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Diophantine approximation and run-length function on β -expansions

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Abstract. For any $\beta > 1$, denoted by $r_n(x, \beta)$ the maximal length of consecutive zeros amongst the first n digits of the β -expansion of $x \in [0, 1)$. The limit superior (respectively limit inferior) of $\frac{r_n(x, \beta)}{n}$ is linked to the classical Diophantine approximation (respectively uniform Diophantine approximation). We obtain the Hausdorff dimension of the level set

$$E_{a,b} = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = a, \limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = b \right\} \quad (0 \leq a \leq b \leq 1).$$

Furthermore, we show that the extremely divergent set $E_{0,1}$ which is of zero Hausdorff dimension is, however, residual. The same problems in the parameter space are also examined.

Key words and phrases beta-expansion; Diophantine approximation; run-length function; Hausdorff dimension; residual

1 Introduction

Let $\beta > 1$ be a real number. The β -transformation on $[0, 1)$ is defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \xi \rfloor$ means the integer part of ξ . It is well-known (see [17]) that, every real number $x \in [0, 1)$ can be uniquely expanded as a series

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \dots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \dots, \quad (1.1)$$

where $\varepsilon_n(x, \beta) = \lfloor \beta T_\beta^{n-1}(x) \rfloor$ for all $n \geq 1$. We call $\varepsilon_n(x, \beta)$ the n -th digit of x and $\varepsilon(x, \beta) := (\varepsilon_1(x, \beta), \dots, \varepsilon_n(x, \beta), \dots)$ the β -expansion of x .

For each $x \in [0, 1)$ and $n \geq 1$, the run-length function $r_n(x, \beta)$ is defined to be the maximal length of consecutive zeros amongst the prefix $(\varepsilon_1(x, \beta), \dots, \varepsilon_n(x, \beta))$, i.e.,

$$r_n(x, \beta) = \max\{1 \leq j \leq n : \varepsilon_{i+1}(x, \beta) = \dots = \varepsilon_{i+j}(x, \beta) = 0 \text{ for some } 0 \leq i \leq n - j\}.$$

If such j does not exist, we set $r_n(x, \beta) = 0$. In 1970, Erdős and Rényi [5] showed that for Lebesgue almost all $x \in [0, 1)$, we have

$$\lim_{n \rightarrow \infty} \frac{r_n(x, 2)}{\log_2 n} = 1. \quad (1.2)$$

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The result of Erdős and Rényi [5] has been extended to the general case $\beta > 1$ by Tong, Yu and Zhao [20]. Ma, Wen and Wen [13] showed that the exceptional set of points violating (1.2) is of full Hausdorff dimension. Let \mathcal{E} denote the set of increasing functions $\varphi : \mathbb{N} \rightarrow (0, +\infty)$ satisfying $\lim_{n \rightarrow \infty} \varphi(n) = +\infty$ and $\limsup_{n \rightarrow \infty} \frac{\varphi(n)}{n} = 0$. For every $0 \leq a \leq b \leq \infty$ and any function $\varphi \in \mathcal{E}$, define

$$E_{a,b}^\varphi := E_{a,b}^\varphi(\beta) = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{\varphi(n)} = a, \limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{\varphi(n)} = b \right\}.$$

The set $E_{a,b}^\varphi$ has been proved to have full Hausdorff dimension by Li and Wu (see [10, 11]) for the case $\beta = 2$ and by Zheng, Wu and Li [22] for the general case $\beta > 1$.

Remark that the above \mathcal{E} does not contain the function $\varphi(n) = n$. In fact, the asymptotic behavior of $\frac{r_n(x, \beta)}{n}$ is directly related to the Diophantine approximation of β -expansions. For all $x \in [0, 1)$, Amou and Bugeaud [1] defined the exponent $v_\beta(x)$ to be the supremum of the real numbers v for which the equation

$$T_\beta^n x \leq \beta^{-nv}$$

has infinitely many positive solutions with integer n . Bugeaud and Liao [7] defined the exponent $\hat{v}_\beta(x)$ to be the supremum of the real numbers \hat{v} for which, for all $N \gg 1$, there is a solution with $1 \leq n \leq N$, such that

$$T_\beta^n x \leq \beta^{-N\hat{v}}.$$

We will see (Lemmas 3.1 and 3.2) that for all $0 < a < 1$, $0 < b < 1$,

$$\liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = a \Leftrightarrow \hat{v}_\beta(x) = \frac{a}{1-a} \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = b \Leftrightarrow v_\beta(x) = \frac{b}{1-b}.$$

For all $0 \leq a \leq b \leq 1$, let

$$E_{a,b} := E_{a,b}(\beta) = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = a, \limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = b \right\}. \quad (1.3)$$

Denote the Hausdorff dimension by \dim_{H} . For more information about the Hausdorff dimension, we refer to [6]. We establish the following theorem.

Theorem 1.1 *The set $E_{0,0}$ has full Lebesgue measure. If $\frac{b}{1+b} < a \leq 1$, $0 < b \leq 1$, then $E_{a,b} = \emptyset$. Otherwise, we have*

$$\dim_{\text{H}} E_{a,b} = 1 - \frac{b^2(1-a)}{b-a}.$$

Let $0 \leq a \leq 1$ and $0 \leq b \leq 1$. We can further study the level sets

$$E_a := E_a(\beta) = \left\{ x \in [0, 1) : \liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = a \right\}$$

and

$$F_b := F_b(\beta) = \left\{ x \in [0, 1) : \limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = b \right\}. \quad (1.4)$$

Using Theorem 1.1, we obtain the following results of the Hausdorff dimensions of E_a and F_b .

Corollary 1.2 (1) *When $0 \leq a \leq \frac{1}{2}$, we have*

$$\dim_{\text{H}} E_a = (1 - 2a)^2.$$

Otherwise, $E_a = \emptyset$.

(2) *For all $0 \leq b \leq 1$, we have*

$$\dim_{\text{H}} F_b = 1 - b.$$

We remark that the statement (2) of Theorem 1.2 was also been obtained in [9, Theorem 1.1] (see [21] for the case $\beta = 2$).

A set R is called *residual* if its complement is meager (i.e., of the first category). In a complete metric space, a set is residual if it contains a dense G_δ set, i.e., a countable intersection of open dense sets (see [14]). Similar to the results of [10, 11, 22], the set of extremely divergent points is residual, and thus is large in the sense of topology.

Theorem 1.3 *The set $E_{0,1}$ is residual in $[0, 1]$.*

It is worth noting that the set $E_{0,1}$ is negligible with respect to the Lebesgue measure and Hausdorff dimension. However, the sets considered in [10, 11, 22] have Hausdorff dimension 1. Let $\varphi \in \mathcal{E}$. Since the intersection of two residual sets is still residual, by combining Theorems 1.3 and [22, Theorem 1.2], we deduce that the smaller set

$$E := E(\varphi, \beta) = \left\{ x \in [0, 1] : \liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{\varphi(n)} = 0, \limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = 1 \right\}$$

is also residual in $[0, 1]$.

The β -expansion of 1 completely characterizes all of the admissible words in the β -dynamical system (see Theorem 2.1 in Section 2 for more details). We also study the run-length function $r_n(\beta)$ of the β -expansion of 1 as β varies in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$, i.e.,

$$r_n(\beta) = \max\{1 \leq j \leq n : \varepsilon_{i+1}(\beta) = \cdots = \varepsilon_{i+j}(\beta) = 0 \text{ for some } 0 \leq i \leq n - j\}.$$

There are some results on $r_n(\beta)$ which are similar to those of $r_n(x, \beta)$. In [8], Hu, Tong and Yu proved that for Lebesgue almost all $1 < \beta < 2$, we have

$$\lim_{n \rightarrow \infty} \frac{r_n(\beta)}{\log_\beta n} = 1. \quad (1.5)$$

Cao and Chen [4] showed that for any $\varphi \in \mathcal{E}$ and for all $0 \leq a \leq b \leq +\infty$, the set

$$\left\{ \beta \in (1, 2) : \liminf_{n \rightarrow \infty} \frac{r_n(\beta)}{\varphi(n)} = a, \limsup_{n \rightarrow \infty} \frac{r_n(\beta)}{\varphi(n)} = b \right\}$$

is of full Hausdorff dimension. Remark that the results of [8] and [4] can be easily generalized to the whole parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$. For simplicity, in this paper, we will also consider the parameter space $(1, 2)$. For all $0 \leq a \leq b \leq 1$, let

$$E_{a,b}^P = \left\{ \beta \in (1, 2) : \liminf_{n \rightarrow \infty} \frac{r_n(\beta)}{n} = a, \limsup_{n \rightarrow \infty} \frac{r_n(\beta)}{n} = b \right\}. \quad (1.6)$$

We have the following theorem.

Theorem 1.4 *The set $E_{0,0}^P$ has full Lebesgue measure. If $\frac{b}{1+b} < a \leq 1$, $0 < b \leq 1$, then $E_{a,b}^P = \emptyset$. Otherwise, we have*

$$\dim_{\text{H}} E_{a,b}^P = 1 - \frac{b^2(1-a)}{b-a}.$$

Similarly, for every $0 \leq a \leq 1$ and $0 \leq b \leq 1$, we consider the set

$$E_a^P = \left\{ \beta \in (1, 2) : \liminf_{n \rightarrow \infty} \frac{r_n(\beta)}{n} = a \right\},$$

and

$$F_b^P = \left\{ \beta \in (1, 2) : \limsup_{n \rightarrow \infty} \frac{r_n(\beta)}{n} = b \right\}.$$

Corollary 1.5 (1) When $0 \leq a \leq \frac{1}{2}$, we have

$$\dim_{\mathbb{H}} E_a^P = (1 - 2a)^2.$$

Otherwise, $E_a = \emptyset$.

(2) For every $0 \leq b \leq 1$, we have

$$\dim_{\mathbb{H}} F_b^P = 1 - b.$$

In addition, similar to Theorem 1.3, we have the following theorem.

Theorem 1.6 The set $E_{0,1}^P$ is residual in $[1, 2]$.

We end this introduction by depicting the organization of our paper. In Section 2, we review some standard facts on the β -expansions without proofs. Theorem 1.1 and corollary 1.2 are proved in Section 3. We give the proof of Theorem 1.3 in Section 4. Section 5 contains a summary of some classical results of β -expansion in the parameter space. The proofs of Theorems 1.4 and 1.6 are given in Sections 6 and 7 respectively.

2 Fundamental results of β -expansion

Throughout this section, we set up some notations and terminologies on β -expansions. Meanwhile, we give some basic results on β -expansion directly. For more properties on β -expansions, we refer the readers to [3, 7, 15, 17].

Let $\mathcal{A} = \{0, 1, \dots, \lceil \beta \rceil - 1\}$ where $\lceil \xi \rceil$ stands for the smallest integer larger than ξ . The definition of β -expansion gives the fact that every digit $\varepsilon_n(x, \beta)$ lies in the set \mathcal{A} . A word $(\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{A}^n$ is called *admissible* with respect to β if there exists an $x \in [0, 1)$ such that the β -expansion of x begins with $(\varepsilon_1, \dots, \varepsilon_n)$. Similarly, an infinite sequence $(\varepsilon_1, \dots, \varepsilon_n, \dots)$ is called *admissible* with respect to β if there exists an $x \in [0, 1)$ whose β -expansion is $(\varepsilon_1, \dots, \varepsilon_n, \dots)$. Denote by Σ_{β}^n the set of all β -admissible words of length n , i.e.,

$$\Sigma_{\beta}^n = \{(\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{A}^n : \exists x \in [0, 1), \text{ s.t. } \varepsilon_j(x, \beta) = \varepsilon_j, \forall 1 \leq j \leq n\}.$$

Denote by Σ_{β}^* the set of all β -admissible words of finite length, i.e., $\Sigma_{\beta}^* = \bigcup_{n=0}^{\infty} \Sigma_{\beta}^n$. The set of β -admissible sequences is denoted by Σ_{β} , i.e.,

$$\Sigma_{\beta} = \{(\varepsilon_1, \varepsilon_2, \dots) \in \mathcal{A}^{\mathbb{N}} : \exists x \in [0, 1), \text{ s.t. } \varepsilon(x, \beta) = (\varepsilon_1, \varepsilon_2, \dots)\}.$$

The β -expansion of the unit 1 plays an important role in the research of admissible words and admissible sequences. By extending the definition of $T_{\beta}(x) = \beta x - \lfloor \beta x \rfloor$ to $x = 1$, the β -expansion of 1 is given by

$$1 = \frac{\varepsilon_1(1, \beta)}{\beta} + \dots + \frac{\varepsilon_n(1, \beta)}{\beta^n} + \dots.$$

where $\varepsilon_n(1, \beta) = \lfloor \beta T_{\beta}^{n-1} 1 \rfloor$. We call β a *simple Parry number* if the β -expansion of 1 is finite. That is, there exists an integer $m \geq 1$ such that $\varepsilon_m \neq 0$ and $\varepsilon_k(\beta) = 0$ for every $k > m$. In this case, we let

$$\varepsilon^*(\beta) := (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots) = (\varepsilon_1(\beta), \varepsilon_2(\beta), \dots, \varepsilon_m(\beta) - 1)^{\infty},$$

where ω^{∞} is the infinite periodic sequence (ω, ω, \dots) . If the β -expansion of 1 is not finite, let

$\varepsilon^*(\beta) = \varepsilon(1, \beta)$. In both cases, we can check that

$$1 = \frac{\varepsilon_1^*(\beta)}{\beta} + \dots + \frac{\varepsilon_n^*(\beta)}{\beta^n} + \dots$$

The sequence $\varepsilon^*(\beta)$ is therefore called *the infinite β -expansion of 1*.

We endow the space $\mathcal{A}^{\mathbb{N}}$ with the *lexicographical order* $<_{\text{lex}}$:

$$(\omega_1, \omega_2, \dots) <_{\text{lex}} (\omega'_1, \omega'_2, \dots)$$

if $\omega_1 < \omega'_1$ or there exists an integer $j > 1$, such that, for all $1 \leq k < j$, $\omega_k = \omega'_k$ but $\omega_j < \omega'_j$. The symbol \leq_{lex} means $=$ or $<_{\text{lex}}$. Moreover, for all $n, m \geq 1$, $(\omega_1, \dots, \omega_n) <_{\text{lex}} (\omega'_1, \dots, \omega'_m)$ stands for $(\omega_1, \dots, \omega_n, 0^\infty) <_{\text{lex}} (\omega'_1, \dots, \omega'_m, 0^\infty)$.

The following theorem due to Parry [15] yields that the β -dynamical system is totally determined by the infinite β -expansion of 1. Let $\omega = (\omega_1, \dots, \omega_n) \in \mathcal{A}^n$ for all $n \geq 1$. Let σ be the shift transformation such that $\sigma\omega = (\omega_2, \dots, \omega_n)$.

Theorem 2.1 (Parry [15]) *Let $\beta > 1$.*

(1) *For every $n \geq 1$, $\omega = (\omega_1, \dots, \omega_n) \in \Sigma_\beta^n$ if and only if $\sigma^j\omega \leq_{\text{lex}} (\varepsilon_1^*, \dots, \varepsilon_{n-j}^*)$ for all $0 \leq j < n$.*

(2) *For all $k \geq 1$, $\sigma^k\varepsilon(1, \beta) <_{\text{lex}} \varepsilon(1, \beta)$.*

(3) *For each $1 < \beta_1 < \beta_2$, it holds that $\varepsilon^*(\beta_1) <_{\text{lex}} \varepsilon^*(\beta_2)$. Consequently, for every $n \geq 1$, we have*

$$\Sigma_{\beta_1}^n \subseteq \Sigma_{\beta_2}^n \quad \text{and} \quad \Sigma_{\beta_1} \subseteq \Sigma_{\beta_2}.$$

The estimation of the cardinality of the set Σ_β^n was given by Rényi [17]. We will use the symbol \sharp to denote the cardinality of a finite set in the remainder of this paper.

Theorem 2.2 (Rényi [17]) *For all $n \geq 1$,*

$$\beta^n \leq \sharp\Sigma_\beta^n \leq \frac{\beta^{n+1}}{\beta - 1}.$$

For an admissible word $\omega = (\omega_1, \dots, \omega_n)$, the associated *cylinder* of order n is defined by

$$I_n(\omega) := I_n(\omega, \beta) = \{x \in [0, 1) : \varepsilon_j(x, \beta) = \omega_j, \text{ for all } 1 \leq j \leq n\}.$$

The cylinder $I_n(\omega)$ is a left-closed and right-open interval (see [7, Lemma 2.3]). Denote by $|I_n(\omega)|$ the length of $I_n(\omega)$. We immediately get $|I_n(\omega)| \leq \beta^{-n}$. We write $I_n(x, \beta)$ as the cylinder of order n containing the point $x \in [0, 1)$ and write $|I_n(x, \beta)|$ as its length. For simplicity, $I_n(x)$ means $I_n(x, \beta)$ in the rest of this paper without otherwise specified. A cylinder of order n is called *full* if $|I_n(\omega)| = \beta^{-n}$ and the corresponding word of the full cylinder is said to be *full*.

Now we give some characterizations and properties of full cylinders.

Theorem 2.3 (Fan and Wang [7]) *For any integer $n \geq 1$, let $\omega = (\omega_1, \dots, \omega_n)$ be an admissible word.*

(1) *The cylinder $I_n(\omega)$ is full if and only if $T_\beta^n(I_n(\omega)) = [0, 1)$, if and only if for any $m \geq 1$ and $\omega' = (\omega'_1, \dots, \omega'_m) \in \Sigma_\beta^m$, the concatenation $\omega * \omega' = (\omega_1, \dots, \omega_n, \omega'_1, \dots, \omega'_m)$ is still admissible.*

(2) *If $(\omega_1, \dots, \omega_{n-1}, \omega'_n)$ with $\omega'_n > 0$ is admissible, then the cylinder $I_n(\omega_1, \dots, \omega_{n-1}, \omega_n)$ is full for every $0 \leq \omega_n < \omega'_n$.*

(3) *If $I_n(\omega)$ is full, then for any $(\omega'_1, \dots, \omega'_m) \in \Sigma_\beta^m$, we have*

$$|I_{n+m}(\omega_1, \dots, \omega_n, \omega'_1, \dots, \omega'_m)| = \beta^{-n} \cdot |I_m(\omega'_1, \dots, \omega'_m)|.$$

In order to construct full words, we introduce a variable Γ_n which is defined as follows. Recall that the infinite β -expansion of 1 is $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots)$. For every integer $n \geq 1$, define

$$t_n = t_n(\beta) := \max\{k \geq 1 : \varepsilon_{n+1}^*(\beta) = \dots = \varepsilon_{n+k}^*(\beta) = 0\}.$$

If such k does not exist, let $t_n = 0$. Now let

$$\Gamma_n = \Gamma_n(\beta) := \max_{1 \leq k \leq n} t_k(\beta). \quad (2.7)$$

Then we can check that Γ_n is a finite integer for all $n \geq 1$. Theorem 2.3 implies the following results which are important for construction of full words.

Proposition 2.4 (Fan and Wang [7]) (1) *If both admissible words $(\omega_1, \dots, \omega_n)$ and $(\omega'_1, \dots, \omega'_m)$ are full, then the concatenation word $(\omega_1, \dots, \omega_n, \omega'_1, \dots, \omega'_m)$ is still full.*

(2) *For all $\ell \geq 1$, the word $0^\ell := \underbrace{(0, \dots, 0)}_\ell$ is full. For any full word $(\omega_1, \dots, \omega_n)$, the word $(\omega_1, \dots, \omega_n, 0^\ell)$ is also full.*

(3) *For any admissible word $(\omega_1, \dots, \omega_n)$, the word $(\omega_1, \dots, \omega_n, 0^{\Gamma_n+1})$ is full.*

Furthermore, Bugeaud and Wang [3] provided the following modified mass distribution principle which is of great importance in estimating the lower bound of the Hausdorff dimension of $E_{a,b}$.

Theorem 2.5 (Bugeaud and Wang [3]) *Let μ be a Borel measure and E be a Borel measurable set with $\mu(E) > 0$. Assume that there exist a constant $c > 0$ and an integer $N \geq 1$ such that for all $n \geq N$ and each cylinder I_n , the inequality $\mu(I_n) \leq c|I_n|^s$ is valid. Then, $\dim_{\mathbb{H}} E \geq s$.*

Now we will introduce some results on Diophantine approximation. We first give the following exponents of approximation.

Shen and Wang [19] obtained the following theorem which gives the dimensional results of the set of points with classical Diophantine property.

Theorem 2.6 (Shen and Wang [19]) *Let $\beta > 1$. Let $0 \leq v \leq +\infty$. Then*

$$\dim_{\mathbb{H}}\{x \in [0, 1) : v_\beta(x) \geq v\} = \frac{1}{1+v}.$$

Bugeaud and Liao [2] studied the set of points with uniform Diophantine properties and established the theorem as follows.

Theorem 2.7 (Bugeaud and Liao [2]) *Let $\beta > 1$. Let $0 < \hat{v} < 1$ and $v > 0$. If $v < \frac{\hat{v}}{1-\hat{v}}$, then the set*

$$U_\beta(\hat{v}, v) := \{x \in [0, 1) : \hat{v}_\beta(x) = \hat{v}, v_\beta(x) = v\}$$

is empty. Otherwise, we have

$$\dim_{\mathbb{H}} U_\beta(\hat{v}, v) = \frac{v - (1+v)\hat{v}}{(1+v)(v-\hat{v})}.$$

Moreover,

$$\dim_{\mathbb{H}}\{x \in [0, 1) : \hat{v}_\beta(x) = \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}}\right)^2.$$

3 Proofs of Theorem 1.1 and Corollary 1.2

Notice that for all $\beta > 1$, we have

$$\left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{r_n(x, \beta)}{\log_\beta n} = 1 \right\} \subseteq E_{0,0}.$$

In [20], Tong, Yu and Zhao showed that the set $\left\{ x \in [0, 1) : \lim_{n \rightarrow \infty} \frac{r_n(x, \beta)}{\log_\beta n} = 1 \right\}$ is of full Lebesgue measure. As a result, the set $E_{0,0}$ has full Lebesgue measure. Hence, we only need to study the case that $0 \leq a \leq 1$, $0 < b \leq 1$. Before we give the proof of Theorem 1.1, we uncover the relationship between run-length function and Diophantine approximation.

3.1 Run-length function and Diophantine approximation

Lemma 3.1 *Let $\beta > 1$. For all $x \in [0, 1)$, for any $0 < a < 1$, we have $\liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = a$ if and only if $\hat{v}_\beta(x) = \frac{a}{1-a}$.*

Proof. It suffices to show the following two cases. The details are left to the readers.

On the one hand, suppose that $\hat{v}_\beta(x) < \frac{a}{1-a}$, then we have $v_0 = \frac{a}{2(1-a)} + \frac{\hat{v}_\beta(x)}{2} > \hat{v}_\beta(x)$. By the definition of $\hat{v}_\beta(x)$, there is a sequence $\{n_k\}_{k=1}^\infty$ such that, for all $1 \leq n \leq n_k$,

$$T_\beta^n x > \beta^{-v_0 n} \geq \beta^{-(\lfloor v_0 n \rfloor + 1)}.$$

So it holds that

$$r_{n_k + \lfloor v_0 n_k \rfloor}(x, \beta) < \lfloor v_0 n_k \rfloor + 1.$$

This implies that

$$\liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} \leq \lim_{k \rightarrow \infty} \frac{r_{n_k + \lfloor v_0 n_k \rfloor}(x, \beta)}{n_k + \lfloor v_0 n_k \rfloor} \leq \lim_{k \rightarrow \infty} \frac{\lfloor v_0 n_k \rfloor + 1}{n_k + \lfloor v_0 n_k \rfloor} = \frac{v_0}{1 + v_0} = \frac{2a + \hat{v}_\beta(x)(1-a) - a}{2 + \hat{v}_\beta(x)(1-a) - a} < a,$$

where the last inequality follows from

$$\frac{a-x}{b-x} < \frac{a}{b}, \text{ for all } 0 \leq a < b, x > 0.$$

On the other hand, suppose that $\hat{v}_\beta(x) > \frac{a}{1-a}$, then $v_0 = \frac{a}{2(1-a)} + \frac{\hat{v}_\beta(x)}{2} < \hat{v}_\beta(x)$. The definition of $\hat{v}_\beta(x)$ implies that for all $N \gg 1$, there exists $1 \leq n \leq N$, such that

$$T_\beta^n x \leq \beta^{-v_0 N}.$$

Then for all $k = N + \lfloor v_0 N \rfloor + 1 \gg 1$, we have

$$r_k(x, \beta) \geq \lfloor v_0 N \rfloor.$$

This implies that

$$\liminf_{k \rightarrow \infty} \frac{r_k(x, \beta)}{k} \geq \lim_{N \rightarrow \infty} \frac{\lfloor v_0 N \rfloor}{N + \lfloor v_0 N \rfloor + 1} = \frac{v_0}{1 + v_0} = \frac{2a + \hat{v}_\beta(x)(1-a) - a}{2 + \hat{v}_\beta(x)(1-a) - a} > a,$$

where the last inequality follows from

$$\frac{a+x}{b+x} > \frac{a}{b} \text{ for all } 0 \leq a < b, x > 0. \quad (3.8)$$

This contradicts with $\liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = a$. Consequently, $\hat{v}_\beta(x) \leq \frac{a}{1-a}$.

Thus, we conclude that $\liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = a$. \square

Lemma 3.2 *Let $\beta > 1$. For all $x \in [0, 1)$, for each $0 < b < 1$, we have $\limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = b$ if and only if $v_\beta(x) = \frac{b}{1-b}$.*

Proof. It can be deduced by the same arguments as the proof of Lemma 3.1. \square

Now we can give part of the proof of Theorem 1.1.

We will first show when $a > \frac{b}{1+b}$, $0 < b \leq 1$, $E_{a,b} = \emptyset$. In fact, if $\limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = b$, then for all $\delta > 0$, there exists a sequence $\{n_k\}_{k=1}^\infty$ such that $r_{n_k}(x, \beta) \leq \lfloor (b + \delta)n_k \rfloor$ and $\varepsilon_{n_k}(x, \beta) > 0$. Thus, when we consider the prefix at the position $n_k + \lfloor bn_k \rfloor$, there are at most $\lfloor (b + \delta)n_k \rfloor$ consecutive 0's. Immediately, $r_{n_k + \lfloor bn_k \rfloor}(x, \beta) \leq \lfloor (b + \delta)n_k \rfloor$. Hence,

$$a = \liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} \leq \lim_{k \rightarrow \infty} \frac{r_{n_k + \lfloor bn_k \rfloor}(x, \beta)}{n_k + \lfloor bn_k \rfloor} \leq \lim_{k \rightarrow \infty} \frac{(b + \delta)n_k}{n_k + \lfloor bn_k \rfloor} = \frac{b + \delta}{1 + b}.$$

Letting $\delta \rightarrow 0$, we have

$$a \leq \frac{b}{1 + b}. \quad (3.9)$$

Therefore, $E_{a,b}$ is empty when $a > \frac{b}{1+b}$, $0 < b \leq 1$.

When $0 < a \leq \frac{b}{1+b}$, $0 < b < 1$, Lemmas 3.1 and 3.2 give the fact that the sets we consider here are essentially the same as the sets studied in Bugeaud and Liao [2], that is

$$E_{a,b} = \left\{ x \in [0, 1) : \hat{v}_\beta(x) = \frac{a}{1-a}, v_\beta(x) = \frac{b}{1-b} \right\} = U_\beta \left(\frac{a}{1-a}, \frac{b}{1-b} \right).$$

Consequently, we can apply Theorem 2.7 to obtain

$$\dim_{\text{H}} E_{a,b} = \dim_{\text{H}} U_\beta \left(\frac{a}{1-a}, \frac{b}{1-b} \right) = 1 - \frac{b^2(1-a)}{b-a}.$$

However, Theorem 2.7 cannot be applied for the cases $a = 0$, $0 < b < 1$ and $0 < a \leq \frac{1}{2}$, $b = 1$. Remark that $E_{a,1} \subseteq F_1$ where F_1 is defined by (1.4) and $\dim_{\text{H}} F_1 = 0$ by [9, Theorem 1.1]. So $\dim_{\text{H}} E_{a,1} = 0$ ($0 < a \leq \frac{1}{2}$) and there is nothing to prove. For the other case, we have

$$E_{0,b} \subseteq \left\{ x \in [0, 1) : v_\beta(x) \geq \frac{b}{1-b} \right\}.$$

Then we can use Theorem 2.6 to obtain the upper bound of $\dim_{\text{H}} E_{0,b}$ which is $1-b$ for all $0 < b < 1$. Hence it remains to give the lower bound of $\dim_{\text{H}} E_{0,b}$ for all $0 < b < 1$.

3.2 Lower bound of $\dim_{\text{H}} E_{0,b}$ ($0 < b < 1$)

Now we give the lower bound of $\dim_{\text{H}} E_{0,b}$ for the case $0 < b < 1$. In fact, we can also include the proof of the lower bound of $\dim_{\text{H}} E_{a,b}$ for the case $0 < a \leq \frac{b}{1+b}$, $0 < b < 1$, though the later case has already been given in the end of Section 3.1.

Let $\beta > 1$. Recall that the infinite β -expansion of 1 is $\varepsilon^*(\beta) = (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots)$. We will apply the approximation of β to construct the Cantor subset as follows. For all N with $\varepsilon_N^* > 0$, let $\beta_N > 1$ be the unique solution of the equation:

$$1 = \frac{\varepsilon_1^*(\beta)}{x} + \dots + \frac{\varepsilon_N^*(\beta)}{x^N}.$$

Then

$$\varepsilon^*(\beta_N) = (\varepsilon_1^*(\beta), \dots, \varepsilon_N^*(\beta) - 1)^\infty.$$

Hence $1 < \beta_N < \beta$ and β_N is increasing to β as N goes to infinity. The number β_N is called *an approximation of β* . Moreover, by Theorem 2.1(3), $\Sigma_{\beta_N}^n \subseteq \Sigma_\beta^n$ for all $n \geq 1$ and $\Sigma_{\beta_N} \subseteq \Sigma_\beta$. We therefore have the following facts.

Proposition 3.3 (Shen and Wang [19]) *For every $\omega \in \Sigma_{\beta_N}^n$, when regarding ω as an element of Σ_β^n , we have*

$$\beta^{-(n+N)} \leq |I_n(\omega, \beta)| \leq \beta^{-n}. \quad (3.10)$$

Moreover, every $\omega \in \Sigma_{\beta_N}^n$ ($n \geq N$) end with 0^N is full when regarding ω as an element of Σ_β^n .

For all $k \geq 1$ and $N > 1$ with $\varepsilon_N^*(\beta) > 0$, choose two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ which satisfy $n_k < m_k < n_{k+1}$ with $n_1 > 2N$, and $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$. Moreover, $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ can be chosen to satisfy

$$\lim_{k \rightarrow \infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = a \quad (3.11)$$

and

$$\lim_{k \rightarrow \infty} \frac{m_k - n_k}{m_k} = b. \quad (3.12)$$

In fact, such sequences do exist by the following arguments.

(1) If $0 < a \leq \frac{b}{1+b}$, $0 < b < 1$, let

$$n'_k = \left\lfloor \left(\frac{b(1-a)}{a(1-b)} \right)^k \right\rfloor \quad \text{and} \quad m'_k = \left\lfloor \frac{1}{1-b} \left(\frac{b(1-a)}{a(1-b)} \right)^k \right\rfloor.$$

Note that $a < b$, so $\frac{b(1-a)}{a(1-b)} > 1$. Then both sequences $\{n'_k\}_{k=1}^\infty$ and $\{m'_k\}_{k=1}^\infty$ are increasing to infinity as k tends to infinity. A small adjustment can attain the required sequences.

(2) If $a = 0$, $0 < b < 1$, let

$$n'_k = k^k \quad \text{and} \quad m'_k = \left\lfloor \frac{1}{1-b} k^k \right\rfloor.$$

We can adjust these sequences to make sure that $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$.

Now let us construct a Cantor subset of $E_{a,b}$.

For all $d > 2N$, let

$$\mathcal{M}_d = \{\omega = (1, 0^{N-1}, \omega_1, \dots, \omega_{d-2N}, 0^N) : (\omega_1, \dots, \omega_{d-N}) \in \Sigma_{\beta_N}^{d-2N}\}. \quad (3.13)$$

Remark that $(1, 0^{N-1}, \omega_1, \dots, \omega_{d-N}, 0^N) \in \Sigma_{\beta_N}^d$ ends with 0^N . Thus, by Proposition 3.3, every word belonging to \mathcal{M}_d is full when regarding it as an element of Σ_β^d . Now let $G_1 = \{\omega : \omega \in \mathcal{M}_{n_1}\}$. Next, for all $k \geq 1$, let $n_{k+1} = (m_k - n_k)t_k + m_k + p_k$ where $0 \leq p_k < m_k - n_k$. Define

$$G_{k+1} = \{u_{k+1} = (1, 0^{m_k - n_k - 1}, u_k^{(1)}, \dots, u_k^{(t_k)}, u_k^{(t_k+1)}) : u_k^{(i)} \in \mathcal{M}_{m_k - n_k} \text{ for all } 1 \leq i \leq t_k\}$$

where

$$u_k^{(t_k+1)} = \begin{cases} 0^{p_k}, & \text{when } p_k \leq 2N; \\ \omega \in \mathcal{M}_{p_k}, & \text{when } p_k > 2N. \end{cases}$$

It follows from Propositions 3.3 and 2.4(2) that every $u_k \in G_k$ is full. Hence, we can define the set D_k as:

$$D_k = \{(u_1, \dots, u_k) : u_i \in G_i, \text{ for all } 1 \leq i \leq k\}. \quad (3.14)$$

Notice that the length of $u_k \in G_k$ satisfies $|u_k| = n_k - n_{k-1}$. For each $u = (u_1, \dots, u_k) \in D_k$, we have

$$|u| = |u_1| + |u_2| + \dots + |u_k| = n_1 + (n_2 - n_1) + \dots + (n_k - n_{k-1}) = n_k.$$

Define

$$E_N = \bigcap_{k=1}^{\infty} \bigcup_{u \in D_k} I_{n_k}(u).$$

The following lemma shows that E_N is a subset of $E_{a,b}$.

Lemma 3.4 *We have $E_N \subseteq E_{a,b}$ for every $0 \leq a \leq \frac{b}{1+b}$ and $0 < b < 1$.*

Proof. For every integer $n \geq 1$, there exists a $k \geq 1$ such that $n_k < n \leq n_{k+1}$. We distinguish three cases.

(1) If $n_k < n \leq n_k + m_{k-1} - n_{k-1} + 2N$, we have $m_{k-1} - n_{k-1} - 1 \leq r_n(x, \beta) \leq m_{k-1} - n_{k-1} + 2N$ by the construction of E_N . It follows that

$$\frac{m_{k-1} - n_{k-1} - 1}{n_k + m_{k-1} - n_{k-1} + 2N} \leq \frac{r_n(x, \beta)}{n} \leq \frac{m_{k-1} - n_{k-1} + 2N}{n_k}.$$

(2) If $n_k + m_{k-1} - n_{k-1} + 2N < n \leq m_k$, the construction of E_N gives $r_n(x, \beta) = n - n_k$. By (3.8), we have

$$\frac{m_{k-1} - n_{k-1} + 2N}{n_k + m_{k-1} - n_{k-1} + 2N} \leq \frac{r_n(x, \beta)}{n} \leq \frac{m_k - n_k}{m_k}.$$

(3) If $m_k \leq n \leq n_{k+1}$, we deduce from the construction of E_N that $m_k - n_k - 1 \leq r_n(x, \beta) \leq m_k - n_k + 2N$. Consequently,

$$\frac{m_k - n_k - 1}{n_{k+1}} \leq \frac{r_n(x, \beta)}{n} \leq \frac{m_k - n_k + 2N}{m_k}.$$

Combining the above three cases, by (3.11) and (3.12), we have

$$\liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} \geq a \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} \leq b.$$

Now we complete our proof by finding the subsequences such that the limit inferior and limit superior are reached. In fact, by (3.11), we get

$$\lim_{k \rightarrow \infty} \frac{r_{n_k + m_{k-1} - n_{k-1}}}{n_k + m_{k-1} - n_{k-1}} \leq \lim_{k \rightarrow \infty} \frac{m_{k-1} - n_{k-1} + 2N}{n_k + m_{k-1} - n_{k-1}} = a.$$

It follows from (3.12) that

$$\lim_{k \rightarrow \infty} \frac{r_{m_k}}{m_k} = \lim_{k \rightarrow \infty} \frac{m_k - n_k - 1}{m_k} = b.$$

□

Now we estimate the cardinality of the set D_k defined by (3.14). Write $q_k := \#D_k$.

Lemma 3.5 *Let $\beta > 1$. Let β_N be an approximation of β . For every $\bar{\beta} < \beta_N$, there exist an integer $k(\bar{\beta}, \beta_N)$ and real numbers $c(\bar{\beta}, \beta_N)$, $c'(\bar{\beta}, \beta_N)$ such that, for all $k \geq k(\bar{\beta}, \beta_N)$, we have*

$$q_k \geq c'(\bar{\beta}, \beta_N) c(\bar{\beta}, \beta_N)^k \bar{\beta}^{\sum_{i=1}^{k-1} (n_{i+1} - m_i)}. \quad (3.15)$$

Proof. Recall the definition of \mathcal{M}_d as (3.13). Theorem 2.2 implies

$$\#\mathcal{M}_d \geq \beta_N^{d-2N}$$

for all $d \geq N$. Since $\bar{\beta} < \beta_N$, there exists an integer d' which depends on $\bar{\beta}$ and β_N such that, for every $d \geq d'$, we have

$$\beta_N^{d-2N} \geq \bar{\beta}^d. \quad (3.16)$$

Moreover, the fact that $m_k - n_k$ is increasing and tends to $+\infty$ as $k \rightarrow +\infty$ ensures that we can find a large enough integer $k(\bar{\beta}, \beta_N)$ satisfying that, for all $k \geq k(\bar{\beta}, \beta_N)$,

$$\#\mathcal{M}_{m_k - n_k} \geq \beta_N^{m_k - n_k - 2N} \geq \bar{\beta}^{m_k - n_k}. \quad (3.17)$$

Then, when $p_k \leq 2N$, we have

$$\#G_{k+1} \geq (\#\mathcal{M}_{m_k - n_k})^{t_k} \geq \bar{\beta}^{(m_k - n_k)t_k} \geq \frac{1}{\bar{\beta}^{2N}} \bar{\beta}^{n_{k+1} - m_k}.$$

When $p_k > 2N$, we deduce that

$$\#G_{k+1} \geq (\#\mathcal{M}_{m_k - n_k})^{t_k} \cdot \#\mathcal{M}_{p_k} \geq \bar{\beta}^{(m_k - n_k)t_k} \cdot \beta_N^{p_k - 2N} = \frac{1}{\beta_N^{d'}} \bar{\beta}^{(m_k - n_k)t_k} \beta_N^{p_k - 2N + d'}.$$

Note that $p_k - 2N + d' > d'$. By (3.16), we have

$$\#G_{k+1} \geq \frac{1}{\beta_N^{d'}} \bar{\beta}^{(m_k - n_k)t_k} \bar{\beta}^{p_k + d'} = \frac{\bar{\beta}^{d'}}{\beta_N^{d'}} \bar{\beta}^{n_{k+1} - m_k}.$$

Let $c(\bar{\beta}, \beta_N) := \min\{\frac{1}{\bar{\beta}^{2N}}, \frac{\bar{\beta}^{d'}}{\beta_N^{d'}}\}$. It follows that for all $k \geq k(\bar{\beta}, \beta_N)$,

$$\#G_{k+1} \geq c(\bar{\beta}, \beta_N) \bar{\beta}^{n_{k+1} - m_k}.$$

Immediately, by the relationship between D_k and G_k , for any $k \geq k(\bar{\beta}, \beta_N)$, it comes to the conclusion that

$$\begin{aligned} q_k = \#D_k &= \prod_{i=1}^k \#G_i \geq \prod_{i=k(\bar{\beta}, \beta_N)}^k \#G_i \geq c(\bar{\beta}, \beta_N)^{k - k(\bar{\beta}, \beta_N)} \bar{\beta}^{\sum_{i=k(\bar{\beta}, \beta_N)}^{k-1} (n_{i+1} - m_i)} \\ &\geq c'(\bar{\beta}, \beta_N) c(\bar{\beta}, \beta_N)^k \bar{\beta}^{\sum_{i=1}^{k-1} (n_{i+1} - m_i)}, \end{aligned}$$

where

$$c'(\bar{\beta}, \beta_N) = \bar{\beta}^{-\sum_{i=1}^{k(\bar{\beta}, \beta_N) - 1} (n_{i+1} - m_i)}.$$

□

Now we divide into three parts to complete our proof of the lower bound of $\dim_{\mathbb{H}} E_{a,b}$ by using the modified mass distribution principle (Theorem 2.5).

(1) Define a probability measure μ supported on E_N . Set

$$\mu([0, 1]) = 1 \quad \text{and} \quad \mu(I_{n_1}(u)) = \frac{1}{\#G_1}, \quad \text{for } u \in D_1.$$

For each $k \geq 1$, and $u = (u_1, \dots, u_{k+1}) \in D_{k+1}$, let

$$\mu(I_{n_{k+1}}(u)) = \frac{\mu(I_{n_k}(u_1, \dots, u_k))}{\#G_{k+1}}. \quad (3.18)$$

For any $u \notin D_k$ ($k \geq 1$), let $\mu(I_{n_k}(u)) = 0$. It is routine to check that μ is well defined on E_N and it can be extended to a probability measure on $[0, 1]$.

(2) Calculate the local dimension $\liminf_{n \rightarrow \infty} \frac{\log \mu(I_n)}{\log |I_n|}$ for any $x \in E_N$. For convenience, we denote $I_n(x)$ by I_n without ambiguity. Then we have

$$\mu(I_{n_i}) = \frac{1}{q_i} \leq \frac{1}{c'(\beta_N, \bar{\beta})c(\beta_N, \bar{\beta})^i \bar{\beta}^{\sum_{j=1}^{i-1} (n_{j+1} - m_j)}} \quad (3.19)$$

for every $i > k(\beta_N, \bar{\beta})$, where $k(\beta_N, \bar{\beta})$ is an integer given in Lemma 3.5. For all $n \geq 1$, there is an integer $k \geq 1$ such that $n_k < n \leq n_{k+1}$. By the construction of E_N and the definition of μ , it is natural to estimate the lower bound of $\frac{\log \mu(I_n)}{\log |I_n|}$ by dividing into the following three cases.

Case 1. $n_k < n \leq m_k$. It follows from (3.19) that

$$\mu(I_n) = \mu(I_{n_k}) \leq c'(\beta_N, \bar{\beta})^{-1} c(\beta_N, \bar{\beta})^{-k} \bar{\beta}^{-\sum_{j=1}^{k-1} (n_{j+1} - m_j)}.$$

Furthermore, Theorem 2.3(3) implies

$$|I_n(x)| \geq |I_{m_k}(x)| = \frac{1}{\beta^{m_k}}.$$

As a consequence,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\sum_{j=1}^{k-1} (n_{j+1} - m_j) \log \bar{\beta} + k \log c(\beta_N, \bar{\beta}) + \log c'(\beta_N, \bar{\beta})}{m_k \log \beta}.$$

Case 2. $n = m_k + i(m_k - n_k) + \ell$ for some $0 \leq i < t_k$ and $0 \leq \ell < m_k - n_k$. In this case, when $0 \leq \ell \leq N$, by (3.19) and (3.17), we have

$$\begin{aligned} \mu(I_n) &= \mu(I_{m_k + i(m_k - n_k) + \ell}) \leq \mu(I_{m_k}) \cdot \frac{1}{(\#\mathcal{M}_{m_k - n_k})^i} \\ &\leq c'(\beta_N, \bar{\beta})^{-1} c(\beta_N, \bar{\beta})^{-k} \bar{\beta}^{-\left(\sum_{j=1}^{k-1} (n_{j+1} - m_j) + i(m_k - n_k)\right)}. \end{aligned}$$

When $N < \ell < m_k - n_k$, we similarly see that

$$\begin{aligned} \mu(I_n) &= \mu(I_{m_k + i(m_k - n_k) + \ell}) \leq \mu(I_{m_k}) \cdot \frac{1}{(\#\mathcal{M}_{m_k - n_k})^i} \cdot \frac{1}{\sum_{\beta_N}^{\ell - 2N}} \\ &\leq c'(\beta_N, \bar{\beta})^{-1} c(\beta_N, \bar{\beta})^{-k} \bar{\beta}^{-\left(\sum_{j=1}^{k-1} (n_{j+1} - m_j) + i(m_k - n_k)\right)} \beta_N^{-\ell + 2N}. \end{aligned}$$

Moreover, by (3.10), it holds that

$$|I_n| \geq |I_{m_k + i(m_k - n_k) + \ell}| \geq \frac{1}{\beta^{m_k + i(m_k - n_k) + \ell + 2N}}.$$

Therefore,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\left(\sum_{j=1}^{k-1} (n_{j+1} - m_j) + i(m_k - n_k)\right) \log \bar{\beta} + (\ell - N) \log \beta_N + k \log c(\beta_N, \bar{\beta}) + \log c'(\beta_N, \bar{\beta})}{(m_k + i(m_k - n_k) + \ell + N) \log \beta}.$$

Case 3. $n = m_k + t_k(m_k - n_k) + \ell$ where $0 \leq \ell \leq p_k$. When $0 \leq \ell \leq 2N$, we have

$$\begin{aligned} \mu(I_n) &= \mu(I_{m_k+t_k(m_k-n_k)}) = \mu(I_{m_k}(x)) \cdot \frac{1}{(\#\mathcal{M}_{m_k-n_k})^{t_k}} \\ &\leq c'(\beta_N, \bar{\beta})^{-1} c(\beta_N, \bar{\beta})^{-k} \bar{\beta}^{-\left(\sum_{j=1}^{k-1} (n_{j+1}-m_j) + t_k(m_k-n_k)\right)}. \end{aligned}$$

When $2N < \ell \leq p_k$, we get

$$\begin{aligned} \mu(I_n) &= \mu(I_{m_k+t_k(m_k-n_k)+\ell}) \leq \mu(I_{m_k}(x)) \cdot \frac{1}{(\#M_{m_k-n_k})^{t_k}} \cdot \frac{1}{\sum_{\beta_N}^{\ell-2N}} \\ &\leq c'(\beta_N, \bar{\beta})^{-1} c(\beta_N, \bar{\beta})^{-k} \bar{\beta}^{-\left(\sum_{j=1}^{k-1} (n_{j+1}-m_j) + t_k(m_k-n_k)\right)} \beta_N^{-\ell+2N}. \end{aligned}$$

In addition, by (3.10),

$$|I_n| \geq |I_{m_k+t_k(m_k-n_k)+\ell}| \geq \frac{1}{\beta^{m_k+t_k(m_k-n_k)+\ell+N}}.$$

Hence,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\left(\sum_{j=1}^{k-1} (n_{j+1}-m_j) + t_k(m_k-n_k)\right) \log \bar{\beta} + (\ell-2N) \log \beta_N + k \log c(\beta_N, \bar{\beta}) + \log c'(\beta_N, \bar{\beta})}{(m_k + t_k(m_k - n_k) + \ell + N) \log \beta}.$$

In all three cases, using (3.8), we obtain

$$\liminf_{n \rightarrow \infty} \frac{\log \mu(I_n)}{\log |I_n|} \geq \lim_{k \rightarrow \infty} \frac{\sum_{j=1}^{k-1} (n_{j+1}-m_j)}{m_k} \frac{\log \bar{\beta}}{\log \beta}.$$

By (3.11) and (3.12), it immediately holds that

$$\lim_{k \rightarrow \infty} \frac{n_k}{m_k} = 1 - b, \quad \lim_{k \rightarrow \infty} \frac{n_{k+1}}{m_k} = \frac{b(1-a)}{a} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{m_{k+1}}{m_k} = \frac{b(1-a)}{a(1-b)}.$$

By the Stolz-Cesàro Theorem, we have

$$\lim_{k \rightarrow \infty} \frac{\sum_{j=1}^{k-1} (n_{j+1}-m_j)}{m_k} = \lim_{k \rightarrow \infty} \frac{n_{k+1}-m_k}{m_{k+1}-m_k} = \lim_{k \rightarrow \infty} \frac{\frac{n_{k+1}}{m_k} - 1}{\frac{m_{k+1}}{m_k} - 1} = 1 - \frac{b^2(1-a)}{b-a}.$$

As a consequence,

$$\liminf_{n \rightarrow \infty} \frac{\log \mu(I_n)}{\log |I_n|} \geq \left(1 - \frac{b^2(1-a)}{b-a}\right) \frac{\log \bar{\beta}}{\log \beta}.$$

(3) Use the modified mass distribution principle (Theorem 2.5). We first let $\bar{\beta} \rightarrow \beta_N$, and then let $N \rightarrow \infty$. Applying Theorem 2.5, we finish our proof.

3.3 Proof of Corollary 1.2

Note that when $\frac{1}{2} < a \leq 1$, the inequality (3.9) implies $E_a = \emptyset$. We only need to consider the case $0 \leq a \leq \frac{1}{2}$. By Lemma 3.1, we have

$$E_a = \left\{ x \in [0, 1) : \hat{v}_\beta(x) = \frac{a}{1-a} \right\}.$$

Thus, applying Theorem 2.7, we have, for all $0 < a \leq \frac{1}{2}$,

$$\dim_{\text{H}} E_a = \dim_{\text{H}} \left\{ x \in [0, 1) : \hat{v}_\beta(x) = \frac{a}{1-a} \right\} = (1-2a)^2.$$

When $a = 0$, by noting that $E_{0,0} \subseteq E_0$, we deduce that E_0 has full Lebesgue measure and thus has Hausdorff dimension 1.

4 Proof of Theorem 1.3

The key point to prove Theorem 1.3 is constructing a set U with the following properties: (1) U is a subset of $E_{0,1}$; (2) U is dense in the interval $[0, 1]$; (3) U is a G_δ set, i.e., a countable intersection of open sets.

Let $\beta > 1$. Define $M = \min\{i > 1 : \varepsilon_i^*(\beta) > 0\}$. For all $k \geq 1$, let Γ_k be defined by (2.7). We choose two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $n_k < m_k < n_{k+1}$ with $n_k > 2k + \Gamma_k$ and $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$. In addition, $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ satisfy

$$\lim_{k \rightarrow \infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = 0,$$

and

$$\lim_{k \rightarrow \infty} \frac{m_k - n_k}{m_k} = 1.$$

In fact, let

$$n'_k = (2k + \Gamma_k)^{2k} \quad \text{and} \quad m'_k = (2k + 2 + \Gamma_{k+1})^{2k+1}.$$

Then by small adjustments, we can obtain the required sequences.

For all $k \geq 1$, write $n_{k+1} = (m_k - n_k)t_k + n_k + p_k$ where $0 \leq p_k < m_k - n_k$. Now we define

$$U := \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty \bigcup_{(\epsilon_1, \dots, \epsilon_k) \in \Sigma_\beta^k} \text{int}(I_{n_{k+1}}(\epsilon_1, \dots, \epsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k-1})^{t_k}, 0^{p_k})),$$

where $\text{int}(I_{|\epsilon|}(\epsilon))$ stands for the interior of $I_{|\epsilon|}(\epsilon)$ for all $\epsilon \in \Sigma_\beta^*$.

Remark 1 For all $(\epsilon_1, \dots, \epsilon_k) \in \Sigma_\beta^k$, it follows from Proposition 2.4(3) that $(\epsilon_1, \dots, \epsilon_k, 0^{n_k-k})$ is full since $n_k > 2k + \Gamma_k$. Note that $m_k - n_k \geq M$. Then the word $(1, 0^{m_k-n_k-1})$ is full. By Proposition 2.4(2), the word 0^{p_k} is full. Thus U is well defined.

The set $\text{int}(I_{|\epsilon|}(\epsilon))$ is open which implies that U is a G_δ set. Consequently, it suffices to show that U is a subset of $E_{0,1}$ and is dense in $[0, 1]$.

Lemma 4.1 *The set U is a subset of $E_{0,1}$.*

Proof. For any $x \in U$, it follows from the construction of U that there exist infinitely many k such that $\varepsilon(x, \beta) = (\epsilon_1, \dots, \epsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k-1})^{t_k}, 0^{p_k})$ for some $(\epsilon_1, \dots, \epsilon_k) \in \Sigma_\beta^k$. Now we are going to give the upper limit and lower limit of $\frac{r_n(x, \beta)}{n}$.

Let $n = n_{k+1} + m_k - n_k - 1$. Since $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$, we obtain

$$r_{n_{k+1}+m_k-n_k-1}(x, \beta) = m_k - n_k - 1.$$

As a result,

$$\liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} \leq \lim_{k \rightarrow \infty} \frac{r_{n_{k+1}+m_k-n_k-1}(x, \beta)}{n_{k+1} + m_k - n_k - 1} = \lim_{k \rightarrow \infty} \frac{m_k - n_k - 1}{n_{k+1} + m_k - n_k - 1} = 0.$$

Let $n = m_k$. Note that $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$. The definition of $r_n(x, \beta)$ shows that

$$r_{m_k}(x, \beta) = m_k - n_k - 1.$$

It therefore follows that

$$\limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} \geq \lim_{k \rightarrow \infty} \frac{r_{m_k}(x, \beta)}{m_k} = \lim_{k \rightarrow \infty} \frac{m_k - n_k - 1}{m_k} = \lim_{k \rightarrow \infty} \frac{m_k - n_k - 1}{m_k} = 1.$$

By the above discussion, we conclude that

$$\liminf_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \frac{r_n(x, \beta)}{n} = 1.$$

Hence, $x \in E_{0,1}$ which gives $U \subseteq E_{0,1}$. □

Proof of Theorem 1.3 It remains to show that for all $n \geq 1$, the set

$$U_n = \bigcup_{k=n}^{\infty} \bigcup_{(\epsilon_1, \dots, \epsilon_k) \in \Sigma_{\beta}^k} \text{int} (I_{n_{k+1}} (\epsilon_1, \dots, \epsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k})^{t_k}, 0^{p_k}))$$

is dense in $[0,1]$. Now we will concentrate on finding a real number $y \in U$ such that $|x - y| \leq r$ for every $x \in [0, 1]$ and $r > 0$. Suppose that $\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \dots)$. Let ℓ' be an integer satisfying $\beta^{-\ell'} \leq r$. Let $\ell = \max\{n, \ell'\}$. Since $(\varepsilon_1(x, \beta), \dots, \varepsilon_{\ell}(x, \beta)) \in \Sigma_{\beta}^{\ell}$, we choose a point

$$y \in \text{int} (I_{n_{\ell+1}} (\epsilon_1, \dots, \epsilon_{\ell}, 0^{n_{\ell}-\ell}, (1, 0^{m_{\ell}-n_{\ell}})^{t_{\ell}}, 0^{p_{\ell}})).$$

Then it holds that $|x - y| \leq \beta^{-\ell} \leq r$ and $y \in U_n$. To sum up, the set

$$\bigcup_{k=n}^{\infty} \bigcup_{(\epsilon_1, \dots, \epsilon_k) \in \Sigma_{\beta}^k} \text{int} (I_{n_{k+1}} (\epsilon_1, \dots, \epsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k})^{t_k}, 0^{p_k}))$$

is dense in $[0, 1]$.

Thus, we can conclude by the Baire Category Theorem that U is residual in $[0, 1]$. Then, $E_{0,1}$ is residual in $[0, 1]$ by Lemma 4.1. □

5 Classical results of β -expansion in the parameter space

In this section, we recall some important results of β -expansion in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$. The readers can refer to [4, 8, 12, 15, 18] for more information.

Definition 5.1 We call a word $\omega = (\omega_1, \dots, \omega_n)$ self-admissible if for all $1 \leq i < n$,

$$\sigma^i \omega \leq_{\text{lex}} (\omega_1, \dots, \omega_{n-i}).$$

An infinite sequence $\omega = (\omega_1, \omega_2, \dots)$ is called *self-admissible* if $\sigma^i \omega <_{\text{lex}} \omega$ for all $i \geq 1$.

Denote by Λ_n the set of all self-admissible words with length n , i.e.,

$$\Lambda_n = \{\omega = (\omega_1, \omega_2, \dots, \omega_n) : \text{for every } 1 \leq i < n, \sigma^i \omega \leq_{\text{lex}} (\omega_1, \dots, \omega_{n-i})\}.$$

For convenience, for all $1 < \beta_1 < \beta_2$, let

$$\Lambda_n(\beta_1, \beta_2) = \{\omega = (\omega_1, \dots, \omega_n) \in \Lambda_n : \exists \beta \in (\beta_1, \beta_2] : \text{s.t. } \varepsilon_1(\beta) = \omega_1, \dots, \varepsilon_n(\beta) = \omega_n\}. \quad (5.20)$$

The definition of self-admissible word immediately gives the following result. The proof is evident and will be omitted.

Proposition 5.1 *For any $m \geq n \geq 1$, let $\omega \in \Lambda_n$. Let $\beta > 1$ whose infinite β -expansion of 1 satisfy $(\varepsilon_1^*(\beta), \dots, \varepsilon_n^*(\beta)) <_{\text{lex}} \omega$. Then for all $v_1, v_2, \dots, v_i \in \Sigma_\beta^m$ ($i \geq 1$), the concatenation $\omega * v_1 * \dots * v_j$ is still self-admissible for all $1 \leq j \leq i$.*

The characterization of the the β -expansion of 1 was given by Parry [15].

Theorem 5.2 (Parry [15]) *An infinite sequence $(\omega_1, \omega_2, \dots)$ is the β -expansion of 1 for some $\beta > 1$ if and only if it is self-admissible.*

Now we consider the *cylinders in the parameter space* $\{\beta \in \mathbb{R} : \beta > 1\}$.

Definition 5.2 *For any $\omega = (\omega_1, \dots, \omega_n) \in \Lambda_n$. The cylinder $I_n^P(\omega)$ associated to ω in the parameter space is the set of $\beta \in (1, +\infty)$ whose β -expansion of 1 has the prefix $(\omega_1, \dots, \omega_n)$, i.e.*

$$I_n^P(\omega) := \{\beta \in (1, +\infty) : \varepsilon_1(\beta) = \omega_1, \dots, \varepsilon_n(\beta) = \omega_n\}.$$

The cylinders in the parameter space are intervals (see [18, Lemma 4.1]). The length of the cylinders of $\omega \in \Lambda_n$ in the parameter space is denoted by $|I_n^P(\omega)|$. For simplicity, the left endpoint and right endpoint of $I_n^P(\omega)$ are written as $\underline{\beta}(\omega)$ and $\overline{\beta}(\omega)$ respectively.

To estimate the length of cylinders in the parameter space, we need the notion of *recurrence time* $\tau(\omega)$ (see [12]) of the self-admissible word $\omega = (\omega_1, \dots, \omega_n) \in \Lambda_n$. Define

$$\tau(\omega) := \inf\{1 \leq k < n : \sigma^k(\omega_1, \dots, \omega_n) = (\omega_1, \dots, \omega_{n-k})\}.$$

If we cannot find such an integer k , we set $\tau(\omega) = n$. In this case, the self-admissible word ω is said to be *non-recurrent*.

The above definition of recurrence time immediately provides the following properties.

Remark 2 (1) *Write*

$$t(\omega) := n - \left\lfloor \frac{n}{\tau(\omega)} \right\rfloor \tau(\omega).$$

Then we have

$$(\omega_1, \dots, \omega_n) = \left((\omega_1, \dots, \omega_{\tau(\omega)})^{\lfloor \frac{n}{\tau(\omega)} \rfloor}, \omega_1, \dots, \omega_{t(\omega)} \right).$$

(2) *If $\omega = (\omega_1, \dots, \omega_n)$ is non-recurrent, then the word $(\omega_1, \dots, \omega_n, 0^\ell)$ is still non-recurrent for all $\ell \geq 1$.*

The following result gives the upper and lower bounds of the length of the cylinder $I_n^P(\omega)$.

Lemma 5.3 (Schemling [18], Li, Persson, Wang and Wu [12]) *Let $\omega = (\omega_1, \dots, \omega_n) \in \Lambda_n$. We have the following inequalities:*

$$(1) |I_n^P(\omega)| \leq \overline{\beta}(\omega)^{-n+1};$$

(2)

$$|I_n^P(\omega)| \geq \begin{cases} C(\omega)\bar{\beta}(\omega)^{-n}, & \text{when } t(\omega) = 0; \\ C(\omega)\bar{\beta}(\omega)^{-n} \left(\frac{\omega_{t(\omega)+1}}{\bar{\beta}(\omega)} + \dots + \frac{\omega_{\tau(\omega)+1}}{\bar{\beta}(\omega)^{\tau(\omega)-t(\omega)}} \right), & \text{otherwise,} \end{cases}$$

where

$$C(\omega) := \frac{(\beta(\omega) - 1)^2}{\underline{\beta}(\omega)}. \quad (5.21)$$

The study of the parameter space usually concerns on the set of parameters with respect to which the approximation properties of the orbit of 1 are prescribed. Persson and Schmeling [16] proved the following result.

Theorem 5.4 (Persson and Schmeling [16]) *Let $v \geq 0$. Then*

$$\dim_{\text{H}}\{\beta \in (1, 2) : v_{\beta}(1) \geq v\} = \frac{1}{1+v}.$$

Analogous to Theorem 2.7, Bugeaud and Liao [2] obtained the following theorem in the parameter space.

Theorem 5.5 (Bugeaud and Liao [2]) *Let $0 < \hat{v} < 1$ and $v > 0$. If $v < \frac{\hat{v}}{1-\hat{v}}$, then the set*

$$U(\hat{v}, v) := \{\beta \in (1, 2) : \hat{v}_{\beta}(1) = \hat{v}, v_{\beta}(1) = v\}$$

is empty. Otherwise, we have

$$\dim_{\text{H}} U(\hat{v}, v) = \frac{v - (1+v)\hat{v}}{(1+v)(v-\hat{v})}.$$

Moreover,

$$\dim_{\text{H}}\{\beta \in (1, 2) : \hat{v}_{\beta}(1) = \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}} \right)^2.$$

6 Proof of Theorem 1.4

As the same discussion at the first part of Section 3, it holds that $\dim_{\text{H}} E_{0,0}^P$ is of full Lebesgue measure by using the result of Cao and Chen [4] that the set

$$\left\{ \beta \in (1, 2) : \lim_{n \rightarrow \infty} \frac{r_n(\beta)}{\log_{\beta} n} = 1 \right\}$$

is of full Lebesgue measure. By the same argument as the proof of Theorem 1.1 for the case $a > \frac{b}{1+b}$, $0 < b \leq 1$ in the end of Section 3.1, we get that $E_{a,b}^P$ is empty when $a > \frac{b}{1+b}$, $0 < b \leq 1$.

When $0 < a \leq \frac{b}{1+b}$, $0 < b < 1$, Lemmas 3.1 and 3.2 imply that

$$E_{a,b}^P = \left\{ \beta \in (1, 2) : \hat{v}_{\beta}(1) = \frac{a}{1-a}, v_{\beta}(1) = \frac{b}{1-b} \right\} = U\left(\frac{a}{1-a}, \frac{b}{1-b} \right).$$

Then by Theorem 5.5, it holds that

$$\dim_{\text{H}} E_{a,b}^P = \dim_{\text{H}} U\left(\frac{a}{1-a}, \frac{b}{1-b} \right) = 1 - \frac{b^2(1-a)}{b-a}.$$

But Theorem 5.5 is not applicable for the case of $a = 0$, $0 < b < 1$ and $0 < a \leq \frac{1}{2}$, $b = 1$. Note that $E_{a,1}^P \subseteq F_1^P$ where F_1^P is defined by (1.4). So we first give the Hausdorff dimension of F_1^P .

Since

$$F_1^P = \{\beta \in (1, 2) : v_\beta(1) = +\infty\} \subseteq \{\beta \in (1, 2) : v_\beta(1) \geq v\}$$

for all $v > 0$, we have

$$\dim_{\mathbb{H}} F_1^P \leq \dim_{\mathbb{H}} \{\beta \in (1, 2) : v_\beta(1) \geq v\} = \frac{1}{1+v}$$

where the last equality follows from Theorem 5.4. Letting $v \rightarrow +\infty$, we have $\dim_{\mathbb{H}} F_1^P \leq 0$. This implies $\dim_{\mathbb{H}} E_{a,1}^P \leq 0$. In conclusion, $\dim_{\mathbb{H}} E_{a,1}^P = 0$ for any $0 < a \leq \frac{1}{2}$. For the other case, we have

$$E_{0,b}^P \subseteq \left\{ \beta \in (1, 2) : v_\beta(1) \geq \frac{b}{1-b} \right\}.$$

By Theorem 5.4, we deduce that the upper bound of $\dim_{\mathbb{H}} E_{0,b}^P$ is $1-b$ for all $0 < b < 1$. Hence, we only need to give the lower bound of $\dim_{\mathbb{H}} E_{0,b}^P$ for all $0 < b < 1$. We also include our proof of the case $0 < a \leq \frac{b}{1+b}$, $0 < b < 1$.

For every $1 < \beta_1 < \beta_2 < 2$, instead of dealing with the Hausdorff dimension of the set $E_{a,b}^P$ directly, we will technically investigate the Hausdorff dimension of the following set. For all $0 \leq a \leq \frac{b}{1+b}$, $0 < b \leq 1$, let

$$E_{a,b}^P(\beta_1, \beta_2) = \left\{ \beta \in [\beta_1, \beta_2) : \liminf_{n \rightarrow \infty} \frac{r_n(\beta)}{n} = a, \limsup_{n \rightarrow \infty} \frac{r_n(\beta)}{n} = b \right\}. \quad (6.22)$$

For all $1 < \beta_1 < \beta_2 < 2$, throughout this section, we assume that both β_1 and β_2 are not simple Parry number. We will give the lower bound of $\dim_{\mathbb{H}} E_{a,b}^P(\beta_1, \beta_2)$ for all $1 < \beta_1 < \beta_2 < 2$.

Suppose that N is a large enough integer such that $\varepsilon_N(\beta_2) > 0$ and

$$(\varepsilon_1(\beta_1), \dots, \varepsilon_N(\beta_1)) <_{\text{lex}} (\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2))$$

Let $\tilde{\beta}_N$ be the unique solution of the equation:

$$1 = \frac{\varepsilon_1(\beta_2)}{x} + \dots + \frac{\varepsilon_N(\beta_2)}{x^N}.$$

Then

$$\varepsilon^*(\tilde{\beta}_N) = (\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2) - 1)^\infty.$$

An observation of the lexicographical order of $\varepsilon^*(\beta_1)$, $\varepsilon^*(\beta_2)$ and $\varepsilon^*(\tilde{\beta}_N)$ implies $\beta_1 < \tilde{\beta}_N < \beta_2$ and $\tilde{\beta}_N \rightarrow \beta_2$ as N tends to infinity.

For every $k \geq 1$, similar to what we did in Section 3.2, we take two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $n_k < m_k < n_{k+1}$ with $n_1 > 2N$ and $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$. In addition,

$$\lim_{k \rightarrow \infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = a \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{m_k - n_k}{m_k} = b.$$

We can choose such two sequences by the same way in Section 3.2.

Now let us construct a Cantor set contained in $E_{a,b}^P(\beta_1, \beta_2)$ as follows.

For any integer $d > 2N$, we set

$$\mathcal{M}'_d = \{\omega = (\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2) - 1, \omega_1, \dots, \omega_{d-2N}, 0^N) : (\omega_1, \dots, \omega_{d-2N}) \in \Sigma_{\tilde{\beta}_N}^{d-2N}\}. \quad (6.23)$$

Let

$$G'_1 = \{(\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2), \omega_1, \dots, \omega_{d-2N}, 0^N) : (\omega_1, \dots, \omega_{d-2N}) \in \Sigma_{\tilde{\beta}_N}^{d-2N}\}.$$

Note that $(\varepsilon_1^*(\tilde{\beta}_N), \dots, \varepsilon_N^*(\tilde{\beta}_N)) <_{\text{lex}} (\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2))$. Now we give some observations on

the elements in G'_1 as follows.

Remark 3 (1) For all $\omega \in G'_1$, since $(\omega_1, \dots, \omega_{d-2N}, 0^N) \in \Sigma_{\tilde{\beta}_N}^{d-N}$ ($d > 2N$), by Proposition 5.1, ω is self-admissible.

(2) For every $u \in \mathcal{M}'_d$ ($d > 2N$), we have $u \in \Sigma_{\tilde{\beta}_N}^d$. By Lemma 5.1, the word $\omega * u$ is still self-admissible for every all $\omega \in G'_1$.

For every $k \geq 1$, write $n_{k+1} = (m_k - n_k)t_k + m_k + p_k$ with $0 \leq p_k < m_k - n_k$, then define $G'_{k+1} = \{u_{k+1} = (\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2) - 1, 0^{m_k - n_k - N}, u_k^{(1)}, \dots, u_k^{(t_k)}, u_k^{(t_k+1)}) : u_k^{(i)} \in \mathcal{M}'_{m_k - n_k}, 1 \leq i \leq t_k\}$, where

$$u_k^{(t_k+1)} = \begin{cases} 0^{p_k}, & \text{when } p_k \leq 2N; \\ \omega \in \mathcal{M}'_{p_k}, & \text{when } p_k > 2N. \end{cases}$$

Let

$$D'_k = \{(u_1, \dots, u_k) : u_i \in G'_i, 1 \leq i \leq k\}. \quad (6.24)$$

Notice that every $u_k \in G'_k$ ends with 0^N . This guarantees that (u_1, \dots, u_k) can concatenate with any u_{k+1} to be a new self-admissible word. As a result, the set D'_k is well-defined.

As the classical technique of constructing a Cantor set, let

$$E(\beta_1, \beta_2) = \bigcap_{k=1}^{\infty} \bigcup_{u \in D'_k} I_{n_k}^P(u).$$

Similar to the process of Section 3, we now give the following result which means that $E(\beta_1, \beta_2)$ is a subset of $E_{a,b}^P(\beta_1, \beta_2)$.

Lemma 6.1 For every $1 < \beta_1 < \beta_2 < 2$, $E(\beta_1, \beta_2) \subset E_{a,b}^P(\beta_1, \beta_2)$ for all $0 \leq a \leq \frac{b}{1+b}$ and $0 < b < 1$.

Proof. The proof is just as the same as the proof of Lemma 3.4 by dividing into three cases. We omit it here. \square

Analogously, we now focus on the estimation of the cardinality of the set D'_k . Let $q'_k := \#D'_k$. We obtain the following lemma.

Lemma 6.2 For every $1 < \beta_1 < \beta_2 < 2$, let $\tilde{\beta}_N$ be the real number defined in this section. Then there exist an integer $k(\beta_1, \tilde{\beta}_N)$ and real numbers $c(\beta_1, \tilde{\beta}_N), c'(\beta_1, \tilde{\beta}_N)$ such that, for every $k \geq k(\beta_1, \tilde{\beta}_N)$, we have

$$q'_k \geq c'(\beta_1, \tilde{\beta}_N) c(\beta_1, \tilde{\beta}_N)^k \beta_1^{\sum_{i=1}^{k-1} (n_{i+1} - m_i)}. \quad (6.25)$$

Proof. We use the similar method as Lemma 3.5, the details are left to the readers. \square

Let

$$C(\beta_1) = \frac{(\beta_1 - 1)^2}{\beta_1}.$$

Notice that $\underline{\beta}(u) \geq \beta_1 > 1$ for any $u = (u_1, \dots, u_n) \in \Lambda_n(\beta_1, \beta_2)$ where $\Lambda_n(\beta_1, \beta_2)$ is defined by (5.20). Then

$$C(\underline{\beta}(u)) = \frac{(\underline{\beta}(u) - 1)^2}{\underline{\beta}(u)} \geq C(\beta_1).$$

The following lemma gives the estimation of the length of the cylinders with non-empty intersection with the Cantor set $E(\beta_1, \beta_2)$ which will be useful to estimate the local dimension $\liminf_{n \rightarrow \infty} \frac{\log \mu(B(\beta, r))}{\log |r|}$ for any $r > 0$ and $\beta \in E(\beta_1, \beta_2)$.

Lemma 6.3 For any $\beta \in E(\beta_1, \beta_2)$, suppose $\varepsilon(1, \beta) = (u_1, u_2, \dots)$. Then we have

$$|I_n^P(u_1, \dots, u_n)| \geq C(\beta_1)\beta_2^{-(n+N)}$$

for any $n \geq 1$.

Proof. For any $n \geq 1$, we are going to take the word $(u_1, \dots, u_n, 0^N)$ into account. We claim that the word $(u_1, \dots, u_n, 0^N)$ is non-recurrent.

In fact, by the construction of $E(\beta_1, \beta_2)$, for any $1 \leq i < n$, we have

$$\sigma^i(u_1, \dots, u_n, 0^N) \in \Sigma_{\tilde{\beta}_N}^{n-i+N}.$$

Notice that $\omega \leq_{\text{lex}} (\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2) - 1) <_{\text{lex}} (\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2))$ for any $\omega \in \Sigma_{\tilde{\beta}_N}^n$ with $n \geq N$. It comes to the conclusion that $\sigma^i(u_1, \dots, u_n, 0^N) <_{\text{lex}} (\varepsilon_1(\beta_2), \dots, \varepsilon_N(\beta_2))$ for any $1 \leq i < n + N$ which implies that $(u_1, \dots, u_n, 0^N)$ is non-recurrent. Thus, by Lemma 5.3(2), we have

$$|I_n^P(u_1, \dots, u_n)| \geq |I_{n+N}^P(u_1, \dots, u_n, 0^N)| \geq C(u_1, \dots, u_n, 0^N) \bar{\beta}(u_1, \dots, u_n, 0^N)^{-(n+N)}.$$

It follows from the fact $\bar{\beta}(u_1, \dots, u_n, 0^N) \leq \beta_2$ that

$$|I_n^P(u_1, \dots, u_n)| \geq C(\beta_1)\beta_2^{-(n+N)}.$$

□

Let us now concentrate on giving the lower bound of $\dim_{\text{H}} E(\beta_1, \beta_2)$. As the conventional process, we define a measure supported on $E(\beta_1, \beta_2)$ which is similar to Section 3.2 by distributing the mass uniformly. We will give the local dimension $\liminf_{n \rightarrow \infty} \frac{\log \mu(I_n^P(u))}{\log |I_n^P(u)|}$ for any cylinder $I_n^P(u)$ which has non-empty intersection with $E(\beta_1, \beta_2)$. Without any confusion, here and subsequently, I_n^P stands for the cylinder $I_n^P(u)$ for all $u \in \Lambda_n$.

(1) Define a probability measure supported on $E(\beta_1, \beta_2)$. Let

$$\mu([\beta_1, \beta_2]) = 1 \quad \text{and} \quad \mu(I_{n_1}^P(u)) = \frac{1}{\#G'_1}, \quad \text{for } u \in D'_1.$$

For all $k \geq 1$, and $u = (u_1, \dots, u_{k+1}) \in D'_{k+1}$, define

$$\mu(I_{n_{k+1}}^P(u)) = \frac{\mu(I_{n_k}^P(u_1, \dots, u_k))}{\#G'_{k+1}}.$$

(2) Estimate the local dimension $\liminf_{n \rightarrow \infty} \frac{\log \mu(I_n^P)}{\log |I_n^P|}$ where $I_n^P \cap E(\beta_1, \beta_2) \neq \emptyset$. It follows from the definition of the measure that

$$\mu(I_{n_i}^P) = \frac{1}{q'_i} \leq \frac{1}{c'(\beta_1, \tilde{\beta}_N) c(\beta_1, \tilde{\beta}_N)^i \beta_1^{\sum_{j=1}^{i-1} (n_{j+1} - m_j)}}, \quad (6.26)$$

for every $i > k(\beta_1, \tilde{\beta}_N)$. For any $\beta \in E(\beta_1, \beta_2)$, suppose $\varepsilon(1, \beta) = (u_1, u_2, \dots)$. For each $n \geq 1$, there exists an integer $k \geq 1$ such that $n_k < n \leq n_{k+1}$. It falls naturally into three cases.

Case 1. $n_k < n \leq m_k$. It follows from (6.26) that

$$\mu(I_n^P) = \mu(I_{n_k}^P) \leq c'(\beta_1, \tilde{\beta}_N)^{-1} c(\beta_1, \tilde{\beta}_N)^{-k} \beta_1^{-\sum_{j=1}^{k-1} (n_{j+1} - m_j)}.$$

Furthermore, by the construction of $E(\beta_1, \beta_2)$, the word (u_1, \dots, u_{m_k}) is non-recurrent. Thus, by Lemma 5.3, we have

$$|I_n^P| \geq |I_{m_k}^P| \geq C(u_1, \dots, u_{m_k}) \bar{\beta}(u_1, \dots, u_{m_k})^{-m_k} \geq c(\beta_1) \beta_2^{-m_k}.$$

As a consequence,

$$\frac{\log \mu(I_n^P)}{\log |I_n^P|} \geq \frac{\log c'(\beta_1, \tilde{\beta}_N) + k \log c(\beta_1, \tilde{\beta}_N) + \sum_{j=1}^{k-1} (n_{j+1} - m_j) \log \beta_1}{\log c(\beta_1) + m_k \log \beta_1}.$$

Case 2. $n = m_k + i(m_k - n_k) + \ell$ for some $0 \leq i < t_k$ and $0 \leq \ell < m_k - n_k$. On the one hand, when $0 \leq \ell \leq 2N$, we have

$$\begin{aligned} \mu(I_n^P) &= \mu(I_{m_k+i(m_k-n_k)+\ell}^P) \leq \mu(I_{m_k}^P) \cdot \frac{1}{(\#M_{m_k-n_k})^i} \\ &\leq c'(\beta_1, \tilde{\beta}_N)^{-1} c(\beta_1, \tilde{\beta}_N)^{-k} \beta_1^{-\left(\sum_{j=1}^{k-1} (n_{j+1}-m_j)+i(m_k-n_k)\right)}. \end{aligned}$$

On the other hand, when $2N < \ell < m_k - n_k$, we have

$$\begin{aligned} \mu(I_n^P) &= \mu(I_{m_k+i(m_k-n_k)+\ell}^P) \leq \mu(I_{m_k}^P) \cdot \frac{1}{(\#M_{m_k-n_k})^i} \cdot \frac{1}{\sum_{\tilde{\beta}_N}^{\ell-2N}} \\ &\leq c'(\beta_1, \tilde{\beta}_N)^{-1} c(\beta_1, \tilde{\beta}_N)^{-k} \beta_1^{-\left(\sum_{j=1}^{k-1} (n_{j+1}-m_j)+i(m_k-n_k)\right)} \tilde{\beta}_N^{-\ell+2N}. \end{aligned}$$

Moreover, by Lemma 6.3,

$$|I_n^P| = |I_{m_k+i(m_k-n_k)+\ell}^P| \geq C(\beta_1) \beta_2^{-(m_k+i(m_k-n_k)+\ell+N)}.$$

Hence,

$$\begin{aligned} &\frac{\log \mu(I_n^P)}{\log |I_n^P|} \\ &\geq \frac{\left(\sum_{j=1}^{k-1} (n_{j+1} - m_j) + i(m_k - n_k)\right) \log \beta_1 + (\ell - 2N) \log \tilde{\beta}_N + k \log c(\beta_1, \tilde{\beta}_N) + \log c'(\beta_1, \tilde{\beta}_N)}{\log C(\beta_1) + (m_k + i(m_k - n_k) + \ell + N) \log \beta_2}. \end{aligned}$$

Case 3. $n = m_k + t_k(m_k - n_k) + \ell$ for some $0 \leq \ell \leq p_k$. Similarly, when $0 \leq \ell \leq 2N$, we have

$$\begin{aligned} \mu(I_n^P) &= \mu(I_{m_k+t_k(m_k-n_k)+\ell}^P) \leq \mu(I_{m_k}^P) \cdot \frac{1}{(\#M_{m_k-n_k})^{t_k}} \\ &\leq c'(\beta_1, \tilde{\beta}_N)^{-1} c(\beta_1, \tilde{\beta}_N)^{-k} \beta_1^{-\left(\sum_{j=1}^{k-1} (n_{j+1}-m_j)+t_k(m_k-n_k)\right)}. \end{aligned}$$

When $2N < \ell \leq p_k$, it follows that

$$\begin{aligned} \mu(I_n^P) &= \mu(I_{m_k+t_k(m_k-n_k)+\ell}^P) \leq \mu(I_{m_k}^P) \cdot \frac{1}{(\#M_{m_k-n_k})^{t_k}} \cdot \frac{1}{\sum_{\tilde{\beta}_N}^{\ell-2N}} \\ &\leq c'(\beta_1, \tilde{\beta}_N)^{-1} c(\beta_1, \tilde{\beta}_N)^{-k} \beta_1^{-\left(\sum_{j=1}^{k-1} (n_{j+1}-m_j)+i(m_k-n_k)\right)} \tilde{\beta}_N^{-\ell+2N}. \end{aligned}$$

Furthermore, we conclude from Lemma 6.3 that

$$|I_n^P| = |I_{m_k+t_k(m_k-n_k)+\ell}^P| \geq C(\beta_1)\beta_2^{-(m_k+t_k(m_k-n_k)+\ell+N)}.$$

Therefore, we have

$$\begin{aligned} & \frac{\log \mu(I_n^P)}{\log |I_n^P|} \\ & \geq \frac{\left(\sum_{j=1}^{k-1} (n_{j+1} - m_j) + t_k(m_k - n_k) \right) \log \beta_1 + (\ell - 2N) \log \tilde{\beta}_N + k \log c(\beta_1, \tilde{\beta}_N) + \log c'(\beta_1, \tilde{\beta}_N)}{\log C(\beta_1) + (m_k + t_k(m_k - n_k) + \ell + N) \log \beta_2}. \end{aligned}$$

Just proceeding as the same analysis in Section 3.2, for all the above three cases, we obtain

$$\liminf_{n \rightarrow \infty} \frac{\log \mu(I_n^P)}{\log |I_n^P|} \geq \left(1 - \frac{b^2(1-a)}{b-a} \right) \frac{\log \beta_1}{\log \beta_2}.$$

(3) Use the mass distribution principle (see [6, Page 60]). Now we take any $B(\beta, r)$ with center $\beta \in E(\beta_1, \beta_2)$ and sufficiently small enough r verifying

$$|I_{n+1}^P| \leq r < |I_n^P| \leq \beta_1^{-n+1}, \quad (6.27)$$

where the last inequality is guaranteed by the fact that $\bar{\beta}(\omega) \geq \beta_1$ for any $\omega \in \Lambda_n(\beta_1, \beta_2)$. By Lemma 6.3, we have

$$|I_n^P| \geq C(\beta_1)\beta_2^{-(n+N)}.$$

As a result, the ball $B(\beta, r)$ intersects no more than $2 \left[C(\beta_1)^{-1} \beta_2^N \left(\frac{\beta_2}{\beta_1} \right)^{n-1} \right] + 2$ cylinders of order n . Moreover, it follows from Lemma 6.3 that

$$r \geq |I_{n+1}^P| \geq C(\beta_1)\beta_2^{-(n+1+N)}. \quad (6.28)$$

Immediately, the combination of (6.27) and (6.28) gives

$$\begin{aligned} & \liminf_{r \rightarrow 0} \frac{\log \mu(B(\beta, r))}{\log r} \\ & \geq \liminf_{n \rightarrow \infty} \frac{\log \left(2 \left[C(\beta_1)^{-1} \beta_2^N \left(\frac{\beta_2}{\beta_1} \right)^{n-1} \right] + 2 \right) + \log \mu(I_n^P)}{\log |I_n^P|} \cdot \frac{\log |I_n^P|}{-\log C(\beta_1) + (n+1+N) \log \beta_2} \\ & \geq \liminf_{n \rightarrow \infty} \left(\frac{(n-1)(\log \beta_2 - \log \beta_1)}{-\log C(\beta_1) + (n+N) \log \beta_2} + \frac{\log \mu(I_n^P)}{\log |I_n^P|} \right) \cdot \frac{(n-1) \log \beta_1}{-\log C(\beta_1) + (n+1+N) \log \beta_2} \\ & \geq \left(\frac{\log \beta_2 - \log \beta_1}{\log \beta_2} + \left(1 - \frac{b^2(1-a)}{b-a} \right) \frac{\log \beta_1}{\log \beta_2} \right) \frac{\log \beta_1}{\log \beta_2}. \end{aligned}$$

Therefore, by the mass distribution principle and letting $\beta_1 \rightarrow \beta_2$, we get our desired result.

7 Proof of Theorem 1.6

Akin to Section 5, we need to find a subset of $E_{0,1}^P$ which is a dense G_δ set in the interval $[1, 2]$. Since the process of our proof is almost the same as Section 5. We only provide the construction of the required set V' in this section.

For all $k \geq 1$, we first choose the sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $m_k - n_k >$

$\max\{2(m_{k-1} - n_{k-1}), n_k - k\}$ and $n_k < m_k < n_{k+1}$. In addition, the sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ is chosen to satisfy

$$\lim_{k \rightarrow \infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = 0,$$

and

$$\lim_{k \rightarrow \infty} \frac{m_k - n_k}{m_k} = 1.$$

Actually, let

$$n'_k = k^{2k} \quad \text{and} \quad m'_k = (k+1)^{2k+1}.$$

We can obtain the required sequences with some adjustments.

For all $k \geq 1$, denote $n_{k+1} = (m_k - n_k)t_k + m_k + p_k$ where $0 \leq p_k < m_k - n_k$. We now define

$$V' = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{(\epsilon_1, \dots, \epsilon_k) \in \Lambda_k(1,2)} \text{int} \left(I_{n_{k+1}}^P (\epsilon_1, \dots, \epsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k})^{t_k}, 0^{p_k}) \right)$$

where $\Lambda_k(1, 2)$ is defined by (5.20). Since $m_k - n_k > n_k - k$, we get $(1, 0^{m_k-n_k}) <_{\text{lex}} (\epsilon_1, \dots, \epsilon_k, 0^{n_k-k})$ for all $k \geq 1$. By Lemma 5.1, the set V' is well defined.

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