*.
\end{aligned}$$

Since $f^* \prec u^* + v^*$, then by Fact 2.6 there exists a decomposition $f = u' + v'$ with $u' \prec u^*$, $v' \prec v^*$. Then from $u \in L_1$ and $v \in M_{L_\infty,w}$ it follows that

$$u' \in L_1,\ v' \in M_W \ \text{ and } \ \|f\|_{L_1+M_W} \le \|u'\|_1 + \|v'\|_{M_W} \le \|u^*\|_1 + \|v^*\|_{M_{L_\infty,w}}.$$

Thus $Q_{L_1+L_\infty,w} \subset L_1 + M_W$.

Moreover, in view of the above paragraph if $\|f\|_{Q_{L_1+L_\infty,w}} < 1$ we may choose $g \in M_{L_1+L_\infty,w} = L_1 + M_{L_\infty,w}$ with $f \prec g$ and $\|g\|_{L_1+M_{L_\infty,w}} < 1$. Thus there is a decomposition $g = u + v$ with $\|u\|_1 + \|v\|_{M_{L_\infty,w}} < 1$. Hence $\|f\|_{L_1+M_W} \le \|g\|_{L_1+M_W} \le \|u\|_1 + \|v\|_{M_W} \le \|u\|_1 + \|v\|_{M_{L_\infty,w}} < 1$. Hence $\|f\|_{L_1+M_W} \le \|f\|_{Q_{L_1+L_\infty,w}}$.

As for the converse direction, let $f \in L_1 + M_W$ with $\|f\|_{L_1+M_W} < 1$ and $f = k + h$ be a decomposition with $\|k\|_1 + \|h\|_{M_W} < 1$. Then

$$f^* \prec k^* + h^* \prec k^* + Cw \ \text{ with } C = \|h^*\|_{M_W} = \|h\|_{M_W}.$$

Setting $g = k^* + Cw$, we have $f \prec g$, and $\|Cw\|_{M_{L_\infty,w}} = C\|w/w\|_{(L_\infty)_w} = \|h\|_{M_W}$. Hence

$$\|g\|_{L_1+M_{L_\infty,w}} \le \|k\|_1 + \|Cw\|_{M_{L_\infty,w}} = \|k\|_1 + \|h\|_{M_W} < 1.$$

Since g is decreasing and $\|\cdot\|_{L_1+M_{L_\infty,w}} = \|\cdot\|_{M_{L_1+L_\infty,w}}$, we have $g = g^* \in M_{L_1+L_\infty,w}$ with $\|g\|_{M_{L_1+L_\infty,w}} < 1$, hence $f \in Q_{M_{L_1+L_\infty,w}}$ with $\|f\|_{Q_{M_{L_1+L_\infty,w}}} < 1$. Thus $L_1 + M_W \subset Q_{M_{L_1+L_\infty,w}}$ and $\|f\|_{L_1+M_W} \ge \|f\|_{Q_{L_1+L_\infty,w}}$.

Consequently $L_1 + M_W = Q_{M_{L_1+L_\infty,w}}$ with equality of norms. □

Appendix

We give here a self-contained and simple proof of Fact 8.6.

For $x, y > 0$ set

$$D(x,y) = \begin{cases} \frac{\psi(x)-\psi(y)}{x-y} & \text{if } x \ne y, \\ \psi'_+(x) & \text{if } x = y, \end{cases}$$

where ψ'_+ is the right derivative of ψ. Observe that if $x_1 \le x_2$ and $y_1 \le y_2$ then $D(x_1,y_1) \le D(x_2,y_2)$, by convexity of the function ψ. Indeed if we set $a_i = \min(x_i,y_i)$ and $b_i = \max(x_i,y_i)$, $i = 1,2$, then $a_1 \le a_2$, $b_1 \le b_2$ and $D(x_1,y_1) = D(a_1,b_1)$ and $D(x_2,y_2) = D(a_2,b_2)$ are the respective slopes of the chords of the graph of ψ corresponding to the intervals $[a_1,b_1]$ and $[a_2,b_2]$, or the slopes of the right tangent lines in the case $a_i = b_i$, $i = 1,2$.

Since for any $f \in L^0(\Omega)$, $f^{*,\mu} \in L^0(0,\mu(\Omega))$, we may assume without loss of generality that $f, g : [0,\infty) \to [0,\infty)$ are decreasing non-negative functions. Then the function $D(f,g) : t \mapsto D(f(t),g(t))$ is also decreasing. Assuming that $f \prec g$, we want to show that $\psi(f) \prec \psi(g)$. We note that for $x \ge 0$,

$$0 \le \int_0^x g\,dm - \int_0^x f dm = \int_0^x (g-f)\,dm = \int_0^x (g-f)_+\,dm - \int_0^x (g-f)_-\,dm,$$

thus for $x \ge 0$,

$$\int_0^x (g-f)_-\,dm \le \int_0^x (g-f)_+\,dm.$$

By Hardy's Lemma [1, Proposition 3.6, Ch.2] this implies that

$$\int_0^x (g-f)_- D(g,f)\,dm \le \int_0^x (g-f)_+ D(g,f)\,dm$$

for $x \geq 0$. We have $\psi(g) - \psi(f) = (g - f)D(g, f)$ and since $D(g, f) \geq 0$ it follows that $(\psi(g) - \psi(f))_\pm = (g - f)_\pm D(g, f)$. Hence the preceding inequality may be rewritten as

$$\int_0^x (\psi(g) - \psi(f))_- \, dm \leq \int_0^x (\psi(g) - \psi(f))_+ \, dm. \tag{9.3}$$

Supposing that $\psi(f)$ is integrable on finite intervals, it implies the same for $(\psi(g) - \psi(f))_-$ since $(\psi(g) - \psi(f))_- \leq \psi(f)$. Then for any $x > 0$,

$$\begin{aligned}\int_0^x \psi(g) \, dm - \int_0^x \psi(f) \, dm &= \int_0^x (\psi(g) - \psi(f)) \, dm \\ &= \int_0^x (\psi(g) - \psi(f))_+ \, dm - \int_0^x (\psi(g) - \psi(f))_- \, dm \geq 0,\end{aligned}$$

which implies that $\psi(f) \prec \psi(g)$. If we have no information on the local integrability of $\psi(f)$, we may apply the above to the couple $(f \wedge n, g)$, where $n \in \mathbb{N}$. Indeed we have $f \wedge n \leq f \prec g$, $f \wedge n$ is decreasing, and $\psi(f \wedge n) = \psi(f) \wedge \psi(n)$ is bounded, and thus integrable on finite intervals. Hence for all $n \in \mathbb{N}$, $x \geq 0$,

$$\int_0^x \psi(f) \wedge \psi(n) \, dm \leq \int_0^x \psi(g) \, dm,$$

and passing to the limit $n \to \infty$ we obtain that $\psi(f) \prec \psi(g)$. □

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