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About the validity of the parabolic approximation in Kerr lensing effect

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Abstract:

The usual *parabolic approximation* used for modelling the focusing properties of a *Kerr lens* induced by a Gaussian laser beam, through optical Kerr effect, consists to roughly approximate a Gaussian profile by a parabola. The Kerr focal length deduced from the *ABCD* formalism is found to not correctly provide the focal plane position. The latter has been numerically determined from a numerical diffraction analysis, and we have empirically found that the right focal length is obtained by multiplying, by a factor equal to 3, the focal length given by the usual parabolic approximation. The expression of the focal length associated with the Kerr lens that we have obtained is in a good agreement with the expression of the focal length determined on the basis of a Zernike decomposition. In addition, it is demonstrated that the emerging beam from the Kerr lens is no longer Gaussian since its propagation factor M^2 is larger than unity. It is found that the M^2 factor increases linearly with the on axis phase shift.

1.Introduction

Generation of light pulses in the range of picoseconds or less is typically based on mode-locked lasers. This technique can be described as *active* (Pockels or acoustooptic cells) or *passive* (saturable absorber) mode-locking, and has been developed since the 1960's [1,2]. More later in 1990's, a new concept of mode-locking technique has been experimented by using the optical Kerr effect (OKE) leading to a nonlinear lensing effect inducing time-dependent losses when combined with an intracavity iris [3]. This peculiar regime of laser oscillation is usually called as Kerr-lens-mode-locking (KLM). Its theoretical modelling has been proposed at the beginning of the ninety decade [4,5]. Since these pioneer works various studies dedicated to KLM lasers can be found in literature and well summarised and analysed in [6].

Two contradictory necessities are seen when designing a KLM laser in which the laser medium acts as both the gain and the Kerr media:

- the laser mode has to be sufficiently tight for making the light intensity to be high in order to induce the adequate Kerr lensing effect,
- the size of the laser mode has to be sufficiently large in order to fulfil the required good overlap with the pumped region of the active medium for achieving a high laser efficiency.

A possible solution of this dilemma should be to separate the gain medium and Kerr medium. This is currently the trend of the laser cavity design for achieving a high performing KLM laser [7]. The knowledge of the focusing action of the nonlinear Kerr lens effect is fundamental for the design procedure of a KLM laser, especially since the laser cavity is usually operated close the stability limit. In addition, the Kerr lens has to achieve the required loss discrimination between the continuous wave (CW), and the mode-locking regimes in favour of the pulsed operation as shown in Fig. 1. These differential losses are generated by introducing an intracavity aperture which may be termed as *hard* (diaphragm) or *soft* (radial gain profile). In both cases, the resulting self-amplitude-modulation (SAM) is organised by the time varying Kerr lens effect.

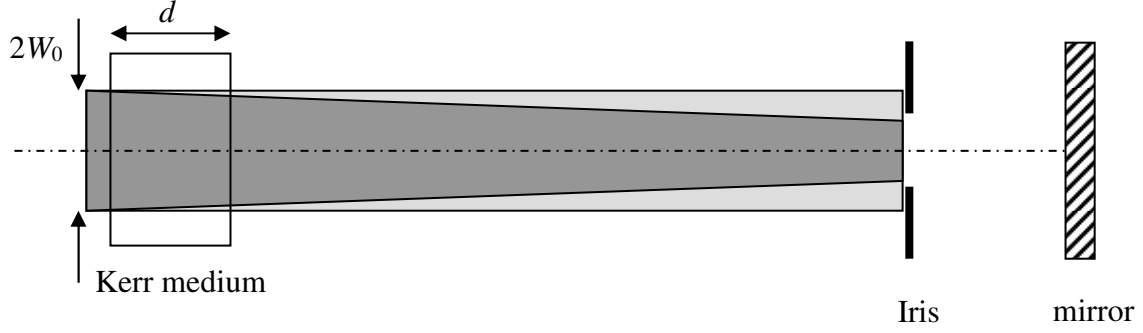


Figure 1: High (low) transmission through the iris due to self focusing by nonlinear Kerr medium for pulsed (CW) operation represented by the dark (light) beam.

2. Assessment of the parabolic approximation adequacy

Let us recall the concept of Kerr lens effect by considering a collimated Gaussian laser beam of width W_0 and having a transverse intensity profile given by:

$$I(\rho) = I_0 \exp[-2\rho^2 / W_0^2], \quad (1)$$

which is incident on a slice of transparent Kerr material characterised by its thickness d and refractive index $n = n_1 + n_2 \cdot I$; where n_2 is the nonlinear refractive index of the material.

The phase shift $\Delta\phi$ associated with the Gaussian beam emerging from the Kerr medium writes as follows

$$\Delta\phi = knd = k \cdot n_1 \cdot d + k \cdot n_2 \cdot d \cdot I_0 \exp[-2\rho^2 / W_0^2] \quad (2)$$

where $k = 2\pi / \lambda$. Note that the first term in Eq. (2) is usually omitted since it does not contribute to the lensing effect under study. Finally, the phase shift given in Eq. (2) reduces to $\Delta\phi_1$ given in Eq. (3) by setting $\phi_0 = k \cdot n_2 \cdot d \cdot I_0$.

$$\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2 / W_0^2] \quad (3)$$

Knowing the relationship $P = (\pi W_0^2 I_0 / 2)$ between the power P and the on-axis intensity I_0 of the Gaussian beam of width W_0 , we obtain the following expression for the on-axis phase shift

$$\phi_0 = \frac{4n_2 P d}{\lambda W_0^2} \quad (4)$$

The phase distribution expressed in Eq. (3) has to be considered as a phase aberration behaving as an aberrated lens. The later is usually called as *Kerr lens* which definition is not unique as recently shown [8]. However, the key point of KLM lasers modeling involves the Kerr lens in term of *ABCD* formalism based on approximation of the Gaussian term in Eq. (3) into a parabolic term. This technique is known as *parabolic approximation* and consists to develop at first order the exponential function in Eq. (3) leading to the following approximate phase distribution:

$$\Delta\phi_2(\rho) = \phi_0 \left[1 - 2 \frac{\rho^2}{W_0^2} \right] \quad (5)$$

The phase distribution given in Eq. (5) corresponds to a pure distributed lens (without aberration). This case has been already considered in [9] by defining the equivalent focal length f of a transparent medium of length d having a refractive index $n(\rho)$ distributed along a parabola given by

$$n(\rho) = n_0 \left[1 - 2 \frac{\rho^2}{\gamma^2} \right], \quad (6)$$

$$f = \frac{\gamma}{2n_0 \sin\left(\frac{2d}{\gamma}\right)}. \quad (7)$$

In the peculiar case where $d \ll \gamma$, Eq. (7) reduces to the well known result given by

$$f = \frac{\gamma^2}{4n_0 d} \quad (8)$$

which takes the following form in the literature addressing the KLM modelling

$$\frac{1}{f} = \frac{8n_2 \cdot P \cdot d}{\pi W_0^4} \quad (9)$$

The context of the *parabolic approximation* which has been presented above is questionable at least on two main points given below:

- At first sight, it is difficult to have confidence in the foundation of the *parabolic approximation* which consist to approximate $\Delta\phi_1(\rho)$ by $\Delta\phi_2(\rho)$. For that, we have only to examine the radial distributions of $\Delta\phi_1$ and $\Delta\phi_2$ shown in Fig. 2. It is seen that the *parabolic approximation* is valid only very close the axis $\rho = 0$ while the incident

Gaussian beam has 99% of its total power in a circle of radius $1.5W_0$. Note that the *parabolic approximation* is of a common use in optical thermal effect studies because the refractive index distribution due to thermal effects is very close to a parabola for lateral pumping [10] or end-pumping [11].

- In fact, the approximation $d \ll \gamma$ leading to Eq. (9) cannot be justified in most cases since it assumes $d \ll W_0$. This condition is generally not fulfilled in KLM lasers since the length d is about some millimeters either in the case where the amplifying medium plays the role of Kerr medium [12], or when the gain medium and Kerr medium are separated [7]. On the other hand, the size W_0 of the phase profile $\Delta\phi_1$ has a magnitude of some tens of micrometers so that the condition $d \ll W_0$ cannot be fulfilled. As a consequence, the Kerr lens focal length given by Eq. (9), although of common use in KLM laser literature, seems to be not appropriate for describing the Kerr lens effect.

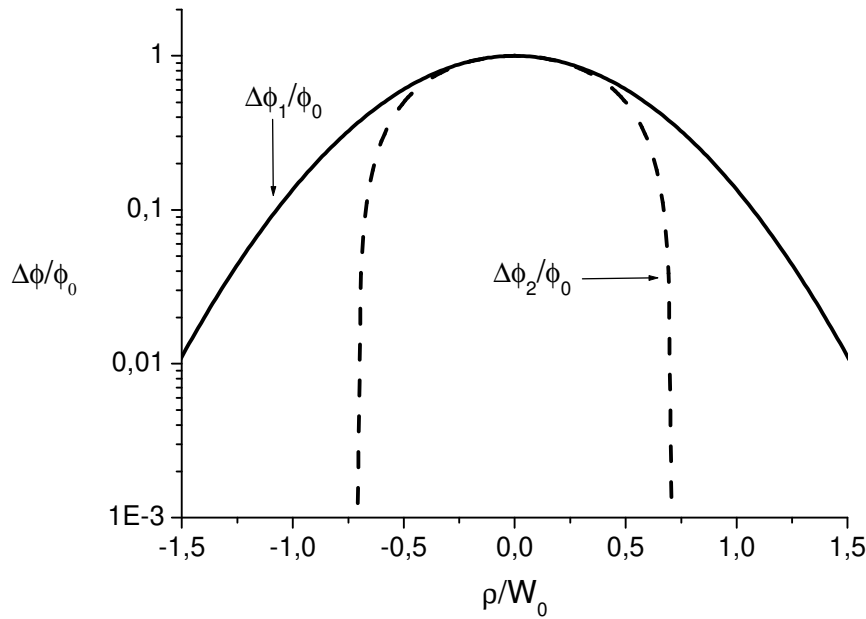


Figure 2: Transverse distribution of the normalised phase shift: Gaussian phase shift (solid line), parabolic phase shift (dashed line)

Now, we have to evaluate the accuracy of the usual *parabolic approximation* for predicting the losses induced by the hard aperture in Fig. 1. For that we will determine the aperture transmission T_D by two different ways:

- First: The light incident on the aperture results from the diffraction of the collimated Gaussian beam of width W_0 upon the phase profile $\Delta\phi_1(\rho)$. First, we determine the diffracted field $E_d(r, L)$ in the aperture plane $z=L$ in accordance with Fig. 3 by using the FRESNEL-KIRCHHOFF integral:

$$E_d(r, L) = \frac{2\pi}{\lambda L} \int_0^\infty E_{in}(\rho) \exp[-i\Delta\phi_1(\rho)] \exp\left[-\frac{i\pi\rho^2}{\lambda L}\right] J_0\left[\frac{2\pi}{\lambda L} r \cdot \rho\right] \rho d\rho \quad (10)$$

with $E_{in}(\rho) = E_0 \exp[-\rho^2 / W_0^2]$,

where r is the radial coordinate in the aperture plane, J_0 the zero-order Bessel function of first order, $\lambda=1064\text{nm}$ is the light wavelength.

The transmission T_{D1} of the diaphragm of radius r_D is then deduced from the following ratio

$$T_{D1} = \frac{\int_0^{r_D} |E_d(r, L)|^2 r dr}{\int_0^\infty |E_d(r, L)|^2 r dr} \quad (11)$$

In the following, T_{D1} will be considered as a reference since the calculation of the diffracted field $E_d(r, L)$ is made without any assumption except that the Kerr medium is assumed to be thin, i.e. we ignore propagation effects in the Kerr medium.

- Second: The transmission of the diaphragm, denoted T_{D2} is calculated by assuming that the beam emerging the Kerr lens remains Gaussian. The Kerr medium is characterised by a phase distribution $\Delta\phi_2(\rho)$ given by Eq. (5), and in the framework of the *parabolic approximation* is equivalent to a thin lens of focal length given by Eq. (9) which after rearrangement writes as follows

-

$$\frac{1}{f} = \frac{2\lambda\phi_0}{\pi W_0^2} \quad (12)$$

We are dealing with a simple transformation of a collimated Gaussian beam of width W_0 by a lens of focal length f given by Eq. (12) into a new Gaussian beam [9] having a beam-waist of width W_0' given by

$$W_0' = \frac{W_0}{\sqrt{1 + \left(\frac{\pi W_0^2}{\lambda f} \right)^2}} \quad (13)$$

This new beam-waist occurs at a distance, denoted z_f , from the Kerr medium.

$$z_f = \frac{f}{1 + \left(\frac{\lambda f}{\pi W_0^2} \right)^2} \quad (14)$$

Finally, the width W_L of the Gaussian beam incident on the aperture is given by

$$W_L = W_0' \sqrt{1 + \left(\frac{\lambda(L - z_f)}{\pi W_0'^2} \right)^2} \quad (15)$$

In accordance with **Fig. 4**, then the transmission T_{D2} of the aperture of the above Gaussian beam writes as follows:

$$T_{D2} = 1 - \exp \left[-\frac{2r_D^2}{W_L^2} \right] \quad (16)$$

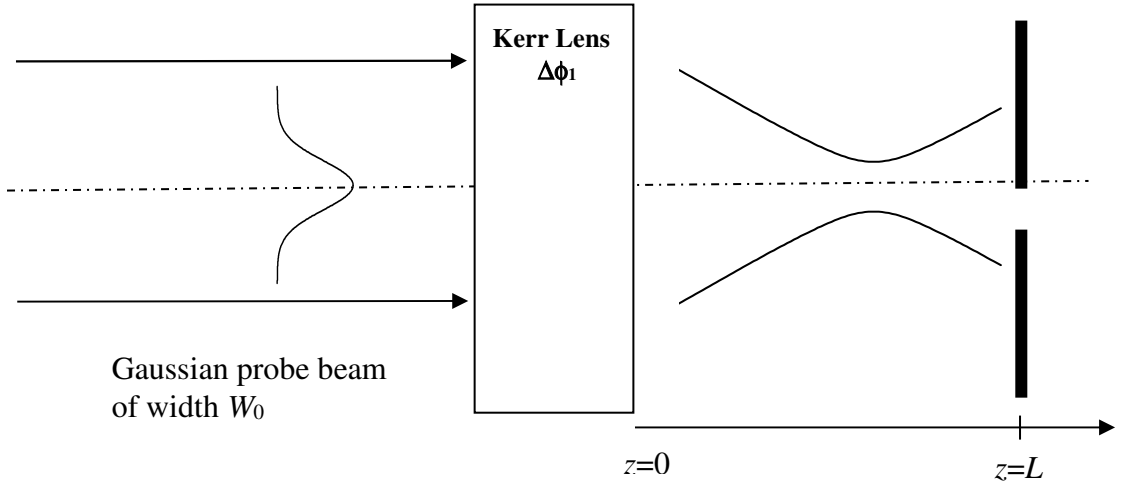


Figure 3: Set-up showing the geometry of the diffraction of a Gaussian beam of width W_0 on a Kerr phase shift $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$. The diaphragm of radius r_D is set at a distance $z=L$ from the Kerr medium assumed to be thin (thickness very small in comparison with the associated equivalent focal length).

Let us introduce the “initial transmission” of the diaphragm, denoted T_{ini} , which corresponds to the diaphragm transmission without Kerr lens effect, and which is supposed to be the

transmission of the diaphragm, inside a KLM laser, for the CW regime . Note that T_{ini} has to be sufficiently low in order to prevent the CW laser oscillation. The variations of T_{D1} and T_{D2} as a function of the normalised on-axis phase shift ϕ_0/π are shown in Fig. 5 and 6 where the diaphragm transmissions are plotted with a logarithmic scale. Note that in Fig. 6 the combination (Kerr lens+diaphragm) acts like a “saturable absorber” for $L=10\text{mm}$ and for the lower values of ϕ_0 , while it is rather an “inverse saturable absorber” for higher values of ϕ_0 .

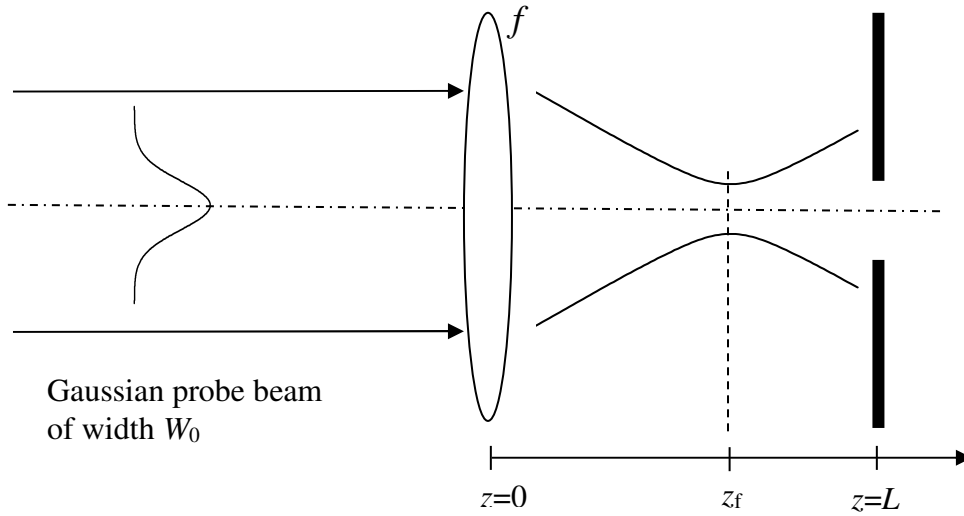


Figure 4: Set-up showing the geometry of the focusing of the Gaussian beam of width W_0 through a thin lens of focal length $f = \pi W_0^2 / (2\lambda\phi_0)$ corresponding to the quadratic phase shift profile $\Delta\phi_2(\rho) = \phi_0 [1 - 2\rho^2 / W_0^2]$. The diaphragm of radius r_d is set at a distance $z=L$ from the Kerr medium (thickness very small in comparison with the associated equivalent focal length).

In Fig. 5 it is seen that for $L=150\text{mm}$ the behaviour is that of an inverse saturable absorber since the aperture transmission is essentially decreasing with ϕ_0 , i.e. with the light intensity. The role of distance L is relatively simple to understand since before (after) the focal plane associated with the Kerr lens the beam width reduces (increases), and thus the aperture transmission increases (decreases). This is the case in Fig. 5 for all values of ϕ_0/π . In contrast, it should be recognised that in Fig. 6 the Kerr focal length is probably larger (smaller) than the distance L for the lower (higher) values of ϕ_0 .

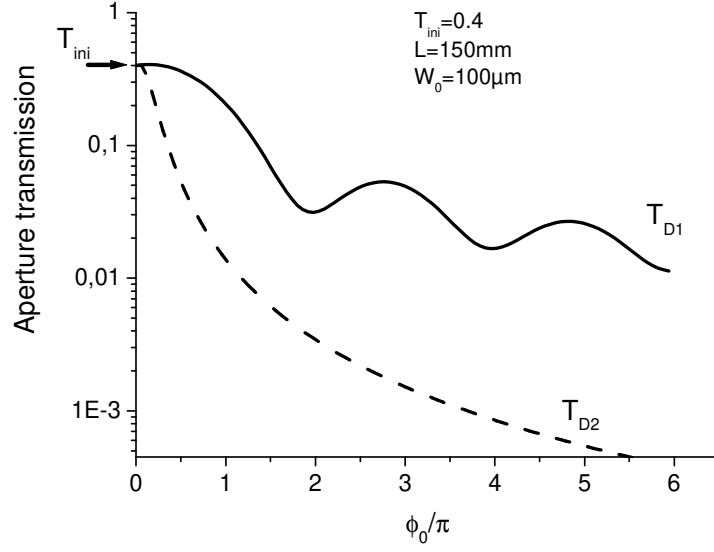


Figure 5: Variations of diaphragm transmissions T_{D1} (Gaussian phase shift) and T_{D2} (*parabolic approximation*) versus the normalised on-axis Kerr phase shift ϕ_0/π . The distance between the Kerr lens and the hard aperture is $L=150\text{mm}$. The transmission of the diaphragm without the Kerr lens is $T_{\text{ini}}=40\%$, and the width of the incident beam is $W_0=100\mu\text{m}$.

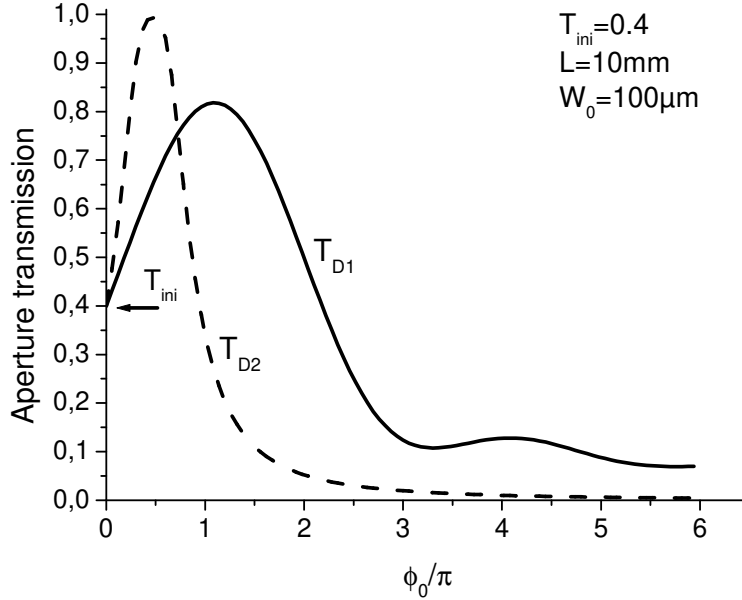


Figure 6: Variations of diaphragm transmissions T_{D1} (Gaussian phase shift) and T_{D2} (*parabolic approximation*) versus the normalised on-axis Kerr phase shift ϕ_0/π . The distance between the Kerr lens and the hard aperture is $L=10\text{mm}$. The transmission of the diaphragm without the Kerr lens is $T_{\text{ini}}=40\%$, and the width of the incident beam is $W_0=100\mu\text{m}$.

Before to examine the validity of the focal length given by Eq. (12), let us consider a second important hypothesis on which relies the parabolic approximation. This hypothesis consists to assume that the beam emerging from the Kerr lens, i.e. the Kerr aberration described by Eq. (3), keeps its Gaussian nature. For evaluating the relevance of this assumption, let us consider the beam factor quality M^2 of the Gaussian beam passing through the Kerr lens. In the case where the beam crossing through the Kerr lens should be Gaussian, it is expected to have $M^2 \approx 1$. In contrast, a value for M^2 larger than unity is a signature of an aberrated beam having lost its Gaussian character. Note that the diffracted field $E_d(r, L)$ is calculated by using the Kirchhoff's diffraction integral given by Eq. (10) whose integrand is multiplied by the phase term $\exp[ik\rho^2/(2f_L)]$ for accounting the presence of a focusing thin lens of focal length f_L .

In practice, the collimated Gaussian beam of width $W_0=0.25\text{mm}$ crosses two adjoining lenses: the first one is the Kerr lens, and the second one is the thin lens of focal length $f_L=125\text{mm}$. The M^2 factor of the emerging beam through the combination (Kerr lens+thin lens of focal length f_L) is deduced from the longitudinal variations of the diffracted beam width W_2 based on the second-order intensity moment [13] defined as follows

$$W_2^2(z) = \frac{2 \int_0^\infty |E_d(r, z)|^2 r^3 dr}{\int_0^\infty |E_d(r, z)|^2 r dr} \quad (17)$$

The calculation of $W_2(z)$ has been done for 60 values of coordinate z , the distance from the combination (Kerr lens+thin lens of focal length f_L). The longitudinal distribution of W_2 is characterised by two quantities:

- (i) the minimum value of W_2 , denoted $W_{2\min}$
- (ii) the longitudinal position, denoted z_{\min} , where the beam focuses, i.e. where

$$W_2 = W_{2\min}.$$

The value of the beam propagation factor M^2 is deduced from a fit of the plot W_2 versus z with the following parabola

$$W_2^2(z) = W_{2\min}^2 \left[1 + \left(\frac{M^2 \lambda (z - z_{\min})}{\pi W_{2\min}} \right)^2 \right] \quad (18)$$

The results are shown in Fig. 7, and it is seen that the M^2 factor increases roughly linearly with the on-axis phase shift ϕ_0 . In addition, we have established that the M^2 factor does not depend upon W_0 , the value of the incident Gaussian beam. In Fig. 7, it is seen that the M^2 factor increases rapidly with on-axis phase shift ϕ_0 , and this calls into question the common assumption of the Gaussian nature of the beam emerging from a Kerr lens and that as soon as ϕ_0 exceeds a fraction of π . We will come back to this point below. The results shown in Fig. 7 indicates prospectively that the M^2 factor of a KLM laser beam should have a time-dependent M^2 factor in the pico or femtoseconde scale. Such a situation of a time-dependent M^2 factor [14] or far-field beam spreading has been already studied in a Q-switching regime of a laser showing a mechanism of intensity-dependent refractive index which is different from the optical Kerr effect [15,16]. In any case, it is probably impossible or very difficult to determine experimentally the time-dependence of the M^2 factor of ultra-short laser pulses.

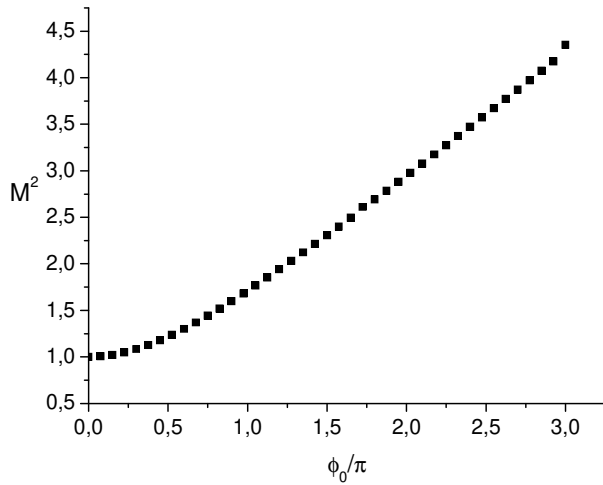


Figure 7: Variations of the beam quality factor M^2 of the Gaussian beam passing through the Gaussian Kerr lens described by $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$ as a function of the normalised on-axis phase shift ϕ_0/π for $W_0=0.25\text{mm}$.

Let us now return on the discussion about the validity of the *parabolic approximation* for expressing (see Eq. (12)) the equivalent focal length of a Kerr lens characterised by the phase shift distribution $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$. For that we need a certain reference with regard to the focusing properties of the Kerr lens which can be deduced from the true position of the

beam focus position z_{\min} where is located the beam-waist of the beam passing through the combination (Kerr lens of focal length f_K + thin lens of focal length f_L). These two lenses assumed to be in contact can be replaced by a lens having a focal length noted f_{eq} and given by

$$\frac{1}{f_{eq}} = \frac{1}{f_K} + \frac{1}{f_L} \quad (19)$$

In the framework of the Gaussian optics, the actual focal spot of a laser beam having a width W_0 incident on a lens of focal length f_{eq} located in its beam-waist plane gives rise to a focal spot which is not exactly located at the geometrical focus f_{eq} but is shifted towards the lens at position z_{\min} given by

$$z_{\min} = \frac{f_{eq}}{1 + \left(\frac{f_{eq}}{z_R}\right)^2} \quad (20)$$

where $z_R = \pi W_0^2 / \lambda$ is the Rayleigh distance of the incident Gaussian beam. In the case where $z_R \gg f_{eq}$ then Eq. (20) reduces to $z_{\min} \approx f_{eq}$.

The knowledge of the variations of the beam-width $W_2(z)$ as function coordinate z , used previously for M^2 calculation, gives the true location z_{\min} of the beam focus. Its identification with Eq.(20) leads to a quadratic equation from which we select the solution $f_{eq} \leq f_L$. Finally, we deduce the value of f_K from Eq. (19). The results are given in Fig. 8 showing that the focal length f_K describing the “true” focal plane of the beam passing through the Kerr lens increases continuously as a function of (π/ϕ_0) with values larger than the focal length $f = \pi W_0^2 / (2\lambda\phi_0)$ resulting from the *parabolic approximation* (see Eq. (12)), and this difference increases for lower values of ϕ_0 .

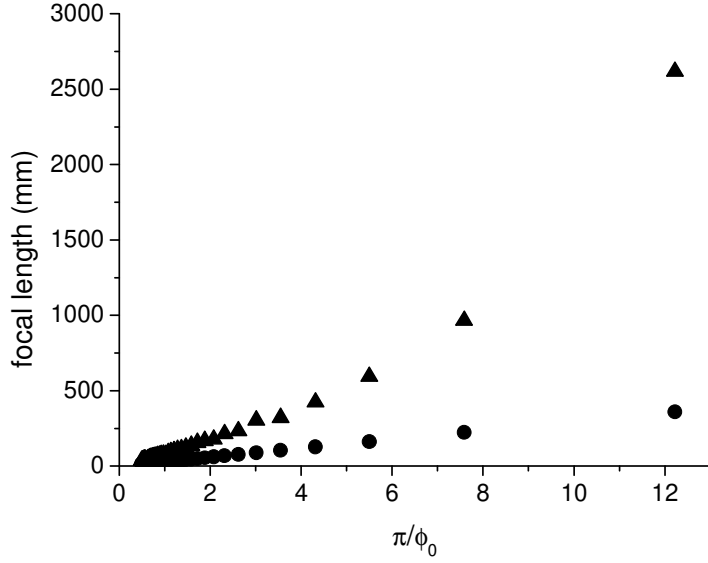


Figure 8: Variations of the two focal length of interest versus the normalised phase shift (π/ϕ_0):

- (a) Triangles: Kerr focal length f_K deduced from the beam-waist position of the Gaussian beam of width $W_0 = 0.25\text{mm}$ focused by the combination (Kerr lens+thin lens of focal length $f_L=125\text{mm}$).
- (b) Circles: Focal length f obtained from the *parabolic approximation* (Eq. 12)) for $W_0 = 0.25\text{mm}$

Now, in order to check that this agreement between f_K and f observed in Fig. 8 for $W_0 = 0.25\text{mm}$ occurs also for other values of width W_0 , we have plotted in Fig. 9 the variations of f_K and f for $W_0 = 1\text{mm}$.

From Fig. 8 it is seen that the focal length f (Eq. 12)) deduced from the usual *parabolic approximation* of the Gaussian phase shift distribution does not describe the location of focus plane associated with a Gaussian Kerr phase shift distribution.

We have empirically found that the focal length f multiplied by a factor equal to 3 allows to get values slightly equal to the focal length f_K . Consequently, we can adopt as a focal length describing the focusing properties of a Gaussian Kerr lens, characterised by a phase shift distribution $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$, the focal length noted f_p and given by

$$f_p = \frac{3\pi W_0^2}{2\lambda\phi_0} = \frac{4.7W_0^2}{\lambda\phi_0} \quad (21)$$

The plots of f_K and f_p shown in Fig. 9 demonstrate that Eq. (21) is a good compromise for describing the focal length of a Gaussian Kerr lens, and can be considered as resulting from a *corrected parabolic approximation*.

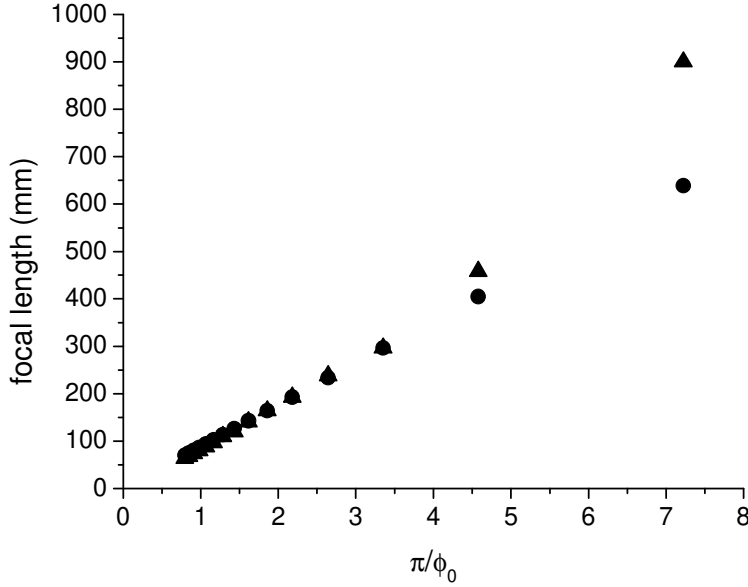


Figure 9: Variations of the two focal length of interest, f_K (triangle) and f_p (circle), versus the normalised phase shift (π/ϕ_0):

- (c) Triangles: Kerr focal length f_K deduced from the beam-waist position of the Gaussian beam of width $W_0 = 0.25\text{mm}$ focused by the combination (Kerr lens+thin lens of focal length $f_L=125\text{mm}$).
- (d) Circles: Focal length f_p obtained from the *corrected parabolic approximation* (Eq. 21)) for $W_0 = 0.25\text{mm}$

However, for the sake of completeness, we have to check if the Kerr focal length f_K is really proportional to W_0^2 like the focal length f_p . The result is shown in Fig. 10 which displays the variations of f_K with W_0^2 for $\phi_0 = \pi$ which is found to be linear and allows to consider the focal length f_p given by Eq. (21) as worthy of confidence for describing the location of the focusing plane of a Gaussian Kerr lens. Similar results are obtained with other values of ϕ_0 , the on-axis phase shift.

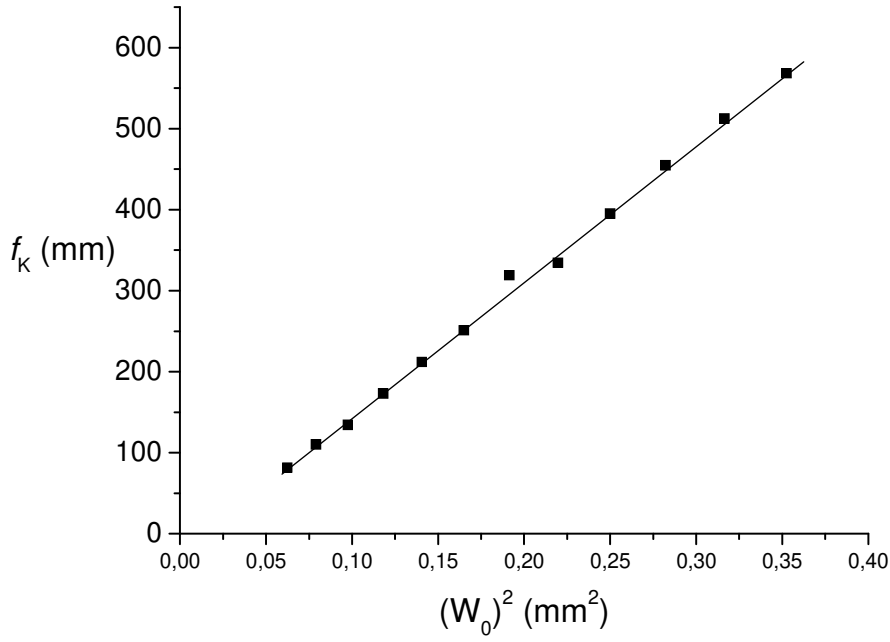


Figure 10: Variations of Kerr focal length f_K versus W_0^2 for an on-axis phase shift $\phi_0 = \pi$.

Now, it should be interesting to compare the analytical formulation of the focal length given by Eq. (21) to another one resulting from a Zernike decomposition. The wavefront $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$ can be expanded as a linear combination of Zernike polynomials, and the coefficients of this decomposition are precisely the aberration coefficients. The later are calculated in Appendix-1 by taking into account the Gaussian beam intensity profile. The result is distinct from the usual plane wave calculation. The focal length f_K of the Kerr lens can be expressed as a function of the defocus term by Eq. (A-7) in Appendix-1 associated with the aberration coefficient A_4 given in Table (A-2) for the Gaussian beam.

$$f_Z = \frac{\sqrt{3}R^2}{12\lambda A_4} \quad (22)$$

where R is the radius of the unit circle (Appendix-1) which is chosen so that the circle of radius R contains 99% of the incident Gaussian beam. Consequently, the radius of the unit circle is set to $R = 1.5W_0$ and the effective focal length noted f_Z is expressed as follows

$$f_Z = \frac{4.2W_0^2}{\lambda\phi_0} \quad (23)$$

It is seen that the focal lengths f_p and f_z given by Eqs. (21) and (23) are very close, and this strengthens the present study of the Kerr lensing, and validates the concept of *corrected parabolic approximation* described above. The *corrected parabolic approximation* giving the focal length f_p (Eq. (21)) is able to provide the right location of the focus z_{\min} associated with the Kerr lens. It remains now to examine if the focal length f_p allows to get the right beam transmission through a diaphragm (see Fig. 5 and 6), on the usual hypothesis of keeping the Gaussian nature of the beam. For that, let us determine the transmission T_{D3} of the diaphragm calculated along similar lines used previously when calculating T_{D2} , but by replacing the focal length f by f_p . The result is shown in Fig. 11 and it is seen that the focal length f_p allows an accurate determination of the aperture transmission ($T_{D3} \approx T_{D1}$) only if $\phi_0 \leq 0.6\pi$. Beyond this value, T_{D3} does not describe faithfully the aperture transmission because the beam is no longer Gaussian since its M^2 factor is beginning to be important as shown in Fig. 7.

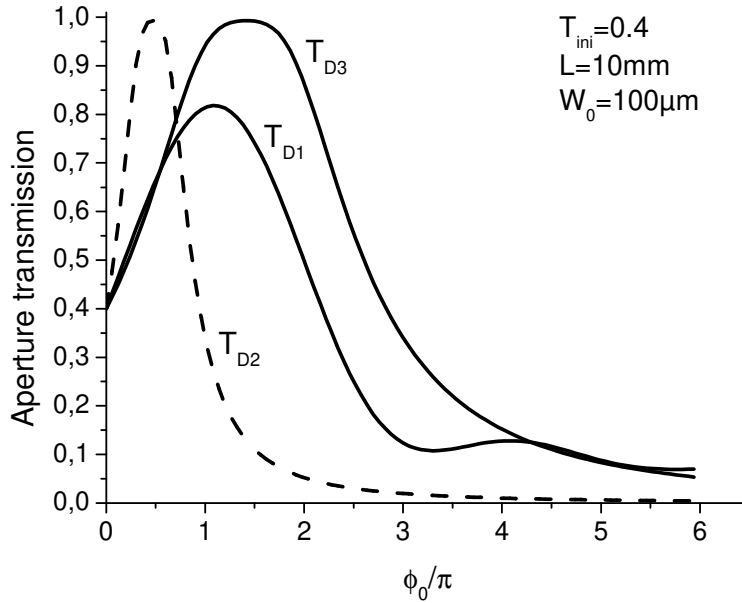


Figure 11: Variations of diaphragm transmissions T_{D1} (Gaussian phase shift), T_{D2} (f : *parabolic approximation*) and T_{D3} (f_p : *corrected parabolic approximation*) versus the normalised on-axis Kerr phase shift ϕ_0/π . The distance between the Kerr lens and the hard

aperture is $L=10\text{mm}$. The transmission of the diaphragm without the Kerr lens is $T_{\text{ini}}=40\%$, and the width of the incident beam is $W_0=100\mu\text{m}$.

3. Conclusion

We have considered the accuracy of the *parabolic approximation*, widely used in KLM lasers modelling, to express the lensing effect associated to a Kerr phase shift Gaussian in shape. A priori this approximation could be considered as rough in view of the fact that a parabola is a very bad approximation of a Gaussian function (see Fig. 2). To evaluate the accuracy of the equivalent focal length of a Kerr lens determined in the framework of this parabolic approximation we need a focal length of reference. For that, in a first step, we have defined a “true” focal length for the Kerr lens characterised by the phase shift distribution $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$ probed by a collimated Gaussian beam of width W_0 . The resulting diffracted field distribution has been determined without any approximation by using the usual Fresnel-Kirchhoff integral allowing the knowledge of the true focal plane z_{min} where the width, based on the second-order intensity moment, of the diffracted beam is minimum.

The first important conclusion is that the usual *parabolic approximation* is not able to provide the right location of the focus associated with the Kerr lens. Note that we have recently shown [8] that the different possible modellings of a distributed lensing effect characterised by the phase shift $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$ lead to different effective focal lengths without being able to identify the best one because neither focal length could be considered as a reference. In the present work, we believe that we have stated unambiguously the focal length f_K of a Kerr lens from the position z_{min} of the true focal plane, and we have found that the *parabolic approximation* through the focal length $f = \pi W_0^2 / (2\lambda\phi_0)$ underestimates the value of the Kerr lens focal plane. We have demonstrated that the focal length $f_p = 3f$ describe correctly the Kerr lens in the sense that $f_K \approx f_p = 3\pi W_0^2 / (\lambda\phi_0)$ in a large range of values for ϕ_0 until 1.6π . The expression of the focal length f_p of the Kerr lens resulting from the *corrected parabolic approximation* has been confirmed by the focal length f_Z resulting from the Zernike decomposition of the phase term $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$.

The second important conclusion that can be drawn concerns the second assumption of the *parabolic approximation* which states that the beam emerging from the Kerr lens remains

Gaussian, has been demonstrated to be wrong. Indeed, a probe Gaussian beam of width W_0 passing through a Kerr lens characterised by the phase shift $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$ has a beam propagation factor M^2 greater than unity, approximately proportional to ϕ_0 and independent from W_0 . It is found that the M^2 factor increases from 1 to 4.5 when the on-axis phase shift ϕ_0 increases from 0 to 3π . The Optical Kerr effect induced by a Gaussian laser beam should not be reduced only to a simple pure thin lens. It is important to keep in mind that a Kerr lens is above all basically a phase aberration. As a result, it has been found that the *parabolic approximation* is not able to estimate the losses due to a simple diaphragm by assuming a Gaussian nature of the beam. However, the *corrected parabolic approximation* introduced in this work, i.e. the focal length f_p , is able to describe the losses due to a hard aperture on the hypothesis of a Gaussian beam but only for $\phi_0 \leq 0.6\pi$. This should be a crucial issue for modelling accurately the laser cavity operating in KLM regime. Since the parabolic approximation gives a Kerr focal length deduced from the *ABCD* formalism unable to correctly provide the focal plane, this raises serious doubts about the stability limits of a KLM laser cavity obtained in the context of the usual *parabolic approximation*. We believe that reducing the Kerr lens effect to a thin lens of focal length $f_p = 3\pi W_0^2 / (\lambda \phi_0)$ obtained in the framework of the so-called *corrected parabolic approximation* is credible to describe the stability limits of a KLM laser cavity, and the losses induced by the intracavity hard aperture only if $\phi_0 \leq 0.6\pi$, i.e. the aberrations associated with the Kerr effect are neglected. Otherwise, these aberrations have to be taken into account so that the fundamental mode of the laser cavity including a Kerr lens is probably no longer Gaussian, and its determination should be complicated.

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Appendix-1: Zernike polynomial decomposition of the OKE

The propagating term of the collimated incident Gaussian beam emerging from the distributed lens writes $\exp[ikS(\rho, \theta)]$, where $kS(\rho, \theta) = -\Delta\phi_1(\rho)$ and $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$. The phase distribution $S(\rho, \theta)$ is independent of angle θ , and is usually called as wave aberration function (WAF) according to the terminology of optical aberration modelling [17]. As usual, the WAF is expanded as a linear combination of Zernike polynomials (ZP), Z_j , as follows

$$S(\bar{\rho}, \theta) = \sum_{j=1}^{\infty} a_j Z_j(\bar{\rho}, \theta), \quad (\text{A-1})$$

where the index j is a polynomial-ordering number, and a_j the aberration coefficients. The normalised radial coordinate is defined as $\bar{\rho} = \rho/R$, where R is the radius of the unit circle. In the following we set R equal to $1.5W_0$ since this circle contains 99% of the incident Gaussian beam power. The Zernike polynomials $Z_j(\bar{\rho}, \theta)$ are a set of orthogonal functions over the unit circle ($0 \leq \bar{\rho} \leq 1$).

The sum in Eq. (A-1) is infinite, but is usually truncated. Here, we will work arbitrarily until $j=22$, and because of the rotational symmetry of $\Delta\phi_1$ most of coefficients a_j are equal to zero except four of them: a_1, a_4, a_{11} and a_{22} . Table A-1 gives the corresponding ZP's [18].

j	Z_j	Type of aberration
1	1	Piston
4	$\sqrt{3}(2\bar{\rho}^2 - 1)$	Defocus
11	$\sqrt{5}(6\bar{\rho}^4 - 6\bar{\rho}^2 + 1)$	Primary spherical
22	$\sqrt{7}(20\bar{\rho}^6 - 30\bar{\rho}^4 + 12\bar{\rho}^2 - 1)$	Secondary spherical

Table A-1: Zernike polynomials allowing to calculate the four non-zero aberration coefficients a_1, a_4, a_{11} and a_{22} .

The aberration coefficient a_j takes a different form depending on the type of incident wave: plane wave of amplitude A_0 , or Gaussian beam of amplitude $A(\rho) = A_0 \exp[-\rho^2/W_0^2]$. The expression of a_j 's write as follows:

For a plane wave [18] :

$$a_j = -\frac{1}{k\pi} \int_0^1 \int_0^{2\pi} \Delta\phi_1(\bar{\rho}, \theta) Z_j(\bar{\rho}, \theta) \bar{\rho} d\bar{\rho} d\theta \quad (\text{A-2})$$

For a Gaussian beam [19]:

$$a_j = \frac{-\int_0^1 \int_0^{2\pi} A(\bar{\rho}) \Delta\phi_1(\bar{\rho}, \theta) Z_j(\bar{\rho}, \theta) \bar{\rho} d\bar{\rho} d\theta}{k \int_0^1 \int_0^{2\pi} A(\bar{\rho}) \bar{\rho} d\bar{\rho} d\theta} \quad (\text{A-3})$$

It is convenient to use the dimensionless aberration coefficients A_j which are expressed in unit of wavelength, and defined as

$$A_j = \frac{a_j}{\lambda} \quad (\text{A-4})$$

The integral in Eqs. (A-2) and (A-4) implements the product of a Gaussian function and a polynomial. Its calculation can be made analytically by using the table of integrals in REF. [20]. The results are shown in Table (A-2) proving that the phase aberration $\Delta\phi_1(\rho)$ is characterised by aberrations coefficients which are very different according to the probe beam (plane wave or Gaussian beam).

Incident wave	A_1	A_4	A_{11}	A_{22}
Plane wave	$-2 \times 10^{-2} \phi_0$	$+2.5 \times 10^{-2} \phi_0$	$-4.7 \times 10^{-2} \phi_0$	$+1 \times 10^{-2} \phi_0$
Gaussian beam	$-5.3 \times 10^{-2} \phi_0$	$+7.7 \times 10^{-2} \phi_0$	$-7 \times 10^{-2} \phi_0$	$+4.9 \times 10^{-2} \phi_0$

Table A-2: Dimensionless aberration coefficients associated to a phase aberration $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$ when the incident wave is plane or Gaussian of width $W=1\text{mm}$ and $\lambda = 1064\text{nm}$.

It is worth noting that the aberration term for $j=4$ is known as “defocus term” which has to be considered as a pure lensing effect which we will hereafter use for defining the equivalent focal length associated with phase shift $\Delta\phi_1(\rho) = \phi_0 \exp[-2\rho^2/W_0^2]$. Let us consider the WAF, noted S_L , of a collimated laser beam after crossing a thin lens of focal length f :

$$S_L = \frac{\rho^2}{2f} = \frac{R^2 \bar{\rho}^2}{2f} \quad (\text{A-5})$$

Let us determine the coefficient a'_4 associated with the WAF given by Eq. (A-5). For that, we have to calculate the following integral:

$$a'_4 = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} S_L(\bar{\rho}, \theta) Z_4(\bar{\rho}, \theta) \bar{\rho} d\bar{\rho} d\theta = \frac{\sqrt{3}R^2}{12f} \quad (\text{A-6})$$

By identifying a'_4 given by Eq. (A-6) to the defocus coefficient a_4 , we find the expression of the equivalent focal length, noted f_z , resulting from the Zernike decomposition:

$$f_z = \frac{\sqrt{3}R^2}{12\lambda A_4} \quad (\text{A-7})$$