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Abstract

The aim of this paper is to present an algorithm to solve frictional contact problems by considering the Coulomb's criterion. As it is known, the frictional contact problem using Coulomb's criterion has no minimization principle behind. However in order to use algorithms based on minimization methods, the frictional contact problem is written as an optimization one, more specifically as a sequence of Tresca contact problems. Moreover a family of regularization functions is introduced in order to regularize the non-smooth character of the Tresca criterion, which in some cases can have an experimental justifications. As each minimization problem becomes smooth enough, the interior point method is used to solve the generated optimization problem.

Keywords— Frictional contact, regularization , optimization, fixed point, interior point method, symmetric algorithm

1 Introduction

The contact problems with friction were studied in many papers, for example in [10] the existence of solutions for elastic static contact problems with a small coefficient of friction was proved in the case of Coulomb's friction, where a penalty method was used. In the paper [3], existence of solutions for static Signorini's problem with Coulomb friction was proved, where the normal component of the stress was replaced by a regularized one, in addition, the uniqueness of the solution was proved for a small friction coefficient. In the chapter 3 of [9] several studies for elastic frictional problems were done in addition to the formulation into a variational inequality for the Signorini's problem with Coulomb's friction.

In [23] and in the case of linear elastic material, the Coulomb frictional contact problem is transformed into a sequence of Tresca frictional contact problems. Indeed the problem is transformed into a fixed point method where at each iteration a minimization problem over a convex set is solved. The bi-potential method was used for frictional dynamic problem in [11, 12], indeed the contact forces are computed by a process of prediction and correction, by a projection on the Coulomb cone. Then the contact forces are considered as an external loading.

A penalty formulation based on the integration points was considered in [13], to solve frictional contact problems. The integration in [13] was done on the non-mortar segments (or the slave contact segments), with a fixed integration points on the non-mortar segment, and not on the overlapping regions between mortar and non-mortar areas, so the computational effort is reduced. In [1, 25, 26] the elastic problems were considered, where a non-classical friction law was used in [25, 26], the idea was to fix the normal pressure at each iteration and update it for the next iteration.

The returning mapping method is a method used for the frictional contact problems, this method was already been used in plasticity problems (see [5, 6]), where like the yield surface, a surface which is dependent on the normal and on the tangential stress is created (see [14]). We can cite for example [28] where the penalty and the augmented Lagrangian methods were used. The return mapping method and the penalty method were also presented in [30] for contact problems with large deformations.

In this paper, we present an algorithm to solve frictional contact problems using Coulomb's criterion for elastic and finite deformation problems. As we know, the frictional contact problem using Coulomb's criterion has no minimization principle behind. However, expressing the contact problem in a minimization form and solve it with optimization methods can be a robust way to solve it, indeed we can use several optimization techniques (for example line search method) in order to converge faster to the solution which is a minimum. Therefore in order to use optimization algorithms, the frictional contact problem is written as an optimization one, more specifically as a sequence of Tresca contact problems until convergence (a fixed point method).

Each Tresca contact problem is equivalent to a minimization one, unfortunately the energy to minimize becomes not smooth enough, therefore we introduce a family of regularization functions in order to regularize the non-smooth part. In addition, in some cases, regularization can be justified because tangential slip always occurs, even for a small tangential stress [4, 24]. We can also cite [17], where a micro-displacement is produced between a hard steel ball and the flat end of a hard steel roller in contact, when the tangential force applied on the ball is less than the value necessary to produce slip.

In the case of large deformations, the non-penetration constraints are non-linear, therefore the contact problem is written into a sequence of problems with linear constraints, more specifically as a fixed point (more like [7, 8, 20]). In addition the non-penetration constraints are imposed in a symmetrical manner, in other words, non-penetration constraints are prescribed on the slave body to forbid the penetration of the latter into the master one, in addition to non-penetration constraints which are prescribed on the master body to forbid the penetration of the latter into the slave one. Generally, the symmetrical non-penetration constraints can be redundant or linearly dependent therefore numerical difficulties can be generated, we can see in [16] how these numerical difficulties are avoided when using the interior point method. In short, we have two loops, the exterior one is for the Tresca problems and the interior one is for the non-linearity of the constraints in the finite deformation case.

This paper is organized as follows: in section 2, we present several reminders

about frictional contact problems. A family of regularization functions to make the Tresca problem smooth enough in addition to the corresponding frictional regularized problems, are introduced in section 3. The finite deformation case is discussed in section 4. The existence and the uniqueness of a solution in addition to the convergence of the frictional fixed point algorithm in the discretized case are shown in section 5. The algorithm can be found in section 6. Finally, in section 7 our method is validated against several contact examples.

2 Linear elasticity

We consider here two elastic bodies $\Omega^l \subset \mathbb{R}^2$ or $\mathbb{R}^3$ with $l = 1, 2$ initially in contact at the border Γ_C (see Figure 1), the contact area after loading is supposed to be included in Γ_C. Let Γ_0^l be the border of the body Ω^l where a null displacement is imposed, and Γ_1^l where a surface traction $\mathbf{t}^l$ is imposed, in addition $\Omega = \Omega^1 \cup \Omega^2$. We call $\mathbf{n} := \mathbf{n}^1$, $\mathbf{n}^2$ respectively the outward unit normal vector on $\partial\Omega_1$ and on $\partial\Omega_2$. Finally, the body force $\mathbf{f}^l$ is applied on Ω^l.

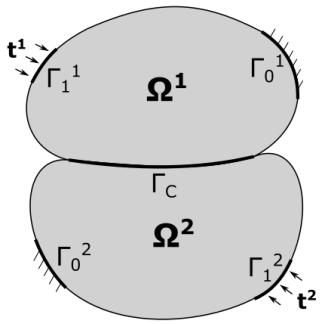

Figure 1: The two bodies in contact

The frictional contact problem using Coulomb's criterion is given as follows

$$\begin{cases} \nabla.\boldsymbol{\sigma}^l + \mathbf{f}^l = 0 & \text{in } \Omega^l \\ \boldsymbol{\sigma}^l = C^l \boldsymbol{\epsilon}^l & \text{in } \Omega^l \quad \text{(Hook's law)} \\ \mathbf{u}^l = \mathbf{0} & \text{on } \Gamma_0^l \\ \boldsymbol{\sigma}^l.\mathbf{n}^l = \mathbf{t}^l & \text{on } \Gamma_1^l \end{cases} \tag{1}$$

with the following contact conditions

$$\begin{cases} [\mathbf{u}.\mathbf{n}] = \mathbf{u}^1.\mathbf{n}^1 + \mathbf{u}^2.\mathbf{n}^2 = (\mathbf{u}^1 - \mathbf{u}^2).\mathbf{n} \leq 0 & \text{on } \Gamma_C \\ \sigma_n = (\boldsymbol{\sigma}^1.\mathbf{n}^1).\mathbf{n}^1 = (\boldsymbol{\sigma}^2.\mathbf{n}^2).\mathbf{n}^2 \leq 0 & \text{on } \Gamma_C \\ \sigma_n.[\mathbf{u}.\mathbf{n}] = 0 & \text{on } \Gamma_C \end{cases} \tag{2}$$

Here the normal vector $\mathbf{n}$ is considered to be equal to $\mathbf{n}^1$, in addition at the contact area we have $\mathbf{n}^1 = -\mathbf{n}^2$. Given a friction coefficient μ, the static Coulomb criterion

on Γ_C states

$$\begin{cases} \boldsymbol{\sigma}_T^1 = -\boldsymbol{\sigma}_T^2 \\ |\boldsymbol{\sigma}_T^1| \leq \mu|\sigma_n| \\ \text{if } |\boldsymbol{\sigma}_T^1| < \mu|\sigma_n| \Rightarrow \mathbf{u}_T^1 - \mathbf{u}_T^2 = \mathbf{0} \\ \text{if } |\boldsymbol{\sigma}_T^1| = \mu|\sigma_n| \Rightarrow \exists \lambda \geq 0 \text{ s.t } \mathbf{u}_T^1 - \mathbf{u}_T^2 = -\lambda \boldsymbol{\sigma}_T^1 \end{cases} \tag{3}$$

where the subscript T means the tangential part, otherwise speaking, for a vector $\boldsymbol{v}^l$, $\boldsymbol{v}_T^l = \boldsymbol{v}^l - (\boldsymbol{v}^l.\boldsymbol{n}^l)\boldsymbol{n}^l$. Moreover the symbol $|\cdot|$ for a vector means its module.

2.1 Tresca criterion

Let $\tau \in L^2(\Gamma_C) \geq 0$, be the sliding limit of the Tresca criterion, the governing equations are the same, except the system (3), which becomes

$$\begin{cases} \boldsymbol{\sigma}_T^1 = -\boldsymbol{\sigma}_T^2 \\ |\boldsymbol{\sigma}_T^1| \leq \tau \\ \text{if } |\boldsymbol{\sigma}_T^1| < \tau \Rightarrow \mathbf{u}_T^1 - \mathbf{u}_T^2 = \mathbf{0} \\ \text{if } |\boldsymbol{\sigma}_T^1| = \tau \Rightarrow \exists \lambda \geq 0 \text{ s.t } \mathbf{u}_T^1 - \mathbf{u}_T^2 = -\lambda \boldsymbol{\sigma}_T^1 \end{cases} \tag{4}$$

where $\mu|\sigma_n|$ was replaced by the sliding limit τ.

2.2 Variational formulation for Tresca criterion

The displacement field $\mathbf{u}$ is defined by $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2)$. We define the admissible set as $\mathbf{V} = \mathbf{V}^1 \times \mathbf{V}^2$ where

$$\mathbf{V}^l = \{\mathbf{v} \in \mathbf{H}^1(\Omega^l) = H^1(\Omega^l) \times H^1(\Omega^l) \,|\, \mathbf{v} = 0 \text{ a.e on } \Gamma_0^l\} \tag{5}$$

where $H^1(\Omega^l)$ is the Sobolev space endowed with the norm $\|\cdot\|_1$. The space $\mathbf{V}$ is endowed with the broken norm:

$$\|\mathbf{u}\|_1 = \|(\mathbf{u}^1, \mathbf{u}^2)\|_1 = \left(\|\mathbf{u}^1\|_1^2 + \|\mathbf{u}^2\|_1^2\right)^{\frac{1}{2}} \tag{6}$$

where $\|\mathbf{u}^l\|_1$ is the broken norm of the space $\mathbf{V}^l$. Let the applications $a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $f : \mathbf{V} \to \mathbb{R}$ be defined by

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) = a^1(\mathbf{u}, \mathbf{v}) + a^2(\mathbf{u}, \mathbf{v}) \\ f(\mathbf{v}) = f^1(\mathbf{v}) + f^2(\mathbf{v}) \end{cases} \tag{7}$$

where for $l = 1, 2$

$$\begin{cases} a^l(\mathbf{u}, \mathbf{v}) = \displaystyle\int_{\Omega^l} \boldsymbol{\sigma}(\mathbf{u}^l) : \boldsymbol{\epsilon}(\mathbf{v}^l)\, dv \\ f^l(\mathbf{v}) = \displaystyle\int_{\Omega^l} \mathbf{f}^l.\mathbf{v}^l\, dv + \int_{\Gamma_1^l} \mathbf{t}^l.\mathbf{v}^l\, ds \end{cases} \tag{8}$$

We also consider the application $j_\tau : \mathbf{V} \to \mathbb{R}_+$ defined by

$$j_\tau(\mathbf{v}) = \int_{\Gamma_C} \tau|\mathbf{v}_T^1 - \mathbf{v}_T^2|\, ds \tag{9}$$

The convex and closed set $\mathbf{K}$ describes the non-penetration between the two bodies, and is defined by

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} \,|\, [\mathbf{v}.\mathbf{n}] \leq 0 \text{ a.e on } \Gamma_C\} \tag{10}$$

where $[\mathbf{v}.\mathbf{n}] = (\mathbf{v}^1 - \mathbf{v}^2).\mathbf{n} = (\mathbf{v}^1 - \mathbf{v}^2).\mathbf{n}^1$. The variational formulation of the frictional contact problem using Tresca's criterion can be proven to be equal to

Find $\mathbf{u} \in \mathbf{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_\tau(\mathbf{v}) - j_\tau(\mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{11}$$

2.3 Minimization formulation for the Tresca criterion

Before giving the minimization formulation, let's recall a theorem which can be found in [19].

Theorem 2.1. *Let $\mathbf{K}$ be a nonempty, closed and convex, subset of the normed linear space $\mathbf{V}$, and consider a function $F : \mathbf{K} \subset \mathbf{V} \to \mathbb{R}$ of the form $F = F_1 + \Phi$ where F_1 and Φ are convex and lower semicontinuous and F_1 is Gâteaux differentiable on $\mathbf{K}$. Then $\mathbf{u}$ is a minimizer of F on $\mathbf{K}$ if and only if,*

$$< DF_1(\mathbf{u}), \mathbf{v} - \mathbf{u} > + \Phi(\mathbf{v}) - \Phi(\mathbf{u}) \geq 0 \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{12}$$

Let $\mathcal{E}_p$ denotes the total potential energy of the two bodies, $\mathcal{E}_p$ can be given by

$$\mathcal{E}_p(\mathbf{v}) := \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - f(\mathbf{v}) \tag{13}$$

Consider the energy functional J_τ given by

$$J_\tau(\mathbf{v}) := \mathcal{E}_p(\mathbf{v}) + j_\tau(\mathbf{v}) \tag{14}$$

Then by applying the theorem 2.1 above, by taking $F = J_\tau$, $F_1 = \mathcal{E}_p$ and $\Phi = j_\tau$, the frictional problem (11) is equivalent to the following minimization problem

Find $\mathbf{u} \in \mathbf{K}$ such that

$$J_\tau(\mathbf{u}) \leq J_\tau(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{15}$$

2.4 Coulomb's criterion as a fixed point problem

The idea to study the Tresca criterion, is that the Coulomb criterion can be equivalent to the fixed point of the following application (see [22, 27])

$$T(\tau) = -\mu \sigma_N(\mathbf{u}_\tau) \tag{16}$$

where σ_N the normal stress and $\mathbf{u}_\tau$ the solution of the Tresca problem with the sliding limit $\tau \geq 0$, otherwise speaking, solution of the following problem

Find $\mathbf{u}_\tau \in \mathbf{K}$ such that

$$a(\mathbf{u}_\tau, \mathbf{v} - \mathbf{u}_\tau) + j_\tau(\mathbf{v}) - j_\tau(\mathbf{u}_\tau) \geq f(\mathbf{v} - \mathbf{u}_\tau) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{17}$$

or equivalently

Find $\mathbf{u}_\tau \in \mathbf{K}$ such that

$$J_\tau(\mathbf{u}_\tau) \leq J_\tau(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{18}$$

Otherwise speaking, if τ^* is the fixed point of the application T, $\tau^* = -\mu \sigma_N(\mathbf{u}_{\tau^*})$, then $\mathbf{u}_{\tau^*}$ is the solution of the frictional problem using Coulomb's criterion.

2.5 The quasi-static problem for Coulomb's criterion

In reality the friction depends on the history of the loading, indeed the Coulomb criterion depends on the velocity rather than the displacement, and therefore the friction depends on the state of the previous time step. However the static criterion is very useful to treat the quasi-static case, because as we will see the quasi-static criterion can be written as a sequence of a static criterion when the velocity is discretized.

The quasi-static criterion is given by the following

$$\begin{cases} |\boldsymbol{\sigma}_T^1| \leq \mu|\sigma_n| \\ \text{if } |\boldsymbol{\sigma}_T^1| < \mu|\sigma_n| \Rightarrow \dot{\mathbf{u}}_T^1 - \dot{\mathbf{u}}_T^2 = \mathbf{0} \\ \text{if } |\boldsymbol{\sigma}_T^1| = \mu|\sigma_n| \Rightarrow \exists \lambda \geq 0 \text{ s.t } \dot{\mathbf{u}}_T^1 - \dot{\mathbf{u}}_T^2 = -\lambda \boldsymbol{\sigma}_T^1 \end{cases} \tag{19}$$

where $\dot{\mathbf{u}}$ denotes the velocity.

For a time step Δt the velocities $\dot{\mathbf{u}}_T^1$ and $\dot{\mathbf{u}}_T^2$ are discretized as follows

$$\begin{cases} \dot{\mathbf{u}}_T^1 = \dfrac{\mathbf{u}_{T,i+1}^1 - \mathbf{u}_{T,i}^1}{\Delta t} \\ \dot{\mathbf{u}}_T^2 = \dfrac{\mathbf{u}_{T,i+1}^2 - \mathbf{u}_{T,i}^2}{\Delta t} \end{cases} \tag{20}$$

where $i+1$ and i denotes respectively the actual and the previous state. Therefore the quasi-static criterion becomes

$$\begin{cases} |\boldsymbol{\sigma}_T^1| \leq \mu|\sigma_n| \\ \text{if } |\boldsymbol{\sigma}_T^1| < \mu|\sigma_n| \Rightarrow (\mathbf{u}_{T,i+1}^1 - \mathbf{u}_{T,i}^1) - (\mathbf{u}_{T,i+1}^2 - \mathbf{u}_{T,i}^2) = \mathbf{0} \\ \text{if } |\boldsymbol{\sigma}_T^1| = \mu|\sigma_n| \Rightarrow \exists \lambda \geq 0 \text{ s.t } (\mathbf{u}_{T,i+1}^1 - \mathbf{u}_{T,i}^1) - (\mathbf{u}_{T,i+1}^2 - \mathbf{u}_{T,i}^2) = -\lambda \boldsymbol{\sigma}_T^1 \end{cases} \tag{21}$$

3 Regularization of the Tresca frictional problem

Recall that the frictional problem using Tresca's criterion is given by

Find $\mathbf{u} \in \mathbf{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_\tau(\mathbf{v}) - j_\tau(\mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{K} \tag{22}$$

The application j_τ is not differentiable because of its module term. For this reason and for an algorithmic point of view, in order to obtain a smooth problem, the module vector $|\cdot|$ is approximated by an application η_α.

We suppose that the regularization function η_α approximating the module of a vector in $\mathbb{R}^d$ $(d = 2, 3)$, belongs to the set Ξ_α, defined below.

Definition 3.1. *Define Ξ_α, for $\alpha > 0$, the set of functions such that*

$$\eta_\alpha \in \Xi_\alpha \iff \begin{cases} \eta_\alpha \in C^2(\mathbb{R}^d) \\ \eta_\alpha \textit{ is convex} \\ \eta_\alpha(\mathbf{v}) = \eta_\alpha(-\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{R}^d \\ \eta_\alpha(\mathbf{v}) \geq 0 \quad \forall \mathbf{v} \in \mathbb{R}^d \\ |\eta_\alpha(\mathbf{v}) - |\mathbf{v}|| \leq \alpha \quad \forall \mathbf{v} \in \mathbb{R}^d \\ |\eta_\alpha(\mathbf{v}_1) - \eta_\alpha(\mathbf{v}_2)| \leq ||\mathbf{v}_1| - |\mathbf{v}_2|| \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d \end{cases} \tag{23}$$

Let's give an example of a regularization function belonging to Ξ_α.

Example 3.1. *For $\alpha > 0$, the function $\bar{\eta}_\alpha : \mathbb{R}^d \to [0, \infty[$ defined by*

$$\bar{\eta}_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2} \quad \forall\, \mathbf{v} \in \mathbb{R}^d \tag{24}$$

belongs to Ξ_α.

In the following we take a regularization function η_α such that $\eta_\alpha \in \Xi_\alpha$. Our regularized frictional problem becomes

Find $\mathbf{u} \in \mathbf{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_{\alpha,\tau}(\mathbf{v}) - j_{\alpha,\tau}(\mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{25}$$

where the application $j_{\alpha,\tau}$ is given by

$$j_{\alpha,\tau}(\mathbf{v}) = \int_{\Gamma_C} \tau.\eta_\alpha(\mathbf{v}_T^1 - \mathbf{v}_T^2)\, ds \tag{26}$$

Note that the regularization of frictional problems was considered in many papers like [1, 21, 24].

3.1 Frictional criterion generated from the regularized problem

In most papers and for Signorini's contact problem, one can cite [9, 19], the variational inequality (11) is proved to be equivalent to the contact problem equations (1), (2), with Tresca's frictional criterion (4). Otherwise, regularizing the frictional criterion can be found in [27], and in [24] where a nonlocal friction is used (the normal stress is replaced by a weighed average of the normal stress in the friction criterion) and where we can find a physical interpretation for the regularization, as the elastic and elasto-plastic deformation of the junctions (a region in the contact area where an adhesion take place). Here we prove formally, in the case of contact between two bodies, that if the variational inequality (25) is satisfied then the equations of the contact problem (1), (2) are satisfied with a special regularized frictional criterion.

In order to obtain the frictional criterion generated by the regularized problem, we present the following theorem

Theorem 3.1. *Let $\mathbf{u} \in \mathbf{K}$ be sufficiently regular ($\mathbf{H}^2$), and satisfying the following variational inequality*

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_{\alpha,\tau}(\mathbf{v}) - j_{\alpha,\tau}(\mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{27}$$

Then $\mathbf{u}$ satisfies the following equations for $l = 1, 2$

$$\begin{cases} \nabla.\boldsymbol{\sigma}^l + \mathbf{f}^l = 0 & \text{in } \Omega^l \\ \boldsymbol{\sigma}^l = C^l \boldsymbol{\epsilon}^l & \text{in } \Omega^l \quad \textit{(Hook's law)} \\ \mathbf{u}^l = \mathbf{0} & \text{on } \Gamma_0^l \\ \boldsymbol{\sigma}^l.\mathbf{n}^l = \mathbf{t}^l & \text{on } \Gamma_1^l \end{cases} \tag{28}$$

with the following contact conditions:

$$\begin{cases} [\mathbf{u}.\mathbf{n}] = \mathbf{u}^1.\mathbf{n}^1 + \mathbf{u}^2.\mathbf{n}^2 = (\mathbf{u}^1 - \mathbf{u}^2).\mathbf{n} \leq 0 & \text{on } \Gamma_C \\ \sigma_n := (\boldsymbol{\sigma}^1.\mathbf{n}^1).\mathbf{n}^1 = (\boldsymbol{\sigma}^2.\mathbf{n}^2).\mathbf{n}^2 \leq 0 & \text{on } \Gamma_C \\ \sigma_n.[\mathbf{u}.\mathbf{n}] = 0 & \text{on } \Gamma_C \end{cases} \tag{29}$$

with the following regularized frictional criterion on Γ_C

$$\begin{cases} \boldsymbol{\sigma}_T^1 &= -\boldsymbol{\sigma}_T^2 \\ \boldsymbol{\sigma}_T^1 &= -\tau.\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2) \\ &= -\tau\frac{(\mathbf{u}_T^1-\mathbf{u}_T^2)}{\sqrt{|\mathbf{u}_T^1-\mathbf{u}_T^2|^2+\alpha^2}} \text{ if } \eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2+\alpha^2} \end{cases} \tag{30}$$

where $\nabla\eta_\alpha(\mathbf{u}_T^1-\mathbf{u}_T^2)$ *is supposed to be in the same tangent space of* $\boldsymbol{\sigma}_T^1$ *(true if* $\eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2+\alpha^2}$*).*

Proof. We recall the Green formula which will be useful in the sequel. For $l = 1$ or 2

$$\int_{\Omega^l} \boldsymbol{\sigma}^l(\mathbf{u}^l) : \boldsymbol{\epsilon}(\mathbf{v}^l)\, dv = -\int_{\Omega^l} \nabla.\boldsymbol{\sigma}^l(\mathbf{u}^l).\mathbf{v}^l\, dv + \int_{\partial\Omega^l} \boldsymbol{\sigma}^l(\mathbf{u}^l).\mathbf{n}^l.\mathbf{v}^l\, ds \tag{31}$$

Therefore

$$a(\mathbf{u}, \mathbf{v}-\mathbf{u}) = \sum_{l=1,2}\left(-\int_{\Omega^l} \nabla.\boldsymbol{\sigma}^l(\mathbf{u}^l).(\mathbf{v}^l-\mathbf{u}^l)\, dv + \int_{\partial\Omega^l} \boldsymbol{\sigma}^l(\mathbf{u}^l).\mathbf{n}^l.(\mathbf{v}^l-\mathbf{u}^l)\, ds\right) \tag{32}$$

First consider an application $\boldsymbol{\phi} \in \boldsymbol{\mathcal{D}}^2(\Omega^1)$ ($\mathbf{C}^2(\Omega^1)$ with a compact support in Ω^1), we will take the test function $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ such that $\mathbf{v}^1 = \mathbf{u}^1 \pm \boldsymbol{\phi}$ and $\mathbf{v}^2 = \mathbf{u}^2$. Then using the variational inequality (27), the Green formula (32) and the fact that $\boldsymbol{\phi}$ is equal to zero on the borders, one obtains

$$-\int_{\Omega^1} \nabla.\boldsymbol{\sigma}^1(\mathbf{u}^1).\boldsymbol{\phi}\, dv - \int_{\Omega^1} \mathbf{f}^1.\boldsymbol{\phi}\, dv \geq 0 \tag{33}$$

and

$$\int_{\Omega^1} \nabla.\boldsymbol{\sigma}^1(\mathbf{u}^1).\boldsymbol{\phi}\, dv + \int_{\Omega^1} \mathbf{f}^1.\boldsymbol{\phi}\, dv \geq 0 \tag{34}$$

Thus

$$\int_{\Omega^1} (\nabla.\boldsymbol{\sigma}^1(\mathbf{u}^1) + \mathbf{f}^1).\boldsymbol{\phi}\, dv = 0 \tag{35}$$

Otherwise speaking

$$\nabla.\boldsymbol{\sigma}^1(\mathbf{u}^1) + \mathbf{f}^1 = 0 \text{ a.e on } \Omega^1 \tag{36}$$

In the same manner if we take $\mathbf{v}$ such that $\mathbf{v}^1 = \mathbf{u}^1$ and $\mathbf{v}^2 = \mathbf{u}^2 \pm \boldsymbol{\phi}$, we obtain

$$\nabla.\boldsymbol{\sigma}^2(\mathbf{u}^2) + \mathbf{f}^2 = 0 \text{ a.e on } \Omega^2 \tag{37}$$

Consider the test function $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ such that $\mathbf{v}^1 = \mathbf{u}^1 \pm \boldsymbol{\phi}$ and $\mathbf{v}^2 = \mathbf{u}^2$, then using the two equilibrium equations (36) and (37) the variational inequality becomes

$$\int_{\partial\Omega^1} \boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1.(\pm\boldsymbol{\phi})\, ds - \int_{\Gamma_1^1} \mathbf{t}^1.(\pm\boldsymbol{\phi})\, ds + \int_{\Gamma_C} \tau.(\eta_\alpha(\mathbf{v}_T^1-\mathbf{u}_T^2) - \eta_\alpha(\mathbf{u}_T^1-\mathbf{u}_T^2))\, ds \geq 0 \tag{38}$$

In the inequality (38) we can take $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\partial\Omega^1)$ with $\text{supp}(\boldsymbol{\phi}) \subset \Gamma_1^1$, and we will obtain then

$$-\int_{\Gamma_1^1} (\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1 - \mathbf{t}^1).\boldsymbol{\phi}\, ds \geq 0 \tag{39}$$

and

$$\int_{\Gamma_1^1} (\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1 - \mathbf{t}^1).\boldsymbol{\phi}\, ds \geq 0 \tag{40}$$

Otherwise speaking

$$\int_{\Gamma_1^1} (\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1 - \mathbf{t}^1).\boldsymbol{\phi}\, ds = 0 \tag{41}$$

Therefore

$$\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1 = \mathbf{t}^1 \text{ a.e on } \Gamma_1^1 \tag{42}$$

In the same manner if we take $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ such that $\mathbf{v}^1 = \mathbf{u}^1$ and $\mathbf{v}^2 = \mathbf{u}^2 \pm \boldsymbol{\phi}$ we obtain

$$\boldsymbol{\sigma}^2(\mathbf{u}^2).\mathbf{n}^2 = \mathbf{t}^2 \text{ a.e on } \Gamma_1^2 \tag{43}$$

Because $\mathbf{u} \in \mathbf{K}$, then by definition $[\mathbf{u}.\mathbf{n}] \leq 0$ on Γ_C and $\mathbf{u}^l = \mathbf{0}$ on Γ_0^l. So it remains to verify the last two equations of (29) and the two equations of (30).

Using the equations (36), (37), (42), (43) and the Green formula, the variational inequality (27) becomes

$$\sum_{l=1,2} \int_{\Gamma_C} \boldsymbol{\sigma}^l(\mathbf{u}^l).\mathbf{n}^l.(\mathbf{v}^l - \mathbf{u}^l)\, ds + \int_{\Gamma_C} \tau.(\eta_\alpha(\mathbf{v}_T^1 - \mathbf{v}_T^2) - \eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2))\, ds \geq 0 \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{44}$$

Taking $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ such that $\mathbf{v}^1 = \phi.\mathbf{n}^1 + \mathbf{u}^1$ and $\mathbf{v}^2 = -\phi.\mathbf{n}^2 + \mathbf{u}^2$, we have the fact that $\mathbf{v} \in \mathbf{K}$ because $[\mathbf{v}.\mathbf{n}] = \mathbf{v}^1.\mathbf{n}^1 + \mathbf{v}^2.\mathbf{n}^2 = \mathbf{u}^1.\mathbf{n}^1 + \mathbf{u}^2.\mathbf{n}^2 = [\mathbf{u}.\mathbf{n}] \leq 0$. Thus if we inject $\mathbf{v}$ in the inequality (44) one obtains

$$\int_{\Gamma_C} \left((\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1).\mathbf{n}^1 - (\boldsymbol{\sigma}^2(\mathbf{u}^2).\mathbf{n}^2).\mathbf{n}^2\right) \phi\, ds \geq 0 \tag{45}$$

In addition if we consider $-\phi$ instead of ϕ, we can obtain thus

$$\int_{\Gamma_C} \left((\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1).\mathbf{n}^1 - (\boldsymbol{\sigma}^2(\mathbf{u}^2).\mathbf{n}^2).\mathbf{n}^2\right) \phi\, ds \leq 0 \tag{46}$$

Therefore

$$\int_{\Gamma_C} \left((\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1).\mathbf{n}^1 - (\boldsymbol{\sigma}^2(\mathbf{u}^2).\mathbf{n}^2).\mathbf{n}^2\right) \phi\, ds = 0 \tag{47}$$

We can deduce that $\sigma_n := (\boldsymbol{\sigma}^1.\mathbf{n}^1).\mathbf{n}^1 = (\boldsymbol{\sigma}^2.\mathbf{n}^2).\mathbf{n}^2$ a.e on Γ_C which is the second equation of (29).

Now we take $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ such that $\mathbf{v}^1 = \lambda.\phi.\mathbf{n}^1 + \mathbf{u}_T^1$ and $\mathbf{v}^2 = \mathbf{u}_T^2$, where $\lambda \geq 0$ and $\phi \in H^{1/2}(\partial\Omega^1) \leq 0$ with $\text{supp}(\phi) \subset \Gamma_C$. Clearly $\mathbf{v} \in \mathbf{K}$ because $[\mathbf{v}.\mathbf{n}] = \mathbf{v}^1.\mathbf{n}^1 + \mathbf{v}^2.\mathbf{n}^2 = \lambda.\phi \leq 0$. Injecting $\mathbf{v}$ in the inequality (44) and using the fact that $\mathbf{u}^1 = (\mathbf{u}^1.\mathbf{n}^1)\mathbf{n}^1 + \mathbf{u}_T^1$ and $\mathbf{u}^2 = (\mathbf{u}^2.\mathbf{n}^2)\mathbf{n}^2 + \mathbf{u}_T^2$, one obtains

$$\int_{\Gamma_C} (\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1).\lambda.\phi\mathbf{n}^1\, ds - \int_{\Gamma_C} (\boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1).(\mathbf{u}^1.\mathbf{n}^1)\mathbf{n}^1\, ds - \int_{\Gamma_C} (\boldsymbol{\sigma}^2(\mathbf{u}^2).\mathbf{n}^2).(\mathbf{u}^2.\mathbf{n}^2)\mathbf{n}^2\, ds \geq 0 \tag{48}$$

Then

$$\int_{\Gamma_C} \sigma_n.\lambda.\phi\, ds - \int_{\Gamma_C} \sigma_n.(\mathbf{u}^1.\mathbf{n}^1)\, ds - \int_{\Gamma_C} \sigma_n.(\mathbf{u}^2.\mathbf{n}^2)\, ds \geq 0 \tag{49}$$

which is equivalent to

$$\lambda \int_{\Gamma_C} \sigma_n.\phi\, ds - \int_{\Gamma_C} \sigma_n.[\mathbf{u}.\mathbf{n}]\, ds \geq 0 \tag{50}$$

If $\lambda \to 0$ then

$$\int_{\Gamma_C} \sigma_n.[\mathbf{u}.\mathbf{n}]\, ds \leq 0 \tag{51}$$

If we divide the inequality (50) by λ and taking $\lambda \to +\infty$ then

$$\int_{\Gamma_C} \sigma_n.\phi \, ds \geq 0 \tag{52}$$

From the inequality (52) and from the fact that $\phi \leq 0$ we deduce that $\sigma_n \leq 0$. By definition $[\mathbf{u}.\mathbf{n}] \leq 0$, thus $\sigma_n.[\mathbf{u}.\mathbf{n}] \geq 0$, therefore from the inequality (51) we deduce that $\sigma_n.[\mathbf{u}.\mathbf{n}] = 0$. Therefore the equations of (29) are verified and it remains to prove the two equations of (30).

Let $\boldsymbol{\phi} = \phi_n \mathbf{n}^1 + \boldsymbol{\phi}_T \in \mathbf{H}^{1/2}(\partial\Omega^1)$ with $\text{supp}(\boldsymbol{\phi}) \subset \Gamma_C$. Take $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ such that $\mathbf{v}^1 = \mathbf{u}^1 \pm \epsilon\boldsymbol{\phi}_T$ and $\mathbf{v}^2 = \mathbf{u}^2$, where $\epsilon \in [0,1]$. Clearly $\mathbf{v} \in \mathbf{K}$, thus injecting $\mathbf{v}$ in (44) one obtains

$$\int_{\Gamma_C} \boldsymbol{\sigma}^1(\mathbf{u}^1).\mathbf{n}^1.(\pm\epsilon\boldsymbol{\phi}_T)\, ds + \int_{\Gamma_C} \tau.(\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2 \pm \epsilon\boldsymbol{\phi}_T) - \eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2))\, ds \geq 0 \tag{53}$$

thus and after dividing by ϵ

$$\int_{\Gamma_C} \boldsymbol{\sigma}_T^1.(\pm\boldsymbol{\phi}_T)\, ds + \int_{\Gamma_C} \tau.\frac{\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2 \pm \epsilon\boldsymbol{\phi}_T) - \eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2)}{\epsilon}\, ds \geq 0 \tag{54}$$

Thanks to the differentiability of η_α, we obtain the following inequality when $\epsilon \to 0$

$$\int_{\Gamma_C} \boldsymbol{\sigma}_T^1.(\pm\boldsymbol{\phi}_T)\, ds + \int_{\Gamma_C} \tau.\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2).(\pm\boldsymbol{\phi}_T)\, ds \geq 0 \tag{55}$$

We deduce that

$$\int_{\Gamma_C} \boldsymbol{\sigma}_T^1.\boldsymbol{\phi}_T\, ds + \int_{\Gamma_C} \tau.\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2).\boldsymbol{\phi}_T\, ds = 0 \tag{56}$$

Otherwise $\boldsymbol{\sigma}_T^1.\boldsymbol{\phi}_T = \boldsymbol{\sigma}_T^1.\boldsymbol{\phi}$ and $\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2).\boldsymbol{\phi}_T = \nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2).\boldsymbol{\phi}$, (indeed $\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2)$ is supposed to be in the same tangent space, it's true if $\eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2}$), thus

$$\int_{\Gamma_C} (\boldsymbol{\sigma}_T^1 + \tau.\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2)).\boldsymbol{\phi}\, ds = 0 \tag{57}$$

We conclude that

$$\boldsymbol{\sigma}_T^1 = -\tau.\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2) = -\tau\frac{\mathbf{u}_T^1 - \mathbf{u}_T^2}{\sqrt{\|\mathbf{u}_T^1 - \mathbf{u}_T^2\|^2 + \alpha^2}} \text{ a.e on } \Gamma_C \tag{58}$$

In the same manner, we take $\mathbf{v} = (\mathbf{v}^1, \mathbf{v}^2)$ such that $\mathbf{v}^1 = \mathbf{u}^1$ and $\mathbf{v}^2 = \mathbf{u}^2 \pm \epsilon\boldsymbol{\phi}_T$, where $\epsilon \in [0,1]$, and we inject $\mathbf{v}$ in (44). We obtain

$$\int_{\Gamma_C} \boldsymbol{\sigma}^2(\mathbf{u}^2).\mathbf{n}^2.(\pm\epsilon\boldsymbol{\phi}_T)\, ds + \int_{\Gamma_C} \tau.(\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2 \mp \epsilon\boldsymbol{\phi}_T) - \eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2))\, ds \geq 0 \tag{59}$$

As before we conclude that

$$\boldsymbol{\sigma}_T^2 = \tau.\nabla\eta_\alpha(\mathbf{u}_T^1 - \mathbf{u}_T^2) = -\boldsymbol{\sigma}_T^1 \text{ a.e on } \Gamma_C \tag{60}$$

□

3.2 Minimization formulation for the regularized problem

The variational inequality of the problem is recalled below.

Find $\mathbf{u} \in \mathbf{K}$ such that

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_{\alpha,\tau}(\mathbf{v}) - j_{\alpha,\tau}(\mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{61}$$

As before, let $\mathcal{E}_p$ denotes the total potential energy of the two bodies

$$\mathcal{E}_p(\mathbf{v}) := \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - f(\mathbf{v}) \tag{62}$$

Consider the energy functional $J_{\alpha,\tau}$ given by

$$J_{\alpha,\tau}(\mathbf{v}) := \mathcal{E}_p(\mathbf{v}) + j_{\alpha,\tau}(\mathbf{v}) \tag{63}$$

where as before, the functional is given by

$$j_{\alpha,\tau}(\mathbf{v}) = \int_{\Gamma_C} \tau.\eta_\alpha(\mathbf{v}_T^1 - \mathbf{v}_T^2)\, ds \tag{64}$$

The functional $j_{\alpha,\tau}$ is lower semicontinuous, and because η_α is convex then $j_{\alpha,\tau}$ is convex. Therefore by applying the theorem 2.1 above, by taking $F = J_{\alpha,\tau}$, $F_1 = \mathcal{E}_p$ and $\Phi = j_{\alpha,\tau}$, the frictional problem (61) is equivalent to the following minimization problem

Find $\mathbf{u} \in \mathbf{K}$ such that

$$J_{\alpha,\tau}(\mathbf{u}) \leq J_{\alpha,\tau}(\mathbf{v}) \quad \forall\, \mathbf{v} \in K \tag{65}$$

Otherwise the energy functional $J_{\alpha,\tau}$ is convex, Gâteaux differentiable (or continuous) because η_α is differentiable (or continuous), and coercive because $\mathcal{E}_p$ is coercive and the functional $j_{\alpha,\tau}$ is positive. We conclude that there exits a solution of the minimization problem (65), in addition this minimizer is unique because $J_{\alpha,\tau}$ is strictly convex.

3.3 Error between Tresca's solution and regularized Tresca's solution

Theorem 3.2. *Let $\mathbf{u} \in \mathbf{K}$ be the Tresca solution, in other words solution of*

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + j_\tau(\mathbf{v}) - j_\tau(\mathbf{u}) \geq f(\mathbf{v} - \mathbf{u}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{66}$$

and let $\mathbf{u}_\alpha \in \mathbf{K}$ be the regularized Tresca solution, otherwise speaking solution of

$$a(\mathbf{u}_\alpha, \mathbf{v} - \mathbf{u}_\alpha) + j_{\alpha,\tau}(\mathbf{v}) - j_{\alpha,\tau}(\mathbf{u}_\alpha) \geq f(\mathbf{v} - \mathbf{u}_\alpha) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{67}$$

then there exists a constant $C \geq 0$ such that

$$\|\mathbf{u}_\alpha - \mathbf{u}\|_1 \leq C\alpha \tag{68}$$

Proof. Replacing $\mathbf{v}$ by $\mathbf{u}_\alpha$ in the equation (66), and $\mathbf{v}$ by $\mathbf{u}$ in the equation (67), one obtains

$$\begin{cases} a(\mathbf{u}, \mathbf{u}_\alpha - \mathbf{u}) + j_\tau(\mathbf{u}_\alpha) - j_\tau(\mathbf{u}) \geq f(\mathbf{u}_\alpha - \mathbf{u}) \\ a(\mathbf{u}_\alpha, \mathbf{u} - \mathbf{u}_\alpha) + j_{\alpha,\tau}(\mathbf{u}) - j_{\alpha,\tau}(\mathbf{u}_\alpha) \geq f(\mathbf{u} - \mathbf{u}_\alpha) \end{cases} \tag{69}$$

Adding these two equations, we obtain

$$a(\mathbf{u}-\mathbf{u}_\alpha,\mathbf{u}_\alpha-\mathbf{u})+j_\tau(\mathbf{u}_\alpha)-j_{\alpha,\tau}(\mathbf{u}_\alpha)+j_{\alpha,\tau}(\mathbf{u})-j_\tau(\mathbf{u})\geq 0 \tag{70}$$

Thus

$$\begin{aligned} a(\mathbf{u}_\alpha-\mathbf{u},\mathbf{u}_\alpha-\mathbf{u}) &\leq j_\tau(\mathbf{u}_\alpha)-j_{\alpha,\tau}(\mathbf{u}_\alpha)+j_{\alpha,\tau}(\mathbf{u})-j_\tau(\mathbf{u}) \\ &\leq \int_{\Gamma_C}\tau.|\eta_\alpha(\mathbf{u}^1_{\alpha,T}-\mathbf{u}^2_{\alpha,T})-|\mathbf{u}^1_{\alpha,T}-\mathbf{u}^2_{\alpha,T}||\,ds \\ &\quad+\int_{\Gamma_C}\tau.|\eta_\alpha(\mathbf{u}^1_T-\mathbf{u}^2_T)-|\mathbf{u}^1_T-\mathbf{u}^2_T||\,ds \\ &\leq 2\|\tau\|_{L^2(\Gamma_C)}.\sqrt{\mathrm{meas}(\Gamma_C)}.\alpha \quad (\eta_\alpha\in\Xi_\alpha) \end{aligned} \tag{71}$$

Because a is elliptic then

$$\|\mathbf{u}_\alpha-\mathbf{u}\|_1\leq C\sqrt{\alpha} \tag{72}$$

Note that the differentiability of η_α is not needed. □

Corollary 3.1. *If* $\eta_\alpha(\mathbf{v})=\sqrt{|\mathbf{v}|^2+\alpha^2}$ *then*

$$\|\mathbf{u}_\alpha-\mathbf{u}\|_1\leq C\alpha^{1/2} \tag{73}$$

3.4 Coulomb's criterion as a fixed point problem for the regularized problem

Given a regularized parameter $\alpha>0$, consider the following application

$$T(\tau)=-\mu\sigma^1_N(\mathbf{u}_\tau) \tag{74}$$

where σ^1_N is the normal stress and $\mathbf{u}_\tau$ the solution of the Tresca problem with the sliding limit $\tau\geq 0\in L^2(\Gamma_C)$, otherwise speaking, solution of the following problem

Find $\mathbf{u}_\tau\in\mathbf{K}$ such that

$$a(\mathbf{u}_\tau,\mathbf{v}-\mathbf{u}_\tau)+j_{\alpha,\tau}(\mathbf{v})-j_{\alpha,\tau}(\mathbf{u}_\tau)\geq f(\mathbf{v}-\mathbf{u}_\tau)\quad\forall\,\mathbf{v}\in\mathbf{K} \tag{75}$$

or equivalently

Find $\mathbf{u}_\tau\in\mathbf{K}$ such that

$$J_{\alpha,\tau}(\mathbf{u}_\tau)\leq J_{\alpha,\tau}(\mathbf{v})\quad\forall\,\mathbf{v}\in\mathbf{K} \tag{76}$$

Let $\tau^*=T(\tau^*)=-\mu\sigma^1_N(\mathbf{u}_{\tau^*})$ be the fixed point of the application T and let $\mathbf{u}=\mathbf{u}_{\tau^*}$, the corresponding displacement which is supposed to be sufficiently regular, therefore according to the theorem 3.1, the equations (28) and (29) are verified and the regularized Coulomb's criterion becomes

$$\begin{cases} \boldsymbol{\sigma}^1_T &= -\boldsymbol{\sigma}^2_T \\ \boldsymbol{\sigma}^1_T &= \mu\sigma^1_N(\mathbf{u}).\nabla\eta_\alpha(\mathbf{u}^1_T-\mathbf{u}^2_T) \\ &= \mu\sigma^1_N(\mathbf{u})\frac{(\mathbf{u}^1_T-\mathbf{u}^2_T)}{\sqrt{|\mathbf{u}^1_T-\mathbf{u}^2_T|^2+\alpha^2}} \text{ if } \eta_\alpha(\mathbf{v})=\sqrt{|\mathbf{v}|^2+\alpha^2} \end{cases} \tag{77}$$

In the section 5 we prove for the discretized case, the existence and the uniqueness of a fixed point for the application T (where $\sigma^1_N\in H^{-1/2}(\Gamma_C)$, the dual of $H^{1/2}(\Gamma_C)$), in addition to the convergence of the fixed point algorithm to solve the frictional contact problem.

3.5 Quasi-static problem for the regularized one

Given the displacement solution $\mathbf{u}_i = (\mathbf{u}_i^1, \mathbf{u}_i^2)$ of the previous step, then for each sliding limit $\tau \geq 0 \in L^2(\Gamma_C)$ of the fixed point algorithm, the following problem is considered

Find $\mathbf{u}_{\tau,i+1} \in \mathbf{K}$ such that

$$a(\mathbf{u}_{\tau,i+1}, \mathbf{v} - \mathbf{u}_{\tau,i+1}) + j_{\alpha,\tau}(\mathbf{v} - \mathbf{u}_i) - j_{\alpha,\tau}(\mathbf{u}_{\tau,i+1} - \mathbf{u}_i) \geq f(\mathbf{v} - \mathbf{u}_{\tau,i+1}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{78}$$

or equivalently

Find $\mathbf{u}_{\tau,i+1} \in \mathbf{K}$ such that

$$J_{\alpha,\tau}(\mathbf{u}_{\tau,i+1}) \leq J_{\alpha,\tau}(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{K} \tag{79}$$

where $j_{\alpha,\tau}(\mathbf{v})$ is replaced by $j_{\alpha,\tau}(\mathbf{v} - \mathbf{u}_i)$ which is given by

$$j_{\alpha,\tau}(\mathbf{v} - \mathbf{u}_i) = \int_{\Gamma_C} \tau.\eta_\alpha((\mathbf{v}_T^1 - \mathbf{u}_{T,i}^1) - (\mathbf{v}_T^2 - \mathbf{u}_{T,i}^2))\, ds \tag{80}$$

As $\mathbf{u}_{T,i}^1$ and $\mathbf{u}_{T,i}^2$ are given, then using the same proof of the theorem 3.1, the two problems (78) and (79) will generate the same equations (28) and (29) of the theorem 3.1, except the friction criterion (30) which will be slightly changed to

$$\begin{cases} \boldsymbol{\sigma}_T^1 &= -\boldsymbol{\sigma}_T^2 \\ \boldsymbol{\sigma}_T^1 &= -\tau.\nabla\eta_\alpha((\mathbf{u}_{T,\tau,i+1}^1 - \mathbf{u}_{T,i}^1) - (\mathbf{u}_{T,\tau,i+1}^2 - \mathbf{u}_{T,i}^2)) \\ &= -\tau \dfrac{(\mathbf{u}_{T,\tau,i+1}^1 - \mathbf{u}_{T,i}^1) - (\mathbf{u}_{T,\tau,i+1}^2 - \mathbf{u}_{T,i}^2)}{\sqrt{|(\mathbf{u}_{T,\tau,i+1}^1 - \mathbf{u}_{T,i}^1) - (\mathbf{u}_{T,\tau,i+1}^2 - \mathbf{u}_{T,i}^2)|^2 + \alpha^2}} \text{ if } \eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2} \end{cases} \tag{81}$$

Let $\mathbf{u}_{i+1}$ be the displacement corresponding to the fixed point of the application T already defined, the regularized Coulomb's criterion for the quasi-static problem becomes

$$\begin{cases} \boldsymbol{\sigma}_T^1 &= -\boldsymbol{\sigma}_T^2 \\ \boldsymbol{\sigma}_T^1 &= \mu\sigma_N^1(\mathbf{u}_{i+1}).\nabla\eta_\alpha((\mathbf{u}_{T,i+1}^1 - \mathbf{u}_{T,i}^1) - (\mathbf{u}_{T,i+1}^2 - \mathbf{u}_{T,i}^2)) \\ &= \mu\sigma_N^1(\mathbf{u}_{i+1}) \dfrac{(\mathbf{u}_{T,i+1}^1 - \mathbf{u}_{T,i}^1) - (\mathbf{u}_{T,i+1}^2 - \mathbf{u}_{T,i}^2)}{\sqrt{|(\mathbf{u}_{T,i+1}^1 - \mathbf{u}_{T,i}^1) - (\mathbf{u}_{T,i+1}^2 - \mathbf{u}_{T,i}^2)|^2 + \alpha^2}} \text{ if } \eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2} \end{cases} \tag{82}$$

4 Finite deformation

We consider here the Signorini's problem for simplicity, indeed the contact between more than one body can be treated in the same manner. As before a fixed point algorithm is used in order to write the frictional contact problem as a sequence of Tresca contact problems until convergence, here the regularization of the Tresca frictional problem is considered. In the following, and for the sake of convenience, our unknown will be the actual position $\boldsymbol{\phi}$ instead of the displacement $\mathbf{u}$, which is not very different because $\boldsymbol{\phi} = \mathbf{X} + \mathbf{u}$.

In the following theorem, Ω is the body domain in $\mathbb{R}^3$ (also works for $\mathbb{R}^2$). In addition, let the borders $\Gamma_0, \Gamma_1, \Gamma_C$ be disjoint relatively to $\partial\Omega$, and $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_C$. The area of the border Γ_C is supposed to be strictly positive.

Γ_0 is the border where a displacement is imposed, Γ_1 is the border where a surface traction is applied, finally Γ_C is the potential contact area, otherwise speaking if $\boldsymbol{\phi}(\Gamma_C)$ is Γ_C in the actual configuration, then the actual contact area is included in $\boldsymbol{\phi}(\Gamma_C)$.

The body force $\boldsymbol{f}$ is applied over the body Ω, the surface traction $\boldsymbol{g}$ is applied over Γ_1, and finally $\boldsymbol{\phi}_0$ is the imposed position on Γ_0.

The obstacle is described by the open set $\mathcal{C} \subset \mathbb{R}^3$, the strain energy function is denoted by $\hat{W}$, and the first Piola-Kirchhoff stress by $\mathbf{P}$. We have that $\mathbf{P} = \frac{\partial \hat{W}}{\partial \mathbf{F}}$, where $\mathbf{F}$ is the deformation gradient tensor.

The admissible solutions set $\boldsymbol{\Phi}$ is defined by:

$$\boldsymbol{\Phi} = \{\boldsymbol{\psi} : \bar{\Omega} \to \mathbb{R}^3; \det(\nabla \boldsymbol{\psi}) > 0 \text{ in } \bar{\Omega}; \boldsymbol{\psi} = \boldsymbol{\phi}_0 \text{ on } \Gamma_0 \text{ with } \boldsymbol{\psi}(\Gamma_C) \subseteq \mathcal{C}^c\} \tag{83}$$

The condition $\boldsymbol{\psi}(\Gamma_C) \subseteq \mathcal{C}^c$ (the complement of $\mathcal{C}$) describes the non-penetration of the body into the obstacle.

The potential energy of the body is given by

$$\mathcal{E}_p(\boldsymbol{\psi}) = \int_\Omega \hat{W}(\nabla \boldsymbol{\psi})\, dx - \int_\Omega \boldsymbol{f}.\boldsymbol{\psi}\, dx - \int_{\Gamma_1} \boldsymbol{g}.\boldsymbol{\psi}\, dS \tag{84}$$

We define the mapping $T : \boldsymbol{\Phi} \to \boldsymbol{\Phi}$, which for a given $\boldsymbol{\zeta} \in \boldsymbol{\Phi}$, $T(\boldsymbol{\zeta})$ is the solution of the following constrained minimization problem

$$\min_{\boldsymbol{\psi} \in \boldsymbol{\Phi}} \left(\mathcal{E}_p(\boldsymbol{\psi}) + \int_{\boldsymbol{\zeta}(\Gamma_C) \cap \partial \mathcal{C}} \tau.\eta_\alpha(\mathbf{v}_T)\, ds \right) \tag{85}$$

where $\mathbf{v}(\mathbf{x}) = \boldsymbol{\psi}(\boldsymbol{\zeta}^{-1}(\mathbf{x})) - \boldsymbol{\zeta}^{-1}(\mathbf{x})$, $\forall\, \mathbf{x} \in \zeta(\Gamma_C) \cap \partial\mathcal{C}$. Moreover $\tau(\mathbf{x}) = \tau_0(\boldsymbol{\zeta}^{-1}(\mathbf{x}))$ where $\tau_0 \in L^2(\Gamma_C) \geq 0$ is the sliding limit of the Tresca criterion and $\eta_\alpha : \mathbb{R}^3 \to \mathbb{R}_+$ a regularization function belonging to Ξ_α, defined before (for example $\eta_\alpha(x, y, z) = \sqrt{x^2 + y^2 + z^2 + \alpha^2}$ with $\alpha > 0$). Finally $\mathbf{v}_T$ is the tangential part of $\mathbf{v}$, and can be given by $\mathbf{v}_T = \mathbf{v} - v_n \mathbf{n}$ with v_n its normal component and $\mathbf{n}$ the normal vector at $\boldsymbol{\zeta}(\Gamma_C) \cap \partial\mathcal{C}$.

In the following theorem, the elements proof where the friction is not taken into account, are taken from [2].

Theorem 4.1. *Let $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ be the fixed point of the application T defined above, otherwise speaking $\boldsymbol{\phi} = T(\boldsymbol{\phi})$, which means also that $\boldsymbol{\phi}$ is solution of*

$$\min_{\boldsymbol{\psi} \in \boldsymbol{\Phi}} \mathcal{E}(\boldsymbol{\psi}) \tag{86}$$

where

$$\mathcal{E}(\boldsymbol{\psi}) = \mathcal{E}_p(\boldsymbol{\psi}) + \int_{\boldsymbol{\phi}(\Gamma_C) \cap \partial\mathcal{C}} \tau.\eta_\alpha(\mathbf{v}_T)\, ds \tag{87}$$

If $\boldsymbol{\phi}$ is smooth enough, then $\boldsymbol{\phi}$ satisfies formally the following properties, corresponding to the frictional contact between a body and an obstacle, with a regularized Tresca

criterion.

$$\begin{cases} -\operatorname{div}\,\mathbf{P} = \boldsymbol{f} \ \textit{in}\ \Omega \\ \boldsymbol{\phi} = \boldsymbol{\phi}_0 \ \textit{on}\ \Gamma_0 \\ \mathbf{P}.\mathbf{N} = \mathbf{g} \ \textit{on}\ \Gamma_1 \\ \boldsymbol{\phi}(\Gamma_C) \subseteq \mathcal{C}^c \\ \mathbf{P}.\mathbf{N} = \mathbf{0} \ \textit{if}\ \mathbf{X} \in \Gamma_C \ \textit{and}\ \boldsymbol{\phi}(\mathbf{X}) \in \overline{\mathcal{C}}^c \quad (\notin \mathcal{C} \cup \partial\mathcal{C}) \\ (\mathbf{P}.\mathbf{N}).\mathbf{n} = \lambda_n \ \textit{if}\ \mathbf{X} \in \Gamma_C \ \textit{and}\ \mathbf{x} = \boldsymbol{\phi}(\mathbf{X}) \in \partial\mathcal{C} \ \textit{where}\ \lambda_n \leq 0 \\ \boldsymbol{\sigma}_T = -\tau\nabla\eta_\alpha(\mathbf{u}_T) = -\tau\frac{\mathbf{u}_T}{\sqrt{|\mathbf{u}_T|^2+\alpha^2}} \ \textit{on}\ \gamma_C = \boldsymbol{\phi}(\Gamma_C) \cap \partial\mathcal{C} \ \ (\textit{if}\ \eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2+\alpha^2}) \end{cases} \tag{88}$$

where $\boldsymbol{\sigma}_T$,$\mathbf{u}_T$ are respectively the tangential stress and displacement, $\mathbf{N}$ and $\mathbf{n}$ are respectively the unit outer normal vector on the initial and on the deformed surface of the body. $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{\sigma}\mathbf{n}$ has the same direction of $\mathbf{PN}$. Finally $\nabla\eta_\alpha(\mathbf{u}_T)$ is supposed to belong to the tangent plane (it is true if $\eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2+\alpha^2}$).

Proof. The function $\boldsymbol{\phi}$ is a solution of the minimization problem (86), therefore

$$\mathcal{E}(\boldsymbol{\phi}) \leq \mathcal{E}(\boldsymbol{\psi}) \quad \forall\, \boldsymbol{\psi} \in \boldsymbol{\Phi} \tag{89}$$

In the following we need the Green formula, which for a smooth enough tensor $\mathbf{T}$ states

$$\int_\Omega \mathbf{T} : \nabla\boldsymbol{\theta}\, dx = -\int_\Omega \operatorname{div}(\mathbf{T}).\boldsymbol{\theta}\, dx + \int_\Gamma \mathbf{TN}.\boldsymbol{\theta}\, dS \quad \forall\boldsymbol{\theta} \tag{90}$$

Consider $\boldsymbol{\theta}$ a sufficient smooth function that vanishes in a neighborhood of $\Gamma_0 \cup \Gamma_C$. There exists $\epsilon_0 = \epsilon(\boldsymbol{\theta}) > 0$ such that $\boldsymbol{\phi}_\epsilon = \boldsymbol{\phi} + \epsilon\boldsymbol{\theta} \in \boldsymbol{\Phi} \quad \forall\, |\epsilon| \leq \epsilon_0$.

$$\mathcal{E}(\boldsymbol{\phi}_\epsilon) - \mathcal{E}(\boldsymbol{\phi}) \geq 0 \tag{91}$$

$$\begin{aligned} \mathcal{E}(\boldsymbol{\phi}_\epsilon) - \mathcal{E}(\boldsymbol{\phi}) &= \mathcal{E}(\boldsymbol{\phi} + \epsilon\boldsymbol{\theta}) - \mathcal{E}(\boldsymbol{\phi}) \\ &= \int_\Omega \left(\hat{W}(\nabla\boldsymbol{\phi} + \epsilon\nabla\boldsymbol{\theta}) - \hat{W}(\nabla\boldsymbol{\phi})\right) dx - \epsilon\left(\int_\Omega \boldsymbol{f}.\boldsymbol{\theta}\, dx + \int_{\Gamma_1} \boldsymbol{g}.\boldsymbol{\theta}\, dS\right) \end{aligned} \tag{92}$$

Otherwise we have:

$$\begin{aligned} \hat{W}(\nabla\boldsymbol{\phi} + \epsilon\nabla\boldsymbol{\theta}) - \hat{W}(\nabla\boldsymbol{\phi}) &= \epsilon\frac{\partial\hat{W}}{\partial\mathbf{F}} : \nabla\boldsymbol{\theta} + o(\epsilon) \\ &= \epsilon\mathbf{P} : \nabla\boldsymbol{\theta} + o(\epsilon) \end{aligned} \tag{93}$$

Using the Green formula in equation (90) and the fact that $\boldsymbol{\theta}$ vanishes in a neighborhood of $\Gamma_0 \cup \Gamma_C$, we obtain :

$$\begin{aligned} \int_\Omega \left(\hat{W}(\nabla\boldsymbol{\phi} + \epsilon\nabla\boldsymbol{\theta}) - \hat{W}(\nabla\boldsymbol{\phi})\right) dx &= \epsilon\int_\Omega \mathbf{P} : \nabla\boldsymbol{\theta}\, dx + o(\epsilon) \\ &= -\epsilon\int_\Omega \operatorname{div}(\mathbf{P}).\boldsymbol{\theta}\, dx + \epsilon\int_{\Gamma_1} \mathbf{PN}.\boldsymbol{\theta}\, dS + o(\epsilon) \end{aligned} \tag{94}$$

Thus

$$\begin{aligned}\mathcal{E}(\boldsymbol{\phi}_\epsilon) - \mathcal{E}(\boldsymbol{\phi}) =& -\epsilon \int_\Omega \operatorname{div}(\mathbf{P}).\boldsymbol{\theta}\, dx + \epsilon \int_{\Gamma_1} \mathbf{PN}.\boldsymbol{\theta}\, dS \\ & - \epsilon \left(\int_\Omega \boldsymbol{f}.\boldsymbol{\theta}\, dx + \int_{\Gamma_1} \boldsymbol{g}.\boldsymbol{\theta}\, dS \right) + o(\epsilon) \\ =& \epsilon \left\{ \int_\Omega \left(-\operatorname{div}(\mathbf{P}) - \boldsymbol{f}\right).\boldsymbol{\theta}\, dx + \int_{\Gamma_1} \left(\mathbf{PN} - \boldsymbol{g}\right).\boldsymbol{\theta}\, dS + \frac{o(\epsilon)}{\epsilon} \right\} \geq 0\end{aligned} \tag{95}$$

Taking $\epsilon > 0$ and taking the limit $\epsilon \to 0^+$ we have:

$$\int_\Omega \left(-\operatorname{div}(\mathbf{P}) - \boldsymbol{f}\right).\boldsymbol{\theta}\, dx + \int_{\Gamma_1} \left(\mathbf{PN} - \boldsymbol{g}\right).\boldsymbol{\theta}\, dS \geq 0 \tag{96}$$

Let $\mathbf{X} \in \Omega$ and $B(\mathbf{X}, r) \subset \Omega$ the open ball of center $\mathbf{X}$ with a small radius $r > 0$ (see Figure 2). Consider $\boldsymbol{\theta}$ a sufficient smooth function with support in $B(\mathbf{X}, r)$, thus we deduce that $-\operatorname{div}(\mathbf{P}) = \boldsymbol{f}$ in $B(\mathbf{X}, r)$, and then it's true in Ω.

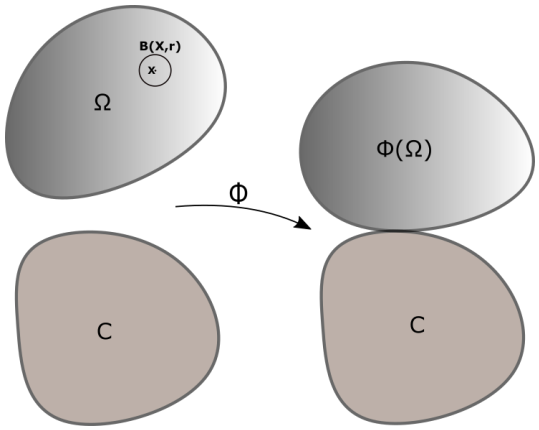

Figure 2: The initial and actual configuration

The equation (96) can always be used, thus using the fact that $-\operatorname{div}(\mathbf{P}) = \boldsymbol{f}$ in Ω, we obtain:

$$\int_{\Gamma_1} \left(\mathbf{PN} - \boldsymbol{g}\right).\boldsymbol{\theta}\, dS \geq 0 \tag{97}$$

We deduce that $\mathbf{PN} = \boldsymbol{g}$ on Γ_1.

Let $\mathbf{X} \in \Gamma_C$ such that $\boldsymbol{\phi}(\mathbf{X}) \in \overline{\mathcal{C}}^c$ (see Figure 3), consider any smooth function $\boldsymbol{\theta} : \bar{\Omega} \to \mathbb{R}^3$ with a support in $B(\mathbf{X}, r) \cap \bar{\Omega}$ where $r > 0$ and small, there exists a $\epsilon_2(\boldsymbol{\theta}) > 0$ such that $\boldsymbol{\phi}_\epsilon = \boldsymbol{\phi} + \epsilon\boldsymbol{\theta} \in \boldsymbol{\Phi} \quad \forall\, |\epsilon| \leq \epsilon_2$. As before, $\mathcal{E}(\boldsymbol{\phi}_\epsilon) - \mathcal{E}(\boldsymbol{\phi}) \geq 0$, thus using Green's formula and the equations of equilibrium we obtain:

$$\epsilon \left\{ \int_{\Gamma_C} \mathbf{PN}.\boldsymbol{\theta}\, dS + \frac{o(\epsilon)}{\epsilon} \right\} \geq 0 \tag{98}$$

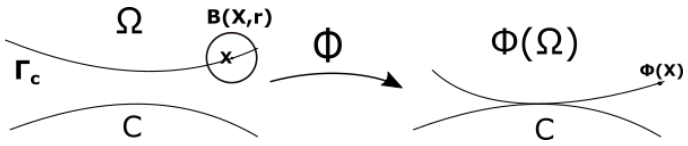

Figure 3: The case where the stress is zero for the point $\mathbf{X}$

We deduce that $\mathbf{PN} = \mathbf{0}$ for $\mathbf{X} \in \Gamma_C$ such that $\boldsymbol{\phi}(\mathbf{X}) \in \overline{\mathcal{C}}^c$.

Let $\gamma_C = \boldsymbol{\phi}(\Gamma_C) \cap \partial\mathcal{C}$. We will prove next that the coefficient λ_n in the equation (88) is negative. We consider $\mathbf{Y} \in \Gamma_C$ such that $\mathbf{y} = \boldsymbol{\phi}(\mathbf{Y}) \in \gamma_C$ (see Figure 4). Consider any positive smooth function $\theta : \bar{\Omega} \to \mathbb{R}_+$ with support in $B(\mathbf{Y}, r)$, then $\exists\, \epsilon_4(\theta) > 0$ such that:

$$\boldsymbol{\phi}_\epsilon = \boldsymbol{\phi} - \epsilon\theta\mathbf{n} \in \boldsymbol{\Phi} \quad \forall\, 0 \leq \epsilon \leq \epsilon_4 \tag{99}$$

Thus by the same procedure we obtain:

$$\epsilon \left\{ \int_{\Gamma_C} -\mathbf{PN}.\mathbf{n}\theta \, dS + \frac{o(\epsilon)}{\epsilon} + \frac{j(\mathbf{u}_\epsilon) - j(\mathbf{u})}{\epsilon} \right\} \geq 0 \tag{100}$$

where $\mathbf{u}_\epsilon$, $\mathbf{u}$ are respectively the displacement fields of $\boldsymbol{\phi}_\epsilon$ and $\boldsymbol{\phi}$, j is defined by

$$j(\mathbf{v}) = \int_{\gamma_C} \tau.\eta_\alpha(\mathbf{v}_T) \, ds \tag{101}$$

As the normal direction is considered then $j(\mathbf{u}_\epsilon) - j(\mathbf{u}) = 0$. If $\epsilon \to 0^+$ then we have:

$$\int_{\Gamma_C} \mathbf{PN}.\mathbf{n}\theta \, dS \leq 0 \tag{102}$$

Thus $\lambda_n = \mathbf{PN}.\mathbf{n} \leq 0$ at $\mathbf{X} \in \Gamma_C$ where $\boldsymbol{\phi}(\mathbf{X}) \in \gamma_C$.

We still have the last equation of the problem (88) to demonstrate. We also consider $\mathbf{Y} \in \Gamma_C$ such that $\mathbf{y} = \boldsymbol{\phi}(\mathbf{Y}) \in \gamma_C$, supposing that the boundaries of $\boldsymbol{\phi}(\Omega)$, $\mathcal{C}$ are smooth enough, then we can assume that $\boldsymbol{\phi}(\Gamma_C)$ and $\partial\mathcal{C}$ have the same tangent space at the point $\mathbf{y} = \boldsymbol{\phi}(\mathbf{Y})$. Let $V(\mathbf{Y})$ be a neighborhood of $\mathbf{y}$ and $\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}$ a 3 smooth fields, such that $\mathbf{t}_1$, $\mathbf{t}_2$ span the tangent space at $V \cap \gamma_C$ and $\|\mathbf{t}_1\| = \|\mathbf{t}_2\| = 1$, $\mathbf{n}$ is the outer normal vector on the body. Consider the ball $B(\mathbf{Y}, r)$ such that $B(\mathbf{Y}, r) \cap \Gamma \subset \Gamma_C$ and $\boldsymbol{\phi}(B(\mathbf{Y}, r)) \subset V$ (see Figure 4), therefore given two smooth functions $\theta_1, \theta_2 : \bar{\Omega} \to \mathbb{R}$ with support in $B(\mathbf{Y}, r)$, there exist $\epsilon_3(\theta_1, \theta_2) > 0$ and two functions $\lambda_1^\epsilon, \lambda_2^\epsilon : \bar{\Omega} \to \mathbb{R}$ with support in $B(\mathbf{Y}, r)$ such that:

For $\beta = 1, 2$

$$\begin{cases} \boldsymbol{\phi}^\epsilon = \boldsymbol{\phi} + \epsilon(\theta_\beta \mathbf{t}_\beta + \lambda_\beta^\epsilon \mathbf{n}) \in \boldsymbol{\Phi} \ \forall\, |\epsilon| \leq \epsilon_3 \\ |\lambda_\beta^\epsilon| = o(\epsilon) \end{cases} \tag{103}$$

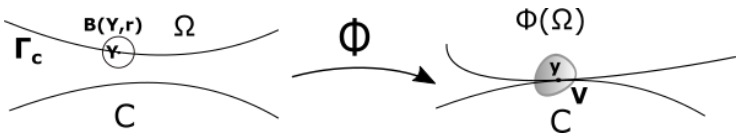

Figure 4: The point $\mathbf{y} = \boldsymbol{\phi}(\mathbf{Y})$ and its neighborhood

Taking $\mathcal{E}(\boldsymbol{\phi}^\epsilon) - \mathcal{E}(\boldsymbol{\phi}) \geq 0$ and repeating the same procedure as before, we obtain:

$$\epsilon \left\{ \int_{\Gamma_C} \mathbf{PN}.\theta_\beta \mathbf{t}_\beta \, dS + \frac{1}{\epsilon}(j(\mathbf{u}_\epsilon) - j(\mathbf{u})) + o(\epsilon) + \frac{o(\epsilon)}{\epsilon} \right\} \geq 0 \tag{104}$$

Now the stress tensor $\mathbf{P}$ is transformed into the Cauchy one $\boldsymbol{\sigma}$, which acts on the actual configuration. We can use also the fact that $\mathbf{P}.\mathbf{N}dS = \boldsymbol{\sigma}\mathbf{n}ds$, where dS and ds are respectively the area measures in the initial and actual configuration. Therefore

$$\epsilon \left\{ \int_{\gamma_C} \boldsymbol{\sigma}\mathbf{n}.\theta_\beta \mathbf{t}_\beta \, ds + \int_{\gamma_C} \tau.\frac{1}{\epsilon}\left(\eta_\alpha(\mathbf{u}_T + \epsilon\theta_\beta \mathbf{t}_\beta + \epsilon \mathbf{o}(\epsilon)) - \eta_\alpha(\mathbf{u}_T)\right) ds + o(\epsilon) + \frac{o(\epsilon)}{\epsilon} \right\} \geq 0 \tag{105}$$

Using Taylor's theorem we obtain

$$\int_{\gamma_C} \boldsymbol{\sigma}\mathbf{n}.\theta_\beta \mathbf{t}_\beta \, ds + \int_{\gamma_C} \tau.\nabla\eta_\alpha(\mathbf{u}_T).(\theta_\beta \mathbf{t}_\beta) \, ds + o(\epsilon) + \frac{o(\epsilon)}{\epsilon} \geq 0 \tag{106}$$

By taking $\epsilon \to 0$ we obtain that

$$\int_{\gamma_C} \boldsymbol{\sigma}\mathbf{n}.\theta_\beta \mathbf{t}_\beta \, ds + \int_{\gamma_C} \tau.\nabla\eta_\alpha(\mathbf{u}_T).(\theta_\beta \mathbf{t}_\beta) \, ds \geq 0 \tag{107}$$

which is equivalent to

$$\int_{\gamma_C} \boldsymbol{\sigma}_T.\theta_\beta \mathbf{t}_\beta \, ds + \int_{\gamma_C} \tau.\nabla\eta_\alpha(\mathbf{u}_T).(\theta_\beta \mathbf{t}_\beta) \, ds \geq 0 \tag{108}$$

and thus we have

$$\int_{\gamma_C} \boldsymbol{\sigma}_T.\theta_\beta \mathbf{t}_\beta \, ds + \int_{\gamma_C} \tau.\nabla\eta_\alpha(\mathbf{u}_T).(\theta_\beta \mathbf{t}_\beta) \, ds = 0 \tag{109}$$

We conclude as before that for $\beta = 1, 2$ we have

$$\boldsymbol{\sigma}_T.\mathbf{t}_\beta = -\tau\nabla\eta_\alpha(\mathbf{u}_T).\mathbf{t}_\beta \tag{110}$$

$$= -\tau \frac{\mathbf{u}_T}{\sqrt{|\mathbf{u}_T|^2 + \alpha^2}}.\mathbf{t}_\beta \text{ if } \eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2} \tag{111}$$

$\nabla\eta_\alpha(\mathbf{u}_T)$ is supposed to belong to the tangent plane (it's true if $\eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2}$). Therefore for any tangential vector $\mathbf{t}$ we have

$$(\boldsymbol{\sigma}_T + \tau\nabla\eta_\alpha(\mathbf{u}_T)).\mathbf{t} = 0 \tag{112}$$

We deduce then that $\boldsymbol{\sigma}_T = -\tau\nabla\eta_\alpha(\mathbf{u}_T) = -\tau\frac{\mathbf{u}_T}{\sqrt{|\mathbf{u}_T|^2+\alpha^2}}$. □

Remark 4.1. *From an algorithmic point of view, we remark that the problem (88) can be solved as a fixed point algorithm. Thus at each step k*

$$\boldsymbol{\phi}_{k+1} = T(\boldsymbol{\phi}_k) \tag{113}$$

5 Fixed point algorithm convergence for the frictional regularized discretized problem

In the paper [23], the discretized frictional Signorini's problem, using P_1 finite elements was written in term of a fixed point algorithm, and it was proven that there exit a solution for this latter and this algorithm converges for small friction coefficient. In [15], a mixed finite element method was considered, and the friction coefficient threshold for the uniqueness of the solution, depends on the mesh size and on the regularization parameter α. However we will follow the most part of the proof of [23] with several modifications, in order to treat our regularized problem also for the Signorini's case. Like [23], the friction coefficient threshold for the uniqueness of the solution depends only on the mesh size.

First consider the following finite element spaces for the body $\Omega_h \subset \mathbb{R}^d$, where $d = 2, 3$

$$\begin{cases} X_h &= \left\{ v \in C^0(\Omega_h) \mid v_{|T_i} \in P_1, \, \forall \, T_i \text{ triangle of } \Omega_h \right\} \\ \mathbf{V}_h &= \left\{ \mathbf{v} = (v_1, v_2) \in (X_h)^d \mid \mathbf{v} = 0 \text{ on } \Gamma_0 \right\} \\ \mathbf{K}_h &= \left\{ \mathbf{v} \in \mathbf{V}_h \, | \, v_n = \mathbf{v}.\mathbf{n} \leq 0 \text{ on } \Gamma_C \right\} \\ \hat{X}_h &= \text{ the trace space of } X_h \text{ on } \Gamma_C \\ \hat{\mathbf{V}}_h &= \text{ the trace space of } \mathbf{V}_h \text{ on } \Gamma_C \end{cases}$$

where as before Γ_C denotes the contact potential area, $\mathbf{V}_h$ the admissible set and $\mathbf{K}_h$ the set describing the non-penetration between the body and the obstacle with $\mathbf{n}$ the outward unit normal vector on Γ_C.

Let $\{\hat{w}_i \in \hat{X}_h \, | \, i = 1, \ldots, n_C\}$ be a basis of $\hat{X}_h$, otherwise speaking each vector of this basis is the non-zero trace of a vector of the basis of X_h on Γ_C. The linear application $R : \hat{\mathbf{V}}_h \to \mathbf{V}_h$ is defined such that, it associates to $\hat{\mathbf{v}} \in \hat{\mathbf{V}}_h$, a unique vector $\mathbf{v} = R\hat{\mathbf{v}} \in \mathbf{V}_h$ such that this latter is equal to zero at all nodes outside Γ_C.

Let Π_h denotes the interpolation operator on $\hat{X}_h$, Π_h has the following property (see [23])

$$|\Pi_h(|\mathbf{v}^h|)|_{L^2(\Gamma_C)} \leq c(h)|\mathbf{v}^h|_{L^2(\Gamma_C)} \quad \forall \, \mathbf{v}^h \in \hat{\mathbf{V}}_h \tag{114}$$

where $c(h)$ a constant depending on h. In addition Π_h has the following useful properties

Lemma 5.1. *Let $\phi, \phi_1, \phi_2 \in L^2(\Gamma_C)$, we have*

$$\begin{cases} \Pi_h(\phi) \geq 0 \quad \text{if } \phi \geq 0 \\ \Pi_h(\phi_1) \leq \Pi_h(\phi_2) \quad \text{if } \phi_1 \leq \phi_2 \\ |\Pi_h(\phi)| \leq \Pi_h(|\phi|) \end{cases} \tag{115}$$

Proof. First of all $\Pi_h(\phi)$ is given by

$$\Pi_h(\phi) = \sum_i \phi_i \hat{w}_i \tag{116}$$

The shape functions $\hat{w}_i \geq 0$ because we use P_1 finite elements, thus if $\phi \geq 0$, then $\Pi_h(\phi) \geq 0$. In addition if $\phi_1 \leq \phi_2$, then $\Pi_h(\phi_2 - \phi_1) \geq 0$ and we obtain the second equation of (115). Finally

$$|\Pi_h(\phi)| \leq \sum_i |\phi_i| \hat{w}_i = \Pi_h(|\phi|) \tag{117}$$

□

According to [23], one can defines two applications

$$\begin{cases} < \boldsymbol{\sigma}(\mathbf{v}), \hat{\mathbf{v}} > & = a(\mathbf{v}, R\hat{\mathbf{v}}) - f(R\hat{\mathbf{v}}) \quad \forall\, \mathbf{v} \in \mathbf{V}_h \text{ and } \forall\, \hat{\mathbf{v}} \in \hat{\mathbf{V}}_h \\ < \sigma_n(\mathbf{v}), \hat{w} > & =< \boldsymbol{\sigma}(\mathbf{v}), \hat{w}\mathbf{n} > \quad \forall\, \mathbf{v} \in \mathbf{V}_h \text{ and } \forall\, \hat{w} \in \hat{X}_h \end{cases} \tag{118}$$

The first one describes the stress vector on Γ_C and the second one describes the normal stress on Γ_C corresponding to a displacement test vector $\mathbf{v}$.

Remark 5.1. *If $\mathbf{v}$ is sufficiently regular (let's say $\mathbf{v} \in H^2$), then the Green formula can be used as in theorem 3.1, to obtain*

$$< \sigma_n(\mathbf{v}), \hat{w} >= \int_{\Gamma_C} \sigma_n(\mathbf{v}) \hat{w}\, ds \tag{119}$$

In the following, the regularization function η_α approximating the module of a vector, belongs to the set Ξ_α, defined before.

Our regularized frictional problem, approximating Coulomb's criterion is given by

Find $\mathbf{u}^h \in \mathbf{K}_h$ such that

$$a(\mathbf{u}^h, \mathbf{v} - \mathbf{u}^h) - < \mu\sigma_n(\mathbf{u}^h), \Pi_h(\eta_\alpha(\mathbf{v}_T) - \eta_\alpha(\mathbf{u}^h_T)) > \geq f(\mathbf{v} - \mathbf{u}^h) \quad \forall\, \mathbf{v} \in \mathbf{K}_h \tag{120}$$

where μ is the friction coefficient.

Let H denotes the set of all positive linear applications on $\hat{X}_h$, otherwise speaking, the set of applications τ such that

$$< \tau, \hat{w} > \geq 0 \quad \forall\, \hat{w} \in \hat{X}_h \geq 0 \tag{121}$$

For $\tau \in H$, the application T is defined by

$$< T(\tau), \hat{w} >= - < \mu\sigma_n(\mathbf{u}^h_\tau), \hat{w} > \tag{122}$$

where $\mathbf{u}^h_\tau$ is the solution of the frictional regularized problem seen before with the sliding limit τ, otherwise speaking $\mathbf{u}^h_\tau$ is the solution of the following problem

Find $\mathbf{u}^h_\tau \in \mathbf{K}_h$ such that

$$a(\mathbf{u}^h_\tau, \mathbf{v} - \mathbf{u}^h_\tau) + < \tau, \Pi_h(\eta_\alpha(\mathbf{v}_T) - \eta_\alpha(\mathbf{u}^h_{\tau,T})) > \geq f(\mathbf{v} - \mathbf{u}^h_\tau) \quad \forall\, \mathbf{v} \in \mathbf{K}_h \tag{123}$$

Remark 5.2. *The problem (123) has a unique solution, indeed consider the following energy*

$$E(\mathbf{v}) := \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - f(\mathbf{v}) + < \tau, \Pi_h(\eta_\alpha(\mathbf{v}_T)) > \tag{124}$$

The functional $\mathbf{v} \rightarrow < \tau, \Pi_h(\eta_\alpha(\mathbf{v}_T)) >$ is positive, convex and continuous, therefore the problem (123) is equivalent to the minimization of E over the closed and convex set $\mathbf{K}_h$, which assure the existence and the uniqueness of the solution.

If $T(\tau) \in H\,, \forall\, \tau \in H$, then we can deduce that the frictional problem (120) is equivalent to

Find a fixed point of the application T

$$T(\tau) = \tau \tag{125}$$

So we want to prove that $T(\tau) \in H$. Let $\hat{w} \geq 0 \in \hat{X}_h$, using the definition 118 one obtains

$$< \sigma_n(\mathbf{u}^h_\tau), \hat{w} >= a(\mathbf{u}^h_\tau, R(\hat{w}\mathbf{n})) - f(R(\hat{w}\mathbf{n})) \quad \forall\, \hat{w} \in \hat{X}_h \tag{126}$$

where R is a linear application previously defined. As $\hat{w} \geq 0$ then $\mathbf{v} = \mathbf{u}_\tau^h - R(\hat{w}\mathbf{n}) \in \mathbf{K}_h$, so we can inject it in the variational inequality (123) to obtain

$$< \sigma_n(\mathbf{u}_\tau^h), \hat{w} >= a(\mathbf{u}_\tau^h, R(\hat{w}\mathbf{n})) - f(R(\hat{w}\mathbf{n})) \leq 0 \tag{127}$$

Therefore $< T(\tau), \hat{w} > \geq 0$ and $T(\tau) \in H$.

In the following we will present some results in order to prove that the application T has a fixed point, which implies the existence of a solution of the problem (120). In addition it will be shown that for a small friction coefficient μ, the application T is a contraction which implies a unique fixed point of T, and the uniqueness of the solution can be proven. Finally the fixed point algorithm can be easily given in order to solve the problem (120) as a sequence of the problem (123).

Lemma 5.2.

$$|T(\tau_1) - T(\tau_2)|_* \leq \mu\, C(h)\, |\tau_1 - \tau_2|_* \quad \forall\, \tau_1, \tau_2 \in H \tag{128}$$

where $C(h)$ a constant which depends on the mesh size, and the dual norm $|\cdot|_*$ is defined as follows

$$|\tau|_* = \sup_{\phi \in \hat{X}_h} \frac{| < \tau, \phi > |}{|\phi|_{H^{1/2}(\Gamma_C)}} \tag{129}$$

Proof. Let $\tau_1, \tau_2 \in H$, and $\mathbf{u}_1$, $\mathbf{u}_2$ respectively the solutions of the equation (123) for $\tau = \tau_1$ and $\tau = \tau_2$. Taking $\mathbf{v} = \mathbf{u}_2$ in the equation (123) for $\tau = \tau_1$ and $\mathbf{v} = \mathbf{u}_1$ in the equation (123) for $\tau = \tau_2$, one obtains

$$\begin{cases} a(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + < \tau_1, \Pi_h(\eta_\alpha(\mathbf{u}_{2T}) - \eta_\alpha(\mathbf{u}_{1T})) > \geq f(\mathbf{u}_2 - \mathbf{u}_1) \\ a(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) + < \tau_2, \Pi_h(\eta_\alpha(\mathbf{u}_{1T}) - \eta_\alpha(\mathbf{u}_{2T})) > \geq f(\mathbf{u}_1 - \mathbf{u}_2) \end{cases} \tag{130}$$

equivalently

$$\begin{cases} a(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + < \tau_1, \Pi_h(\eta_\alpha(\mathbf{u}_{2T}) - \eta_\alpha(\mathbf{u}_{1T})) > \geq f(\mathbf{u}_2 - \mathbf{u}_1) \\ a(-\mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_1) + < \tau_2, \Pi_h(\eta_\alpha(\mathbf{u}_{1T}) - \eta_\alpha(\mathbf{u}_{2T})) > \geq f(\mathbf{u}_1 - \mathbf{u}_2) \end{cases} \tag{131}$$

Adding these two equations, one obtains

$$a(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_2 - \mathbf{u}_1) + < \tau_1 - \tau_2, \Pi_h(\eta_\alpha(\mathbf{u}_{2T}) - \eta_\alpha(\mathbf{u}_{1T})) > \geq 0 \tag{132}$$

Thus

$$a(\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) \leq < \tau_1 - \tau_2, \Pi_h(\eta_\alpha(\mathbf{u}_{2T}) - \eta_\alpha(\mathbf{u}_{1T})) > \tag{133}$$

Hence

$$\begin{aligned} a(\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) &\leq |\tau_1 - \tau_2|_* \, |\Pi_h(\eta_\alpha(\mathbf{u}_{2T}) - \eta_\alpha(\mathbf{u}_{1T}))|_{H^{1/2}(\Gamma_C)} \\ &\leq C|\tau_1 - \tau_2|_* \, |\Pi_h(\eta_\alpha(\mathbf{u}_{2T}) - \eta_\alpha(\mathbf{u}_{1T}))|_{L^2(\Gamma_C)} \end{aligned} \tag{134}$$

The last inequality is due to the equivalence between the norms $|\cdot|_{L^2}$ and $|\cdot|_{H^{1/2}}$ on a finite dimensional space.

Using the fact $\eta_\alpha \in \Xi_\alpha$ (see the definition 3.1) and the properties of Π_h, one obtains

$$\begin{aligned} a(\mathbf{u}_2-\mathbf{u}_1,\mathbf{u}_2-\mathbf{u}_1) &\leq C|\tau_1-\tau_2|_* \,|\Pi_h(\eta_\alpha(\mathbf{u}_{2T})-\eta_\alpha(\mathbf{u}_{1T}))|_{L^2(\Gamma_C)} \\ &\leq C|\tau_1-\tau_2|_* \,|\Pi_h(|\eta_\alpha(\mathbf{u}_{2T})-\eta_\alpha(\mathbf{u}_{1T})|)|_{L^2(\Gamma_C)} \\ &\leq C|\tau_1-\tau_2|_* \,|\,\Pi_h(|\,|\mathbf{u}_{2T}|-|\mathbf{u}_{1T}|\,|)\,|_{L^2(\Gamma_C)} \\ &\leq C|\tau_1-\tau_2|_* \,|\,\Pi_h(|\mathbf{u}_{2T}-\mathbf{u}_{1T}|)\,|_{L^2(\Gamma_C)} \\ &\leq C|\tau_1-\tau_2|_* \,|\,\Pi_h(|\mathbf{u}_2-\mathbf{u}_1|)\,|_{L^2(\Gamma_C)} \\ &\leq c.|\tau_1-\tau_2|_* \,|\,\mathbf{u}_2-\mathbf{u}_1\,|_{L^2(\Gamma_C)} \end{aligned} \tag{135}$$

Using the fact that a is elliptic, the above equation becomes

$$|\mathbf{u}_2-\mathbf{u}_1|_1^2 \leq C_1|\tau_1-\tau_2|_* \,|\,\mathbf{u}_2-\mathbf{u}_1\,|_{L^2(\Gamma_C)} \tag{136}$$

Considering the trace theorem, we obtain

$$|\mathbf{u}_2-\mathbf{u}_1|_1^2 \leq C_2|\tau_1-\tau_2|_* \,|\mathbf{u}_2-\mathbf{u}_1|_1 \tag{137}$$

Finally

$$|\mathbf{u}_2-\mathbf{u}_1|_1 \leq C_2|\tau_2-\tau_1|_* \tag{138}$$

Besides, from the definition 122, for $\hat{w} \in \hat{X}_h$

$$< T(\tau_2)-T(\tau_1),\hat{w} >= \mu < \sigma_n(\mathbf{u}_1)-\sigma_n(\mathbf{u}_2),\hat{w} > \tag{139}$$

Otherwise, using the definition 118 one obtains

$$\begin{cases} < \sigma_n(\mathbf{u}_1),\hat{w} >= a(\mathbf{u}_1,R(\hat{w}\mathbf{n}))-f(R(\hat{w}\mathbf{n})) & \forall\,\hat{w}\in\hat{X}_h \\ < \sigma_n(\mathbf{u}_2),\hat{w} >= a(\mathbf{u}_2,R(\hat{w}\mathbf{n}))-f(R(\hat{w}\mathbf{n})) & \forall\,\hat{w}\in\hat{X}_h \end{cases} \tag{140}$$

where R is a linear application previously defined. Hence

$$< T(\tau_2)-T(\tau_1),\hat{w} >= \mu.a(\mathbf{u}_1-\mathbf{u}_2,R(\hat{w}\mathbf{n})) \tag{141}$$

Therefore

$$\begin{aligned} | < T(\tau_2)-T(\tau_1),\hat{w} > | &\leq \mu.|\mathbf{u}_1-\mathbf{u}_2|_1|R(\hat{w}\mathbf{n})|_1 \text{ (continuity of } a) \\ &\leq \mu.C_2|\tau_2-\tau_1|_*|R(\hat{w}\mathbf{n})|_1 \text{ (equation (138))} \\ &\leq \mu.C_3|\tau_2-\tau_1|_*|\hat{w}\mathbf{n}|_{L^2} \text{ (continuity of } R) \\ &= \mu.C_3|\tau_2-\tau_1|_*|\hat{w}|_{L^2} \\ &\leq \mu.C_4|\tau_2-\tau_1|_*|\hat{w}|_{H^{1/2}} \end{aligned} \tag{142}$$

We conclude that

$$|T(\tau_2)-T(\tau_1)|_* \leq \mu\, C_4\,|\tau_2-\tau_1|_* \quad \forall\,\tau_1,\tau_2\in H \tag{143}$$

□

The existence and the uniqueness of the solution of the regularized frictional problem (120) depend on the existence and the uniqueness of the fixed point of the application T. We have the following theorem

Theorem 5.1. *If $\mu < \frac{1}{C(h)}$, the application T has a unique fixed point, and the following fixed point algorithm converges to the fixed point*

$$\tau_{n+1} = T(\tau_n) \tag{144}$$

In addition if $\mathbf{u}_{n+1}$ is the solution of the problem (123) for $\tau = \tau_{n+1}$ then

$$\mathbf{u}_{n+1} \underset{n\to\infty}{\longrightarrow} \mathbf{u}^* \tag{145}$$

where $\mathbf{u}^*$ is the unique solution of (120).

Proof. From the lemma 5.2, T is a contraction mapping, then using the Banach fixed-point theorem, T admits a unique fixed point τ^* and τ_{n+1} converges to τ^*.

Moreover from the equation (138) we have

$$|\mathbf{u}_{n+1} - \mathbf{u}^*|_1 \leq C_2 |\tau_{n+1} - \tau^*|_* \tag{146}$$

Because $\tau_{n+1} \underset{n\to\infty}{\longrightarrow} \tau^*$ then $\mathbf{u}_{n+1} \underset{n\to\infty}{\longrightarrow} \mathbf{u}^*$. □

Next we want to prove only the existence of a solution for the problem (120) without any restriction on the friction coefficient. First let's introduce the following lemma

Lemma 5.3. *There exists a constant $C > 0$ such that*

$$|T(\tau)|_* \leq C \quad \forall\, \tau \in H \tag{147}$$

Proof. Let $\mathbf{u}$ be a solution of the problem (123) for the sliding limit $\tau \in H$, therefore taking $\mathbf{v} = \mathbf{0} \in \mathbf{K}_h$ in the inequality (123) one obtains

$$a(\mathbf{u}, -\mathbf{u}) + < \tau, \Pi_h(\eta_\alpha(\mathbf{0}) - \eta_\alpha(\mathbf{u}_T)) > \geq f(-\mathbf{u}) \tag{148}$$

Hence

$$a(\mathbf{u}, \mathbf{u}) \leq < \tau, \Pi_h(\eta_\alpha(\mathbf{0}) - \eta_\alpha(\mathbf{u}_T)) > + f(\mathbf{u}) \tag{149}$$

Because $\tau \in H$ and $\eta_\alpha \in \Xi_\alpha$ (see the definition 3.1), we have $< \tau, \Pi_h(\eta_\alpha(\mathbf{0}) - \eta_\alpha(\mathbf{u}_T)) > \leq 0$, and thus

$$a(\mathbf{u}, \mathbf{u}) \leq f(\mathbf{u}) \tag{150}$$

Using the fact that a is elliptic and f is continuous, we deduce the existence of a constant $C_2 \geq 0$ such that

$$|\mathbf{u}|_1 \leq C_2 \tag{151}$$

Besides

$$\begin{aligned} |T(\tau)|_* &= \sup_{\phi \in \hat{X}_h} \frac{| < T(\tau), \phi > |}{|\phi|_{H^{1/2}(\Gamma_C)}} \\ &= \mu \sup_{\phi \in \hat{X}_h} \frac{|a(\mathbf{u}, R(\phi\mathbf{n})) - f(R(\phi\mathbf{n}))|}{|\phi|_{H^{1/2}(\Gamma_C)}} \text{ (equation (140))} \end{aligned} \tag{152}$$

From the continuity of a and f we have

$$\begin{aligned} |a(\mathbf{u}, R(\phi\mathbf{n})) - f(R(\phi\mathbf{n}))| &\leq C_3 |\mathbf{u}|_1 |R(\phi\mathbf{n})|_1 + C_4 |R(\phi\mathbf{n})|_1 \\ &\leq C_5 |\mathbf{u}|_1 |\phi|_{H^{1/2}} + C_6 |\phi|_{H^{1/2}} \text{ (like equation (142))} \\ &= (C_5 |\mathbf{u}|_1 + C_6) |\phi|_{H^{1/2}} \\ &\leq C_7 |\phi|_{H^{1/2}(\Gamma_C)} \text{ (equation (151))} \end{aligned} \tag{153}$$

Therefore

$$|T(\tau)|_* \leq \mu C_7 \tag{154}$$

□

Finally we have the following theorem

Theorem 5.2. *There exists a fixed point for the application T.*

Proof. We are in a finite dimensional space, thus if we take $M = H \cap \bar{B}(0, C)$ as the intersection of H with the closed ball $\bar{B}(0, C)$, M is compact and convex of the dual of $\hat{X}_h$. From the lemma 5.3 we deduce that $T(M) \subseteq M$ and we know that T is continuous, therefore by applying Brouwer's fixed-point theorem we conclude that T admits a fixed point. □

Note that all the above constants do not depend on the regularization parameter α and all previous results can be generalized for the case of contact between two bodies.

5.1 Error between Tresca's discretized solution and regularized Tresca's discretized solution

For the sake of clarity we consider only the Signorini case. We have the following theorem

Theorem 5.3. *Let $\mathbf{u} \in \mathbf{K}_h$ be the Tresca solution, in other words, solution of*

$$a(\mathbf{u}, \mathbf{v} - \mathbf{u}) + < \tau, \Pi_h(|\mathbf{v}_T| - |\mathbf{u}_T|) > \geq f(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in \mathbf{K}_h \tag{155}$$

and let $\mathbf{u}_\alpha \in \mathbf{K}_h$ be the regularized Tresca solution, otherwise speaking, solution of

$$a(\mathbf{u}_\alpha, \mathbf{v} - \mathbf{u}_\alpha) + < \tau, \Pi_h(\eta_\alpha(\mathbf{v}_T) - \eta_\alpha(\mathbf{u}_{\alpha,T})) > \geq f(\mathbf{v} - \mathbf{u}_\alpha) \quad \forall \mathbf{v} \in \mathbf{K}_h \tag{156}$$

then there exists a constant $C \geq 0$ such that

$$|\mathbf{u}_\alpha - \mathbf{u}|_1 \leq C\sqrt{\alpha} \tag{157}$$

Proof. Replacing $\mathbf{v}$ by $\mathbf{u}_\alpha$ in the equation (155), and $\mathbf{v}$ by $\mathbf{u}$ in the equation (156), one obtains

$$\begin{cases} a(\mathbf{u}, \mathbf{u}_\alpha - \mathbf{u}) + < \tau, \Pi_h(|\mathbf{u}_{\alpha,T}| - |\mathbf{u}_T|) > \geq f(\mathbf{u}_\alpha - \mathbf{u}) \\ a(\mathbf{u}_\alpha, \mathbf{u} - \mathbf{u}_\alpha) + < \tau, \Pi_h(\eta_\alpha(\mathbf{u}_T) - \eta_\alpha(\mathbf{u}_{\alpha,T})) > \geq f(\mathbf{u} - \mathbf{u}_\alpha) \end{cases} \tag{158}$$

Adding these two equations, we obtain

$$a(\mathbf{u} - \mathbf{u}_\alpha, \mathbf{u}_\alpha - \mathbf{u}) + < \tau, \Pi_h(|\mathbf{u}_{\alpha,T}| - \eta_\alpha(\mathbf{u}_{\alpha,T}) + \eta_\alpha(\mathbf{u}_T) - |\mathbf{u}_T|) > \geq 0 \tag{159}$$

Thus

$$\begin{aligned} a(\mathbf{u}_\alpha - \mathbf{u}, \mathbf{u}_\alpha - \mathbf{u}) &\leq |\tau|_* \, |\Pi_h(|\mathbf{u}_{\alpha,T}| - \eta_\alpha(\mathbf{u}_{\alpha,T}) + \eta_\alpha(\mathbf{u}_T) - |\mathbf{u}_T|)|_{H^{1/2}(\Gamma_C)} \\ &\leq C|\tau|_* \, |\Pi_h(|\mathbf{u}_{\alpha,T}| - \eta_\alpha(\mathbf{u}_{\alpha,T}) + \eta_\alpha(\mathbf{u}_T) - |\mathbf{u}_T|)|_{L^2(\Gamma_C)} \\ &\leq C|\tau|_* \, |\Pi_h(||\mathbf{u}_{\alpha,T}| - \eta_\alpha(\mathbf{u}_{\alpha,T})|) + \Pi_h(|\eta_\alpha(\mathbf{u}_T) - |\mathbf{u}_T||)|_{L^2(\Gamma_C)} \\ &\leq 2C|\tau|_* |\Pi_h(1)|_{L^2(\Gamma_C)}.\alpha \quad (\eta_\alpha \in \Xi_\alpha) \end{aligned} \tag{160}$$

Because a is elliptic then

$$|\mathbf{u}_\alpha - \mathbf{u}|_1 \leq C_1\sqrt{\alpha} \tag{161}$$

□

Corollary 5.1. *If* $\eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2}$ *then*

$$|\mathbf{u}_\alpha - \mathbf{u}|_1 \leq C\alpha^{1/2} \tag{162}$$

6 The Algorithm

Using the finite element approach, for $l = 1$ or 2, let Ω_h^l be the mesh of the body Ω^l, which is composed from the triangles family $\{T_i^l \,|\, i = 1, \ldots, n_T^l\}$. In addition, consider the following spaces

$$\mathbf{V}_h^l = \left\{ \mathbf{v} = (v_1, v_2) \in C^0(\Omega_h^l) \times C^0(\Omega_h^l) \mid \mathbf{v}_{|T_i^l} \in P_r \times P_r, \, \forall i = 1, \ldots, n_T^l \text{ and } \mathbf{v} = 0 \text{ on } \Gamma_0^l \right\} \tag{163}$$

where $C^0(\Omega_h^l)$ denotes the set of the continuous functions on Ω_h^l, and P_r denotes the linear finite elements for $r = 1$ and the quadratic ones for $r = 2$.

Consider the space $\mathbf{V}_h$ defined as follows

$$\mathbf{V}_h = \mathbf{V}_h^1 \times \mathbf{V}_h^2 \tag{164}$$

Let $\mathbf{u}_h = (\mathbf{u}_h^1, \mathbf{u}_h^2) \in \mathbf{V}_h$, the displacement vector field $\mathbf{u}_h^l$ on the mesh Ω_h^l is given by

$$\mathbf{u}_h^l = \sum_i \begin{pmatrix} U_i^x \\ U_i^y \end{pmatrix} \hat{w}_i^l \tag{165}$$

where $\hat{w}_i^l$ are the shape functions on the mesh Ω_h^l, and $\begin{pmatrix} U_i^x & U_i^y \end{pmatrix}^T$ are the degrees of freedom of $\mathbf{u}_h^l$, otherwise speaking U_i^x and U_i^y represent respectively the horizontal and vertical displacement of the node i in the mesh. In the following $\mathbf{U} \in \mathbb{R}^n$ denotes the vector of all degrees of freedom of $\mathbf{u}_h$, otherwise stated

$$\mathbf{U} = \begin{pmatrix} \ldots & U_i^x & U_i^y & \ldots \end{pmatrix}^T \tag{166}$$

We will present the algorithm to solve frictional contact problem in general case, otherwise speaking in the case of large deformations. For the sake of simplicity and in order to be more clear, the algorithm will be split into several algorithms.

The idea behind the algorithm is first to loop on the sliding limits τ until convergence, more precisely at each iteration k, the regularized frictional problem for a given sliding limit τ_k is solved, which corresponds to a minimization problem, then we retrieve the normal pressure $\sigma_{n,k}$ on the contact area, and the next sliding limit τ_{k+1} is computed via $\tau_{k+1} = -\mu\sigma_{n,k}$. This process continues until the relative error between two successive sliding limits is small enough.

In the sequel, $[\![\tau]\!]$ denotes an array containing the value of τ at the integration points of the contact area.

The algorithm solving the frictional problem is shown in algorithm 1.

Algorithm 1 Regularized frictional algorithm using the fixed point method

Set the error tolerance $\epsilon_{tol} = 10^{-6}$
Compute $\sigma_{n,0}$ the normal stress pressure at the contact area for the frictionless problem
Compute $\tau_0 = -\mu\sigma_{n,0}$, the first sliding limit
while *error* $\geq \epsilon_{tol}$ **do**

1. For a given sliding limit τ_k, solve Tresca's regularized problem, given in the algorithm 2
2. Retrieve the displacement field $\mathbf{u}_h$
3. Compute the normal pressure $\sigma_{n,k}(\mathbf{u}_h)$ on the contact surface
4. Compute the new sliding limit $\tau_{k+1} = -\mu\sigma_{n,k}$
5. error=$\dfrac{\|[\![\tau_{k+1}]\!] - [\![\tau_k]\!]\|_\infty}{\|[\![\tau_k]\!]\|_\infty}$

end while

The resolution of the contact problem without friction, is to solve the following constrained minimization problem

$$\begin{cases} \mathbf{u}_h = \underset{\mathbf{v}\in\mathbf{V}_h}{\arg\min}(\mathcal{E}_p(\mathbf{v})) \text{ s.t} \\ \displaystyle\int_{\Gamma_{C1}} ((\mathbf{x} - \bar{\mathbf{x}}_2)\mathbf{n}).\phi_i^{(1)}\, dS \geq 0 \quad \forall\, i = 1, \ldots, n_{C1} \\ \displaystyle\int_{\Gamma_{C2}} ((\mathbf{x} - \bar{\mathbf{x}}_1)\mathbf{n}).\phi_i^{(2)}\, dS \geq 0 \quad \forall\, i = 1, \ldots, n_{C2} \end{cases} \tag{167}$$

where $\mathbf{x} = \mathbf{X} + \mathbf{v}$ the actual position of a material point, with $\mathbf{X}$ the initial position of this same point. Γ_{Cl} is the initial potential contact area of the body Ω_h^l . $\bar{\mathbf{x}}_l$ is the projection point of $\mathbf{x}$ on the body Ω_h^l, where $l = 1, 2$ and $\mathbf{n}$ is the outward unit normal vector at $\bar{\mathbf{x}}_l$. Note that the two constraints in the problem (167) describe the non-penetration in a weak sense, and it is a symmetric formulation, in other words the user does not need to specify anymore a slave and a master body, see [16] for details. $\phi_i^{(l)}$ are the shape functions on the n_{Cl} nodes of the contact area Γ_{Cl}.

In addition $\mathcal{E}_p$ is the total potential energy defined by

$$\begin{cases} \mathcal{E}_p(\mathbf{v}) = \frac{1}{2}a(\mathbf{v}, \mathbf{v}) - f(\mathbf{v}) \text{ for linear elastic problems} \\ \mathcal{E}_p(\mathbf{v}) = \displaystyle\int_{\Omega_h^1 \cup \Omega_h^2} \hat{W}(\mathbf{v})dv - f(\mathbf{v}) \text{ for large deformations and hyperelastic problems} \end{cases} \tag{168}$$

where $\hat{W}$ is the strain energy function.

Moreover, in the case of large deformations, we can remark that the projection points $\bar{\mathbf{x}}_l$ for $l = 1, 2$ in the non-penetration constraints (167) depend on the actual solution of the problem. Thus we will use a fixed point algorithm to deal with this issue. Indeed in the fixed point algorithm iteration, we will use the displacement of the previous iteration, and based on this displacement we will compute for each point $\mathbf{x}$ its closest segment or triangle in the body Ω_h^l and its projection parameter, and therefore the projection point $\bar{\mathbf{x}}_l$ now depends linearly on the actual displacements. Otherwise speaking we obtain a sequence of minimization problems with linear constraints.

The resolution of the Tresca regularized problem, presented in the algorithm 2, will use the theorem 4.1 and therefore as we saw, we can use a fixed point algorithm.

We used the same fixed point algorithm treating the constraints. Therefore in the fixed point algorithm iteration, let's say $n+1$, we minimize the following energy E_{n+1} submitted to the linear constraints.

$$E_{n+1}(\mathbf{v}) = \mathcal{E}_p(\mathbf{v}) + \int_{\gamma_C^n} \tau_k . \eta_\alpha (\mathbf{v}_T^1 - \bar{\mathbf{v}}_T^2)\, ds \tag{169}$$

where τ_k is the sliding limit at the iteration k of the algorithm 1. $\mathbf{v}^1$ is the admissible displacement field of the first body and $\bar{\mathbf{v}}^2$ is the admissible displacement field of the second body applied on the projection points of the first body on the second one. Finally γ_C^n is the actual contact area based on the displacements of the previous iteration n.

Algorithm 2 Symmetric algorithm using the fixed point method for Tresca's regularized problem

Initialization of the displacement $\mathbf{U}_0$ and setting the tolerance $\epsilon_{tol} = 10^{-6}$
while $error \geq \epsilon_{tol}$ **do**

1. Using the displacement vector $\mathbf{U}_n$ of the previous iteration n:
   - Compute the projection points' parameters $\{\eta_i^* \,|\, i = 1, \ldots, nS\}$ of all slave integration points
   - Compute the normal at the projection points $\{n_i \,|\, i = 1, \ldots, nS\}$ (Using smoothing techniques)
   - Compute the contact area γ_C^n
2. For each integration point, its projection point $\bar{\mathbf{x}}_i$ depends linearly on the actual displacement
3. Reverse the role of the master and the slave bodies
4. Form the Energy E_{n+1} (Equation (169)) and the symmetric linear constraints
5. Use the interior point method in order to solve the minimization problem with linear constraints, and to obtain the actual displacement $\mathbf{U}_{n+1}$
6. error=$\dfrac{\|\mathbf{U}_{n+1} - \mathbf{U}_n\|_\infty}{\|\mathbf{U}_n\|_\infty}$

end while

7 Numerical validations

7.1 Validation of the regularized friction law

In this first example we will try to validate the regularized friction law given in the equation (77). Indeed we will take an elastic rectangular body of dimensions ($40UL \times 20UL$) laid on a rigid rectangle body (see Figure 5). The elastic body has the following material properties, a Young's modulus $E = 10^3 \frac{UF}{UL^2}$ and a poisson's ratio $\nu = 0$, note that UF, UL denote respectively the force and the length unit. A vertical force of $-30\frac{UF}{UL}$ is uniformly distributed along its top area.

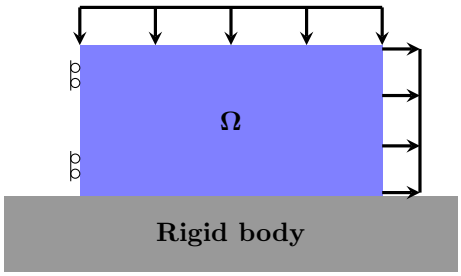

Figure 5: Problem geometry

At the first stage we impose a sliding conditions on its left boundary, and we apply a sequence of an uniformly distributed tangential force on its right boundary pointing to the right, with the following values $2, 5, 10, 20, 30\ \frac{UF}{UL}$. In the second stage the sliding conditions are imposed on its right boundary and we apply a sequence of tangential pressure on its left boundary pointing to the left, with the following values $-2, -5, -10, -20, -30\ \frac{UF}{UL^2}$. Considering the midpoint of the contact area, the goal of this example is to plot the ratio of the tangential and normal stresses $\frac{\sigma_T}{\sigma_n}$ against the tangential displacement u_T, and to compare it with the theoretical one seen in the equation (77).

In all next examples the regularization function will be $\eta_\alpha(\mathbf{v}) = \sqrt{|\mathbf{v}|^2 + \alpha^2}$ for $\mathbf{v} \in \mathbb{R}^2$ or $\mathbb{R}^3$. Taking $\alpha = 10^{-2}$, then according to the equation (77) we have

$$\frac{\sigma_T}{\sigma_n} = \mu \frac{u_T}{\sqrt{u_T^2 + \alpha^2}} \tag{170}$$

We consider a friction coefficient $\mu = 0.1$, and linear finite elements. The normal stress at the contact area is equal to $\sigma_n = \sigma_{yy}$ and the tangential stress on the contact area is equal to $\sigma_T = -\sigma_{xy}$, moreover the tangential displacement is equal to $u_T = u_x$. In the Figure 6, the ratio $\frac{\sigma_T}{\sigma_n}$ against the tangential displacement u_T is plotted for the different loads mentioned above, and is compared with the theoretical one given by the equation (170).

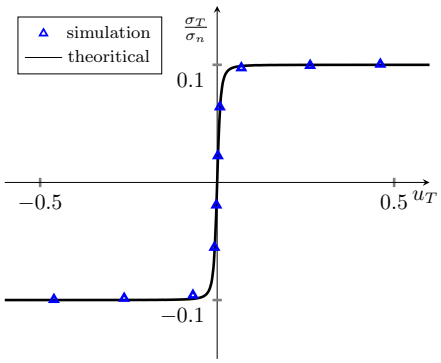

Figure 6: $\frac{\sigma_T}{\sigma_n}$ vs u_T

We can see the consistency between the simulation and the theoretical results, and how regularization can approach Coulomb's law. We saw similar results for quadratic finite elements, three dimensional case and for hyperelastic materials.

7.2 Frictional Hertz contact

We consider a contact between a half elastic cylinder Ω_1 ($E_1 = 200\,\frac{UF}{UL^2}, \nu_1 = 0.3$) and an elastic half-space Ω_2 ($E_2 = 200\,\frac{UF}{UL^2}, \nu_2 = 0.3$), the geometry and the dimensions are shown in the Figure 7 (units in UL). The Frictional coefficient is taken relatively big, $\mu = 0.8$, and the lower area of Ω_2 is fixed. First, we impose a downward vertical displacement of $4.55\,UL$ in 3 steps on the top of the half cylinder, then a total horizontal load of $q = 0.05\,\frac{UF}{UL}$ is applied in 7 steps, again on the top of the half cylinder. Obviously it's a quasi-static study.

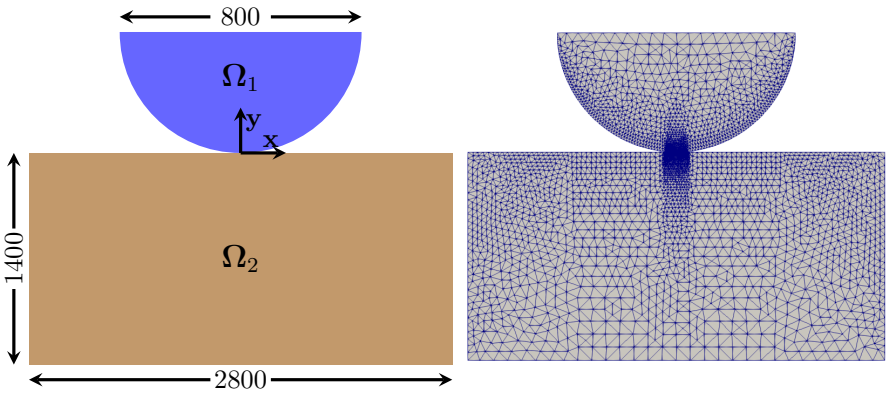

Figure 7: The geometry and the mesh of the problem

The imposed vertical displacement gives an equivalent maximal normal pressure of $p_0 = 2.93\,\frac{UF}{UL}$. Thus the equivalent vertical force is equal to

$$P = \frac{\pi R p_0^2}{E^*} \tag{171}$$

where R is the cylinder radius, and E^* the effective Young modulus given by

$$E^* = \frac{E_1 E_2}{E_1(1-\nu_2^2) + E_2(1-\nu_1^2)} \tag{172}$$

According to [18] and with the small deformations hypothesis, there exist two slip zones $\{c \leq |x| \leq a\}$ and one stick zone $\{|x| \leq c\}$, where

$$\begin{cases} a = \sqrt{\frac{4PR}{\pi E^*}} \\ c = a\sqrt{1 - \frac{Q}{\mu P}} \end{cases} \tag{173}$$

with $Q = 2Rq$ and a the half contact width, moreover the normal and tangential stresses are given by the following.

The normal pressure at the contact zone:

$$p_n = \frac{p_0}{a}\sqrt{a^2 - x^2} \tag{174}$$

The tangential pressure at the contact zone:

$$\begin{cases} p_t = \mu\frac{p_0}{a}(\sqrt{a^2 - x^2} - \sqrt{c^2 - x^2}) & \text{if } |x| \leq c \\ p_t = \mu\frac{p_0}{a}\sqrt{a^2 - x^2} & \text{if } c \leq |x| \leq a \end{cases} \tag{175}$$

The mesh of the two bodies is shown in the Figure 7. Using the quadratic finite elements (P_2), and a regularization parameter $\alpha = 10^{-3}$, the computed normal and tangential stresses, in addition to the theoretical ones, are depicted in the Figure 8.

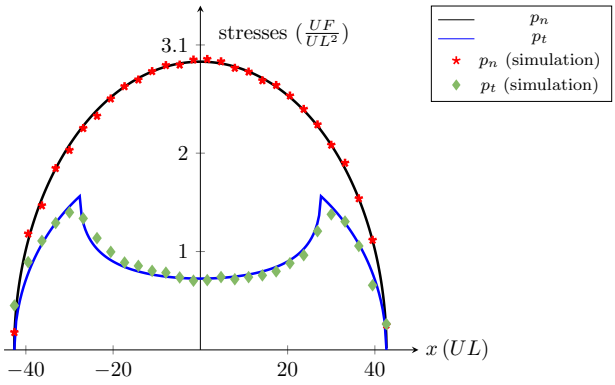

Figure 8: The stresses on the contact area

7.3 Shallow ironing

This example was studied by many papers, we can cite for example [13, 29, 31]. A small indenter Ω_1 is pressed into a less stiffer rectangular body Ω_2 at the first stage , and pulled horizontally to the right at the second stage. The geometries of the two bodies are shown in the Figure 9 (units in UL), in addition the bottom of the rectangular body is fixed. Neo-Hookean material is assumed for the two bodies (see [29]), with ($E_1 = 68.96 \times 10^2\ \frac{UF}{UL^2}$, $\nu = 0.32$) for the small indenter and ($E_2 = 6.896 \times 10^2\ \frac{UF}{UL^2}$, $\nu = 0.32$) for the rectangular body, which is 10 times softer than the indenter.

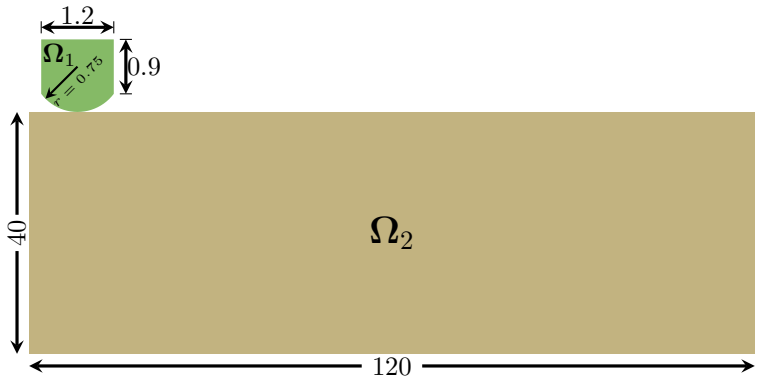

Figure 9: The geometry of the shallow ironing problem

At the first stage a downward vertical displacement of $8\,UL$ is applied in 8 time steps on the top of the indenter, in the second stage a horizontal displacement of $100\,UL$ is applied on the top of the indenter to the right in 500 time steps. This is a quasi-static study with a friction coefficient $\mu = 0.3$.

Using the quadratic finite elements (P_2), and a regularization parameter $\alpha = 10^{-2}$, the mesh of the two bodies and the deformation shapes at some time steps are shown in the Figure 10.

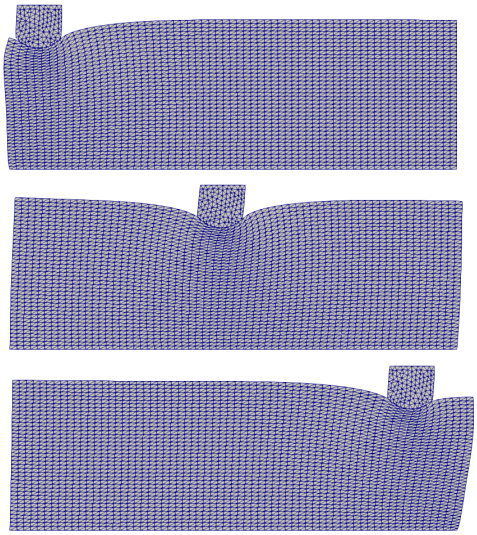

Figure 10: The deformation shapes at $t = 8, 254, 508$

The vertical and horizontal reactions on the indenter, are depicted in the Figure 11.

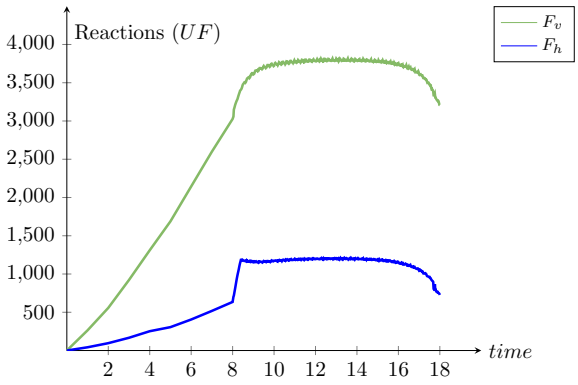

Figure 11: The vertical and horizontal reaction F_v, F_h on the indenter

There was no agreement on the results between the papers which studied this test. However, in order to show that our results are reasonable, we computed the ratio between the horizontal and the vertical reaction, when the indenter slide on the second body. We obtained a ratio approximately equal to 0.32, which is very close to the friction coefficient $\mu = 0.3$.

8 Conclusions

The frictional contact problem was transformed into a sequence of Tresca's problems which are based on a minimization principle, where we can use several optimization techniques in order to converge faster to the solution. In addition, a regularization was used in order to eliminate the non-smooth character of the friction behavior, and the interior point method was employed to solve the generated minimization problems. A fixed point method was used in the case of finite deformation, and symmetrical non-penetration constraints were considered. Finally, our algorithm gives satisfactory results and has been validated on several contact examples.

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