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Inadequacy of the CHSH Bell inequality for an application to its associated photon correlation experiments

Gerrit Coddens (a)

Laboratoire des Solides Irradiés,
Institut Polytechnique de Paris, UMR 7642, CNRS-CEA-Ecole Polytechnique,
28, Route de Saclay, F-91128-Palaiseau CEDEX, France

(a) retired research physicist of LSI

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Abstract. We explain that the CHSH Bell inequality cannot be applied to the photon correlation experiments that have been associated with it. In fact, when we apply the inequality to these experiments its derivation would imply that a single photon can have two different linear polarization states at the same time, while they cannot be measured at the same time. The issue is a consequence of the special experimental context we try to impose the mathematics on. We are able to pinpoint the exact place in the derivation of the inequality where the application of the mathematics to the physics causes a problem. We then derive a Bell inequality that is appropriate for an application to the experiments. The boundaries of this corrected inequality are much less stringent and the experiments do not violate this inequality.

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1 Introduction: commutation relations

In quantum mechanics (QM) we learn that the operators

$$\hat{L}_z = \frac{\hbar}{i} \left[x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right] (\text{cycl.}), \quad (1)$$

are the operators for the components of the angular momentum $\mathbf{L} = (L_x, L_y, L_z)$. They obey the commutation relations:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z (\text{cycl.}). \quad (2)$$

From this result, Bohr drew the conclusion that two components L_x and L_y of the angular momentum \mathbf{L} could not *exist* simultaneously. This is in the least highly counterintuitive. Heisenberg was less radical and in his discussions with Bohr, he defended the viewpoint that the commutation relation would rather imply that L_x and L_y cannot be *measured* simultaneously. Einstein did not accept these conclusions and in order to refute them he considered two identical particles that would travel in opposite directions. On one of them one would measure L_x and on the other L_y . Some clarification is needed here. First of all, there is also a commutation relation for \hat{p}_x and \hat{x} :

$$[\hat{x}, \hat{p}_x] = i\hbar. \quad (3)$$

But in quantum field theory Dirac replaced this by:

$$[\phi(t, \mathbf{r}_1), \chi(t, \mathbf{r}_2)] = i\hbar \delta(\mathbf{r}_1 - \mathbf{r}_2). \quad (4)$$

Here ϕ and χ are operators for conjugated variables. This implies that the position operator in one place \mathbf{r}_1 and the momentum operator in a different place \mathbf{r}_2 commute, but that position and momentum operators in the same place $\mathbf{r}_1 = \mathbf{r}_2$ do not commute. This remark actually undermines Einstein's strategy to rebut Bohr's conclusion based on the EPR argument, because it makes him miss the target, as we can actually expect similarly:

$$[\hat{L}_x(\mathbf{r}_1), \hat{L}_y(\mathbf{r}_2)] = i\hbar \hat{L}_z \delta(\mathbf{r}_1 - \mathbf{r}_2) (\text{cycl.}), \quad (5)$$

such that it is indeed possible to measure the two quantities simultaneously on two correlated particles. But there is another confusion here, because the real meaning of \hat{L}_x is that it is the operator for the angular momentum \mathbf{L} when it is aligned with the x -axis [1]. The angular-momentum commutation relations are then no longer mysterious. In fact, such commutation relations occur trivially in the Lie algebra of every non-abelian Lie group and are entirely classical. They only express that e.g. the angular momentum cannot be simultaneously aligned with the y -axis, when it is already aligned with the x -axis [1]. We have actually very similar commutation operations for electron spin operators.

Let us consider only polarizers whose planes are parallel to a given plane. We can then characterize their orientation by an angle in this plane. We note these orientations by the angles α_j for filter A and β_k for filter B . We do not know what kind of commutation relations one should define for polarization operators. But conceptually we must take into account that we cannot orient simultaneously a filter A along α_2 in \mathbf{r}_1 when it is already aligned along α_1 in \mathbf{r}_1 . However, it is possible to align a filter B in a position \mathbf{r}_2 along β_k when A is already aligned along α_j in \mathbf{r}_1 . In other words we cannot measure simultaneously the polarizations along α_1 and α_2 in \mathbf{r}_1 or along β_1 and β_2 in \mathbf{r}_2 but we can measure simultaneously any combination of α_j in \mathbf{r}_1 and β_k in \mathbf{r}_2 . That is not only true in QM, it is also true in classical physics.

2 Bell inequalities

A part of the present section is drawn from [2]. The subject matter of the Bell inequalities and the experiments of Aspect *et al.* [3,4] can be supposed to be very well-known. For an introduction we refer to [5]. However, the argument has often been blurred by drawing in unnecessary issues, leading to some confusion. We give here an elementary derivation that removes all unnecessary considerations. This will show how elementary the argument is and how very hard it is to question the validity of the inequalities.

We consider 4 variables $a_1 \in S$, $a_2 \in S$, $b_1 \in S$, $b_2 \in S$, where $S = \{0, 1\}$. The idea is that 0 corresponds to absorption in a polarizer, and 1 to transmission. Then a_j will correspond to polarizer settings in one arm of the set-up, and b_k to polarizer settings in the other arm. There are thus 16 possible combinations for the values of (a_1, a_2, b_1, b_2) . By making a table of these 16 combinations it is easy to verify that we always have:

$$\forall (a_1, a_2, b_1, b_2) \in S^4 : \quad Q = a_1 b_1 - a_1 b_2 - a_2 b_1 - a_2 b_2 + a_2 + b_2 \in S. \quad (6)$$

We consider now functions $a_j \in F(V, S)$ and $b_k \in F(V, S)$. The notation $F(V, S)$ stands for the set of functions whose domain is the set V and whose values belong to the set S . Here V is a set of relevant variables for the experiment. We can call the set V the set of hidden variables, even if some of them may not really be hidden. One can imagine that V could be a subset of a vector space \mathbb{R}^n or of a manifold, e.g. a non-abelian Lie group like $\text{SO}(3)$. We have then:

$$\forall \lambda \in V : \quad 0 \leq Q(\lambda) = a_1(\lambda)b_1(\lambda) - a_1(\lambda)b_2(\lambda) - a_2(\lambda)b_1(\lambda) - a_2(\lambda)b_2(\lambda) + a_2(\lambda) + b_2(\lambda) \leq 1. \quad (7)$$

We can now consider a probability density p over V , i.e. $p(\lambda) d\lambda$. The function p belongs then to the set of functions $F(V, [0, \infty])$ with domain V and values in $[0, \infty[$. We further require that $\int_V p(\lambda) d\lambda = 1$. We can now integrate Eq. 7 with p over V . Introducing the notations:

$$p(a_j \wedge b_k) = \int_V a_j(\lambda) b_k(\lambda) p(\lambda) d\lambda, \quad p(a_j) = \int_V a_j(\lambda) p(\lambda) d\lambda, \quad p(b_k) = \int_V b_k(\lambda) p(\lambda) d\lambda, \quad (8)$$

we obtain then:

$$0 \leq p(a_1 \wedge b_1) - p(a_1 \wedge b_2) - p(a_2 \wedge b_1) - p(a_2 \wedge b_2) + p(a_2) + p(b_2) \leq 1. \quad (9)$$

Here \wedge is the logical “and” operator. This is the CHSH Bell inequality used in the photon correlation experiments described in Shimony’s review article. It is a purely mathematical identity and does not depend on any physical considerations. It is also free of any considerations about statistical correlations and statistical independence. We will apply this inequality to the probabilities in the photon correlation experiments. Then $p(a_j \wedge b_k)$ becomes the probability for the event that photon 1 is transmitted by the filter A aligned along α_j while its buddy photon 2 is transmitted by the filter B aligned along β_k . The probabilities are thus identified with the mathematical expressions for the outcomes of the photon polarization experiments:

$$p(a_j \wedge b_k) = \frac{1}{2} \cos^2(\alpha_j - \beta_k), \quad p(a_j) = \frac{1}{2}, \quad p(b_k) = \frac{1}{2}, \quad (10)$$

where α_j and β_k are the angles of the polarizer settings in the two arms of the experiment. There is a loophole in this derivation, viz. that it cannot be taken for granted that there exists a *common* probability distribution $F(V, [0, \infty])$ from which it would be possible to derive the values of the four quantities $p(a_j \wedge b_k)$ simultaneously by integration over Eq.

7 (See Eqs. 7 and 8 in [2]). From the viewpoint of mathematical rigour the introduction of such a common probability distribution would require an existence proof. A physicist might consider this remark as mathematical faultfinding. The existence of a common probability distribution is certainly a necessary condition for the four quantities $p(\alpha_j \wedge \beta_k)$ to exist simultaneously. This is in turn a necessary condition for one being able to derive the Bell inequality for them.

But the considerations in Section 1 show that the simultaneous existence of these four quantities cannot be taken for granted. This is e.g. what quantum mechanics would state when we measure quantities that correspond to non-commuting operators. To many people such a no-go zone is conceptually purely quantum mechanical. But we have seen in Section 1 that this is not true. Commutation relations also occur in Lie algebras. Furthermore a polarizer cannot be aligned in two different orientations at the same time. The latter remark cannot be considered as outside the scope of classical reasoning. In this respect many authors have used the concept of contextuality to signal this remark.

3 The correct inequality

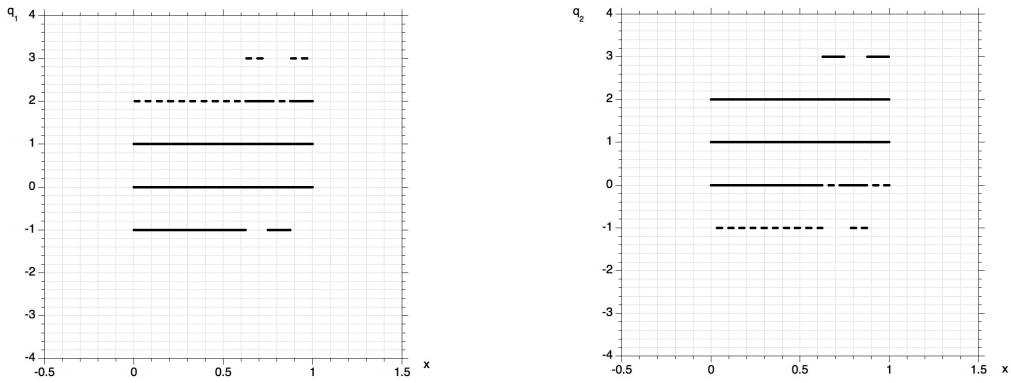


Fig. 1: The quantities q_1 and q_2 as defined in Eq. 11 as a function of the variable $x = m/65536$ which labels the 65536 combinations of the 16 variables $a_1^{(\ell)}, a_2^{(\ell)}, b_1^{(\ell)}, b_2^{(\ell)}, \ell \in \{1, 2, 3, 4\}$. It can be clearly seen that these quantities are no longer confined to the set $\{0, 1\}$.

Note that in Eq. 6 the number a_1 in $a_1 b_1$ is the same as in $a_1 b_2$, implying that $a_1 b_1$ and $a_1 b_2$ must be measured on a same photon pair. The equation implies that all numbers a_1, a_2, b_1, b_2 must be measured on a same photon pair. But as combinations of filter orientations (α_1, β_1) and (α_1, β_2) cannot exist simultaneously, the photon pairs involved in measuring $p(a_1 \wedge b_1)$ and $p(a_1 \wedge b_2)$ must be different. Therefore in Eq. 9 the probabilities $p(a_1 \wedge b_1)$ and $p(a_1 \wedge b_2)$ are forcedly determined by measuring quantities $a_1 b_1$ and $a'_1 b'_2$ on different photon pairs, in contradiction with the assumption expressed in Eq. 6, which requires a photon to have simultaneously two different linear polarizations. One of these polarizations in Eq. 6 is then revealed by measurement, while the (latent) existence of the other one is not revealed but supposed to be granted by determinism. Even if it might be possible to obtain the value of the latent variable by deterministic reasoning it could not possibly contribute to the statistics (as implied by the algebra) because it is not counted in the set-up wherein it remains latent.

The correct Bell inequality for the experiments would consist in considering 16 numbers $a_1^{(\ell)}, a_2^{(\ell)}, b_1^{(\ell)}, b_2^{(\ell)}, \ell \in \{1, 2, 3, 4\}$ where ℓ is an index for the set-up used. This would take into account that the quantities $p(a_j \wedge b_k)$ are all measured with different photon pairs. Instead of Q as defined in Eq. 6 we can define:

$$\begin{aligned}
 q_1 &= a_1^{(1)} b_1^{(1)} - a_1^{(2)} b_2^{(2)} - a_2^{(3)} b_1^{(3)} - a_2^{(4)} b_2^{(4)} + a_2^{(3)} + b_2^{(2)}, \\
 q_2 &= a_1^{(1)} b_1^{(1)} - a_1^{(2)} b_2^{(2)} - a_2^{(3)} b_1^{(3)} - a_2^{(4)} b_2^{(4)} + a_2^{(3)} + b_2^{(4)}, \\
 q_3 &= a_1^{(1)} b_1^{(1)} - a_1^{(2)} b_2^{(2)} - a_2^{(3)} b_1^{(3)} - a_2^{(4)} b_2^{(4)} + a_2^{(4)} + b_2^{(2)}, \\
 q_4 &= a_1^{(1)} b_1^{(1)} - a_1^{(2)} b_2^{(2)} - a_2^{(3)} b_1^{(3)} - a_2^{(4)} b_2^{(4)} + a_2^{(4)} + b_2^{(4)}.
 \end{aligned} \tag{11}$$

Here the four possibilities correspond to four different choices for the measurements from which we use a_2 and b_2 to complete the inequality. There are thus $2^{16} = 65536$ combinations of the numbers $a_1^{(\ell)}, a_2^{(\ell)}, b_1^{(\ell)}, b_2^{(\ell)}, \ell \in \{1, 2, 3, 4\}$ to

be considered. In Fig. 1-2 we represent the graphs of the results of these calculations. The x -axis corresponds each time to $x = m/65536$ where m is the label $m \in [1, 65536] \cap \mathbb{N}$ of the combination. None of the values q_j is limited to the set $\{0, 1\}$. In fact they explore now a larger range because the data occurring in q_j are less strongly correlated than those in Q_ℓ (defined in Eq. 12), which correspond to the pristine Bell inequalities.

When we use the filter orientation α_1 the number a_2 becomes redundant in the sense that it is then not used in Eq. 11. It represents a potentiality that does not become reality by counting. We could therefore have limited ourselves to the 8 parameters $(a_1^{(1)}, b_1^{(1)}, a_1^{(2)}, b_2^{(2)}, a_2^{(3)}, b_1^{(3)}, a_2^{(4)}, b_2^{(4)})$ instead of the 16 numbers, because the other 8 numbers are not being used in Eq. 11. But the formulation with 16 numbers is robust against any permutation of the labels $(1, 2, 3, 4)$ we attribute to the 4 combinations of filter settings. It would only require adapting the definitions of q_j accordingly. Keeping these redundant quantities permits also performing the following cross-check, which we think is enlightening.

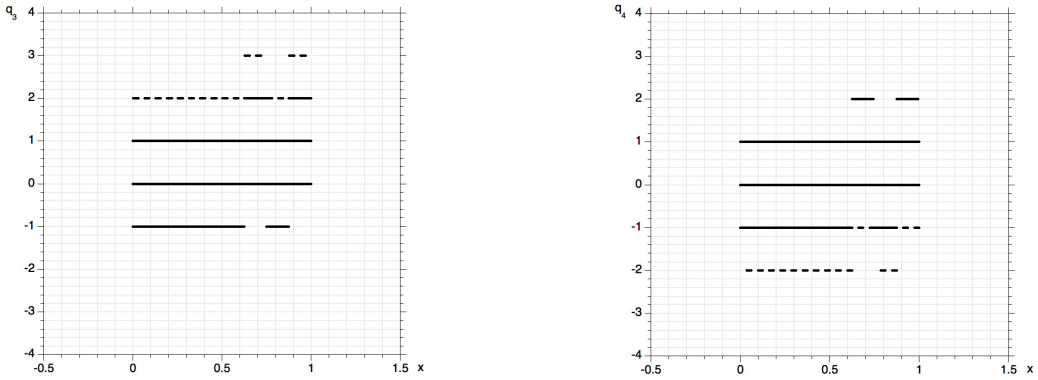


Fig. 2: The quantities q_3 and q_4 as defined in Eq. 11 as a function of the variable $x = m/65536$ which labels the 65536 combinations of the 16 variables $a_1^{(\ell)}, a_2^{(\ell)}, b_1^{(\ell)}, b_2^{(\ell)}, \ell \in \{1, 2, 3, 4\}$. It can be clearly seen that these quantities are no longer confined to the set $\{0, 1\}$.

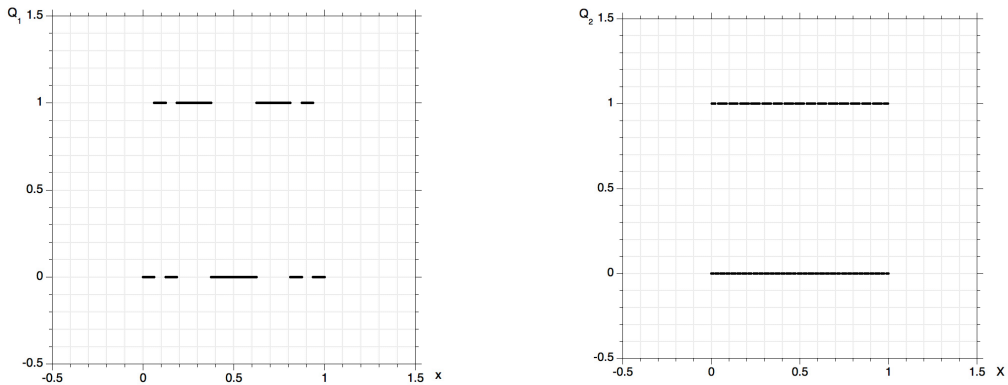


Fig. 3: The quantities Q_1 and Q_2 as defined in Eq. 12 as a function of the variable $x = m/65536$ which labels the 65536 combinations of 4 variables $a_1^{(\ell)}, a_2^{(\ell)}, b_1^{(\ell)}, b_2^{(\ell)}, \ell \in \{1, 2, 3, 4\}$, for $\ell = 1$ and $\ell = 2$. Now the quantities remain confined to the set $\{0, 1\}$, which is the basis for the CHSH pristine Bell inequality.

Let us define Q_ℓ by:

$$Q_\ell = a_1^{(\ell)} b_1^{(\ell)} - a_1^{(\ell)} b_2^{(\ell)} - a_2^{(\ell)} b_1^{(\ell)} - a_2^{(\ell)} b_2^{(\ell)} + a_2^{(\ell)} + b_2^{(\ell)}, \quad \forall \ell \in \{1, 2, 3, 4\}. \quad (12)$$

This corresponds to the traditional Bell inequalities for the four set-ups. For comparison we show in Figs. 3-4 also the graphs for the calculations for Q_ℓ on the same table of 65536 combinations. Now all values Q_ℓ neatly belong to $\{0, 1\}$.

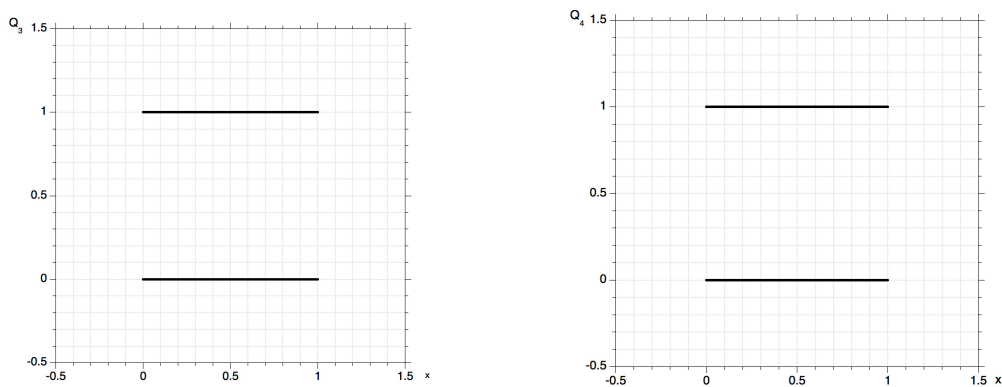


Fig. 4: The quantities Q_3 and Q_4 as defined in Eq. 12 as a function of the variable $x = m/65536$ which labels the 65536 combinations of 4 variables $a_1^{(\ell)}, a_2^{(\ell)}, b_1^{(\ell)}, b_2^{(\ell)}, \ell \in \{1, 2, 3, 4\}$, for $\ell = 3$ and $\ell = 4$. Now the quantities remain confined to the set $\{0, 1\}$, which is the basis for the pristine CHSH Bell inequality.

Hence the traditional Bell inequalities remain correct but they are not tested by the experiments. By just using the simple arguments used in the traditional derivation we can now no longer prove that the boundaries of the testable inequality are as stringent as has been claimed. These results show clearly that a wrong Bell inequality has been applied to the experiments. That the application of the CHSH Bell inequality to the photon correlation experiments was not correct had already been pointed out by many authors, starting with Kupczynski back in 1987 (see [6, 7] and the many references therein). We think that the situation must now be very clear. There is absolutely nothing more to say.

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