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# NUMERICAL CONTROL OF THE WAVE EQUATION AND HUYGENS' PRINCIPLE

CAROLE ROSIER AND LIONEL ROSIER

ABSTRACT. The numerical control of the wave equation is investigated in dimension 2 and 3. Using Huygens' principle in dimension 3 (resp. the method of descent in dimension 2), we show that the exact controllability of the wave equation can be obtained numerically on a cuboid (resp. on a rectangle) in sharp time with a boundary control supported on adjacent sides, by directly solving the wave equation on a larger domain. As a consequence, the numerical control of the wave equation is not affected here by the appearance of high frequency spurious numerical waves, and no filtering is needed.

*This paper is dedicated to Professor Marius Tucsnak for his 60th anniversary.*

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$ , be a bounded open set, and let  $\Gamma_0 \subset \Gamma := \partial\Omega$  be an open set. The classical boundary control problem for the wave equation can be stated as follows: given  $T > 0$  and  $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ , can we find a control input  $g \in L^2(0, T, L^2(\Gamma_0))$  such that the solution of the boundary initial-value problem

$$\square u := u_{tt} - \Delta u = 0 \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

$$u = g \mathbf{1}_{\Gamma_0} \quad \text{on } \Gamma \times (0, T), \quad (1.2)$$

$$u(\cdot, 0) = u^0, \quad u_t(\cdot, 0) = u^1, \quad (1.3)$$

satisfies

$$u(\cdot, T) = u_t(\cdot, T) = 0? \quad (1.4)$$

That issue has been extensively studied, both theoretically and numerically. Using duality arguments, the exact controllability of (1.1)-(1.4) was reduced to an observability estimate for the adjoint equation, that was in turn established with nonharmonic Fourier series in [25, 27, 35], with the multiplier method in [15, 18], and with microlocal analysis in [4, 5]. The study of the numerical control of the wave equation turned to be a delicate issue, due to the presence of high frequency spurious numerical waves. Various remedies (multi-grid methods, filtering, Tychonoff regularization, vanishing viscosity, mixed finite element methods, etc.) were introduced to overcome this problem, see e.g. [2, 3, 8, 12, 13, 22, 26, 29, 33, 37]. We refer the reader to [10, 38, 39] for recent and exhaustive surveys about the numerical control of the wave equation.

The numerical schemes in the references above use finite differences or finite elements methods combined with the Hilbert Uniqueness Method (HUM) introduced by J.-L. Lions. It should be

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*Key words and phrases.* wave equation; Huygens' principle; method of descent; exact controllability, numerical control; Fourier series.

mentioned that some other (more direct) approaches were also introduced since the seventies for the control of PDEs, using domain extensions [19, 35], fundamental solutions compactly supported in time [14, 34], resolution of ill-posed problems [20, 21], flatness outputs [16, 23, 24], or feedback controls achieving a finite-time stabilization [1, 30]. In [6], the numerical control of a semilinear wave equation on the interval  $(0, 1)$  is investigated by using a Picard iterative scheme and a transparent boundary condition at  $x = 1$ .

In this paper, we are interested in developing a *direct* approach (i.e. not based on the observability of the adjoint equation) for the numerical control of (1.1)-(1.4). Our approach rests on the numerical integration (by using spectral methods) of a classical Cauchy problem for the wave equation in an extended domain for which Huygens' principle is valid. By Huygens' principle, we mean here the following property: for any  $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ , we have

$$(v^0(y), v^1(y), \nabla v^0(y)) = 0 \text{ for } |y - x| = |t| \quad \Rightarrow \quad v(x, t) = 0 \quad (1.5)$$

for any smooth solution  $v$  of

$$\square v = 0 \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \quad v(., 0) = v^0, \quad v_t(., 0) = v^1. \quad (1.6)$$

A direct consequence of (1.5) is that if both  $v^0$  and  $v^1$  are supported in a compact set  $K \subset \mathbb{R}^3$ , and if  $T > \text{diam}(K) := \sup_{x_1, x_2 \in K} |x_1 - x_2|$ , then

$$v(x, t) = 0, \quad \text{for } x \in K, \quad t \geq T. \quad (1.7)$$

That property has important implications for the control problem (1.1)-(1.4). Assume first that  $\Gamma_0 = \Gamma$ , and assume given  $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ . Let  $(v^0, v^1) \in L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3)$  denote the extension by 0 of  $(u^0, u^1)$ , and let  $v$  denote the solution of (1.6). Then  $u = v|_{\Omega \times (0, T)}$  and  $g = v|_{\Gamma_0 \times (0, T)}$  solve (1.1)-(1.4) provided that  $T > \text{diam}(\Omega)$ , by (1.7) [19].

When  $\Gamma_0$  is a proper subset of  $\Gamma$ , the additional constraint

$$u(x, t) = 0 \quad (x, t) \in (\Gamma \setminus \Gamma_0) \times (0, T) \quad (1.8)$$

imposes to modify the method as follows. We introduce, when it is possible, some unbounded smooth surface  $\mathcal{S} \subset \mathbb{R}^3$  such that

- (i)  $\mathbb{R}^3 \setminus \mathcal{S} = \Omega_1 \cup \Omega_2$ ,  $\Omega_1$  and  $\Omega_2$  being two unbounded, connected open sets with  $\partial\Omega_1 = \partial\Omega_2 = \mathcal{S}$ ;
- (ii)  $\Omega \subset \Omega_1$ ,  $\Gamma \setminus \Gamma_0 \subset \mathcal{S}$ ;
- (iii) (Huygens' principle in  $\Omega_1$ ) for any compact set  $K \subset \overline{\Omega_1}$ , there exists a time  $T > 0$  such that for any solution  $v$  of

$$\square v = 0 \quad \text{in } \Omega_1 \times \mathbb{R}, \quad (1.9)$$

$$v = 0 \quad \text{in } \mathcal{S} \times \mathbb{R}, \quad (1.10)$$

$$v(., 0) = v^0, \quad v_t(., 0) = v^1 \quad (1.11)$$

with  $v^0, v^1$  supported in  $K$ , it holds

$$v(x, t) = 0 \quad \text{for } (x, t) \in K \times [T, +\infty). \quad (1.12)$$

Next, given  $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ , we let  $(v^0, v^1) \in L^2(\Omega_1) \times H^{-1}(\Omega_1)$  denote the extension by 0 of  $(u^0, u^1)$ , and introduce the solution  $v$  of (1.9)-(1.11). Then  $u = v|_{\Omega \times (0, T)}$  and  $g = v|_{\Gamma_0 \times (0, T)}$  solve (1.1)-(1.4)

Of course, this control method can be carried out at the only condition that Huygens' principle is valid and established. To the best knowledge of the authors, Huygens' principle in open sets as stated in (iii) was not investigated so far. It should be noticed that it is related to the Geometric Control Condition introduced in [4] for the control of the wave equation. Indeed, Huygens' principle in  $\Omega_1$  fails whenever there are some trapped rays in  $\Omega_1$  (see e.g. [32]). The validity of Huygens' principle cannot nevertheless be reduced to the absence of trapped rays in  $\Omega_1$ . Indeed, Huygens' principle is likely false for the complement of a ball in  $\mathbb{R}^3$ , as it was noticed to the authors by G. Lebeau in [17].

Here, we shall assume that  $\Omega_1$  is an unbounded domain in which all the rays of geometric optics can escape to infinity.

In this paper, we shall prove that Huygens' principle as stated in (iii) is valid when  $\Omega_1$  is a half-space. The proof will be done by an extension of the classical solution of the wave equation in  $\mathbb{R}^3$  by spherical means, the spherical wavefronts being replaced by wavefronts in  $\Omega_1$  taking into account the reflection of rays on  $\mathcal{S}$ . As an immediate application of the above method, we shall derive the exact controllability in optimal time of (1.1)-(1.3) when  $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$  and  $\Gamma_0$  is the union of three adjacent sides. We refer to [25, 27] for similar results obtained by an Ingham-type approach, and to [3, 22, 37] for the numerical control of the wave equation on a square.

The main goal of this study is to develop efficient numerical schemes for the control problem (1.1)-(1.4). Even for simple geometries, fast numerical schemes are desirable to study the numerical control of semilinear wave equations (see [6]) or of some equations that may be obtained from the wave equation by the transmutation method (e.g. the heat equation and Schrödinger equation [9, 28, 31]).

In practice, when a pair  $(\Omega_1, \mathcal{S})$  as above can be constructed, one solves the wave equation in an intermediate domain  $\widehat{\Omega}$  such that  $\Omega \subset \widehat{\Omega} \subset \Omega_1$ ,  $\widehat{\Omega}$  is bounded,  $\Gamma \setminus \Gamma_0 \subset \partial\widehat{\Omega}$ , and  $\text{dist}(\partial\widehat{\Omega} \setminus \mathcal{S}, \Omega)$  is large enough so that bounces of waves on  $\partial\widehat{\Omega} \setminus \mathcal{S}$  do not affect the solution  $v$  in  $\Omega \times (0, T)$ . (See Figure 1.)

If  $\Gamma \setminus \Gamma_0$  is flat (examples include a cube, a half-ball, a pyramid, etc.), one can solve the wave equation in  $\widehat{\Omega} = (-L, L)^3$  by using Fourier series.

All the results discussed so far concern domains  $\Omega \subset \mathbb{R}^3$ . It is well known that Huygens' principle fails in  $\mathbb{R}^2$ , so that the above method cannot be applied as it is for domains in  $\mathbb{R}^2$ . Some stabilization results for domains  $\Omega \subset \mathbb{R}^2$  will be obtained by combining the above method with the classical method of descent.

The paper is outlined as follows. Section 2 is devoted to the derivation of Huygens' principle for the half-space. It is based on the extension of solutions of the wave equation to the whole space as odd functions in one coordinate and on the classical solution of the wave equation on  $\mathbb{R}^3$  by spherical means. The same ideas are applied to solve (1.1)-(1.4) when  $\Omega$  is a product of intervals. Section 3 contains some numerical experiments in dimensions 2 and 3. Section 4 provides some concluding remarks.

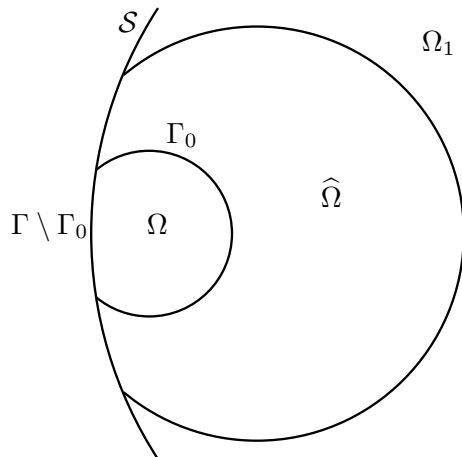


FIGURE 1. The open sets  $\Omega \subset \widehat{\Omega} \subset \Omega_1$  and the surface  $\mathcal{S}$ .

## 2. HUYGENS' PRINCIPLE FOR THE HALF-SPACE

The derivation of Huygens' principle in a half-space  $\Omega_1$  will follow the same pattern as for Huygens' principle in  $\mathbb{R}^3$ . It will be based on some extension of Kirchhoff formula, in which spherical means will be replaced by (algebraic) means over wavefronts.

**2.1. Solution by spherical means and Huygens' principle in  $\mathbb{R}^3$ .** For the reader's convenience, we briefly recall the main steps in the derivation of Kirchhoff formula for the solution of the wave equation in  $\mathbb{R}^3$  (see e.g. [11] for the details). Let  $(v^0, v^1) \in C^2(\mathbb{R}^3)^2$  be given, and let  $v = v(x, t)$  denote the classical solution to (1.6). For given  $x \in \mathbb{R}^3$ ,  $r > 0$  and  $t > 0$ , let

$$V(x, r, t) = \frac{1}{4\pi r^2} \int_{\partial B(x, r)} v(y, t) d\sigma(y), \quad (2.1)$$

$$V^0(x, r) = \frac{1}{4\pi r^2} \int_{\partial B(x, r)} v^0(y) d\sigma(y), \quad V^1(x, r) = \frac{1}{4\pi r^2} \int_{\partial B(x, r)} v^1(y) d\sigma(y). \quad (2.2)$$

It can be proved by a direct computation that

$$V_r = \frac{1}{4\pi r^2} \int_{B(x, r)} \Delta v(y, t) dy. \quad (2.3)$$

Using (1.6) this yields

$$r^2 V_r = \frac{1}{4\pi} \int_{B(x, r)} v_{tt}(y, t) dy,$$

and hence

$$(r^2 V_r)_r = r^2 V_{tt}. \quad (2.4)$$

Let  $\tilde{V} := rV$ ,  $\tilde{V}^0 := rV^0$ , and  $\tilde{V}^1 := rV^1$ . Then  $\tilde{V}$  solves the 1D wave equation

$$\tilde{V}_{tt} - \tilde{V}_{rr} = 0 \quad \text{in } (0, +\infty)_r \times (0, +\infty)_t, \quad (2.5)$$

$$\tilde{V} = 0 \quad \text{in } \{0\}_r \times (0, +\infty)_t, \quad (2.6)$$

$$(\tilde{V}(\cdot, 0), \tilde{V}_t(\cdot, 0)) = (\tilde{V}^0, \tilde{V}^1) \quad (2.7)$$

so that, by d'Alembert formula applied to odd solutions of the 1D wave equation

$$\tilde{V}(x, r, t) = \frac{1}{2}(\tilde{V}^0(x, t+r) - \tilde{V}^0(x, t-r)) + \frac{1}{2} \int_{t-r}^{t+r} \tilde{V}^1(x, z) dz \quad \text{for } 0 \leq r \leq t.$$

This yields with (2.1)-(2.2)

$$\begin{aligned} v(x, t) &= \lim_{r \rightarrow 0} \frac{\tilde{V}(x, r, t)}{r} \\ &= \tilde{V}_r^0(x, t) + \tilde{V}^1(x, t) \\ &= \frac{\partial}{\partial t} \left( \frac{1}{4\pi t} \int_{\partial B(x, t)} v^0(y) d\sigma(y) \right) + \frac{1}{4\pi t} \int_{\partial B(x, t)} v^1(y) d\sigma(y). \end{aligned} \quad (2.8)$$

Computing the derivative in (2.8) leads to Kirchhoff formula in  $\mathbb{R}^3$ :

$$v(x, t) = \frac{1}{4\pi t^2} \int_{\partial B(x, t)} [\nabla v^0(y) \cdot (y - x) + v^0(y) + tv^1(y)] d\sigma(y), \quad (x, t) \in \mathbb{R}^3 \times (0, +\infty). \quad (2.9)$$

Then (1.5) follows at once from (2.9).

Let us turn now to the solution of the wave equation in  $\mathbb{R}^2$ . In order to solve

$$\square w = 0 \quad \text{in } \mathbb{R}^2 \times \mathbb{R}, \quad (2.10)$$

$$(w(\cdot, 0), w_t(\cdot, 0)) = (w^0, w^1) \quad (2.11)$$

the method of descent consists in introducing the solution  $v$  of (1.6) for

$$v^0(x_1, x_2, x_3) = w^0(x_1, x_2), \quad v^1(x_1, x_2, x_3) = w^1(x_1, x_2).$$

Since  $v$  does not depend on  $x_3$ , the solution  $w$  of (2.10)-(2.11) can be written as  $w(x_1, x_2, t) = v(x_1, x_2, 0, t)$ . Combined with (2.9), this gives after some computations [11] Poisson's formula:

$$w(x, t) = \frac{1}{2\pi t^2} \int_{B(x, t)} \frac{t \nabla w^0(y) \cdot (y - x) + tw^0(y) + t^2 w^1(y)}{(t^2 - |y - x|^2)^{\frac{1}{2}}} dy, \quad (x, t) \in \mathbb{R}^2 \times (0, +\infty). \quad (2.12)$$

From (2.12), it is clear (taking e.g.  $w^0 \equiv 0$  and  $w^1 = 1_{B(0,1)}$ ) that Huygens' principle fails in  $\mathbb{R}^2$ .

**2.2. Solution of the wave equation on the half-space.** Assume that  $\Omega_1 = (0, +\infty) \times \mathbb{R}^2$ , so that  $\mathcal{S} = \{x_1 = 0\}$ . To solve (1.9)-(1.11), we use an odd reflection w.r.t.  $x_1$ , i.e. we set

$$\begin{aligned} \tilde{v}^0(x_1, x_2, x_3) &:= \text{sign}(x_1) v^0(|x_1|, x_2, x_3), \quad x \in \mathbb{R}^3, \\ \tilde{v}^1(x_1, x_2, x_3) &:= \text{sign}(x_1) v^1(|x_1|, x_2, x_3), \quad x \in \mathbb{R}^3, \\ \tilde{v}(x_1, x_2, x_3, t) &:= \text{sign}(x_1) v(|x_1|, x_2, x_3, t), \quad (x, t) \in \mathbb{R}^3 \times (0, +\infty). \end{aligned}$$

Since  $\tilde{v}$  solves (1.6) (with  $(v^0, v^1)$  replaced by  $(\tilde{v}^0, \tilde{v}^1)$ ), then  $\tilde{v}$  is given by (2.8) (or, equivalently (2.9)).

As Huygens' principle holds in  $\mathbb{R}^3$ , it holds as well in  $\Omega_1$ . Indeed, if  $v^0$  and  $v^1$  are some functions which are smooth and supported in some compact set  $K \subset \overline{\Omega_1}$ , then if we pick  $R > 0$  and  $\bar{x} \in \mathcal{S}$  such that  $K \subset B(\bar{x}, R)$ , then for  $T \geq 2R$ ,  $x \in K$  and  $y \in \partial B(x, T)$ , we have  $y \notin K$  (for  $|y - \bar{x}| \geq R$ ), hence with (2.9)

$$v(x, T) = \tilde{v}(x, T) = 0.$$

Note that, by an easy density argument, the same conclusion holds if  $(v^0, v^1) \in L^2(\Omega_1) \times H^{-1}(\Omega_1)$  is supported in the compact set  $K$ .

An extension of Kirchhoff formula can also be derived. For any  $x = (x_1, x_2, x_3) \in \Omega_1$  and any  $t > 0$ , we set  $s_1(x) = (-x_1, x_2, x_3)$  and

$$\Sigma_+(x, t) := \partial B(x, t) \cap \{x_1 > 0\}, \quad \Sigma_-(x, t) := \partial B(s_1(x), t) \cap \{x_1 > 0\},$$

$$\Sigma(x, t) := \begin{cases} \Sigma_+(x, t) & \text{if } 0 < t \leq x_1, \\ \Sigma_+(x, t) \cup \Sigma_-(x, t) & \text{if } t > x_1, \end{cases}$$

and for  $y \in \Sigma(x, t)$

$$\varepsilon(x, y, t) := \begin{cases} 1 & \text{if } y \in \Sigma_+(x, t), \\ -1 & \text{if } y \in \Sigma_-(x, t). \end{cases}$$

(See Figure 2.)

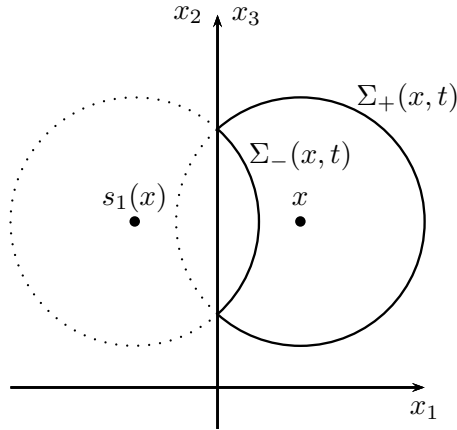


FIGURE 2. The wavefronts  $\Sigma_+(x, t)$  and  $\Sigma_-(x, t)$ .

Since  $\tilde{v}$  is odd w.r.t.  $x_1$ , we infer from (2.8) that

$$v(x, t) = \frac{\partial}{\partial t} \left( \frac{1}{4\pi t} \int_{\Sigma(x, t)} \varepsilon(x, y, t) v^0(y) d\sigma(y) \right) + \frac{1}{4\pi t} \int_{\Sigma(x, t)} \varepsilon(x, y, t) v^1(y) d\sigma(y). \quad (2.13)$$

**2.3. Control problem when  $\Gamma \setminus \Gamma_0$  is flat.** Assume that  $\Omega \subset \mathbb{R}^3$  with  $\Gamma \setminus \Gamma_0 \subset \{x_1 = 0\}$ . Let  $\mathcal{S} = \{x_1 = 0\}$  and  $\Omega_1 = \{x_1 > 0\}$ . Let  $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ . Pick  $\bar{x} = (0, \bar{x}_2, \bar{x}_3)$  and  $R > 0$  such that  $\Omega \subset B(\bar{x}, R)$ , and let  $T = 2R$  and  $L > 3R$ . Set

$$\widehat{\Omega} = (0, L) \times (\bar{x}_2 - L, \bar{x}_2 + L) \times (\bar{x}_3 - L, \bar{x}_3 + L). \quad (2.14)$$

Then the solution  $v$  of

$$\square v = 0 \quad \text{in } \widehat{\Omega} \times \mathbb{R}, \quad (2.15)$$

$$v = 0 \quad \text{in } \partial\widehat{\Omega} \times \mathbb{R} \quad (2.16)$$

$$(v(\cdot, 0), v_t(\cdot, 0)) = (1_\Omega u^0, 1_\Omega u^1) \quad (2.17)$$

satisfies

$$v = v_t = 0 \quad \text{on } \Omega \times \{t = T\}. \quad (2.18)$$

Indeed, since  $L > 3R$ , the solution of (2.15)-(2.17) coincide with those of (1.9)-(1.11) for  $x \in \Omega$  and  $0 \leq t \leq T$ , and hence (2.18) holds according to Huygens' principle in  $\{x_1 > 0\}$ .

Therefore the pair  $(u, g) = (v|_{\Omega \times (0, T)}, v|_{\Gamma_0 \times (0, T)})$  solves the control problem (1.1)-(1.4).

Assume now that  $\Omega \subset \mathbb{R}^2$  is a bounded domain, and that  $\Gamma_0$  is such that  $\partial\Omega \setminus \Gamma_0 \subset \{x_1 = 0\}$ . We shall use the classical *method of descent* (see e.g. [11]). Pick any  $\delta > 0$  and set  $\tilde{\Omega} := \Omega \times (-\delta, \delta)$ . For given  $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ , let

$$v^0(x_1, x_2, x_3) := \begin{cases} u^0(x_1, x_2) & \text{if } (x_1, x_2, x_3) \in \tilde{\Omega}, \\ 0 & \text{otherwise,} \end{cases} \quad (2.19)$$

$$v^1(x_1, x_2, x_3) := \begin{cases} u^1(x_1, x_2) & \text{if } (x_1, x_2, x_3) \in \tilde{\Omega}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.20)$$

Let  $\bar{x} = (0, \bar{x}_2, 0) \in \mathbb{R}^3$  and  $R > 0$  be such that  $\tilde{\Omega} \subset B(\bar{x}, R)$ . Let  $T = 2R$ ,  $L > 3R$ ,  $\widehat{\Omega}$  be as in (2.14), and let  $v$  solve (2.15)-(2.17). Then it is easily seen that

$$u(x_1, x_2, t) := \frac{1}{2\delta} \int_{-L}^L v(x_1, x_2, x_3, t) dx_3 \quad (2.21)$$

satisfies

$$u_{tt} - u_{x_1 x_1} - u_{x_2 x_2} = 0 \quad \text{in } \Omega \times (0, T), \quad (2.22)$$

$$u = 0 \quad \text{on } \{x_1 = 0\} \times (0, T), \quad (2.23)$$

$$(u(\cdot, 0), u_t(\cdot, 0)) = (u^0, u^1). \quad (2.24)$$

(For (2.22), we noticed that  $v(x, t) = 0$  for  $t \in [0, T]$  and  $x$  in a neighborhood of  $\Omega \times \{\pm L\}$ .) Since the property

$$v(x_1, x_2, x_3, T) = 0 \quad \text{for } (x_1, x_2) \in \Omega \text{ and } x_3 \in (-L, L)$$

is not necessarily true, we don't claim that  $u = u_t = 0$  for  $(x_1, x_2) \in \Omega$  and  $t = T$ . What is expected from (2.9) is that  $\|v(\cdot, T)\|_{L^2(\Omega)} = O(T^{-1})$  (resp.  $\|v(\cdot, T)\|_{L^2(\Omega)} = O(T^{-2})$ ) if  $v^1 \neq 0$  (resp.  $v^1 = 0$ ). Indeed, for  $(x_1, x_2, x_3) \in \Omega \times (-L, L)$ , the integration in (2.9) is performed over the set  $\partial B(x, t) \cap \tilde{\Omega}$  whose Lebesgue measure is bounded.

**2.4. Control problem when  $\Omega$  is a box.**



2.4.1.  $N = 3$ . We assume that

$$\begin{aligned}\Omega &= (0, L_1) \times (0, L_2) \times (0, L_3), \quad \Gamma = \partial\Omega, \\ \Gamma \setminus \Gamma_0 &= (\{x_1 = 0\} \cap \Gamma) \cup (\{x_2 = 0\} \cap \Gamma) \cup (\{x_3 = 0\} \cap \Gamma).\end{aligned}$$

For  $k = (k_1, k_2, k_3) \in \mathbb{R}^3$ , we denote  $|k| = \sqrt{k_1^2 + k_2^2 + k_3^2}$  and  $\|k\|_\infty = \max(|k_1|, |k_2|, |k_3|)$ . Pick any  $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$  decomposed in sine Fourier series as

$$\begin{aligned}u^0(x_1, x_2, x_3) &= \sum_{k \in \mathbb{N}^{*3}} u_k^0 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2) \sin(\pi k_3 x_3 / L_3) \\ u^1(x_1, x_2, x_3) &= \sum_{k \in \mathbb{N}^{*3}} u_k^1 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2) \sin(\pi k_3 x_3 / L_3)\end{aligned}$$

with  $\sum_{k \in \mathbb{N}^{*3}} (|u_k^0|^2 + |k|^{-2} |u_k^1|^2) < \infty$ . Let  $A_\Omega = -\Delta$  with domain  $D(A_\Omega) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega)$ . Then for any  $s \geq 0$

$$(u^0, u^1) \in D(A_\Omega^{\frac{s}{2}}) \times D(A_\Omega^{\frac{s-1}{2}}) \iff \sum_{k \in \mathbb{N}^{*3}} |k|^{2s} (|u_k^0|^2 + |k|^{-2} |u_k^1|^2) < \infty.$$

Let  $\varepsilon > 0$  and let  $\rho \in C^\infty(\mathbb{R})$  be an even function such that  $\rho(z) = 1$  for  $|z| \leq 1$  and  $\rho(z) = 0$  for  $|z| \geq 1 + \varepsilon$ . For  $x \in \mathbb{R}^3$ , let

$$\psi(x) := \rho(x_1/L_1) \rho(x_2/L_2) \rho(x_3/L_3) \quad (2.25)$$

and

$$v^0(x_1, x_2, x_3) = \psi(x) \sum_{k \in \mathbb{N}^{*3}} u_k^0 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2) \sin(\pi k_3 x_3 / L_3) \quad (2.26)$$

$$v^1(x_1, x_2, x_3) = \psi(x) \sum_{k \in \mathbb{N}^{*3}} u_k^1 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2) \sin(\pi k_3 x_3 / L_3) \quad (2.27)$$

for  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ . Pick

$$L > 2(1 + \varepsilon) \sqrt{L_1^2 + L_2^2 + L_3^2} + (1 + \varepsilon) \max(L_1, L_2, L_3)$$

and set  $\widehat{\Omega} = (-L, L)^3$ . Then for  $(u^0, u^1) \in D(A_\Omega^{\frac{s}{2}}) \times D(A_\Omega^{\frac{s-1}{2}})$ , we have  $(v^0, v^1) \in D(A_{\widehat{\Omega}}^{\frac{s}{2}}) \times D(A_{\widehat{\Omega}}^{\frac{s-1}{2}})$ , so that the solution  $v$  to

$$\begin{aligned}\square v &= 0 \quad \text{in } \widehat{\Omega} \times \mathbb{R}, \\ v &= 0 \quad \text{on } \partial\widehat{\Omega} \times \mathbb{R}, \\ (v(\cdot, 0), v_t(\cdot, 0)) &= (v^0, v^1)\end{aligned}$$

satisfies  $v \in C(\mathbb{R}, D(A_{\Omega}^{\frac{s}{2}})) \cap C^1(\mathbb{R}, D(A_{\Omega}^{\frac{s-1}{2}}))$ . On the other hand,  $v$  is odd in  $x_1$ ,  $x_2$ , and  $x_3$ . Decomposing  $v^0$  and  $v^1$  as

$$v^0(x_1, x_2, x_3) = \sum_{k \in \mathbb{N}^{*3}} v_k^0 \sin(\pi k_1 x_1 / L) \sin(\pi k_2 x_2 / L) \sin(\pi k_3 x_3 / L), \quad (2.28)$$

$$v^1(x_1, x_2, x_3) = \sum_{k \in \mathbb{N}^{*3}} v_k^1 \sin(\pi k_1 x_1 / L) \sin(\pi k_2 x_2 / L) \sin(\pi k_3 x_3 / L), \quad (2.29)$$

we have that

$$v(x, t) = \frac{1}{2} \sum_{k \in \mathbb{N}^{*3}} \left[ (v_k^0 - i \frac{L v_k^1}{\pi |k|}) e^{i \frac{\pi}{L} |k| t} + (v_k^0 + i \frac{L v_k^1}{\pi |k|}) e^{-i \frac{\pi}{L} |k| t} \right] \sin\left(\frac{\pi k_1 x_1}{L}\right) \sin\left(\frac{\pi k_2 x_2}{L}\right) \sin\left(\frac{\pi k_3 x_3}{L}\right). \quad (2.30)$$

Then  $(u, g) = (v|_{\Omega \times (0, T)}, v|_{\Gamma_0 \times (0, T)})$  solves (1.1)-(1.4) for  $T = 2(1 + \varepsilon)\sqrt{L_1^2 + L_2^2 + L_3^2}$ . Indeed, if  $v$  is viewed as a solution of the wave equation on  $\mathbb{R}^3$  with odd initial data supported in  $\widehat{\Omega}$ , then the above choice of  $T$  gives that  $v(x, T) = v_t(x, T) = 0$  for all  $x \in \Omega$  by (2.13), (2.25), (2.26) and (2.27), while the above choice of  $L$  ensures that the support of  $(v, v_t)$  does not meet the boundary of  $\widehat{\Omega}$  at time  $T$ , so that this solution coincides with the solution of the wave equation with homogeneous Dirichlet boundary conditions on  $\widehat{\Omega}$  given by (2.30).

To address the control problem from a numerical viewpoint, we let  $v_N$  (resp.  $g_N$ ) denote the partial sum of the series in (2.30) (resp. its restriction to  $\Gamma_0 \times [0, T]$ ) restricted to  $k_i \leq N$  for  $i = 1, 2, 3$ .

The following theorems give explicit error estimates in terms of the Fourier coefficients of the initial data.

**Theorem 2.1.** *Let  $s \geq 0$ ,  $(u^0, u^1) \in D(A_{\Omega}^{\frac{s}{2}}) \times D(A_{\Omega}^{\frac{s-1}{2}})$  and  $N \geq 1$ . Then for any  $r \leq s$*

$$\|v - v_N\|_{C([0, T], H^r(\widehat{\Omega}))} \leq \frac{C}{N^{s-r}} \left( \sum_{\|k\|_{\infty} > N} |k|^{2s} (|v_k^0|^2 + |k|^{-2} |v_k^1|^2) \right)^{\frac{1}{2}} \quad (2.31)$$

$$\leq \frac{C}{N^{s-r}} (\|u^0\|_{H^s(\Omega)} + \|u^1\|_{H^{s-1}(\Omega)}) \quad (2.32)$$

where  $C > 0$  is some universal constant.

*Proof.* It follows from (2.30) that

$$\begin{aligned} \|v - v_N\|_{C([0, T], D(A_{\Omega}^{\frac{r}{2}}))}^2 &\leq C \sum_{\|k\|_{\infty} > N} |k|^{2r} (|v_k^0|^2 + |k|^{-2} |v_k^1|^2) \\ &\leq \frac{C}{N^{2(s-r)}} \sum_{\|k\|_{\infty} > N} |k|^{2s} (|v_k^0|^2 + |k|^{-2} |v_k^1|^2) \\ &\leq \frac{C}{N^{2(s-r)}} (\|v^0\|_{D(A_{\Omega}^{\frac{s}{2}})}^2 + \|v^1\|_{D(A_{\Omega}^{\frac{s-1}{2}})}^2) \\ &\leq \frac{C}{N^{2(s-r)}} (\|u^0\|_{D(A_{\Omega}^{\frac{s}{2}})}^2 + \|u^1\|_{D(A_{\Omega}^{\frac{s-1}{2}})}^2). \end{aligned}$$

Since the norms  $\|\cdot\|_{D(A_\Omega^{\frac{s}{2}})}$  and  $\|\cdot\|_{H^s(\Omega)}$  are equivalent in  $D(A_\Omega^{\frac{s}{2}})$ , (2.32) follows.  $\square$

**Remark 2.2.** (1) For  $r = s$ , we infer from (2.31) that  $v_N \rightarrow v$  in  $C([0, T], H^s(\widehat{\Omega}))$ .

(2) By Sobolev embedding, (2.32) yields for  $1 \leq r \leq s$

$$\|g - g_N\|_{C([0, T], H^{r-\frac{1}{2}}(\Gamma_0))} \leq \frac{C}{N^{s-r}} (\|u^0\|_{H^s(\Omega)} + \|u^1\|_{H^{s-1}(\Omega)}).$$

(3) If  $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ , taking merely  $\rho = 1_{(-1, 1)}$  and defining  $v$ ,  $u$  and  $g$  as above, we can solve (1.1)-(1.4) in the sharp time  $T = 2\sqrt{L_1^2 + L_2^2 + L_3^2}$ . Our approach allows us to recover the main results in [25] without using any Ingham inequality. Furthermore, the control input can be expressed explicitly in terms of the Fourier coefficients of the initial data. Note that the same approach gives the same sharp result in  $\Omega = \prod_{i=1}^N (0, L_i)$  for any odd integer  $N \geq 3$ , for Huygens' principle is valid in  $\mathbb{R}^N$  (see e.g. [11, Eq. (31) p. 77]).

2.4.2.  $N = 2$ . We assume here that

$$\begin{aligned} \Omega &= (0, L_1) \times (0, L_2), \\ \Gamma \setminus \Gamma_0 &= (\{x_1 = 0\} \cap \Gamma) \cup (\{x_2 = 0\} \cap \Gamma). \end{aligned}$$

Let  $A_\Omega = -\Delta$  with domain  $D(A_\Omega) = H^2(\Omega) \cap H_0^1(\Omega) \subset L^2(\Omega)$ . Pick any  $(u^0, u^1) \in D(A_\Omega^{\frac{s}{2}}) \times D(A_\Omega^{\frac{s-1}{2}})$  ( $s \geq 0$ ) decomposed in sine Fourier series as

$$\begin{aligned} u^0(x_1, x_2) &= \sum_{k \in \mathbb{N}^{*2}} u_k^0 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2), \\ u^1(x_1, x_2) &= \sum_{k \in \mathbb{N}^{*2}} u_k^1 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2) \end{aligned}$$

with  $\sum_{k \in \mathbb{N}^{*2}} |k|^{2s} (|u_k^0|^2 + |k|^{-2} |u_k^1|^2) < \infty$ . Pick  $\varepsilon > 0$  and let  $\rho \in C^\infty(\mathbb{R})$  be an even function such that  $\rho(z) = 1$  for  $|z| \leq 1$  and  $\rho(z) = 0$  for  $|z| \geq 1 + \varepsilon$ . Pick  $\delta > 0$  and denote  $I_\delta = \int_{\mathbb{R}} \rho(z/\delta) dz$ . For  $x \in \mathbb{R}^2$ , let

$$\psi(x) := \rho(x_1/L_1) \rho(x_2/L_2) \rho(x_3/\delta)$$

and

$$\begin{aligned} v^0(x_1, x_2, x_3) &= \psi(x) \sum_{k \in \mathbb{N}^{*2}} u_k^0 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2) \\ v^1(x_1, x_2, x_3) &= \psi(x) \sum_{k \in \mathbb{N}^{*2}} u_k^1 \sin(\pi k_1 x_1 / L_1) \sin(\pi k_2 x_2 / L_2) \end{aligned}$$

for  $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ . Note that  $v^0$  and  $v^1$  are odd in  $x_1, x_2$ , and even in  $x_3$ . Pick

$$L > 2(1 + \varepsilon) \sqrt{L_1^2 + L_2^2 + \delta^2} + (1 + \varepsilon) \max(L_1, L_2, \delta).$$

It is well-known that the family  $(\sqrt{\frac{2}{\pi}} \cos((k - \frac{1}{2})x))_{k \in \mathbb{N}^*}$  is an orthonormal basis of  $L^2(0, \pi)$ . Decomposing  $v^0$  and  $v^1$  as

$$v^0(x_1, x_2, x_3) = \sum_{k \in \mathbb{N}^{*3}} v_k^0 \sin(\pi k_1 x_1 / L) \sin(\pi k_2 x_2 / L) \cos(\pi(k_3 - \frac{1}{2})x_3 / L), \quad (2.33)$$

$$v^1(x_1, x_2, x_3) = \sum_{k \in \mathbb{N}^{*3}} v_k^1 \sin(\pi k_1 x_1 / L) \sin(\pi k_2 x_2 / L) \cos(\pi(k_3 - \frac{1}{2})x_3 / L) \quad (2.34)$$

and denoting  $\mu_k = \frac{\pi}{L} \sqrt{k_1^2 + k_2^2 + (k_3 - \frac{1}{2})^2}$ , we have that the function

$$\begin{aligned} & w(x_1, x_2, t) \\ &= \frac{1}{2I_\delta} \int_{-L}^L \sum_{k \in \mathbb{N}^{*3}} \left[ (v_k^0 - i \frac{v_k^1}{\mu_k}) e^{i\mu_k t} + (v_k^0 + i \frac{v_k^1}{\mu_k}) e^{-i\mu_k t} \right] \sin(\frac{\pi k_1 x_1}{L}) \sin(\frac{\pi k_2 x_2}{L}) \cos(\pi(k_3 - \frac{1}{2})x_3 / L) dx_3 \\ &= \sum_{k \in \mathbb{N}^{*3}} \left[ (v_k^0 - i \frac{v_k^1}{\mu_k}) e^{i\mu_k t} + (v_k^0 + i \frac{v_k^1}{\mu_k}) e^{-i\mu_k t} \right] \sin(\frac{\pi k_1 x_1}{L}) \sin(\frac{\pi k_2 x_2}{L}) \frac{(-1)^{k_3+1} L}{\pi(k_3 - \frac{1}{2})I_\delta} \end{aligned}$$

solves

$$\begin{aligned} \square w &= 0 && \text{in } (-L, L)^2 \times \mathbb{R}, \\ w &= 0 && \text{on } \partial(-L, L)^2 \times \mathbb{R}, \\ (w(\cdot, 0), w_t(\cdot, 0)) &= (u^0, u^1). \end{aligned}$$

Then  $(u, g) = (w|_{\Omega \times (0, T)}, w|_{\Gamma_0 \times (0, T)})$  satisfies (1.1)-(1.3) for  $T = 2(1 + \varepsilon) \sqrt{L_1^2 + L_2^2 + \delta^2}$ . Indeed, if  $v$  denotes the solution of the system

$$\begin{aligned} \square v &= 0 && \text{in } (-L, L)^3 \times \mathbb{R}, \\ v &= 0 && \text{on } \partial(-L, L)^3 \times \mathbb{R}, \\ (v(\cdot, 0), v_t(\cdot, 0)) &= (v^0, v^1), \end{aligned}$$

then the support of  $(v, v_t)$  does not meet  $\partial(-L, L)^3$  for  $0 \leq t \leq T$ , and hence

$$w_{tt} - w_{x_1 x_1} - w_{x_2 x_2} = \frac{1}{I_\delta} \int_{-L}^L (v_{tt} - v_{x_1 x_1} - v_{x_2 x_2}) dx_3 = \frac{1}{I_\delta} \int_{-L}^L v_{x_3 x_3} dx_3 = 0.$$

We stress that  $u$  is expected to be ‘‘small’’, but not necessarily zero, for  $t = T$ .

Furthermore, we have that for some universal constant  $C > 0$  and for  $r \leq s$

$$\begin{aligned} \|w - w_N\|_{C([0, T], H^r(\Omega))} &\leq \frac{C}{N^{s-r}} \left( \sum_{\|k\|_\infty > N} |k|^{2s} (|v_k^0|^2 + |k|^{-2} |v_k^1|^2) \right)^{\frac{1}{2}} \\ &\leq \frac{C}{N^{s-r}} (\|u^0\|_{H^s(\Omega)} + \|u^1\|_{H^{s-1}(\Omega)}) \end{aligned}$$

where

$$w_N(x, t) = \sum_{\|k\|_\infty \leq N} \left[ (v_k^0 - i \frac{v_k^1}{\mu_k}) e^{i\mu_k t} + (v_k^0 + i \frac{v_k^1}{\mu_k}) e^{-i\mu_k t} \right] \sin(\frac{\pi k_1 x_1}{L}) \sin(\frac{\pi k_2 x_2}{L}) \frac{(-1)^{k_3+1} L}{\pi(k_3 - \frac{1}{2})I_\delta}.$$

## 3. NUMERICAL EXPERIMENTS

3.1. **Case  $\Omega \subset \mathbb{R}^3$ .** For the sake of convenience, to avoid the (easy but tedious) scalings in space variables and in Fourier coefficients, we take

$$\Omega = (0, \frac{1}{2})^3 \subset \widehat{\Omega} = (-\pi, \pi)^3.$$

Thus  $L_1 = L_2 = L_3 = \frac{1}{2}$  and  $L = \pi$ . Let

$$u^0(x_1, x_2, x_3) = \exp\left(-200\left[(x_1 - \frac{1}{4})^2 + (x_2 - \frac{1}{4})^2 + (x_3 - \frac{1}{4})^2\right]\right)$$

and  $u^1(x_1, x_2, x_3) = 0$ . We pick  $\rho = 1_{(-1,1)}$  and  $N = 50$ . The  $(50)^3$  Fourier coefficients  $v_k^0$  in (2.28) are computed by using the trapezoid rule with  $(20)^3$  points. The (free) evolution of the trajectory for  $(x_1, x_2) \in (0, \pi)^2$  and  $x_3 = 1/8$ ,  $x_3 = 1/4$ , and  $x_3 = 3/8$  is given in Figures 3, 4 and 5, respectively.

Note that the sharp control time is  $T = \sqrt{3} \approx 1.73$ . The evolution of  $U(t) := \|u(\cdot, t)\|_{L^\infty(\Omega)}$  is displayed in Figure 6. We used a uniform mesh with  $9^3$  points to estimate  $U(t)$  at each time  $t$ . We found that  $U(1.73) \approx 5.36 \cdot 10^{-5}$ , which shows numerically that the solution vanishes at  $t = T$  for  $(x_1, x_2, x_3) \in (0, \frac{1}{2})^3$ , as desired.

3.2. **Case  $\Omega \subset \mathbb{R}^2$ .** We take

$$\Omega = (0, \frac{1}{2})^2 \quad \text{and} \quad \widehat{\Omega} = (-\pi, \pi)^3,$$

so that  $L_1 = L_2 = \frac{1}{2}$  and  $L = \pi$ . Let  $u^0(x_1, x_2) = \exp(-200[(x_1 - \frac{1}{4})^2 + (x_2 - \frac{1}{4})^2])$  and  $u^1(x_1, x_2) = 0$ . We pick  $\rho = 1_{(-1,1)}$ ,  $\delta = 0.15$  and  $N = 50$ . The  $(50)^3$  Fourier coefficients  $v_k^0$  in (2.33) are computed by using the trapezoid rule with  $(20)^3$  points. Letting  $x_3 = 0$  and  $(x_1, x_2) \in \Omega$  yields the desired controlled trajectory. The evolution of the trajectory in  $(0, \pi)^2$  is given in Figure 7. As expected, we observe the propagation of waves from  $\Omega$  into  $(0, L)^2$ . At time  $t = 1.5$ , waves are localized in  $(0, L)^2 \setminus \Omega$ . The evolution of  $U(t) := \|u(\cdot, t)\|_{L^\infty(\Omega)}$  is displayed in Figure 8. We recall that the sharp control time is  $T = \sqrt{2} \approx 1.44$ . We used a uniform mesh with  $9^2$  points to numerically compute  $U(t)$  at each time  $t$ . We obtained  $U(1.44) \approx 1.5 \cdot 10^{-3}$ , a result which is less good than in the case  $\Omega = (0, \frac{1}{2})^3$  with the same number of Fourier coefficients. This is consistent with the fact that only a stabilization result is obtained in 2D by applying the method of descent.

## 4. CONCLUDING REMARKS

Using Huygens' principle in dimension 3, the control of the wave equation on a cuboid can be reduced to a classical Cauchy problem in a larger domain. A simple numerical scheme, based on Fourier series, can then be used to solve numerically the control problem without incorporating dissipation or filtering.

Our control method amounts to associate with a pair  $(\Omega, \Gamma_0)$  an open set  $\Omega_1 \subset \mathbb{R}^3$  with  $\Omega \subset \Omega_1$ ,  $\Gamma \setminus \Gamma_0 \subset \partial\Omega_1$ , so that Huygens' principle is satisfied in  $\Omega_1$ , and next to extend the initial data by 0 on  $\Omega_1 \setminus \Omega$  and to take the restriction to  $\Omega$  of the free solution of the wave equation on  $\Omega_1$  with Dirichlet boundary conditions.

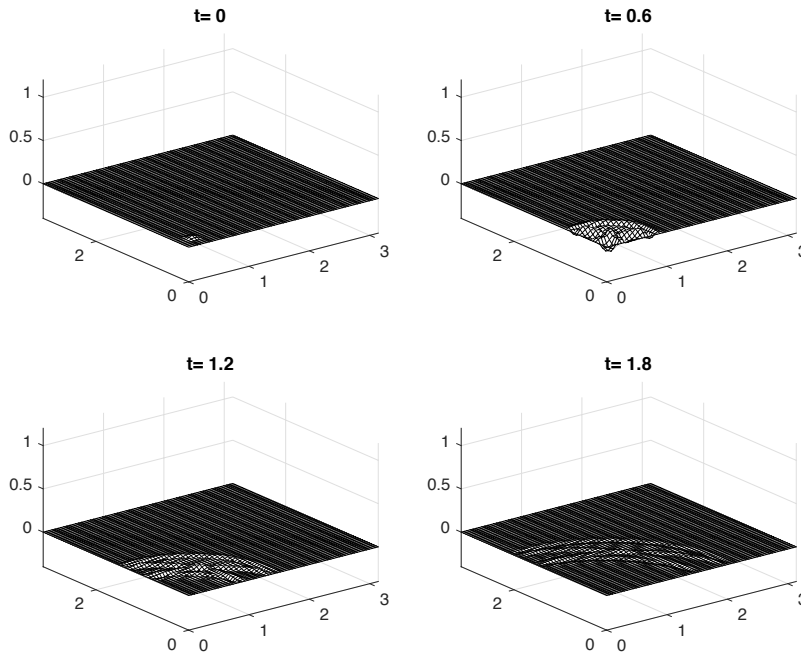


FIGURE 3. Control of the wave equation on  $\Omega = (0, \frac{1}{2})^3$ .  $x_3 = 1/8$ .

So far, the above method is limited to geometries for which the complement of the control region on the boundary is both flat and connected. It would be interesting to remove those assumptions. This requires first to extend Huygens' principle on unbounded domains limited by (convenient) surfaces. One way would be to extend Kirchhoff formula (2.8) as in (2.13) by replacing spheres by wavefronts.

Combining Huygens' principle in dimension 3 with the descent method, we obtain numerically a stabilization result, rather than an exact controllability result. It would be interesting to modify the above approach in order to obtain an exact controllability result in sharp time.

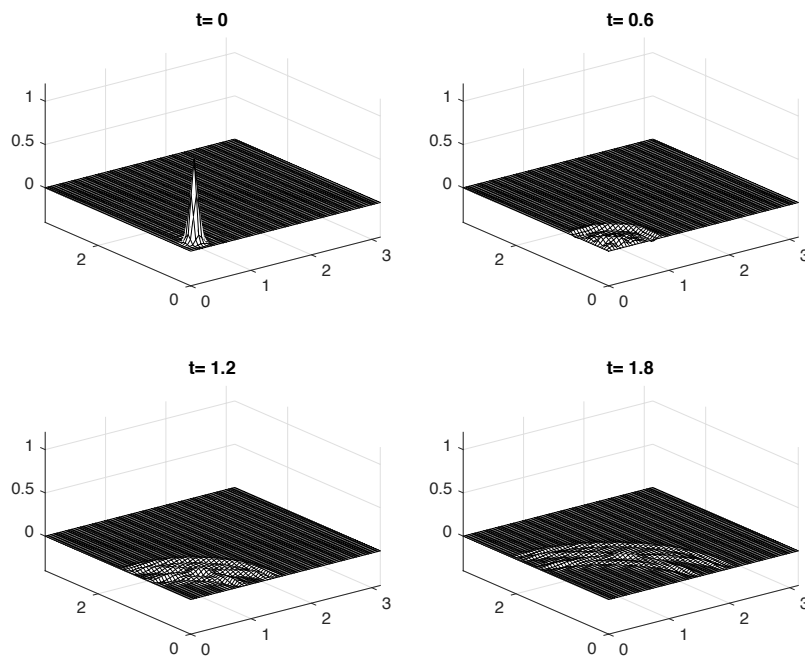


FIGURE 4. Control of the wave equation on  $\Omega = (0, \frac{1}{2})^3$ .  $x_3 = 1/4$ .

We expect that our approach can be applied in some situations when the complement of the control region is connected, but not necessarily flat. This will be investigated elsewhere.

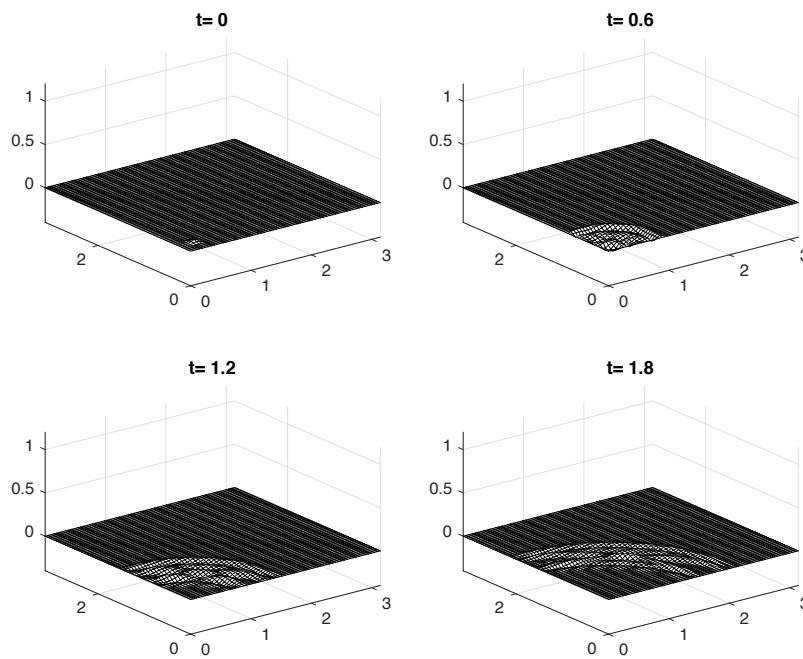


FIGURE 5. Control of the wave equation on  $\Omega = (0, \frac{1}{2})^3$ .  $x_3 = 3/8$ .

#### REFERENCES

- [1] F. Alabau-Boussouira, V. Perrollaz, L. Rosier, *Finite-time stabilization of a network of strings*, Math. Control Relat. Fields **5**, no. 4, 721–742 (2015)
- [2] M. Asch, G. Lebeau, *Geometrical aspects of exact boundary controllability for the wave equation- a numerical study*, ESAIM: COCV, May 1998, Vol. 3, 163–212.
- [3] M. Asch, A. Münch, *Uniformly controllable schemes for the wave equation on the unit square*, J. Optim Theor Appl (2009) 143: 417–438.
- [4] C. Bardos, G. Lebeau, J. Rauch, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control Optim. **30** (1992), no. 5, 1024–1065.



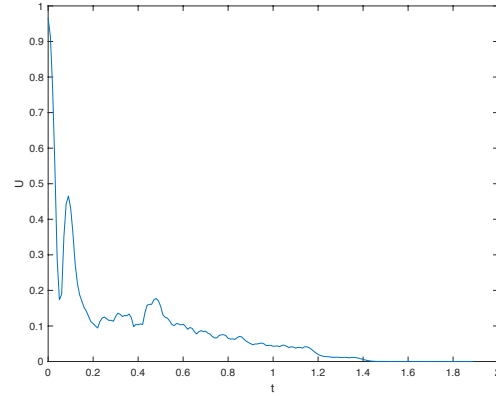


FIGURE 6. Evolution of  $U(t) = \|u(\cdot, t)\|_{L^\infty((0, \frac{1}{2})^3)}$ .

- [5] N. Burq, *Contrôlabilité exacte des ondes dans des ouverts peu réguliers*, *Asymptotic Analysis* **14** (1997), 157–191.
- [6] M.M. Cavalcanti, V.N. Cavalcanti, C. Rosier, L. Rosier, *Numerical control of a semilinear wave equation*. 2021. hal-03369560
- [7] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, *Spectral methods in Fluid Dynamics*, Springer Series in Computational Physics, Springer-Verlag Berlin Heidelberg 1988.
- [8] C. Castro, S. Micu, A. Münch, *Numerical approximation of the boundary control of the 2-D wave equation with mixed finite elements*, *IMA J. Numer. Anal.* **28** (2008), No. 1, 186–214.
- [9] S. Ervedoza, E. Zuazua, *Observability of heat processes by transmutation without geometric restrictions*, *Math. Control Relat. Fields* **1** (2011), no. 2, 177–187.
- [10] S. Ervedoza, E. Zuazua, *Numerical approximation of exact controls for waves*. Springer Briefs in Mathematics. Springer, New York, 2013.
- [11] L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence 1998.
- [12] R. Glowinski, C. Li, J.L. Lions, *A numerical approach to the exact boundary controllability of the wave equation*, *Japan J. Appl. Math.* **3** (1990) 1–76.
- [13] L. Ignat, E. Zuazua, *Convergence of a multi-grid method for the control of waves*, *J. Eur. Math. Soc.* **11** (2009), 351–391.
- [14] B. F. Jones Jr., *A fundamental solution for the heat equation which is supported in a strip*, *J. Math. Anal. Appl.* **60** (1977), 314–324.
- [15] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*. RAM: Research in Applied Mathematics, Masson, Paris, 1994.

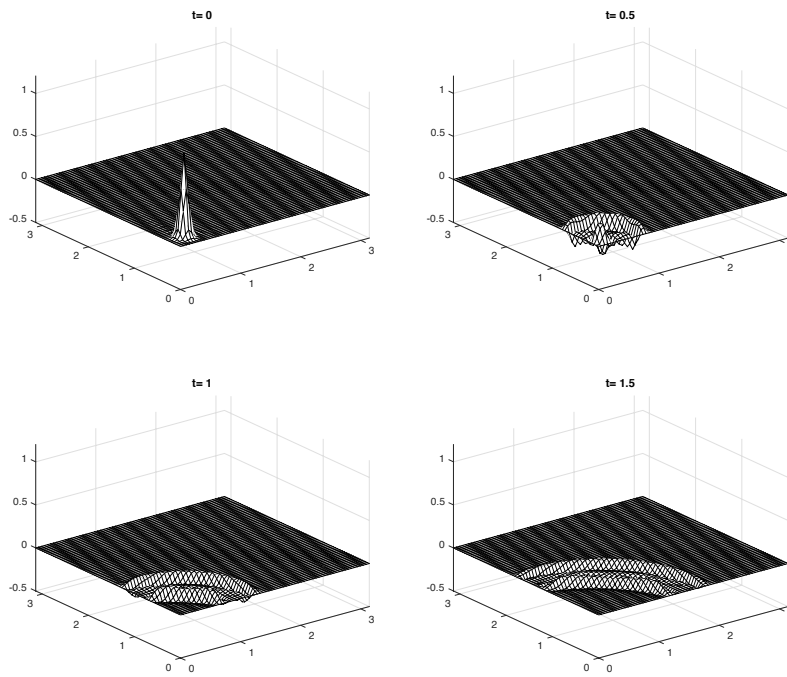


FIGURE 7. Control of the wave equation on  $\Omega = (0, \frac{1}{2})^2$ .

- [16] B. Laroche, P. Martin, P. Rouchon, *Motion planning for the heat equation*, Internat. J. Robust Nonlinear Control **10** (2000), no. 8, 629–643, Nonlinear adaptive and linear systems (Mexico City, 1998).
- [17] G. Lebeau, *personal communication*.
- [18] J.-L. Lions, *Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués*, Tome 1. Masson, Paris (1988)

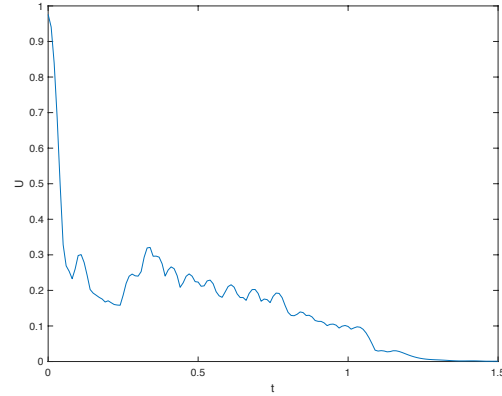


FIGURE 8. Evolution of  $U(t) = \|u(\cdot, t)\|_{L^\infty((0, \frac{1}{2})^2)}$ .

- [19] W. Littman, *Boundary control theory for hyperbolic and parabolic partial differential equations with constant coefficients*, Annali Scuola Normale Superiore, Pisa, 1978, 567–580.
- [20] W. Littman, L. Markus, *Exact boundary controllability of a hybrid system of elasticity*, Arch. Rational Mech. Anal. **103** (1988), no. 3, 193–236.
- [21] W. Littman, S. Taylor, *The heat and Schrödinger equations: boundary control with one shot. Control methods in PDE-dynamical systems*, 293–305, Contemp. Math., 426, Amer. Math. Soc., Providence, RI, 2007.
- [22] P. Loreti, M. Mehrenberger, *An Ingham type proof for a two-grid observability theorem*, ESAIM:COCV **13** (2008), 604–631.
- [23] P. Martin, L. Rosier, P. Rouchon, *Null controllability of the heat equation using flatness*, Automatica **50** (2014), 3067–3076.
- [24] P. Martin, L. Rosier, P. Rouchon, *Controllability of the 1D Schrödinger equation using flatness*, Automatica **91** (2018), 208–216.
- [25] M. Mehrenberger, *An Ingham type proof for the boundary observability of a N-d wave equation*, C. R. Acad. Sci. Paris, Ser. I 347 (2009), 63–68.
- [26] S. Micu, *Uniform boundary controllability of a semidiscrete 1-D wave equation with vanishing viscosity*, SIAM J. Control Optim. 47 (2008), no. 6, 2857–2885.
- [27] S. Micu, L. de Teresa, *A spectral study of the boundary controllability of the linear 2-D wave equation in a rectangle*, Asymptot. Anal. 66 (2010), no. 3-4, 139160.
- [28] A. Münch, E. Zuazua, *Numerical approximation of null controls for the heat equation: ill-posedness and remedies*, Inverse Problems **26** (2010), no. 8, 085018.
- [29] P. Pedregal, F. Periago, J. Villena, *A numerical method of local energy decay for the boundary controllability of time-reversible distributed parameter systems*, Stud. Appl. Math. 121 (2008), no. 1, 27–47.

- [30] V. Perrollaz, L. Rosier, *Finite-time stabilization of  $2 \times 2$  hyperbolic systems on tree-shaped networks*, SIAM J. Control Optim. **52**, no. 1, 143–163 (2014)
- [31] K.-D. Phung, *Observability and control of Schrödinger equations*, SIAM J. Control Optim. **40** (2001), no. 1, 211–230.
- [32] J. V. Ralston, *Solutions of the wave equation with localized energy*, Communications on Pures and Applied Mathematics, **22** (1969) 807–823.
- [33] K. Ramdani, T. Takahashi, M. Tucsnak, *Internal stabilization of the plate equation in a square: the continuous and the semi-discretized problems*, J. Math. Pures Appl (9) **85** (2006), no. 1, 17–37.
- [34] L. Rosier, *A fundamental solution supported in a strip for a dispersive equation*, Computational and Applied Mathematics **21** (2002), 355–367.
- [35] D. L. Russell, *Controllability and stabilization theory for linear partial differential equations: recent progress and open questions*, SIAM Rev. **20** (1978), No. 4, 639–737.
- [36] L. N. Trefethen, Spectral Methods in Matlab, SIAM Philadelphia, 2000.
- [37] E. Zuazua, *Boundary observability for the finite difference space semi-discretizations of the 2-D wave equation in the square*, J. Math. Pures Appl. **78** (1999), 523–563.
- [38] E. Zuazua, *Propagation, observation, control and numerical approximation of waves*, SIAM Rev. **47** (2005), No. 2, 197–243.
- [39] E. Zuazua, *Control and Numerical Approximation of the Wave and Heat Equations*, vol. III, pp. 1389–1417. International Congress of Mathematicians, Madrid, Spain (2006).

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