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Diophantine approximation and run-length function on β-expansions

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Abstract. For any $\beta > 1$, denoted by $r_n(x,\beta)$ the maximal length of consecutive zeros amongst the first n digits of the β-expansion of $x \in [0,1)$. The limit superior (respectively limit inferior) of $\frac{r_n(x,\beta)}{n}$ is linked to the classical Diophantine approximation (respectively uniform Diophantine approximation). We obtain the Hausdorff dimension of the level set

$$E_{a,b} = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \right\} \ (0 \le a \le b \le 1).$$

Furthermore, we show that the extremely divergent set $E_{0,1}$ which is of zero Hausdorff dimension is, however, residual. The same problems in the parameter space are also examined.

Key words and phrases beta-expansion; Diophantine approximation; run-length function; Hausdorff dimension; residual

1 Introduction

Let $\beta > 1$ be a real number. The *β-transformation* on $[0,1)$ is defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \xi \rfloor$ means the integer part of ξ. It is well-known (see [17]) that, every real number $x \in [0,1)$ can be uniquely expanded as a series

$$x = \frac{\varepsilon_1(x,\beta)}{\beta} + \cdots + \frac{\varepsilon_n(x,\beta)}{\beta^n} + \cdots, \tag{1.1}$$

where $\varepsilon_n(x,\beta) = \lfloor \beta T_\beta^{n-1}(x) \rfloor$ for all $n \ge 1$. We call $\varepsilon_n(x,\beta)$ *the n-th digit of x* and $\varepsilon(x,\beta) := (\varepsilon_1(x,\beta), \ldots, \varepsilon_n(x,\beta), \ldots)$ the *β-expansion of x*.

For each $x \in [0,1)$ and $n \ge 1$, the *run-length function* $r_n(x,\beta)$ is defined to be the maximal length of consecutive zeros amongst the prefix $(\varepsilon_1(x,\beta), \ldots, \varepsilon_n(x,\beta))$, i.e.,

$$r_n(x,\beta) = \max\{1 \le j \le n : \varepsilon_{i+1}(x,\beta) = \cdots = \varepsilon_{i+j}(x,\beta) = 0 \text{ for some } 0 \le i \le n-j\}.$$

If such j does not exist, we set $r_n(x,\beta) = 0$. In 1970, Erdös and Rényi [5] showed that for Lebesgue almost all $x \in [0,1)$, we have

$$\lim_{n\to\infty} \frac{r_n(x,2)}{\log_2 n} = 1. \tag{1.2}$$

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The result of Erdös and Rényi [5] has been extended to the general case $\beta > 1$ by Tong, Yu and Zhao [20]. Ma, Wen and Wen [13] showed that the exceptional set of points violating (1.2) is of full Hausdorff dimension. Let $\mathcal{E}$ denote the set of increasing functions $\varphi : \mathbb{N} \to (0, +\infty)$ satisfying $\lim\limits_{n\to\infty} \varphi(n) = +\infty$ and $\limsup\limits_{n\to\infty} \frac{\varphi(n)}{n} = 0$. For every $0 \le a \le b \le \infty$ and any function $\varphi \in \mathcal{E}$, define

$$E^{\varphi}_{a,b} := E^{\varphi}_{a,b}(\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = a,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = b \right\}.$$

The set $E^{\varphi}_{a,b}$ has been proved to have full Hausdorff dimension by Li and Wu (see [10, 11]) for the case $\beta = 2$ and by Zheng, Wu and Li [22] for the general case $\beta > 1$.

Remark that the above $\mathcal{E}$ does not contain the function $\varphi(n) = n$. In fact, the asymptotic behavior of $\frac{r_n(x,\beta)}{n}$ is directly related to the Diophantine approximation of β-expansions. For all $x \in [0,1)$, Amou and Bugeaud [1] defined the exponent $v_\beta(x)$ to be the supremum of the real numbers v for which the equation

$$T^n_\beta x \le \beta^{-nv}$$

has infinitely many positive solutions with integer n. Bugeaud and Liao [7] defined the exponent $\hat{v}_\beta(x)$ to be the supremum of the real numbers $\hat{v}$ for which, for all $N \gg 1$, there is a solution with $1 \le n \le N$, such that

$$T^n_\beta x \le \beta^{-N\hat{v}}.$$

We will see (Lemmas 3.1 and 3.2) that for all $0 < a < 1,\ 0 < b < 1$,

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a \ \Leftrightarrow\ \hat{v}_\beta(x) = \frac{a}{1-a} \quad \text{and} \quad \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \ \Leftrightarrow\ v_\beta(x) = \frac{b}{1-b}.$$

For all $0 \le a \le b \le 1$, let

$$E_{a,b} := E_{a,b}(\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \right\}. \tag{1.3}$$

Denote the Hausdorff dimension by $\dim_{\mathrm{H}}$. For more information about the Hausdorff dimension, we refer to [6]. We establish the following theorem.

Theorem 1.1 *The set $E_{0,0}$ has full Lebesgue measure. If $\frac{b}{1+b} < a \le 1,\ 0 < b \le 1$, then $E_{a,b} = \emptyset$. Otherwise, we have*

$$\dim_{\mathrm{H}} E_{a,b} = 1 - \frac{b^2(1-a)}{b-a}.$$

Let $0 \le a \le 1$ and $0 \le b \le 1$. We can further study the level sets

$$E_a := E_a(\beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a \right\}$$

and

$$F_b := F_b(\beta) = \left\{ x \in [0,1) : \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b \right\}. \tag{1.4}$$

Using Theorem 1.1, we obtain the following results of the Hausdorff dimensions of E_a and F_b.

Corollary 1.2 *(1) When $0 \le a \le \frac{1}{2}$, we have*

$$\dim_{\mathrm{H}} E_a = (1-2a)^2.$$

Otherwise, $E_a = \emptyset$.

(2) For all $0 \le b \le 1$, we have

$$\dim_{\mathrm{H}} F_b = 1 - b.$$

We remak that the statement (2) of Theorem 1.2 was also been obtained in [9, Theorem 1.1] (see [21] for the case $\beta = 2$).

A set R is called *residual* if its complement is meager (i.e., of the first category). In a complete metric space, a set is residual if it contains a dense G_δ set, i.e., a countable intersection of open dense sets (see [14]). Similar to the results of [10, 11, 22], the set of extremely divergent points is residual, and thus is large in the sense of topology.

Theorem 1.3 *The set $E_{0,1}$ is residual in $[0,1]$.*

It is worth noting that the set $E_{0,1}$ is negligible with respect to the Lebesgue measure and Hausdorff dimension. However, the sets considered in [10, 11, 22] have Hausdorff dimension 1. Let $\varphi \in \mathcal{E}$. Since the intersection of two residual sets is still residual, by combining Theorems 1.3 and [22, Theorem 1.2], we deduce that the smaller set

$$E := E(\varphi, \beta) = \left\{ x \in [0,1) : \liminf_{n\to\infty} \frac{r_n(x,\beta)}{\varphi(n)} = 0,\ \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = 1 \right\}$$

is also residual in $[0,1]$.

The β-expansion of 1 completely characterizes all of the admissible words in the β-dynamical system (see Theorem 2.1 in Section 2 for more details). We also study the run-length function $r_n(\beta)$ of the β-expansion of 1 as β varies in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$, i.e.,

$$r_n(\beta) = \max\{1 \le j \le n : \varepsilon_{i+1}(\beta) = \cdots = \varepsilon_{i+j}(\beta) = 0 \text{ for some } 0 \le i \le n-j\}.$$

There are some results on $r_n(\beta)$ which are similar to those of $r_n(x,\beta)$. In [8], Hu, Tong and Yu proved that for Lebesgue almost all $1 < \beta < 2$, we have

$$\lim_{n\to\infty} \frac{r_n(\beta)}{\log_\beta n} = 1. \tag{1.5}$$

Cao and Chen [4] showed that for any $\varphi \in \mathcal{E}$ and for all $0 \le a \le b \le +\infty$, the set

$$\left\{ \beta \in (1,2) : \liminf_{n\to\infty} \frac{r_n(\beta)}{\varphi(n)} = a,\ \limsup_{n\to\infty} \frac{r_n(\beta)}{\varphi(n)} = b \right\}$$

is of full Hausdorff dimension. Remark that the results of [8] and [4] can be easily generalized to the whole parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$. For simplicity, in this paper, we will also consider the parameter space $(1,2)$. For all $0 \le a \le b \le 1$, let

$$E^P_{a,b} = \left\{ \beta \in (1,2) : \liminf_{n\to\infty} \frac{r_n(\beta)}{n} = a,\ \limsup_{n\to\infty} \frac{r_n(\beta)}{n} = b \right\}. \tag{1.6}$$

We have the following theorem.

Theorem 1.4 *The set $E^P_{0,0}$ has full Lebesgue measure. If $\frac{b}{1+b} < a \le 1,\ 0 < b \le 1$, then $E^P_{a,b} = \emptyset$. Otherwise, we have*

$$\dim_{\mathrm{H}} E^P_{a,b} = 1 - \frac{b^2(1-a)}{b-a}.$$

Similarly, for every $0 \le a \le 1$ and $0 \le b \le 1$, we consider the set

$$E^P_a = \left\{ \beta \in (1,2) : \liminf_{n\to\infty} \frac{r_n(\beta)}{n} = a \right\},$$

and

$$F^P_b = \left\{ \beta \in (1,2) : \limsup_{n\to\infty} \frac{r_n(\beta)}{n} = b \right\}.$$

Corollary 1.5 *(1) When* $0 \le a \le \frac{1}{2}$*, we have*

$$\dim_{\mathrm{H}} E_a^P = (1-2a)^2.$$

Otherwise, $E_a = \emptyset$.

(2) For every $0 \le b \le 1$*, we have*

$$\dim_{\mathrm{H}} F_b^P = 1 - b.$$

In addition, similar to Theorem 1.3, we have the following theorem.

Theorem 1.6 *The set* $E_{0,1}^P$ *is residual in* $[1,2]$.

We end this introduction by depicting the organization of our paper. In Section 2, we review some standard facts on the β-expansions without proofs. Theorem 1.1 and corollary 1.2 are proved in Section 3. We give the proof of Theorem 1.3 in Section 4. Section 5 contains a summary of some classical results of β-expansion in the parameter space. The proofs of Theorems 1.4 and 1.6 are given in Sections 6 and 7 respectively.

2 Fundamental results of β-expansion

Throughout this section, we set up some notations and terminologies on β-expansions. Meanwhile, we give some basic results on β-expansion directly. For more properties on β-expansions, we refer the readers to [3, 7, 15, 17].

Let $\mathcal{A} = \{0, 1, \cdots, \lceil\beta\rceil - 1\}$ where $\lceil\xi\rceil$ stands for the smallest integer larger than ξ. The definition of β-expansion gives the fact that every digit $\varepsilon_n(x,\beta)$ lies in the set $\mathcal{A}$. A word $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A}^n$ is called *admissible* with respect to β if there exists an $x \in [0,1)$ such that the β-expansion of x begins with $(\varepsilon_1, \ldots, \varepsilon_n)$. Similarly, an infinite sequence $(\varepsilon_1, \ldots, \varepsilon_n, \ldots)$ is called *admissible* with respect to β if there exists an $x \in [0,1)$ whose β-expansion is $(\varepsilon_1, \ldots, \varepsilon_n, \ldots)$. Denote by Σ_β^n the set of all β-admissible words of length n, i.e.,

$$\Sigma_\beta^n = \{(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{A}^n : \exists\ x \in [0,1),\ \text{s.t. } \varepsilon_j(x,\beta) = \varepsilon_j,\ \forall\ 1 \le j \le n\}.$$

Denote by Σ_β^* the set of all β-admissible words of finite length, i.e., $\Sigma_\beta^* = \bigcup_{n=0}^{\infty} \Sigma_\beta^n$. The set of β-admissible sequences is denoted by Σ_β , i.e.,

$$\Sigma_\beta = \{(\varepsilon_1, \varepsilon_2, \ldots) \in \mathcal{A}^{\mathbb{N}} : \exists\ x \in [0,1),\ \text{s.t. } \varepsilon(x,\beta) = (\varepsilon_1, \varepsilon_2, \ldots)\}.$$

The β-expansion of the unit 1 plays an important role in the research of admissible words and admissible sequences. By extending the definition of $T_\beta(x) = \beta x - \lfloor \beta x \rfloor$ to $x = 1$, the β-expansion of 1 is given by

$$1 = \frac{\varepsilon_1(1,\beta)}{\beta} + \cdots + \frac{\varepsilon_n(1,\beta)}{\beta^n} + \cdots.$$

where $\varepsilon_n(1,\beta) = \lfloor \beta T_\beta^{n-1} 1 \rfloor$. We call β a *simple Parry number* if the β-expansion of 1 is finite. That is, there exists an integer $m \ge 1$ such that $\varepsilon_m \ne 0$ and $\varepsilon_k(\beta) = 0$ for every $k > m$. In this case, we let

$$\varepsilon^*(\beta) := (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots) = (\varepsilon_1(\beta), \varepsilon_2(\beta), \ldots, \varepsilon_m(\beta) - 1)^\infty,$$

where ω^∞ is the infinite periodic sequence $(\omega, \omega, \ldots)$. If the β-expansion of 1 is not finite, let

$\varepsilon^*(\beta) = \varepsilon(1,\beta)$. In both cases, we can check that

$$1 = \frac{\varepsilon_1^*(\beta)}{\beta} + \cdots + \frac{\varepsilon_n^*(\beta)}{\beta^n} + \cdots .$$

The sequence $\varepsilon^*(\beta)$ is therefore called *the infinite β-expansion of* 1.

We endow the space $\mathcal{A}^{\mathbb{N}}$ with the *lexicographical order* $<_{\text{lex}}$:

$$(\omega_1, \omega_2, \ldots) <_{\text{lex}} (\omega_1', \omega_2', \ldots)$$

if $\omega_1 < \omega_1'$ or there exists an integer $j > 1$, such that, for all $1 \le k < j$, $\omega_k = \omega_k'$ but $\omega_j < \omega_j'$. The symbol $\le_{\text{lex}}$ means $=$ or $<_{\text{lex}}$. Moreover, for all $n, m \ge 1$, $(\omega_1, \ldots, \omega_n) <_{\text{lex}} (\omega_1', \ldots, \omega_m')$ stands for $(\omega_1, \ldots, \omega_n, 0^\infty) <_{\text{lex}} (\omega_1', \ldots, \omega_m', 0^\infty)$.

The following theorem due to Parry [15] yields that the β-dynamical system is totally determined by the infinite β-expansion of 1. Let $\omega = (\omega_1, \ldots, \omega_n) \in \mathcal{A}^n$ for all $n \ge 1$. Let σ be the shift transformation such that $\sigma\omega = (\omega_2, \ldots, \omega_n)$.

Theorem 2.1 (Parry [15]) *Let $\beta > 1$.*

(1) For every $n \ge 1$, $\omega = (\omega_1, \ldots, \omega_n) \in \Sigma_\beta^n$ if and only if $\sigma^j\omega \le_{\text{lex}} (\varepsilon_1^*, \ldots, \varepsilon_{n-j}^*)$ for all $0 \le j < n$.

(2) For all $k \ge 1$, $\sigma^k\varepsilon(1,\beta) <_{\text{lex}} \varepsilon(1,\beta)$.

(3) For each $1 < \beta_1 < \beta_2$, it holds that $\varepsilon^*(\beta_1) <_{\text{lex}} \varepsilon^*(\beta_2)$. Consequently, for every $n \ge 1$, we have

$$\Sigma_{\beta_1}^n \subseteq \Sigma_{\beta_2}^n \quad \text{and} \quad \Sigma_{\beta_1} \subseteq \Sigma_{\beta_2}.$$

The estimation of the cardinality of the set Σ_β^n was given by Rényi [17]. We will use the symbol $\sharp$ to denote the cardinality of a finite set in the remainder of this paper.

Theorem 2.2 (Rényi [17]) *For all $n \ge 1$,*

$$\beta^n \le \sharp\Sigma_\beta^n \le \frac{\beta^{n+1}}{\beta - 1}.$$

For an admissible word $\omega = (\omega_1, \ldots, \omega_n)$, the associated *cylinder* of order n is defined by

$$I_n(\omega) := I_n(\omega, \beta) = \{x \in [0,1) : \varepsilon_j(x,\beta) = \omega_j, \text{ for all } 1 \le j \le n\}.$$

The cylinder $I_n(\omega)$ is a left-closed and right-open interval (see [7, Lemma 2.3]). Denote by $|I_n(\omega)|$ the length of $I_n(\omega)$. We immediately get $|I_n(\omega)| \le \beta^{-n}$. We write $I_n(x,\beta)$ as the cylinder of order n containing the point $x \in [0,1)$ and write $|I_n(x,\beta)|$ as its length. For simplicity, $I_n(x)$ means $I_n(x,\beta)$ in the rest of this paper without otherwise specified. A cylinder of order n is called *full* if $|I_n(\omega)| = \beta^{-n}$ and the corresponding word of the full cylinder is said to be *full*.

Now we give some characterizations and properties of full cylinders.

Theorem 2.3 (Fan and Wang [7]) *For any integer $n \ge 1$, let $\omega = (\omega_1, \ldots, \omega_n)$ be an admissible word.*

(1)The cylinder $I_n(\omega)$ is full if and only if $T_\beta^n(I_n(\omega)) = [0,1)$, if and only if for any $m \ge 1$ and $\omega' = (\omega_1', \ldots, \omega_m') \in \Sigma_\beta^m$, the concatenation $\omega * \omega' = (\omega_1, \ldots, \omega_n, \omega_1', \ldots, \omega_m')$ is still admissible.

(2) If $(\omega_1, \ldots, \omega_{n-1}, \omega_n')$ with $\omega_n' > 0$ is admissible, then the cylinder $I_n(\omega_1, \ldots, \omega_{n-1}, \omega_n)$ is full for every $0 \le \omega_n < \omega_n'$.

(3) If $I_n(\omega)$ is full, then for any $(\omega_1', \ldots, \omega_m') \in \Sigma_\beta^m$, we have

$$|I_{n+m}(\omega_1, \ldots, \omega_n, \omega_1', \ldots, \omega_m')| = \beta^{-n} \cdot |I_m(\omega_1', \ldots, \omega_m')|.$$

In order to construct full words, we introduce a variable Γ_n which is defined as follows. Recall that the infinite β-expansion of 1 is $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots)$. For every integer $n \geq 1$, define

$$t_n = t_n(\beta) := \max\{k \geq 1 : \varepsilon_{n+1}^*(\beta) = \cdots = \varepsilon_{n+k}^*(\beta) = 0\}.$$

If such k does not exist, let $t_n = 0$. Now let

$$\Gamma_n = \Gamma_n(\beta) := \max_{1 \leq k \leq n} t_k(\beta). \tag{2.7}$$

Then we can check that Γ_n is a finite integer for all $n \geq 1$. Theorem 2.3 implies the following results which are important for construction of full words.

Proposition 2.4 (Fan and Wang [7]) *(1) If both admissible words $(\omega_1, \ldots, \omega_n)$ and $(\omega_1', \ldots, \omega_m')$ are full, then the concatenation word $(\omega_1, \ldots, \omega_n, \omega_1', \ldots, \omega_m')$ is still full.*

(2)For all $\ell \geq 1$, the word $0^\ell := (\underbrace{0, \ldots, 0}_{\ell})$ is full. For any full word $(\omega_1, \ldots, \omega_n)$, the word $(\omega_1, \ldots, \omega_n, 0^\ell)$ is also full.

(3) For any admissible word $(\omega_1, \ldots, \omega_n)$, the word $(\omega_1, \ldots, \omega_n, 0^{\Gamma_n+1})$ is full.

Furthermore, Bugeaud and Wang [3] provided the following modified mass distribution principle which is of great importance in estimating the lower bound of the Hausdorff dimension of $E_{a,b}$.

Theorem 2.5 (Bugeaud and Wang [3]) *Let μ be a Borel measure and E be a Borel measurable set with $\mu(E) > 0$. Assume that there exist a constant $c > 0$ and an integer $N \geq 1$ such that for all $n \geq N$ and each cylinder I_n, the inequality $\mu(I_n) \leq c|I_n|^s$ is valid. Then, $\dim_{\mathrm{H}} E \geq s$.*

Now we will introduce some results on Diophantine approximation. We fist give the following exponents of approximation.

Shen and Wang [19] obtained the following theorem which gives the dimensional results of the set of points with classical Diophantine property.

Theorem 2.6 (Shen and Wang [19]) *Let $\beta > 1$. Let $0 \leq v \leq +\infty$. Then*

$$\dim_{\mathrm{H}}\{x \in [0,1) : v_\beta(x) \geq v\} = \frac{1}{1+v}.$$

Bugeaud and Liao [2] studied the set of points with uniform Diophantine properties and established the theorem as follows.

Theorem 2.7 (Bugeaud and Liao [2]) *Let $\beta > 1$. Let $0 < \hat{v} < 1$ and $v > 0$. If $v < \frac{\hat{v}}{1-\hat{v}}$, then the set*

$$U_\beta(\hat{v}, v) := \{x \in [0,1) : \hat{v}_\beta(x) = \hat{v},\ v_\beta(x) = v\}$$

is empty. Otherwise, we have

$$\dim_{\mathrm{H}} U_\beta(\hat{v}, v) = \frac{v - (1+v)\hat{v}}{(1+v)(v-\hat{v})}.$$

Moreover,

$$\dim_{\mathrm{H}}\{x \in [0,1) : \hat{v}_\beta(x) = \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}}\right)^2.$$

3 Proofs of Theorem 1.1 and Corollary 1.2

Notice that for all $\beta > 1$, we have

$$\left\{x \in [0,1) : \lim_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = 1\right\} \subseteq E_{0,0}.$$

In [20], Tong, Yu and Zhao showed that the set $\left\{x \in [0,1) : \lim_{n\to\infty} \frac{r_n(x,\beta)}{\log_\beta n} = 1\right\}$ is of full Lebesgue measure. As a result, the set $E_{0,0}$ has full Lebesgue measure. Hence, we only need to study the case that $0 \le a \le 1,\ 0 < b \le 1$. Before we give the proof of Theorem 1.1, we uncover the relationship between run-length function and Diophantine approximation.

3.1 Run-length function and Diophantine approximation

Lemma 3.1 *Let $\beta > 1$. For all $x \in [0,1)$, for any $0 < a < 1$, we have $\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a$ if and only if $\hat{v}_\beta(x) = \frac{a}{1-a}$.*

Proof. It suffices to show the following two cases. The details are left to the readers.

On the one hand, suppose that $\hat{v}_\beta(x) < \frac{a}{1-a}$, then we have $v_0 = \frac{a}{2(1-a)} + \frac{\hat{v}_\beta(x)}{2} > \hat{v}_\beta(x)$. By the definition of $\hat{v}_\beta(x)$, there is a sequence $\{n_k\}_{k=1}^{\infty}$ such that, for all $1 \le n \le n_k$,

$$T_\beta^n x > \beta^{-v_0 n} \ge \beta^{-(\lfloor v_0 n \rfloor + 1)}.$$

So it holds that

$$r_{n_k + \lfloor v_0 n_k \rfloor}(x,\beta) < \lfloor v_0 n_k \rfloor + 1.$$

This implies that

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} \le \lim_{k\to\infty} \frac{r_{n_k+\lfloor v_0 n_k\rfloor}(x,\beta)}{n_k + \lfloor v_0 n_k \rfloor} \le \lim_{k\to\infty} \frac{\lfloor v_0 n_k \rfloor + 1}{n_k + \lfloor v_0 n_k \rfloor} = \frac{v_0}{1+v_0} = \frac{2a + \hat{v}_\beta(x)(1-a) - a}{2 + \hat{v}_\beta(x)(1-a) - a} < a,$$

where the last inequality follows from

$$\frac{a-x}{b-x} < \frac{a}{b},\ \text{for all } 0 \le a < b,\ x > 0.$$

On the other hand, suppose that $\hat{v}_\beta(x) > \frac{a}{1-a}$, then $v_0 = \frac{a}{2(1-a)} + \frac{\hat{v}_\beta(x)}{2} < \hat{v}_\beta(x)$. The definition of $\hat{v}_\beta(x)$ implies that for all $N \gg 1$, there exists $1 \le n \le N$, such that

$$T_\beta^n x \le \beta^{-v_0 N}.$$

Then for all $k = N + \lfloor v_0 N \rfloor + 1 \gg 1$, we have

$$r_k(x,\beta) \ge \lfloor v_0 N \rfloor.$$

This implies that

$$\liminf_{k\to\infty} \frac{r_k(x,\beta)}{k} \ge \lim_{N\to\infty} \frac{\lfloor v_0 N \rfloor}{N + \lfloor v_0 N \rfloor + 1} = \frac{v_0}{1+v_0} = \frac{2a + \hat{v}_\beta(x)(1-a) - a}{2 + \hat{v}_\beta(x)(1-a) - a} > a,$$

where the last inequality follows from

$$\frac{a+x}{b+x} > \frac{a}{b} \text{ for all } 0 \le a < b,\ x > 0. \tag{3.8}$$

This contradicts with $\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a$. Consequently, $\hat{v}_\beta(x) \le \frac{a}{1-a}$.

Thus, we conclude that $\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = a$. □

Lemma 3.2 *Let $\beta > 1$. For all $x \in [0,1)$, for each $0 < b < 1$, we have $\limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b$ if and only if $v_\beta(x) = \frac{b}{1-b}$.*

Proof. It can be deduced by the same arguments as the proof of Lemma 3.1. □

Now we can give part of the proof of Theorem 1.1.

We will first show when $a > \frac{b}{1+b},\ 0 < b \le 1$, $E_{a,b} = \emptyset$. In fact, if $\limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = b$, then for all $\delta > 0$, there exits a sequence $\{n_k\}_{k=1}^{\infty}$ such that $r_{n_k}(x,\beta) \le \lfloor (b+\delta)n_k \rfloor$ and $\varepsilon_{n_k}(x,\beta) > 0$. Thus, when we consider the prefix at the position $n_k + \lfloor bn_k \rfloor$, there are at most $\lfloor (b+\delta)n_k \rfloor$ consecutive 0's. Immediately, $r_{n_k+\lfloor bn_k \rfloor}(x,\beta) \le \lfloor (b+\delta)n_k \rfloor$. Hence,

$$a = \liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} \le \lim_{k\to\infty} \frac{r_{n_k+\lfloor bn_k \rfloor}(x,\beta)}{n_k + \lfloor bn_k \rfloor} \le \lim_{k\to\infty} \frac{(b+\delta)n_k}{n_k + \lfloor bn_k \rfloor} = \frac{b+\delta}{1+b}.$$

Letting $\delta \to 0$, we have

$$a \le \frac{b}{1+b}. \tag{3.9}$$

Therefore, $E_{a,b}$ is empty when $a > \frac{b}{1+b},\ 0 < b \le 1$.

When $0 < a \le \frac{b}{1+b},\ 0 < b < 1$, Lemmas 3.1 and 3.2 give the fact that the sets we consider here are essentially the same as the sets studied in Bugeaud and Liao [2], that is

$$E_{a,b} = \left\{ x \in [0,1) : \hat{v}_\beta(x) = \frac{a}{1-a},\ v_\beta(x) = \frac{b}{1-b} \right\} = U_\beta\left(\frac{a}{1-a}, \frac{b}{1-b} \right).$$

Consequently, we can apply Theorem 2.7 to obtain

$$\dim_{\mathrm{H}} E_{a,b} = \dim_{\mathrm{H}} U_\beta\left(\frac{a}{1-a}, \frac{b}{1-b} \right) = 1 - \frac{b^2(1-a)}{b-a}.$$

However, Theorem 2.7 cannot be applied for the cases $a = 0,\ 0 < b < 1$ and $0 < a \le \frac{1}{2},\ b = 1$. Remark that $E_{a,1} \subseteq F_1$ where F_1 is defined by (1.4) and $\dim_{\mathrm{H}} F_1 = 0$ by [9, Theorem 1.1]. So $\dim_{\mathrm{H}} E_{a,1} = 0$ $(0 < a \le \frac{1}{2})$ and there is nothing to prove. For the other case, we have

$$E_{0,b} \subseteq \left\{ x \in [0,1) : v_\beta(x) \ge \frac{b}{1-b} \right\}.$$

Then we can use Theorem 2.6 to obtain the upper bound of $\dim_{\mathrm{H}} E_{0,b}$ which is $1-b$ for all $0 < b < 1$. Hence it remains to give the lower bound of $\dim_{\mathrm{H}} E_{0,b}$ for all $0 < b < 1$.

3.2 Lower bound of $\dim_{\mathrm{H}} E_{0,b}$ $(0 < b < 1)$

Now we give the lower bound of $\dim_{\mathrm{H}} E_{0,b}$ for the case $0 < b < 1$. In fact, we can also include the proof of the lower bound of $\dim_{\mathrm{H}} E_{a,b}$ for the case $0 < a \le \frac{b}{1+b},\ 0 < b < 1$, though the later case has already been given in the end of Section 3.1.

Let $\beta > 1$. Recall that the infinite β-expansion of 1 is $\varepsilon^*(\beta) = (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots)$. We will apply the approximation of β to construct the Cantor subset as follows. For all N with $\varepsilon_N^* > 0$, let $\beta_N > 1$ be the unique solution of the equation:

$$1 = \frac{\varepsilon_1^*(\beta)}{x} + \cdots + \frac{\varepsilon_N^*(\beta)}{x^N}.$$

Then

$$\varepsilon^*(\beta_N) = (\varepsilon_1^*(\beta), \ldots, \varepsilon_N^*(\beta) - 1)^\infty.$$

Hence $1 < \beta_N < \beta$ and β_N is increasing to β as N goes to infinity. The number β_N is called *an approximation of* β. Moreover, by Theorem 2.1(3), $\Sigma_{\beta_N}^n \subseteq \Sigma_\beta^n$ for all $n \geq 1$ and $\Sigma_{\beta_N} \subseteq \Sigma_\beta$. We therefore have the following facts.

Proposition 3.3 (Shen and Wang [19]) *For every $\omega \in \Sigma_{\beta_N}^n$, when regarding ω as an element of Σ_β^n, we have*

$$\beta^{-(n+N)} \leq |I_n(\omega, \beta)| \leq \beta^{-n}. \tag{3.10}$$

Moreover, every $\omega \in \Sigma_{\beta_N}^n (n \geq N)$ end with 0^N is full when regarding ω as an element of Σ_β^n.

For all $k \geq 1$ and $N > 1$ with $\varepsilon_N^*(\beta) > 0$, choose two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ which satisfy $n_k < m_k < n_{k+1}$ with $n_1 > 2N$, and $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$. Moreover, $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ can be chosen to satisfy

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = a \tag{3.11}$$

and

$$\lim_{k\to\infty} \frac{m_k - n_k}{m_k} = b. \tag{3.12}$$

In fact, such sequences do exist by the following arguments.

(1) If $0 < a \leq \frac{b}{1+b},\ 0 < b < 1$, let

$$n_k' = \left\lfloor \left(\frac{b(1-a)}{a(1-b)} \right)^k \right\rfloor \quad \text{and} \quad m_k' = \left\lfloor \frac{1}{1-b} \left(\frac{b(1-a)}{a(1-b)} \right)^k \right\rfloor.$$

Note that $a < b$, so $\frac{b(1-a)}{a(1-b)} > 1$. Then both sequences $\{n_k'\}_{k=1}^\infty$ and $\{m_k'\}_{k=1}^\infty$ are increasing to infinity as k tends to infinity. A small adjustment can attain the required sequences.

(2) If $a = 0,\ 0 < b < 1$, let

$$n_k' = k^k \quad \text{and} \quad m_k' = \left\lfloor \frac{1}{1-b} k^k \right\rfloor.$$

We can adjust these sequences to make sure that $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$.

Now let us construct a Cantor subset of $E_{a,b}$.

For all $d > 2N$, let

$$\mathcal{M}_d = \{\omega = (1, 0^{N-1}, \omega_1, \ldots, \omega_{d-2N}, 0^N) : (\omega_1, \ldots, \omega_{d-N}) \in \Sigma_{\beta_N}^{d-2N}\}. \tag{3.13}$$

Remark that $(1, 0^{N-1}, \omega_1, \ldots, \omega_{d-N}, 0^N) \in \Sigma_{\beta_N}^d$ ends with 0^N. Thus, by Proposition 3.3, every word belonging to $\mathcal{M}_d$ is full when regarding it as an element of Σ_β^d. Now let $G_1 = \{\omega : \omega \in \mathcal{M}_{n_1}\}$. Next, for all $k \geq 1$, let $n_{k+1} = (m_k - n_k)t_k + m_k + p_k$ where $0 \leq p_k < m_k - n_k$. Define

$$G_{k+1} = \{u_{k+1} = (1, 0^{m_k - n_k - 1}, u_k^{(1)}, \ldots, u_k^{(t_k)}, u_k^{(t_k+1)}) :\ u_k^{(i)} \in \mathcal{M}_{m_k - n_k} \text{ for all } 1 \leq i \leq t_k\}$$

where

$$u_k^{(t_k+1)} = \begin{cases} 0^{p_k}, & \text{when } \ p_k \leq 2N; \\ \omega \in \mathcal{M}_{p_k}, & \text{when } \ p_k > 2N. \end{cases}$$

It follows from Propositions 3.3 and 2.4(2) that every $u_k \in G_k$ is full. Hence, we can define the set D_k as:

$$D_k = \{(u_1, \ldots, u_k) : u_i \in G_i, \text{ for all } 1 \leq i \leq k\}. \tag{3.14}$$

Notice that the length of $u_k \in G_k$ satisfies $|u_k| = n_k - n_{k-1}$. For each $u = (u_1, \ldots, u_k) \in D_k$, we have

$$|u| = |u_1| + |u_2| + \cdots + |u_k| = n_1 + (n_2 - n_1) + \cdots + (n_k - n_{k-1}) = n_k.$$

Define

$$E_N = \bigcap_{k=1}^{\infty} \bigcup_{u \in D_k} I_{n_k}(u).$$

The following lemma shows that E_N is a subset of $E_{a,b}$.

Lemma 3.4 *We have $E_N \subseteq E_{a,b}$ for every $0 \le a \le \frac{b}{1+b}$ and $0 < b < 1$.*

Proof. For every integer $n \ge 1$, there exists a $k \ge 1$ such that $n_k < n \le n_{k+1}$. We distinguish three cases.

(1) If $n_k < n \le n_k + m_{k-1} - n_{k-1} + 2N$, we have $m_{k-1} - n_{k-1} - 1 \le r_n(x, \beta) \le m_{k-1} - n_{k-1} + 2N$ by the construction of E_N. It follows that

$$\frac{m_{k-1} - n_{k-1} - 1}{n_k + m_{k-1} - n_{k-1} + 2N} \le \frac{r_n(x, \beta)}{n} \le \frac{m_{k-1} - n_{k-1} + 2N}{n_k}.$$

(2) If $n_k + m_{k-1} - n_{k-1} + 2N < n \le m_k$, the construction of E_N gives $r_n(x, \beta) = n - n_k$. By (3.8), we have

$$\frac{m_{k-1} - n_{k-1} + 2N}{n_k + m_{k-1} - n_{k-1} + 2N} \le \frac{r_n(x, \beta)}{n} \le \frac{m_k - n_k}{m_k}.$$

(3) If $m_k \le n \le n_{k+1}$, we deduce from the construction of E_N that $m_k - n_k - 1 \le r_n(x, \beta) \le m_k - n_k + 2N$. Consequently,

$$\frac{m_k - n_k - 1}{n_{k+1}} \le \frac{r_n(x, \beta)}{n} \le \frac{m_k - n_k + 2N}{m_k}.$$

Combining the above three cases, by (3.11) and (3.12), we have

$$\liminf_{n \to \infty} \frac{r_n(x, \beta)}{n} \ge a \quad \text{and} \quad \limsup_{n \to \infty} \frac{r_n(x, \beta)}{n} \le b.$$

Now we complete our proof by finding the subsequences such that the limit inferior and limit superior are reached. In fact, by (3.11), we get

$$\lim_{k \to \infty} \frac{r_{n_k + m_{k-1} - n_{k-1}}}{n_k + m_{k-1} - n_{k-1}} \le \lim_{k \to \infty} \frac{m_{k-1} - n_{k-1} + 2N}{n_k + m_{k-1} - n_{k-1}} = a.$$

It follows from (3.12) that

$$\lim_{k \to \infty} \frac{r_{m_k}}{m_k} = \lim_{k \to \infty} \frac{m_k - n_k - 1}{m_k} = b.$$

□

Now we estimate the cardinality of the set D_k defined by (3.14). Write $q_k := \sharp D_k$.

Lemma 3.5 *Let $\beta > 1$. Let β_N be an approximation of β. For every $\overline{\beta} < \beta_N$, there exist an integer $k(\overline{\beta}, \beta_N)$ and real numbers $c(\overline{\beta}, \beta_N)$, $c'(\overline{\beta}, \beta_N)$ such that, for all $k \ge k(\overline{\beta}, \beta_N)$, we have*

$$q_k \ge c'(\overline{\beta}, \beta_N) c(\overline{\beta}, \beta_N)^k \overline{\beta}^{\sum_{i=1}^{k-1} (n_{i+1} - m_i)}. \tag{3.15}$$

Proof. Recall the definition of $\mathcal{M}_d$ as (3.13). Theorem 2.2 implies

$$\sharp \mathcal{M}_d \ge \beta_N^{d-2N}$$

for all $d \geq N$. Since $\overline{\beta} < \beta_N$, there exists an integer d' which depends on $\overline{\beta}$ and β_N such that, for every $d \geq d'$, we have

$$\beta_N^{d-2N} \geq \overline{\beta}^d. \tag{3.16}$$

Moreover, the fact that $m_k - n_k$ is increasing and tends to $+\infty$ as $k \to +\infty$ ensures that we can find a large enough integer $k(\overline{\beta}, \beta_N)$ satisfying that, for all $k \geq k(\overline{\beta}, \beta_N)$,

$$\sharp\mathcal{M}_{m_k-n_k} \geq \beta_N^{m_k-n_k-2N} \geq \overline{\beta}^{m_k-n_k}. \tag{3.17}$$

Then, when $p_k \leq 2N$, we have

$$\sharp G_{k+1} \geq (\sharp\mathcal{M}_{m_k-n_k})^{t_k} \geq \overline{\beta}^{(m_k-n_k)t_k} \geq \frac{1}{\overline{\beta}^{2N}}\overline{\beta}^{n_{k+1}-m_k}.$$

When $p_k > 2N$, we deduce that

$$\sharp G_{k+1} \geq (\sharp\mathcal{M}_{m_k-n_k})^{t_k} \cdot \sharp\mathcal{M}_{p_k} \geq \overline{\beta}^{(m_k-n_k)t_k} \cdot \beta_N^{p_k-2N} = \frac{1}{{\beta_N}^{d'}}\overline{\beta}^{(m_k-n_k)t_k}{\beta_N}^{p_k-2N+d'}.$$

Note that $p_k - 2N + d' > d'$. By (3.16), we have

$$\sharp G_{k+1} \geq \frac{1}{{\beta_N}^{d'}}\overline{\beta}^{(m_k-n_k)t_k}\overline{\beta}^{p_k+d'} = \frac{\overline{\beta}^{d'}}{{\beta_N}^{d'}}\overline{\beta}^{n_{k+1}-m_k}.$$

Let $c(\overline{\beta}, \beta_N) := \min\{\frac{1}{\overline{\beta}^{2N}}, \frac{\overline{\beta}^{d'}}{{\beta_N}^{d'}}\}$. It follows that for all $k \geq k(\overline{\beta}, \beta_N)$,

$$\sharp G_{k+1} \geq c(\overline{\beta}, \beta_N)\overline{\beta}^{n_{k+1}-m_k}.$$

Immediately, by the relationship between D_k and G_k, for any $k \geq k(\overline{\beta}, \beta_N)$, it comes to the conclusion that

$$\begin{aligned} q_k = \sharp D_k = \prod_{i=1}^{k} \sharp G_k \geq \prod_{i=k(\overline{\beta},\beta_N)}^{k} \sharp G_i &\geq c(\overline{\beta}, \beta_N)^{k-k(\overline{\beta},\beta_N)}\overline{\beta}^{\sum_{i=k(\overline{\beta},\beta_N)}^{k-1}(n_{i+1}-m_i)} \\ &\geq c'(\overline{\beta}, \beta_N)c(\overline{\beta}, \beta_N)^k\overline{\beta}^{\sum_{i=1}^{k-1}(n_{i+1}-m_i)}, \end{aligned}$$

where

$$c'(\overline{\beta}, \beta_N) = \overline{\beta}^{-\sum_{i=1}^{k(\overline{\beta},\beta_N)-1}(n_{i+1}-m_i)}.$$

□

Now we divide into three parts to complete our proof of the lower bound of $\dim_{\mathrm{H}} E_{a,b}$ by using the modified mass distribution principle (Theorem 2.5).

(1) Define a probability measure μ supported on E_N. Set

$$\mu([0,1]) = 1 \quad \text{and} \quad \mu(I_{n_1}(u)) = \frac{1}{\sharp G_1}, \text{ for } u \in D_1.$$

For each $k \geq 1$, and $u = (u_1, \ldots, u_{k+1}) \in D_{k+1}$, let

$$\mu(I_{n_{k+1}}(u)) = \frac{\mu(I_{n_k}(u_1, \ldots, u_k))}{\sharp G_{k+1}}. \tag{3.18}$$

For any $u \notin D_k$ $(k \geq 1)$, let $\mu(I_{n_k}(u)) = 0$. It is routine to check that μ is well defined on E_N and it can be extended to a probability measure on $[0,1]$.

(2) Calculate the local dimension $\liminf\limits_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|}$ for any $x \in E_N$. For convenience, we denote $I_n(x)$ by I_n without ambiguity. Then we have

$$\mu(I_{n_i}) = \frac{1}{q_i} \leq \frac{1}{c'(\beta_N,\overline{\beta})c(\beta_N,\overline{\beta})^i \overline{\beta}^{\sum\limits_{j=1}^{i-1}(n_{j+1}-m_j)}} \tag{3.19}$$

for every $i > k(\beta_N,\overline{\beta})$, where $k(\beta_N,\overline{\beta})$ is an integer given in Lemma 3.5. For all $n \geq 1$, there is an integer $k \geq 1$ such that $n_k < n \leq n_{k+1}$. By the construction of E_N and the definition of μ, it is natural to estimate the lower bound of $\frac{\log \mu(I_n)}{\log |I_n|}$ by dividing into the following three cases.

Case 1. $n_k < n \leq m_k$. It follows from (3.19) that

$$\mu(I_n) = \mu(I_{n_k}) \leq c'(\beta_N,\overline{\beta})^{-1} c(\beta_N,\overline{\beta})^{-k} \overline{\beta}^{-\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)}.$$

Furthermore, Theorem 2.3(3) implies

$$|I_n(x)| \geq |I_{m_k}(x)| = \frac{1}{\beta^{m_k}}.$$

As a consequence,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)\log\overline{\beta} + k\log c(\beta_N,\overline{\beta}) + \log c'(\beta_N,\overline{\beta})}{m_k \log\beta}.$$

Case 2. $n = m_k + i(m_k - n_k) + \ell$ for some $0 \leq i < t_k$ and $0 \leq \ell < m_k - n_k$. In this case, when $0 \leq \ell \leq N$, by (3.19) and (3.17), we have

$$\begin{aligned}\mu(I_n) = \mu(I_{m_k+i(m_k-n_k)+\ell}) &\leq \mu(I_{m_k}) \cdot \frac{1}{(\sharp\mathcal{M}_{m_k-n_k})^i} \\ &\leq c'(\beta_N,\overline{\beta})^{-1} c(\beta_N,\overline{\beta})^{-k} \overline{\beta}^{-\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)}.\end{aligned}$$

When $N < \ell < m_k - n_k$, we similarly see that

$$\begin{aligned}\mu(I_n) = \mu(I_{m_k+i(m_k-n_k)+\ell}) &\leq \mu(I_{m_k}) \cdot \frac{1}{(\sharp\mathcal{M}_{m_k-n_k})^i} \cdot \frac{1}{\Sigma_{\beta_N}^{\ell-2N}} \\ &\leq c'(\beta_N,\overline{\beta})^{-1} c(\beta_N,\overline{\beta})^{-k} \overline{\beta}^{-\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)} {\beta_N}^{-\ell+2N}.\end{aligned}$$

Moreover, by (3.10), it holds that

$$|I_n| \geq |I_{m_k+i(m_k-n_k)+\ell}| \geq \frac{1}{\beta^{m_k+i(m_k-n_k)+\ell+2N}}.$$

Therefore,

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j) + i(m_k-n_k)\right)\log\overline{\beta} + (\ell-N)\log\beta_N + k\log c(\beta_N,\overline{\beta}) + \log c'(\beta_N,\overline{\beta})}{(m_k + i(m_k-n_k) + \ell + N)\log\beta}.$$

Case 3. $n = m_k + t_k(m_k - n_k) + \ell$ where $0 \le \ell \le p_k$. When $0 \le \ell \le 2N$, we have

$$\begin{aligned}\mu(I_n) = \mu(I_{m_k+t_k(m_k-n_k)}) &= \mu(I_{m_k}(x)) \cdot \frac{1}{(\sharp \mathcal{M}_{m_k-n_k})^{t_k}} \\ &\le c'(\beta_N, \overline{\beta})^{-1} c(\beta_N, \overline{\beta})^{-k} \overline{\beta}^{-\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)}.\end{aligned}$$

When $2N < \ell \le p_k$, we get

$$\begin{aligned}\mu(I_n) = \mu(I_{m_k+t_k(m_k-n_k)+\ell}) &\le \mu(I_{m_k}(x)) \cdot \frac{1}{(\sharp\ M_{m_k-n_k})^{t_k}} \cdot \frac{1}{\Sigma_{\beta_N}^{\ell-2N}} \\ &\le c'(\beta_N, \overline{\beta})^{-1} c(\beta_N, \overline{\beta})^{-k} \overline{\beta}^{-\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)} {\beta_N}^{-\ell+2N}.\end{aligned}$$

In addition, by (3.10),

$$|I_n| \ge |I_{m_k+t_k(m_k-n_k)+\ell}| \ge \frac{1}{\beta^{m_k+t_k(m_k-n_k)+\ell+N}}.$$

Hence,

$$\frac{\log \mu(I_n)}{\log |I_n|} \ge \frac{\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)\log\overline{\beta} + (\ell-2N)\log\beta_N + k\log c(\beta_N,\overline{\beta}) + \log c'(\beta_N,\overline{\beta})}{(m_k+t_k(m_k-n_k)+\ell+N)\log\beta}.$$

In all three cases, using (3.8), we obtain

$$\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} \ge \lim_{k\to\infty} \frac{\sum_{j=1}^{k-1}(n_{j+1}-m_j)}{m_k} \frac{\log\overline{\beta}}{\log\beta}.$$

By (3.11) and (3.12), it immediately holds that

$$\lim_{k\to\infty} \frac{n_k}{m_k} = 1-b, \quad \lim_{k\to\infty} \frac{n_{k+1}}{m_k} = \frac{b(1-a)}{a} \quad \text{and} \quad \lim_{k\to\infty} \frac{m_{k+1}}{m_k} = \frac{b(1-a)}{a(1-b)}.$$

By the Stolz-Cesàro Theorem, we have

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{k-1}(n_{j+1}-m_j)}{m_k} = \lim_{k\to\infty} \frac{n_{k+1}-m_k}{m_{k+1}-m_k} = \lim_{k\to\infty} \frac{\frac{n_{k+1}}{m_k}-1}{\frac{m_{k+1}}{m_k}-1} = 1 - \frac{b^2(1-a)}{b-a}.$$

As a consequence,

$$\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} \ge \left(1 - \frac{b^2(1-a)}{b-a}\right) \frac{\log\overline{\beta}}{\log\beta}.$$

(3) Use the modified mass distribution principle (Theorem 2.5). We first let $\overline{\beta} \to \beta_N$, and then let $N \to \infty$. Applying Theorem 2.5, we finish our proof.

3.3 Proof of Corollary 1.2

Note that when $\frac{1}{2} < a \le 1$, the inequality (3.9) implies $E_a = \emptyset$. We only need to consider the case $0 \le a \le \frac{1}{2}$. By Lemma 3.1, we have

$$E_a = \left\{ x \in [0,1) : \hat{v}_\beta(x) = \frac{a}{1-a} \right\}.$$

Thus, applying Theorem 2.7, we have, for all $0 < a \le \frac{1}{2}$,

$$\dim_{\mathrm{H}} E_a = \dim_{\mathrm{H}} \left\{ x \in [0,1) : \hat{v}_\beta(x) = \frac{a}{1-a} \right\} = (1-2a)^2.$$

When $a = 0$, by noting that $E_{0,0} \subseteq E_0$, we deduce that E_0 has full Lebesgue measure and thus has Huasdorff dimension 1.

4 Proof of Theorem 1.3

The key point to prove Theorem 1.3 is constructing a set U with the following properties: (1) U is a subset of $E_{0,1}$; (2) U is dense in the interval $[0,1]$; (3) U is a G_δ set, i.e., a countable intersection of open sets.

Let $\beta > 1$. Define $M = \min\{i > 1 : \varepsilon_i^*(\beta) > 0\}$. For all $k \ge 1$, let Γ_k be defined by (2.7). We choose two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $n_k < m_k < n_{k+1}$ with $n_k > 2k + \Gamma_k$ and $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$. In addition, $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ satisfy

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = 0,$$

and

$$\lim_{k\to\infty} \frac{m_k - n_k}{m_k} = 1.$$

In fact, let

$$n_k' = (2k + \Gamma_k)^{2k} \quad \text{and} \quad m_k' = (2k + 2 + \Gamma_{k+1})^{2k+1}.$$

Then by small adjustments, we can obtain the required sequences.

For all $k \ge 1$, write $n_{k+1} = (m_k - n_k)t_k + n_k + p_k$ where $0 \le p_k < m_k - n_k$. Now we define

$$U := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{(\epsilon_1,\ldots,\epsilon_k)\in\Sigma_\beta^k} \mathrm{int}\left(I_{n_{k+1}}(\epsilon_1, \ldots, \epsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k-1})^{t_k}, 0^{p_k}) \right),$$

where $\mathrm{int}(I_{|\epsilon|}(\epsilon))$ stands for the interior of $I_{|\epsilon|}(\epsilon)$ for all $\epsilon \in \Sigma_\beta^*$.

Remark 1 *For all $(\epsilon_1, \ldots, \epsilon_k) \in \Sigma_\beta^k$, it follows from Proposition 2.4(3) that $(\epsilon_1, \ldots, \epsilon_k, 0^{n_k-k})$ is full since $n_k > 2k + \Gamma_k$. Note that $m_k - n_k \ge M$. Then the word $(1, 0^{m_k-n_k-1})$ is full. By Proposition 2.4(2), the word 0^{p_k} is full. Thus U is well defined.*

The set $\mathrm{int}(I_{|\epsilon|}(\epsilon))$ is open which implies that U is a G_δ set. Consequently, it suffices to show that U is a subset of $E_{0,1}$ and is dense in $[0,1]$.

Lemma 4.1 *The set U is a subset of $E_{0,1}$.*

Proof. For any $x \in U$, it follows from the construction of U that there exist infinitely many k such that $\varepsilon(x,\beta) = (\epsilon_1, \ldots, \epsilon_k, 0^{n_k-k}, (1, 0^{m_k-n_k-1})^{t_k}, 0^{p_k})$ for some $(\epsilon_1, \ldots, \epsilon_k) \in \Sigma_\beta^k$. Now we are going to give the upper limit and lower limit of $\frac{r_n(x,\beta)}{n}$.

Let $n = n_{k+1} + m_k - n_k - 1$. Since $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$, we obtain

$$r_{n_{k+1}+m_k-n_k-1}(x,\beta) = m_k - n_k - 1.$$

As a result,

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} \le \lim_{k\to\infty} \frac{r_{n_{k+1}+m_k-n_k-1}(x,\beta)}{n_{k+1}+m_k-n_k-1} = \lim_{k\to\infty} \frac{m_k-n_k-1}{n_{k+1}+m_k-n_k-1} = 0.$$

Let $n = m_k$. Note that $m_k - n_k > \max\{2(m_{k-1} - n_{k-1}), n_k - k, M\}$. The definition of $r_n(x,\beta)$ shows that

$$r_{m_k}(x,\beta) = m_k - n_k - 1.$$

It therefore follows that

$$\limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} \ge \lim_{k\to\infty} \frac{r_{m_k}(x,\beta)}{m_k} = \lim_{k\to\infty} \frac{m_k-n_k-1}{m_k} = \lim_{k\to\infty} \frac{m_k-n_k-1}{m_k} = 1.$$

By the above discussion, we conclude that

$$\liminf_{n\to\infty} \frac{r_n(x,\beta)}{n} = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{r_n(x,\beta)}{n} = 1.$$

Hence, $x \in E_{0,1}$ which gives $U \subseteq E_{0,1}$. □

Proof of Theorem 1.3 It remains to show that for all $n \ge 1$, the set

$$U_n = \bigcup_{k=n}^{\infty} \bigcup_{(\epsilon_1,\ldots,\epsilon_k)\in\Sigma_\beta^k} \operatorname{int}\left(I_{n_{k+1}}\left(\epsilon_1,\ldots,\epsilon_k,0^{n_k-k},(1,0^{m_k-n_k})^{t_k},0^{p_k}\right)\right)$$

is dense in [0,1]. Now we will concentrate on finding a real number $y \in U$ such that $|x-y| \le r$ for every $x \in [0,1]$ and $r > 0$. Suppose that $\varepsilon(x,\beta) = (\varepsilon_1(x,\beta), \varepsilon_2(x,\beta), \ldots)$. Let ℓ' be an integer satisfying $\beta^{-\ell'} \le r$. Let $\ell = \max\{n, \ell'\}$. Since $(\varepsilon_1(x,\beta),\ldots,\varepsilon_\ell(x,\beta)) \in \Sigma_\beta^\ell$, we choose a point

$$y \in \operatorname{int}\left(I_{n_{\ell+1}}\left(\epsilon_1,\ldots,\epsilon_\ell,0^{n_\ell-\ell},(1,0^{m_\ell-n_\ell})^{t_\ell},0^{p_\ell}\right)\right).$$

Then it holds that $|x-y| \le \beta^{-\ell} \le r$ and $y \in U_n$. To sum up, the set

$$\bigcup_{k=n}^{\infty} \bigcup_{(\epsilon_1,\ldots,\epsilon_k)\in\Sigma_\beta^k} \operatorname{int}\left(I_{n_{k+1}}\left(\epsilon_1,\ldots,\epsilon_k,0^{n_k-k},(1,0^{m_k-n_k})^{t_k},0^{p_k}\right)\right)$$

is dense in $[0,1]$.

Thus, we can conclude by the Baire Category Theorem that U is residual in $[0,1]$. Then, $E_{0,1}$ is residual in $[0,1]$ by Lemma 4.1.

□

5 Classical results of β-expansion in the parameter space

In this section, we recall some important results of β-expansion in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$. The readers can refer to [4, 8, 12, 15, 18] for more information.

Definition 5.1 *We call a word $\omega = (\omega_1,\ldots,\omega_n)$ self-admissible if for all $1 \le i < n$,*

$$\sigma^i\omega \le_{\text{lex}} (\omega_1,\ldots,\omega_{n-i}).$$

An infinite sequence $\omega = (\omega_1, \omega_2, \ldots)$ is called self-admissible if $\sigma^i\omega <_{\rm lex} \omega$ for all $i \geq 1$.

Denote by Λ_n the set of all self-admissible words with length n, i.e.,

$$\Lambda_n = \{\omega = (\omega_1, \omega_2, \ldots, \omega_n) : \text{ for every } 1 \leq i < n,\ \sigma^i\omega \leq_{\rm lex} (\omega_1, \ldots, \omega_{n-i})\}.$$

For convenience, for all $1 < \beta_1 < \beta_2$, let

$$\Lambda_n(\beta_1, \beta_2) = \{\omega = (\omega_1, \ldots, \omega_n) \in \Lambda_n : \exists\ \beta \in (\beta_1, \beta_2] : \text{ s.t. } \varepsilon_1(\beta) = \omega_1, \ldots, \varepsilon_n(\beta) = \omega_n\}. \quad (5.20)$$

The definition of self-admissible word immediately gives the following result. The proof is evident and will be omitted.

Proposition 5.1 *For any $m \geq n \geq 1$, let $\omega \in \Lambda_n$. Let $\beta > 1$ whose infinite β-expansion of 1 satisfy $(\varepsilon_1^*(\beta), \ldots, \varepsilon_n^*(\beta)) <_{\rm lex} \omega$. Then for all $v_1, v_2, \ldots, v_i \in \Sigma_\beta^m$ $(i \geq 1)$, the concatenation $\omega * v_1 * \cdots * v_j$ is still self-admissible for all $1 \leq j \leq i$.*

The characterization of the the β-expansion of 1 was given by Parry [15].

Theorem 5.2 (Parry [15]) *An infinite sequence $(\omega_1, \omega_2, \ldots)$ is the β-expansion of 1 for some $\beta > 1$ if and only if it is self-admissible.*

Now we consider the *cylinders in the parameter space* $\{\beta \in \mathbb{R} : \beta > 1\}$.

Definition 5.2 *For any $\omega = (\omega_1, \ldots, \omega_n) \in \Lambda_n$. The cylinder $I_n^P(\omega)$ associated to ω in the parameter space is the set of $\beta \in (1, +\infty)$ whose β-expansion of 1 has the prefix $(\omega_1, \ldots, \omega_n)$, i.e.*

$$I_n^P(\omega) := \{\beta \in (1, +\infty) : \varepsilon_1(\beta) = \omega_1, \ldots, \varepsilon_n(\beta) = \omega_n\}.$$

The cylinders in the parameter space are intervals (see [18, Lemma 4.1]). The length of the cylinders of $\omega \in \Lambda_n$ in the parameter space is denoted by $|I_n^P(\omega)|$. For simplicity, the left endpoint and right endpoint of $I_n^P(\omega)$ are written as $\underline{\beta}(\omega)$ and $\overline{\beta}(\omega)$ respectively.

To estimate the length of cylinders in the parameter space, we need the notion of *recurrence time* $\tau(\omega)$ (see [12]) of the self-admissible word $\omega = (\omega_1, \ldots, \omega_n) \in \Lambda_n$. Define

$$\tau(\omega) := \inf\{1 \leq k < n : \sigma^k(\omega_1, \ldots, \omega_n) = (\omega_1, \ldots, \omega_{n-k})\}.$$

If we cannot find such an integer k, we set $\tau(\omega) = n$. In this case, the self-admissible word ω is said to be *non-recurrent*.

The above definition of recurrence time immediately provides the following properties.

Remark 2 *(1) Write*

$$t(\omega) := n - \left\lfloor \frac{n}{\tau(\omega)} \right\rfloor \tau(\omega).$$

Then we have

$$(\omega_1, \ldots, \omega_n) = \left((\omega_1, \ldots, \omega_{\tau(\omega)})^{\lfloor \frac{n}{\tau(\omega)} \rfloor}, \omega_1, \ldots, \omega_{t(\omega)}\right).$$

(2) If $\omega = (\omega_1, \ldots, \omega_n)$ is non-recurrent, then the word $(\omega_1, \ldots, \omega_n, 0^\ell)$ is still non-recurrent for all $\ell \geq 1$.

The following result gives the upper and lower bounds of the length of the cylinder $I_n^P(\omega)$.

Lemma 5.3 (Schemling [18], Li, Persson, Wang and Wu [12]) *Let $\omega = (\omega_1, \ldots, \omega_n) \in \Lambda_n$. We have the following inequalities:*

(1) $|I_n^P(\omega)| \leq \overline{\beta}(\omega)^{-n+1}$;

(2)

$$|I_n^P(\omega)| \geq \begin{cases} C(\omega)\overline{\beta}(\omega)^{-n}, & \text{when } t(\omega)=0; \\ C(\omega)\overline{\beta}(\omega)^{-n}\left(\dfrac{\omega_{t(\omega)+1}}{\overline{\beta}(\omega)}+\cdots+\dfrac{\omega_{\tau(\omega)}+1}{\overline{\beta}(\omega)^{\tau(\omega)-t(\omega)}}\right), & \text{otherwise,} \end{cases}$$

where

$$C(\omega) := \frac{(\underline{\beta}(\omega)-1)^2}{\underline{\beta}(\omega)}. \tag{5.21}$$

The study of the parameter space usually concerns on the set of parameters with respect to which the approximation properties of the orbit of 1 are prescribed. Persson and Schmeling [16] proved the following result.

Theorem 5.4 (Persson and Schmeling [16]) *Let $v \geq 0$. Then*

$$\dim_{\mathrm{H}}\{\beta \in (1,2) : v_\beta(1) \geq v\} = \frac{1}{1+v}.$$

Analogous to Theorem 2.7, Bugeaud and Liao [2] obtained the following theorem in the parameter space.

Theorem 5.5 (Bugeaud and Liao [2]) *Let $0 < \hat{v} < 1$ and $v > 0$. If $v < \frac{\hat{v}}{1-\hat{v}}$, then the set*

$$U(\hat{v}, v) := \{\beta \in (1,2) : \hat{v}_\beta(1) = \hat{v},\ v_\beta(1) = v\}$$

is empty. Otherwise, we have

$$\dim_{\mathrm{H}} U(\hat{v}, v) = \frac{v-(1+v)\hat{v}}{(1+v)(v-\hat{v})}.$$

Moreover,

$$\dim_{\mathrm{H}}\{\beta \in (1,2) : \hat{v}_\beta(1) = \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}}\right)^2.$$

6 Proof of Theorem 1.4

As the same discussion at the first part of Section 3, it holds that $\dim_{\mathrm{H}} E_{0,0}^P$ is of full Lebesgue measure by using the result of Cao and Chen [4] that the set

$$\left\{\beta \in (1,2) : \lim_{n\to\infty} \frac{r_n(\beta)}{\log_\beta n} = 1\right\}$$

is of full Lebesgue measure. By the same argument as the proof of Theorem 1.1 for the case $a > \frac{b}{1+b},\ 0 < b \leq 1$ in the end of Section 3.1, we get that $E_{a,b}^P$ is empty when $a > \frac{b}{1+b},\ 0 < b \leq 1$.

When $0 < a \leq \frac{b}{1+b},\ 0 < b < 1$, Lemmas 3.1 and 3.2 imply that

$$E_{a,b}^P = \left\{\beta \in (1,2) : \hat{v}_\beta(1) = \frac{a}{1-a},\ v_\beta(1) = \frac{b}{1-b}\right\} = U\left(\frac{a}{1-a}, \frac{b}{1-b}\right).$$

Then by Theorem 5.5, it holds that

$$\dim_{\mathrm{H}} E_{a,b}^P = \dim_{\mathrm{H}} U\left(\frac{a}{1-a}, \frac{b}{1-b}\right) = 1 - \frac{b^2(1-a)}{b-a}.$$

But Theorem 5.5 is not applicable for the case of $a = 0,\ 0 < b < 1$ and $0 < a \leq \frac{1}{2},\ b = 1$. Note that $E_{a,1}^P \subseteq F_1^P$ where F_1^P is defined by (1.4). So we first give the Hausdorff dimension of F_1^P.

Since

$$F_1^P = \{\beta \in (1,2) : v_\beta(1) = +\infty\} \subseteq \{\beta \in (1,2) : v_\beta(1) \geq v\}$$

for all $v > 0$, we have

$$\dim_{\mathrm{H}} F_1^P \leq \dim_{\mathrm{H}} \{\beta \in (1,2) : v_\beta(1) \geq v\} = \frac{1}{1+v}$$

where the last equality follows from Theorem 5.4. Letting $v \to +\infty$, we have $\dim_{\mathrm{H}} F_1^P \leq 0$. This implies $\dim_{\mathrm{H}} E_{a,1}^P \leq 0$. In conclusion, $\dim_{\mathrm{H}} E_{a,1}^P = 0$ for any $0 < a \leq \frac{1}{2}$. For the other case, we have

$$E_{0,b}^P \subseteq \left\{\beta \in (1,2) : v_\beta(1) \geq \frac{b}{1-b}\right\}.$$

By Theorem 5.4, we deduce that the upper bound of $\dim_{\mathrm{H}} E_{0,b}^P$ is $1-b$ for all $0 < b < 1$. Hence, we only need to give the lower bound of $\dim_{\mathrm{H}} E_{0,b}$ for all $0 < b < 1$. We also include our proof of the case $0 < a \leq \frac{b}{1+b},\ 0 < b < 1$.

For every $1 < \beta_1 < \beta_2 < 2$, instead of dealing with the Hausdorff dimension of the set $E_{a,b}^P$ directly, we will technically investigate the Hausdorff dimension of the following set. For all $0 \leq a \leq \frac{b}{1+b},\ 0 < b \leq 1$, let

$$E_{a,b}^P(\beta_1,\beta_2) = \left\{\beta \in [\beta_1,\beta_2) : \liminf_{n\to\infty} \frac{r_n(\beta)}{n} = a,\ \limsup_{n\to\infty} \frac{r_n(\beta)}{n} = b\right\}. \tag{6.22}$$

For all $1 < \beta_1 < \beta_2 < 2$, throughout this section, we assume that both β_1 and β_2 are not simple Parry number. We will give the lower bound of $\dim_{\mathrm{H}} E_{a,b}^P(\beta_1,\beta_2)$ for all $1 < \beta_1 < \beta_2 < 2$.

Suppose that N is a large enough integer such that $\varepsilon_N(\beta_2) > 0$ and

$$(\varepsilon_1(\beta_1),\ldots,\varepsilon_N(\beta_1)) <_{\mathrm{lex}} (\varepsilon_1(\beta_2),\ldots,\varepsilon_N(\beta_2))$$

Let $\widetilde{\beta}_N$ be the unique solution of the equation:

$$1 = \frac{\varepsilon_1(\beta_2)}{x} + \cdots + \frac{\varepsilon_N(\beta_2)}{x^N}.$$

Then

$$\varepsilon^*(\widetilde{\beta}_N) = (\varepsilon_1(\beta_2),\ldots,\varepsilon_N(\beta_2)-1)^\infty.$$

An observation of the lexicographical order of $\varepsilon^*(\beta_1)$, $\varepsilon^*(\beta_2)$ and $\varepsilon^*(\widetilde{\beta}_N)$ implies $\beta_1 < \widetilde{\beta}_N < \beta_2$ and $\widetilde{\beta}_N \to \beta_2$ as N tends to infinity.

For every $k \geq 1$, similar to what we did in Section 3.2, we take two sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $n_k < m_k < n_{k+1}$ with $n_1 > 2N$ and $m_k - n_k > m_{k-1} - n_{k-1}$ with $m_1 - n_1 > 2N$. In addition,

$$\lim_{k\to\infty} \frac{m_k - n_k}{n_{k+1} + m_k - n_k} = a \quad \text{and} \quad \lim_{k\to\infty} \frac{m_k - n_k}{m_k} = b.$$

We can choose such two sequences by the same way in Section 3.2.

Now let us construct a Cantor set contained in $E_{a,b}^P(\beta_1,\beta_2)$ as follows.

For any integer $d > 2N$, we set

$$\mathcal{M}_d' = \{\omega = (\varepsilon_1(\beta_2),\ldots,\varepsilon_N(\beta_2)-1,\omega_1,\ldots,\omega_{d-2N},0^N) : (\omega_1,\ldots,\omega_{d-2N}) \in \Sigma_{\widetilde{\beta}_N}^{d-2N}\}. \tag{6.23}$$

Let

$$G_1' = \{(\varepsilon_1(\beta_2),\ldots,\varepsilon_N(\beta_2),\omega_1,\ldots,\omega_{d-2N},0^N) : (\omega_1,\ldots,\omega_{d-2N}) \in \Sigma_{\widetilde{\beta}_N}^{d-2N}\}.$$

Note that $\left(\varepsilon_1^*(\widetilde{\beta}_N),\ldots,\varepsilon_N^*(\widetilde{\beta}_N)\right) <_{\mathrm{lex}} (\varepsilon_1(\beta_2),\ldots,\varepsilon_N(\beta_2))$. Now we give some observations on

the elements in G_1' as follows.

Remark 3 *(1) For all $\omega \in G_1'$, since $\left(\omega_1, \ldots, \omega_{d-2N}, 0^N\right) \in \Sigma_{\widetilde{\beta}_N}^{d-N}$ $(d > 2N)$, by Proposition 5.1, ω is self-admissible.*

(2) For every $u \in \mathcal{M}_d'$ $(d > 2N)$, we have $u \in \Sigma_{\widetilde{\beta}_N}^d$. By Lemma 5.1, the word $\omega * u$ is still self-admissible for every all $\omega \in G_1'$.

For every $k \geq 1$, write $n_{k+1} = (m_k - n_k)t_k + m_k + p_k$ with $0 \leq p_k < m_k - n_k$, then define $G_{k+1}' =$

$$\{u_{k+1} = \left(\varepsilon_1(\beta_2), \ldots, \varepsilon_N(\beta_2) - 1, 0^{m_k - n_k - N}, u_k^{(1)}, \ldots, u_k^{(t_k)}, u_k^{(t_k+1)}\right) : \, u_k^{(i)} \in \mathcal{M}_{m_k - n_k}', \, 1 \leq i \leq t_k\},$$

where

$$u_k^{(t_k+1)} = \begin{cases} 0^{p_k}, & \text{when } \ p_k \leq 2N; \\ \omega \in \mathcal{M}_{p_k}', & \text{when } \ p_k > 2N. \end{cases}$$

Let

$$D_k' = \{(u_1, \ldots, u_k) : u_i \in G_i', \ 1 \leq i \leq k\}. \tag{6.24}$$

Notice that every $u_k \in G_k'$ ends with 0^N. This guarantees that $(u_1, \ldots, u_k)$ can concatenate with any u_{k+1} to be a new self-admissible word. As a result, the set D_k' is well-defined.

As the classical technique of constructing a Cantor set, let

$$E(\beta_1, \beta_2) = \bigcap_{k=1}^{\infty} \bigcup_{u \in D_k'} I_{n_k}^P(u).$$

Similar to the process of Section 3, we now give the following result which means that $E(\beta_1, \beta_2)$ is a subset of $E_{a,b}^P(\beta_1, \beta_2)$.

Lemma 6.1 *For every $1 < \beta_1 < \beta_2 < 2$, $E(\beta_1, \beta_2) \subset E_{a,b}^P(\beta_1, \beta_2)$ for all $0 \leq a \leq \frac{b}{1+b}$ and $0 < b < 1$.*

Proof. The proof is just as the same as the proof of Lemma 3.4 by dividing into three cases. We omit it here. □

Analogously, we now focus on the estimation of the cardinality of the set D_k'. Let $q_k' := \sharp D_k'$. We obtain the following lemma.

Lemma 6.2 *For every $1 < \beta_1 < \beta_2 < 2$, let $\widetilde{\beta}_N$ be the real number defined in this section. Then there exist an integer $k(\beta_1, \widetilde{\beta}_N)$ and real numbers $c(\beta_1, \widetilde{\beta}_N), c'(\beta_1, \widetilde{\beta}_N)$ such that, for every $k \geq k(\beta_1, \widetilde{\beta}_N)$, we have*

$$q_k' \geq c'(\beta_1, \widetilde{\beta}_N) c(\beta_1, \widetilde{\beta}_N)^k \beta_1^{\sum_{i=1}^{k-1}(n_{i+1} - m_i)}. \tag{6.25}$$

Proof. We use the similar method as Lemma 3.5, the details are left to the readers. □

Let

$$C(\beta_1) = \frac{(\beta_1 - 1)^2}{\beta_1}.$$

Notice that $\underline{\beta}(u) \geq \beta_1 > 1$ for any $u = (u_1, \ldots, u_n) \in \Lambda_n(\beta_1, \beta_2)$ where $\Lambda_n(\beta_1, \beta_2)$ is defined by (5.20). Then

$$C(\underline{\beta}(u)) = \frac{(\underline{\beta}(u) - 1)^2}{\underline{\beta}(u)} \geq C(\beta_1).$$

The following lemma gives the estimation of the length of the cylinders with non-empty intersection with the Cantor set $E(\beta_1, \beta_2)$ which will be useful to estimate the local dimension $\liminf\limits_{n \to \infty} \frac{\log \mu(B(\beta, r))}{\log |r|}$ for any $r > 0$ and $\beta \in E(\beta_1, \beta_2)$.

Lemma 6.3 *For any $\beta \in E(\beta_1, \beta_2)$, suppose $\varepsilon(1,\beta) = (u_1, u_2, \ldots)$. Then we have*

$$|I_n^P(u_1, \ldots, u_n)| \geq C(\beta_1)\beta_2^{-(n+N)}$$

for any $n \geq 1$.

Proof. For any $n \geq 1$, we are going to take the word $(u_1, \ldots, u_n, 0^N)$ into account. We claim that the word $(u_1, \ldots, u_n, 0^N)$ is non-recurrent.

In fact, by the construction of $E(\beta_1, \beta_2)$, for any $1 \leq i < n$, we have

$$\sigma^i(u_1, \ldots, u_n, 0^N) \in \Sigma^{n-i+N}_{\widetilde{\beta}_N}.$$

Notice that $\omega \leq_{\text{lex}} (\varepsilon_1(\beta_2), \ldots, \varepsilon_N(\beta_2) - 1) <_{\text{lex}} (\varepsilon_1(\beta_2), \ldots, \varepsilon_N(\beta_2))$ for any $\omega \in \Sigma^n_{\widetilde{\beta}_N}$ with $n \geq N$. It comes to the conclusion that $\sigma^i(u_1, \ldots, u_n, 0^N) <_{\text{lex}} (\varepsilon_1(\beta_2), \ldots, \varepsilon_N(\beta_2))$ for any $1 \leq i < n+N$ which implies that $(u_1, \ldots, u_n, 0^N)$ is non-recurrent. Thus, by Lemma 5.3(2), we have

$$|I_n^P(u_1, \ldots, u_n)| \geq |I_{n+N}^P(u_1, \ldots, u_n, 0^N)| \geq C(u_1, \ldots, u_n, 0^N)\overline{\beta}(u_1, \ldots, u_n, 0^N)^{-(n+N)}.$$

It follows from the fact $\overline{\beta}(u_1, \ldots, u_n, 0^N) \leq \beta_2$ that

$$|I_n^P(u_1, \ldots, u_n)| \geq C(\beta_1)\beta_2^{-(n+N)}.$$

□

Let us now concentrate on giving the lower bound of $\dim_{\text{H}} E(\beta_1, \beta_2)$. As the conventional process, we define a measure supported on $E(\beta_1, \beta_2)$ which is similar to Section 3.2 by distributing the mass uniformly. We will give the local dimension $\liminf\limits_{n\to\infty} \frac{\log \mu\left(I_n^P(u)\right)}{\log |I_n^P(u)|}$ for any cylinder $I_n^P(u)$ which has non-empty intersection with $E(\beta_1, \beta_2)$. Without any confusion, here and subsequently, I_n^P stands for the cylinder $I_n^P(u)$ for all $u \in \Lambda_n$.

(1) Define a probability measure supported on $E(\beta_1, \beta_2)$. Let

$$\mu([\beta_1, \beta_2)) = 1 \quad \text{and} \quad \mu(I_{n_1}^P(u)) = \frac{1}{\sharp G_1'}, \text{ for } u \in D_1'.$$

For all $k \geq 1$, and $u = (u_1, \ldots, u_{k+1}) \in D_{k+1}'$, define

$$\mu(I_{n_{k+1}}^P(u)) = \frac{\mu(I_{n_k}^P(u_1, \ldots, u_k))}{\sharp G_{k+1}'}.$$

(2) Estimate the local dimension $\liminf\limits_{n\to\infty} \frac{\log \mu\left(I_n^P\right)}{\log |I_n^P|}$ where $I_n^P \cap E(\beta_1, \beta_2) \neq \emptyset$. It follows from the definition of the measure that

$$\mu(I_{n_i}^P) = \frac{1}{q_i'} \leq \frac{1}{c'(\beta_1, \widetilde{\beta}_N)c(\beta_1, \widetilde{\beta}_N)^i \beta_1^{\sum_{j=1}^{i-1}(n_{j+1}-m_j)}}, \tag{6.26}$$

for every $i > k(\beta_1, \widetilde{\beta}_N)$. For any $\beta \in E(\beta_1, \beta_2)$, suppose $\varepsilon(1,\beta) = (u_1, u_2, \ldots)$. For each $n \geq 1$, there exists an integer $k \geq 1$ such that $n_k < n \leq n_{k+1}$. It falls naturally into three cases.

Case 1. $n_k < n \leq m_k$. It follows from (6.26) that

$$\mu(I_n^P) = \mu(I_{n_k}^P) \leq c'(\beta_1, \widetilde{\beta}_N)^{-1} c(\beta_1, \widetilde{\beta}_N)^{-k} \beta_1^{-\sum_{j=1}^{k-1}(n_{j+1}-m_j)}.$$

Furthermore, by the construction of $E(\beta_1,\beta_2)$, the word $(u_1,\dots,u_{m_k})$ is non-recurrent. Thus, by Lemma 5.3, we have

$$|I_n^P| \geq |I_{m_k}^P| \geq C(u_1,\dots,u_{m_k})\overline{\beta}(u_1,\dots,u_{m_k})^{-m_k} \geq c(\beta_1)\beta_2^{-m_k}.$$

As a consequence,

$$\frac{\log \mu(I_n^P)}{\log |I_n^P|} \geq \frac{\log c'(\beta_1,\widetilde{\beta}_N) + k\log c(\beta_1,\widetilde{\beta}_N) + \sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)\log\beta_1}{\log c(\beta_1) + m_k\log\beta_1}.$$

Case 2. $n = m_k + i(m_k-n_k)+\ell$ for some $0\leq i<t_k$ and $0\leq \ell < m_k-n_k$. On the one hand, when $0\leq\ell\leq 2N$, we have

$$\begin{aligned}\mu(I_n^P) = \mu(I^P_{m_k+i(m_k-n_k)+\ell}) &\leq \mu(I_{m_k}^P)\cdot\frac{1}{(\sharp M_{m_k-n_k})^i}\\ &\leq c'(\beta_1,\widetilde{\beta}_N)^{-1}c(\beta_1,\widetilde{\beta}_N)^{-k}\beta_1^{-\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)}.\end{aligned}$$

On the other hand, when $2N<\ell<m_k-n_k$, we have

$$\begin{aligned}\mu(I_n^P) = \mu(I^P_{m_k+i(m_k-n_k)+\ell}) &\leq \mu(I_{m_k}^P)\cdot\frac{1}{(\sharp M_{m_k-n_k})^i}\cdot\frac{1}{\Sigma_{\widetilde{\beta}_N}^{\ell-2N}}\\ &\leq c'(\beta_1,\widetilde{\beta}_N)^{-1}c(\beta_1,\widetilde{\beta}_N)^{-k}\beta_1^{-\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)}\widetilde{\beta}_N^{-\ell+2N}.\end{aligned}$$

Moreover, by Lemma 6.3,

$$|I_n^P| = |I^P_{m_k+i(m_k-n_k)+\ell}| \geq C(\beta_1)\beta_2^{-(m_k+i(m_k-n_k)+\ell+N)}.$$

Hence,

$$\begin{aligned}&\frac{\log\mu(I_n^P)}{\log|I_n^P|}\\ &\geq \frac{\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)\log\beta_1 + (\ell-2N)\log\widetilde{\beta}_N + k\log c(\beta_1,\widetilde{\beta}_N)+\log c'(\beta_1,\tilde{\beta}_N)}{\log C(\beta_1)+(m_k+i(m_k-n_k)+\ell+N)\log\beta_2}.\end{aligned}$$

Case 3. $n = m_k + t_k(m_k-n_k)+\ell$ for some $0\leq\ell\leq p_k$. Similarly, when $0\leq\ell\leq 2N$, we have

$$\begin{aligned}\mu(I_n^P) = \mu(I^P_{m_k+i(m_k-n_k)+\ell}) &\leq \mu(I_{m_k}^P)\cdot\frac{1}{(\sharp M_{m_k-n_k})^{t_k}}\\ &\leq c'(\beta_1,\widetilde{\beta}_N)^{-1}c(\beta_1,\widetilde{\beta}_N)^{-k}\beta_1^{-\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)}.\end{aligned}$$

When $2N<\ell\leq p_k$, it follows that

$$\begin{aligned}\mu(I_n^P) = \mu(I^P_{m_k+t_k(m_k-n_k)+\ell}) &\leq \mu(I_{m_k}^P)\cdot\frac{1}{(\sharp M_{m_k-n_k})^i}\cdot\frac{1}{\Sigma_{\widetilde{\beta}_N}^{\ell-2N}}\\ &\leq c'(\beta_1,\widetilde{\beta}_N)^{-1}c(\beta_1,\widetilde{\beta}_N)^{-k}\beta_1^{-\left(\sum\limits_{j=1}^{k-1}(n_{j+1}-m_j)+i(m_k-n_k)\right)}\widetilde{\beta}_N^{-\ell+2N}.\end{aligned}$$

Furthermore, we conclude from Lemma 6.3 that

$$|I_n^P| = |I^P_{m_k+t_k(m_k-n_k)+\ell}| \geq C(\beta_1)\beta_2^{-(m_k+t_k(m_k-n_k)+\ell+N)}.$$

Therefore, we have

$$\begin{aligned}
&\frac{\log \mu(I_n^P)}{\log |I_n^P|} \\
&\geq \frac{\left(\sum_{j=1}^{k-1}(n_{j+1}-m_j)+t_k(m_k-n_k)\right)\log\beta_1 + (\ell-2N)\log\widetilde{\beta}_N + k\log c(\beta_1,\widetilde{\beta}_N) + \log c'(\beta_1,\widetilde{\beta}_N)}{\log C(\beta_1) + (m_k+t_k(m_k-n_k)+\ell+N)\log\beta_2}.
\end{aligned}$$

Just proceeding as the same analysis in Section 3.2, for all the above three cases, we obtain

$$\liminf_{n\to\infty}\frac{\log\mu(I_n^P)}{\log|I_n^P|} \geq \left(1-\frac{b^2(1-a)}{b-a}\right)\frac{\log\beta_1}{\log\beta_2}.$$

(3) Use the mass distribution principle (see [6, Page 60]). Now we take any $B(\beta,r)$ with center $\beta\in E(\beta_1,\beta_2)$ and sufficiently small enough r verifying

$$|I^P_{n+1}| \leq r < |I_n^P| \leq \beta_1^{-n+1}, \tag{6.27}$$

where the last inequalities is guaranteed by the fact that $\overline{\beta}(\omega)\geq\beta_1$ for any $\omega\in\Lambda_n(\beta_1,\beta_2)$. By Lemma 6.3, we have

$$|I_n^P| \geq C(\beta_1)\beta_2^{-(n+N)}.$$

As a result, the ball $B(\beta,r)$ intersects no more than $2\left\lfloor C(\beta_1)^{-1}\beta_2^N\left(\frac{\beta_2}{\beta_1}\right)^{n-1}\right\rfloor+2$ cylinders of order n. Moreover, it follows from Lemma 6.3 that

$$r\geq|I^P_{n+1}|\geq C(\beta_1)\beta_2^{-(n+1+N)}. \tag{6.28}$$

Immediately, the combination of (6.27) and (6.28) gives

$$\begin{aligned}
&\liminf_{r\to0}\frac{\log\mu\left(B(\beta,r)\right)}{\log r} \\
&\geq \liminf_{n\to\infty}\frac{\log\left(2\left\lfloor C(\beta_1)^{-1}\beta_2^N\left(\frac{\beta_2}{\beta_1}\right)^{n-1}\right\rfloor+2\right)+\log\mu\left(I_n^P\right)}{\log|I_n^P|}\cdot\frac{\log|I_n^P|}{-\log C(\beta_1)+(n+1+N)\log\beta_2} \\
&\geq \liminf_{n\to\infty}\left(\frac{(n-1)(\log\beta_2-\log\beta_1)}{-\log C(\beta_1)+(n+N)\log\beta_2}+\frac{\log\mu\left(I_n^P\right)}{\log|I_n^P|}\right)\cdot\frac{(n-1)\log\beta_1}{-\log C(\beta_1)+(n+1+N)\log\beta_2} \\
&\geq \left(\frac{\log\beta_2-\log\beta_1}{\log\beta_2}+\left(1-\frac{b^2(1-a)}{b-a}\right)\frac{\log\beta_1}{\log\beta_2}\right)\frac{\log\beta_1}{\log\beta_2}.
\end{aligned}$$

Therefore, by the mass distribution principle and letting $\beta_1\to\beta_2$, we get our desired result.

7 Proof of Theorem 1.6

Akin to Section 5, we need to find a subset of $E^P_{0,1}$ which is a dense G_δ set in the interval $[1,2]$. Since the process of our proof is almost the same as Section 5. We only provide the construction of the required set V' in this section.

For all $k\geq1$, we first choose the sequences $\{n_k\}_{k=1}^\infty$ and $\{m_k\}_{k=1}^\infty$ such that $m_k-n_k>$

$\max\{2(m_{k-1}-n_{k-1}), n_k-k\}$ and $n_k < m_k < n_{k+1}$. In addition, the sequences $\{n_k\}_{k=1}^{\infty}$ and $\{m_k\}_{k=1}^{\infty}$ is chosen to satisfy

$$\lim_{k\to\infty} \frac{m_k-n_k}{n_{k+1}+m_k-n_k} = 0,$$

and

$$\lim_{k\to\infty} \frac{m_k-n_k}{m_k} = 1.$$

Actually, let

$$n'_k = k^{2k} \quad \text{and} \quad m'_k = (k+1)^{2k+1}.$$

We can obtain the required sequences with some adjustments.

For all $k \geq 1$, denote $n_{k+1} = (m_k-n_k)t_k + m_k + p_k$ where $0 \leq p_k < m_k - n_k$. We now define

$$V' = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{(\epsilon_1,\ldots,\epsilon_k)\in\Lambda_k(1,2)} \operatorname{int}\left(I^P_{n_{k+1}}\left(\epsilon_1,\ldots,\epsilon_k,0^{n_k-k},(1,0^{m_k-n_k})^{t_k},0^{p_k}\right)\right)$$

where $\Lambda_k(1,2)$ is defined by (5.20). Since $m_k-n_k > n_k-k$, we get $(1,0^{m_k-n_k}) <_{\text{lex}} (\varepsilon_1,\cdots,\varepsilon_k,0^{n_k-k})$ for all $k \geq 1$. By Lemma 5.1, the set V' is well defined.

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