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# On Up-to Context Techniques in the $\pi$ -calculus

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## Abstract

We present a variant of the theory of compatible functions on relations, due to Sangiorgi and Pous. We show that the up-to context proof technique for bisimulation is compatible in this setting for two subsets of the pi-calculus: the asynchronous pi-calculus and a pi-calculus with immediately available names.

Proving that two elements are bisimilar is usually done by relying on a relation that is a bisimulation and also contains the corresponding pair of elements. Up-to techniques provide a powerful way of simplifying such proofs, by requiring that a relation is only included in a bisimulation. One such example is the up-to context technique which allows us to remove contexts when playing along the bisimulation game. A general theory of those techniques is developed in [2], by focusing on the class of compatible functions on relations, that are both sound up-to techniques and have nice compositional properties.

In the  $\pi$ -calculus, up-to context is not a sound technique. In fact, bisimilarity is not even a congruence, due to the input prefix creating substitutions. However, in subcalculi like the Asynchronous  $\pi$ -calculus ( $A\pi$ ), bisimilarity is closed by substitution making it a congruence. Thus, the question of soundness of the up-to context technique for this subcalculus arises again. It is known that up-to substitution is not compatible, and not even below the greatest compatible function (called the *companion* in [2]). Thus it seemed that even if up-to substitution is sound, it could not be used in conjunction with other techniques without having to redo the proofs all over again.

Intuitively, the reason why compatibility fails for up-to substitution is that compatibility assumes the knowledge about one step of transitions in the bisimulation game, while in the proof of the congruence for  $A\pi$ , the substitution is dealt with by looking at two successive transitions to deduce the behaviour of the next step of the program after substitution. More precisely, we need to look at two visible transitions to reason about an internal step. There is thus a distinction to be made between visible and internal steps which leads us to define the usual bisimulation function as the intersection of the two bisimulation functions represented by the diagrams below, with  $\alpha$  ranging over visible actions.

$$\begin{array}{ccc} P & \mathcal{R} & Q \\ \downarrow \alpha & & \downarrow \alpha \\ P' & \mathcal{R} & Q' \end{array} \qquad \begin{array}{ccc} P & \mathcal{R} & Q \\ \downarrow \tau & & \downarrow \tau \\ P' & \mathcal{R} & Q' \end{array}$$

In this paper, we propose a new notion of compatibility for a bisimulation function defined as  $f \cap g$  (above  $f$  would be the visible actions while  $g$  would be the silent ones). The key idea is to impose a stronger condition on  $f$  and a weaker condition on  $g$  while preserving soundness. This allows us to define a framework where standard up-to techniques, including full up-to context, are both compatible and sound.

We show this result for two subcalculi where bisimilarity is a congruence and similarly for weak bisimilarity.

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# 1 Compatibility and Soundness

In this section, we present some standard results about compatibility and their usage to show the soundness of up-to techniques (Section 1.1). Then we introduce *compatibility with a function* that is a generalisation of compatibility (Section 1.2). This notion still provides a soundness result while keeping nice properties of compatible functions (like being composable).

## 1.1 Previous work

Here, we recall standard results for compatibility from [2].

**Definition 1** (Compatibility).  $f$  is  $g$ -compatible if  $f \circ g \subseteq g \circ f$

**Definition 2** (Soundness).  $f$  is  $g$ -sound via  $f'$  if  $f'$  is extensive and  $\mathcal{R} \subseteq (g \circ f)(\mathcal{R})$  implies  $f'(\mathcal{R}) \subseteq b(f'(\mathcal{R}))$

Compatible functions can be composed freely in a modular fashion.

**Lemma 3.** If  $f_1, f_2$  are  $g$ -compatible, then  $f_1 \circ f_2$  is  $g$ -compatible.

If  $g$  is monotone, we also have that  $f_1 \cup f_2$  is  $g$ -compatible.

Compatible functions are useful as they are sound up-to techniques.

**Lemma 4.** If  $f$  is monotone and  $g$ -compatible, then  $f$  is  $g$ -sound via  $f^\omega$ .

However, there are sound up-to techniques that are not exactly compatible. We can recover some of them using compatibility up-to.

**Definition 5** (Compatible up-to).  $f$  is  $g$ -compatible up to  $f'$  when  $f'$  is expansive and  $f \circ g \subseteq g \circ f' \circ f$ .

Compatible functions up to  $f'$  can be related to compatible functions when  $f'$  is also compatible ensuring the soundness of such functions.

**Lemma 6.** If  $f'$  is idempotent, monotone and expansive,  $g$ -compatible and  $f$  is  $g$ -compatible up to  $f'$ , then  $f' \circ f$  is  $g$ -compatible.

*Proof.*

$$\begin{array}{ll}
 f \circ g \subseteq g \circ f' \circ f & \\
 f' \circ f \circ g \subseteq f' \circ g \circ f' \circ f & f' \text{ is monotone} \\
 \subseteq g \circ f' \circ f' \circ f & f' \text{ is } g\text{-compatible} \\
 \subseteq g \circ f' \circ f & f' \text{ is idempotent}
 \end{array}$$

□

In fact, compatible functions are a subset of compatible functions up to  $f'$  for any expansive  $f'$ .

*Remark 7.* If  $f'$  is expansive,  $g$  is monotone and  $f$  is  $g$ -compatible, then  $f$  is also  $g$ -compatible up to  $f'$ .

## 1.2 Compatibility with a function

Unfortunately, substitution is not a compatible function not even up to some compatible  $f'$ . To see why, we call  $b$  the bisimulation function associated to bisimilarity for the  $\pi$ -calculus (see Section 2.1). We need the following lemma where  $\top$  is the universal relation:

**Lemma 8.** Taking notations from CCS, we have  $(a.\bar{c} \mid \bar{c}, a \mid \bar{c}) \in b^2(\top)$  but  $((a.\bar{c} \mid \bar{c})\{a/c\}, (a \mid \bar{c})\{a/c\}) \notin b^2(\top)$

*Proof.* First, by definition  $(\bar{c}, \mathbf{0}) \in \top$  so  $(a, \bar{c}, a), (\bar{c} \mid \bar{c}, \bar{c}) \in b(\top)$ . Therefore,  $(a, \bar{c} \mid \bar{c}, a \mid \bar{c}) \in b^2(\top)$ .

Then, we have  $(a, \bar{a} \mid \bar{a})\{a/c\} = a, \bar{a} \mid \bar{a} \xrightarrow{\tau} \bar{a}$  and the only transition that the second process can do is  $(a \mid \bar{a})\{a/c\} = a \mid \bar{a} \xrightarrow{\tau} \mathbf{0}$ . Thus, as  $\bar{a} \xrightarrow{a}$  but  $\mathbf{0} \not\xrightarrow{a}$ , we have  $(\bar{a}, \mathbf{0}) \notin b(\top)$ , meaning that  $(a, \bar{a} \mid \bar{a}, a \mid \bar{a}) \notin b^2(\top)$ .  $\square$

Being compatible up-to some compatible function implies being smaller than some other compatible function ( $f \subseteq f' \circ f$  in Lemma 5). So it is enough to show that substitution is not included in the companion  $t$ , which is the greatest compatible function. As  $t$  is compatible, we have that  $t \circ b^2(\top) \subseteq b^2(\top)$ . Thus, Lemma 8 implies that  $\text{sub}(b^2(\top)) \not\subseteq t(b^2(\top))$ . This entails that up-to context is not compatible in the  $\pi$ -calculus.

This example is asynchronous and as we will see later, up-to substitution is sound for  $\Lambda\pi$ . Our goal is to adapt the notion of compatibility so that it captures up-to substitution.

The main idea is to proceed in two steps. We first focus on visible actions and establish compatibility with respect to the corresponding bisimulation function. Then, we exploit that result to prove a weaker version of compatibility on internal actions, which we define now.

**Definition 9** (Compatibility with). We say that  $f$  is  $h$ -compatible with  $g$  (or  $g, h$ -compatible) if  $f \circ (g \cap h) \subseteq h \circ f$ .

Intuitively, with the knowledge we have about  $g$ , we are able to prove a sort of compatibility result on  $h$ .

The main use of “compatibility with” is to prove  $g \cap h$ -soundness, in which case we do not need to prove exactly  $g, h$ -compatibility. For instance  $g$ -compatibility and  $g^2, h$ -compatibility is sufficient (see Theorem 15).

In Section 2, we exploit this approach taking  $g$  as the bisimulation function restricted to visible actions and  $h$  the one restricted to internal actions.

“Compatibility with” can be linked back to standard compatibility as follows:

**Lemma 10.** If  $f$  is  $g \cap h$ -compatible, then  $f$  is  $g, h$ -compatible and  $h, g$ -compatible.

If  $f$  is monotone,  $g$ -compatible and  $g, h$ -compatible, then  $f$  is  $g \cap h$ -compatible.

*Proof.*

- $f \circ (g \cap h) \subseteq (g \cap h) \circ f \subseteq g \circ f$  (and  $h \circ f$  respectively)
- The following proof uses Lemma 11, presented below.

$$\begin{aligned}
f \circ (g \cap h) &= f \circ (g \cap (g \cap h)) \\
&\subseteq f \circ g \cap f \circ (g \cap h) && \text{by Lemma 11} \\
&\subseteq g \circ f \cap f \circ (g \cap h) && f \text{ is } g\text{-compatible} \\
&\subseteq g \circ f \cap h \circ f && f \text{ is } g, h\text{-compatible} \\
&= (g \cap h) \circ f
\end{aligned}$$

$\square$

If  $f$  is a compatible function for  $g \cap h$ , like it is the case in Section 2, we cannot show directly that  $f$  is both  $g$ -compatible and  $g, h$ -compatible. Indeed, we only have the latter by Lemma 10. We show in Section 2 that in the case of the  $\pi$ -calculus, things can be handled smoothly.

Since we often use monotone functions and intersections, we rely on the following lemma:

**Lemma 11.** If  $f$  is monotone, then for any set  $A, B$ ,  $f(A \cap B) \subseteq f(A) \cap f(B)$ . Similarly, for any function  $g, h$ ,  $f \circ (g \cap h) \subseteq f \circ h \cap f \circ g$ .

Conversely, for any  $g, h$ ,  $f \circ g \cup f \circ h \subseteq f \circ (g \cup h)$ .

We can also build the composition and union of  $g, h$ -compatible functions under mild assumptions.

**Lemma 12.** If  $f_1, f_2$  are  $g$ -compatible,  $g, h$ -compatible and monotone, then  $f_1 \circ f_2$  is  $g, h$ -compatible.

*Proof.* By Lemma 10,  $f_1$  and  $f_2$  are  $g \cap h$ -compatible, so  $f_1 \circ f_2$  is too, meaning that  $f_1 \circ f_2$  is  $g, h$ -compatible.  $\square$

**Lemma 13.** If  $f_1, f_2, h$  are monotone and  $f_1, f_2$  are both  $g, h$ -compatible, then  $f_1 \cup f_2$  is  $g, h$ -compatible.

*Proof.*

$$\begin{aligned} (f_1 \cup f_2) \circ (g \cap h) &= f_1 \circ (g \cap h) \cup f_2 \circ (g \cap h) \\ &\subseteq h \circ f_1 \cup h \circ f_2 && f_1, f_2 \text{ are } g, h\text{-compatible} \\ &\subseteq h \circ (f_1 \cup f_2) && \text{by Lemma 11} \end{aligned}$$

$\square$

**Corollary 14.** If  $f, h$  are monotone,  $g$ -compatible and  $g, h$ -compatible, then  $f^\omega$  is  $g, h$ -compatible.

Here, we state the theorem that is the equivalent of Lemma 4 for compatibility with. Intuitively, if  $f$  is  $g$ -compatible, then we can use  $g$  any number of times to show the compatibility of  $f$  with respect to  $h$  (i.e.  $f$  is  $g^m, h$ -compatible), and this is enough to prove that  $f$  is a sound up-to technique for  $g \cap h$ .

**Theorem 15.** If  $f, g, h$  are monotone and  $f$  is both  $g$ -compatible,  $g^m, h$ -compatible (with  $m \geq 1$ ) then  $f$  is  $g \cap h$ -sound via  $f'^\omega$  with  $f' = \bigcup_{i \leq m} f^i$ .

*Proof.* First, notice that  $\mathcal{R} \subseteq ((g \cap h) \circ f)(\mathcal{R})$  implies  $\mathcal{R} \subseteq (g \circ f)(\mathcal{R})$  and  $\mathcal{R} \subseteq (h \circ f)(\mathcal{R})$ .

On one side, we then have  $\mathcal{R} \subseteq (g \circ f)^m(\mathcal{R})$ , meaning  $\mathcal{R} \subseteq (g^m \circ f^m)(\mathcal{R})$  by compatibility and so  $\mathcal{R} \subseteq (g^m \circ f')(\mathcal{R})$ . On the other side, we have that  $\mathcal{R} \subseteq (h \circ f')(\mathcal{R})$  as  $h$  is monotone and  $m \geq 1$ . Therefore,  $\mathcal{R} \subseteq ((g^m \cap h) \circ f')(\mathcal{R})$ .

By Lemma 10,  $f$  is  $g^m \cap h$ -compatible, so  $f'$  is too. Thus  $f'$  is  $g^m \cap h$ -sound via  $f'^\omega$  meaning  $f'^\omega(\mathcal{R}) \subseteq (g^m \cap h)(f'^\omega(\mathcal{R}))$ . Therefore we have  $f'^\omega(\mathcal{R}) \subseteq h(f'^\omega(\mathcal{R}))$ . As  $f$  is  $g$ -compatible, we can also prove that  $f'^\omega(\mathcal{R}) \subseteq g(f'^\omega(\mathcal{R}))$ .

By combining the two, we obtain that  $f'^\omega(\mathcal{R}) \subseteq (g \cap h)(f'^\omega(\mathcal{R}))$ .

$\square$

To prove  $g \cap h$ -soundness, we are thus interested in showing  $g$ -compatibility, and  $g^m, h$ -compatibility. By showing a weakening of compatibility with, we are able to compose functions with different  $m$ .

**Lemma 16.** If  $f$  is monotone, is  $g, h$ -compatible, and  $g' \subseteq g$ , then  $f$  is  $g', h$ -compatible.

*Proof.* As  $f$  is monotone,  $f \circ (g' \cap h) \subseteq f \circ (g \cap h)$ . So  $f \circ (g' \cap h) \subseteq h \circ f$ .  $\square$

**Lemma 17.** If  $f_1, f_2$  are monotone,  $g$ -compatible,  $f_1$  is  $g^m, h$ -compatible,  $f_2$  is  $g^n, h$ -compatible with  $m \geq n$  and  $g \subseteq id$ , then  $f_1 \circ f_2$  and  $f_2 \circ f_1$  are  $g^m, h$ -compatible.

*Proof.* As  $g \subseteq id$ , so is  $g^{m-n}$  meaning  $g^m \subseteq g^n$ . Thus, we have that  $f_2$  is  $g^m, h$ -compatible.

Also,  $f_1, f_2$  being  $g$ -compatible, they are also  $g^m$ -compatible.

Therefore, by Lemma 12,  $f_1 \circ f_2$  and  $f_2 \circ f_1$  are  $g^m, h$ -compatible.  $\square$

Compatibility with can also be combined with compatibility up to (see Definition 5):

**Definition 18.**  $f$  is  $g, h$ -compatible up to  $f'$  when  $f'$  is expansive and  $f \circ (g \cap h) \subseteq h \circ f' \circ f$ .

**Lemma 19.** If  $f'$  is idempotent, monotone,  $g, h$ -compatible and  $f$  is  $g$ -compatible up to  $f'$ ,  $g, h$ -compatible up to  $f'$ , then  $f' \circ f$  is  $g, h$ -compatible.

*Proof.*

$$\begin{aligned}
f \circ (g \cap h) &= f \circ (g \cap g \cap h) && \text{by Lemma 11} \\
&\subseteq f \circ g \cap f \circ (g \cap h) \\
&\subseteq g \circ f' \circ f \cap h \circ f' \circ f \\
&= (g \cap h) \circ f' \circ f \\
f' \circ f \circ (g \cap h) &\subseteq f' \circ (g \cap h) \circ f' \circ f && f' \text{ is monotone} \\
&\subseteq h \circ f' \circ f' \circ f && f' \text{ is } g, h\text{-compatible} \\
&\subseteq h \circ f' \circ f && f' \text{ is idempotent}
\end{aligned}$$

□

**Lemma 20.** If  $f'$  is idempotent,  $g, h$ -compatible up to  $f''$  and  $f$  is  $g, h$ -compatible up to  $f'$ , then  $f' \circ f$  is  $g, h$ -compatible up to  $f''$ .

*Proof.*

$$\begin{aligned}
f \circ (g \cap h) &= f \circ (g \cap g \cap h) && \text{by Lemma 11} \\
&\subseteq f \circ g \cap f \circ (g \cap h) \\
&\subseteq g \circ f' \circ f \cap h \circ f' \circ f \\
&= (g \cap h) \circ f' \circ f \\
f' \circ f \circ (g \cap h) &\subseteq f' \circ (g \cap h) \circ f' \circ f && f' \text{ is monotone} \\
&\subseteq h \circ f'' \circ f' \circ f' \circ f && f' \text{ is } g, h\text{-compatible up to } f'' \\
&\subseteq h \circ f'' \circ f' \circ f && f' \text{ is idempotent}
\end{aligned}$$

□

## 2 Up-to context in the $\pi$ -calculus

We apply the theory developed above to the  $\pi$ -calculus. We recall the syntax and operational semantics of the  $\pi$ -calculus in Figure 1 (symmetric transitions have omitted).

We now show the usage of compatibility with to prove the soundness of the up-to context techniques in subcalculi of the  $\pi$ -calculus. For that, we first show it using our framework on non-input contexts (Section 2.1). This result is not new, but along the way, we prove that these up-to context techniques are  $\bar{b}_\alpha$  which will be required to compose it later on. Then, in Section 2.2, we isolate the key property (Definition 26) that is needed to have the soundness of up-to substitution, and show how it gives the soundness result for up-to context. We end by giving two subcalculi, the Asynchronous  $\pi$ -calculus and a  $\pi$ -calculus with immediately available names, where this property holds, thus proving up-to context technique can be used for these calculi.

### 2.1 Up-to behavioural relations and evaluation contexts

We briefly recall the simulation  $(s, \dots)$  and bisimulation functions  $b, \bar{b}$  and introduce their weaker versions  $b_\alpha, b_\tau$  that only impose conditions on visible and internal actions respectively.

$$\begin{aligned}
P, Q &::= !G \mid P \mid Q \mid (\nu a)P \mid G \\
G, G' &::= \mathbf{0} \mid \bar{a}(b).P \mid a(b).P \mid \tau.P \mid G + G' \\
\alpha &::= \bar{a}(b) \mid \bar{a}(b) \mid a(b) \\
\mu &::= \alpha \mid \tau
\end{aligned}$$

$$\begin{array}{c}
\text{INP} \\
\hline
a(b).P \xrightarrow{\bar{a}(c)} P\{c/b\} \\
\\
\text{OUT} \\
\hline
\bar{a}(b).P \xrightarrow{\bar{a}(b)} P \\
\\
\text{REP} \\
\hline
!G \mid G \xrightarrow{\mu} G' \\
!G \xrightarrow{\mu} G' \\
\\
\text{OPEN} \\
\hline
\frac{P \xrightarrow{\bar{a}(b)} P'}{(\nu b)P \xrightarrow{\bar{a}(b)} P'} \text{ if } b \neq a \\
\\
\text{SUM} \\
\hline
\frac{G \xrightarrow{\mu} G'}{G + G'' \xrightarrow{\mu} G'} \\
\\
\text{RES} \\
\hline
\frac{P \xrightarrow{\mu} P'}{(\nu a)P \xrightarrow{\mu} (\nu a)P} \text{ if } a \notin \text{n}(\mu) \\
\\
\text{PAR} \\
\hline
\frac{P \xrightarrow{\mu} P'}{P \mid Q \xrightarrow{\mu} P' \mid Q} \text{ if } \text{bn}(\mu) \cap \text{fn}(Q) = \emptyset \\
\\
\text{COMM} \\
\hline
\frac{P \xrightarrow{a(b)} P' \quad Q \xrightarrow{\bar{a}(b)} Q'}{P \mid Q \xrightarrow{\tau} P' \mid Q'} \\
\\
\text{CLOSE} \\
\hline
\frac{P \xrightarrow{a(b)} P' \quad Q \xrightarrow{\bar{a}(b)} Q'}{P \mid Q \xrightarrow{\tau} (\nu b)(P' \mid Q')} \text{ if } b \notin \text{fn}(P)
\end{array}$$

Figure 1: Syntax and Early Labelled Transition System of the full  $\pi$ -calculus

$$\begin{aligned}
s(\mathcal{R}) &\stackrel{\text{def}}{=} \{(P, Q) \mid \text{for all } \mu, P', P \xrightarrow{\mu} P' \text{ implies there exists } Q' \text{ s.t. } Q \xrightarrow{\mu} Q', P' \mathcal{R} Q'\} \\
s_\alpha(\mathcal{R}) &\stackrel{\text{def}}{=} \{(P, Q) \mid \text{for all } \alpha, P', P \xrightarrow{\alpha} P' \text{ implies there exists } Q' \text{ s.t. } Q \xrightarrow{\alpha} Q', P' \mathcal{R} Q'\} \\
s_\tau(\mathcal{R}) &\stackrel{\text{def}}{=} \{(P, Q) \mid \text{for all } P', P \xrightarrow{\tau} P' \text{ implies there exists } Q' \text{ s.t. } Q \xrightarrow{\tau} Q', P' \mathcal{R} Q'\} \\
b(\mathcal{R}) &\stackrel{\text{def}}{=} s(\mathcal{R}) \cap s(\mathcal{R}^{-1}) \\
b_\alpha(\mathcal{R}) &\stackrel{\text{def}}{=} s_\alpha(\mathcal{R}) \cap s_\alpha(\mathcal{R}^{-1}) \\
b_\tau(\mathcal{R}) &\stackrel{\text{def}}{=} s_\tau(\mathcal{R}) \cap s_\tau(\mathcal{R}^{-1}) \\
\bar{b} &\stackrel{\text{def}}{=} id \cap b \\
\bar{b}_\alpha(\mathcal{R}) &\stackrel{\text{def}}{=} id \cap b_\alpha \\
\bar{b}_\tau(\mathcal{R}) &\stackrel{\text{def}}{=} id \cap b_\tau
\end{aligned}$$

The variant  $\bar{b}(\mathcal{R})$  only contains pairs that are also in  $\mathcal{R}$ , thus corresponding to the notion of respectfulness. As all the up-to techniques we use are monotone, any results for  $b_\alpha$  (resp.  $b_\tau$ ) also holds with their variant  $\bar{b}_\alpha$  (resp.  $\bar{b}_\tau$ ). We note  $\sim$  for the bisimilarity.

*Remark 21.*

- $b = b_\alpha \cap b_\tau$ ,  $\bar{b} = \bar{b}_\alpha \cap \bar{b}_\tau$ .
- All functions are monotone.
- $\bar{b}, \bar{b}_\alpha, \bar{b}_\tau \subseteq id$

We will now define some up-to techniques corresponding to evaluation contexts and prove their  $b_\alpha$ -compatibility and  $b_\tau$ -compatibility with  $b_\alpha$ .

$$\mathcal{F}_S(\mathcal{R}) = S\mathcal{R}S^{-1} \qquad \text{refl}(\mathcal{R}) = \{(P, P)\}$$

**Lemma 22.**  $\mathcal{F}_\sim, id, \mathbf{ref1}$  are  $b_\alpha$ -compatible.

Evaluation contexts contain parallel composition and restriction.

$$\mathbf{res}(\mathcal{R}) = \{((\nu\tilde{a})P, (\nu\tilde{a})Q) \mid P \mathcal{R} Q\} \quad \mathbf{pcomp}(\mathcal{R}) = \{(P \mid P', Q \mid Q') \mid P \mathcal{R} Q, P' \mathcal{R} Q'\}$$

**Lemma 23.**  $\mathbf{res}, \mathbf{pcomp}$  are  $b_\alpha$ -compatible.

By direct application of Lemma 10 and existing results of  $b$ -compatibility [1], we have that  $\mathcal{F}_\sim, id, \mathbf{ref1}, \mathbf{res}$  are  $b_\alpha, \bar{b}_\tau$ -compatible, and  $\mathbf{pcomp}$  is  $b_\alpha, \bar{b}_\tau$ -compatible up to  $\mathbf{res}$ .

Thus, we are able to take the union and compose while remaining sound according to Theorem 15.

**Corollary 24.**  $(id \cup \mathcal{F}_\sim \cup \mathbf{ref1} \cup \mathbf{res} \cup \mathbf{pcomp})^\omega$  is  $b$ -sound.

Corollary 24 is not new, but we obtain it via  $b_\alpha$ -compatibility and  $b_\alpha, \bar{b}_\tau$ -compatibility instead of  $b$ -compatibility. This is used below to compose those techniques with up-to substitution which is not  $b$ -compatible.

In fact, we can already go further and add the remaining non-input contexts.

$$\begin{aligned} \mathbf{tau}(\mathcal{R}) &= \{(\tau.P, \tau.Q) \mid P \mathcal{R} Q\} & \mathbf{out}(\mathcal{R}) &= \{(\bar{a}(b).P, \bar{a}(b).Q) \mid P \mathcal{R} Q\} \\ \mathbf{sum}(\mathcal{R}) &= \{(G_1 + G_2, G'_1 + G'_2) \mid G_1 \mathcal{R} G'_1, G_2 \mathcal{R} G'_2\} & \mathbf{rep}(\mathcal{R}) &= \{(!G, !G') \mid G \mathcal{R} G'\} \end{aligned}$$

**Lemma 25.**

- $\mathbf{tau}$ , is  $b_\alpha$ -compatible.
- $id \cup \mathbf{out}$  is  $\bar{b}_\alpha$ -compatible.
- $id \cup \mathbf{sum}$  is  $b_\alpha$ -compatible.
- $id \cup \mathbf{rep}$  is  $b_\alpha$ -compatible up-to  $\mathbf{pcomp} \cup id$

Similarly, using Lemma 10 and existing results, we have that  $id \cup \mathbf{tau}$  and  $id \cup \mathbf{out}$  are  $\bar{b}_\alpha, \bar{b}_\tau$ -compatible,  $id \cup \mathbf{sum}$  is  $\bar{b}_\alpha, \bar{b}_\tau$ -compatible and  $id \cup \mathbf{rep}$  is  $\bar{b}_\alpha, \bar{b}_\tau$ -compatible up to  $\mathbf{pcomp} \cup id$ .

## 2.2 Up-to substitution and input for subcalculi of $\pi$

Next up, we can add substitution and input related contexts. The substitution makes use of Theorem 15 with  $m > 1$ .

The proof requires an additional property that is not true in general in the  $\pi$ -calculus.

**Definition 26** (Aliased Communication Property). We say that a set of processes  $\mathcal{P}$  satisfies the *aliased communication property* if for all processes  $P$  in  $\mathcal{P}$ , we have the following properties:

- $P \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} P'$  implies  $P\sigma \xrightarrow{\tau} P'\sigma$  for all  $\sigma$  s.t.  $a\sigma = c\sigma$ .
- $P \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} P'$  implies  $P\sigma \xrightarrow{\tau} (\nu b)P'\sigma$  for all  $\sigma$  s.t.  $a\sigma = c\sigma$ .

This property is for instance satisfied in the asynchronous  $\pi$ -calculus and used to show that bisimilarity on asynchronous  $\pi$ -terms is closed by substitution.

$$\mathbf{sub}(\mathcal{R}) = \{(P\sigma, Q\sigma) \mid P \mathcal{R} Q\} \quad \mathbf{inp}(\mathcal{R}) = \{(a(b).P, a(b).Q) \mid P \mathcal{R} Q\}$$

**Lemma 27.**



- **sub** is  $b_\alpha$ -compatible and  $b_\alpha^2, b_\tau$ -compatible up to  $\mathcal{F}_\equiv \circ \text{res}$ .
- **inp** is  $\bar{b}_\alpha, b_\tau$ -compatible.
- $id \cup \text{inp}$  is  $\bar{b}_\alpha$ -compatible up to **sub**.

*Proof.*

- We rely on [4, Lemma 1.4.13]:

**Lemma 28.**

1. If  $P\sigma \xrightarrow{\alpha} P'$  then  $P \xrightarrow{\alpha'} P''$  for some  $\alpha', P''$  with  $\alpha'\sigma = \alpha$  and  $P''\sigma = P'$ .
2. If  $P\sigma \xrightarrow{\tau} P'$  then
  - (a)  $P \xrightarrow{\tau} P''$  for some  $P''$  with  $P''\sigma = P'$ , or
  - (b)  $P \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} P''$  for some  $P'', a, b, c$  with  $a\sigma = c\sigma$  and  $P''\sigma \equiv P'$ , or
  - (c)  $P \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} P''$  for some  $P'', a, b, c$  with  $a\sigma = c\sigma$  and  $(\nu b)P''\sigma \equiv P'$ .

$b_\alpha$ -compatibility follows from the lemma and that if  $Q \xrightarrow{\alpha'} Q'$ , then  $Q\sigma \xrightarrow{\alpha'\sigma} Q'\sigma$ .

Take a relation  $R$  and some processes  $P, Q$  such that  $(P, Q) \in b_\alpha^2 \cap b_\tau(\mathcal{R})$ . We want to show that for all  $\sigma$ ,  $(P\sigma, Q\sigma) \in b_\tau \circ \mathcal{F}_\equiv \circ \text{res} \circ \text{sub}(\mathcal{R})$ .

Take  $P\sigma \xrightarrow{\tau} P'$ , by the lemma, we have three cases:

1. either  $P \xrightarrow{\tau} P''$  for some  $P''$  with  $P''\sigma = P'$ . Then, as  $(P, Q) \in b_\tau(\mathcal{R})$ ,  $Q \xrightarrow{\tau} Q'$  and  $P' \mathcal{R} Q'$ . Thus,  $Q\sigma \xrightarrow{\tau} Q'\sigma$  and  $(P'\sigma, Q'\sigma) \in \text{sub}(\mathcal{R}) \subseteq \mathcal{F}_\equiv \circ \text{res} \circ \text{sub}(\mathcal{R})$ .
  2. or  $P \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} P''$  for some  $P'', a, b, c$  with  $a\sigma = c\sigma$  and  $P''\sigma \equiv P'$ . Then, as  $(P, Q) \in b_\alpha^2(\mathcal{R})$ ,  $Q \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} Q''$  and  $P'' \mathcal{R} Q''$ . Thus, by the Aliased Communication Property  $Q\sigma \xrightarrow{\tau} Q''\sigma$  and  $(P', Q''\sigma) \in \mathcal{F}_\equiv \circ \text{sub}(\mathcal{R}) \subseteq \mathcal{F}_\equiv \circ \text{res} \circ \text{sub}(\mathcal{R})$ .
  3. or  $P \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} P''$  for some  $P'', a, b, c$  with  $a\sigma = c\sigma$  and  $(\nu b)P''\sigma \equiv P'$ . Then, as  $(P, Q) \in b_\alpha^2(\mathcal{R})$ ,  $Q \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} Q''$  and  $P'' \mathcal{R} Q''$ . Thus, by the Aliased Communication Property  $Q\sigma \xrightarrow{\tau} (\nu b)Q''\sigma$  and  $(P', Q''\sigma) \in \mathcal{F}_\equiv \circ \text{res} \circ \text{sub}(\mathcal{R})$ .
- Trivial (no transition)
  - We can prove  $\text{inp} \subseteq b_\alpha \circ \text{sub}$ .

Then,

$$\begin{aligned}
(id \cup \text{inp}) \circ \bar{b}_\alpha &= \bar{b}_\alpha \cup \text{inp} \circ \bar{b}_\alpha \\
&\subseteq b_\alpha \cup b_\alpha \circ \text{sub} \circ \bar{b}_\alpha \\
&\subseteq b_\alpha \circ (id \cup \text{sub} \circ \bar{b}_\alpha) && \text{by Lemma 11}
\end{aligned}$$

Thus we know that  $id \cup \text{sub} \circ \bar{b}_\alpha \subseteq \text{sub} \subseteq \text{sub} \circ (id \cup \text{inp})$ .

On the other hand, we have  $(id \cup \text{inp}) \circ \bar{b}_\alpha \subseteq (id \cup \text{inp}) \subseteq \text{sub} \circ (id \cup \text{inp})$ . So  $(id \cup \text{inp}) \circ \bar{b}_\alpha \subseteq b_\alpha \circ \text{sub} \circ (id \cup \text{inp}) \cap \text{sub} \circ (id \cup \text{inp}) = \bar{b}_\alpha \circ \text{sub} \circ (id \cup \text{inp})$ .

□

The property defined in Definition 26 is only used to show that **sub** is  $b_\alpha^2, b_\tau$ -compatible up to **res**. However, because the compatibility of **inp** is shown up to **sub**, the soundness of the corresponding technique relies on the compatibility result for **sub**.

Finally, if we aggregate all the results:

**Theorem 29.** If the aliased communication property holds, then  $(\mathcal{F}_\sim \cup id \cup \text{refl} \cup \text{sub} \cup \text{res} \cup \text{pcomp} \cup \text{sum} \cup \text{rep} \cup \text{tau} \cup \text{out} \cup \text{inp})^\omega$  is  $\bar{b}$ -sound.

## 2.3 Subcalculi satisfying the aliased communication property

We present two subcalculi satisfying the aliased communication property. The property does not hold in general because of processes like  $\bar{a}.b$ . Thus, we look at  $A\pi$ , where outputs cannot guard processes, and processes with immediately available names, where dually inputs cannot be guarded.

**Asynchronous  $\pi$ -calculus.** The asynchronous  $\pi$ -calculus is defined by imposing that outputs no longer guard a process, meaning that there are forbidden in sums and in  $\bar{a}\langle b \rangle.P$ , we have  $P = \mathbf{0}$ .

**Lemma 30.**  $A\pi$  satisfies the Aliased Communication Property.

*Proof.* This is the direct application of Lemma 5.3.2 (3) and (4) in [4].  $\square$

**Immediately available names.** Immediately available names may only be used in input as soon as the name is created. This is a weaker notion than linear receptiveness or uniform receptiveness [3] which impose that exactly one input (resp. replicated input) must be accessible.

This discipline is formalised by the following typing rules where  $\Gamma$  is the set of name that can be used as input.

$$\frac{\emptyset \vdash P}{\Gamma \vdash \tau.P, \bar{a}\langle b \rangle.P} \quad \frac{\Gamma \vdash G}{\Gamma \vdash !G} \quad \frac{\emptyset \vdash P \quad a \in \Gamma}{\Gamma \vdash a(b).P} \quad \frac{\Gamma, a \vdash P}{\Gamma \vdash (\nu a)P} \quad \frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \mid Q} \quad \frac{\emptyset \vdash G \quad \emptyset \vdash G'}{\emptyset \vdash G + G'}$$

Note that because of the typing rule for sum, inputs are forbidden in sums.

Typable processes form a subcalculus of the  $\pi$ -calculus. Indeed, the set of typable processes is closed by transitions as expressed by the lemma below.

**Lemma 31** (Subject Reduction). If  $\Gamma \vdash P$  and  $P \xrightarrow{\mu} P'$ , then  $\Gamma \cup \text{bn}(\mu) \vdash P'$ .

**Lemma 32.** The set of typable processes satisfies the Aliased Communication Property.

As a consequence of Theorem 29, the up-to context techniques is sound for both subcalculi.

## 2.4 The weak case

We show how these results can be adapted to the weak case. The weak arrows are defined as usual:  $\Rightarrow \stackrel{\text{def}}{=} \tau^*$ ,  $\overset{\alpha}{\Rightarrow} \stackrel{\text{def}}{=} \alpha \Rightarrow$ ,  $\widehat{\alpha} \stackrel{\text{def}}{=} \alpha$  and  $\widehat{\tau} \stackrel{\text{def}}{=} \tau$ .

We define the simulation functions for the weak case  $ws, ws_\alpha, ws_\tau$ , the corresponding bisimulation functions  $wb, wb_\alpha, wb_\tau$ , and their variant  $\widehat{wb}, \widehat{wb}_\alpha, \widehat{wb}_\tau$  follow as expected.

$$\begin{aligned} ws(\mathcal{R}) &\stackrel{\text{def}}{=} \{(P, Q) \mid \text{for all } \mu, P', P \xrightarrow{\mu} P' \text{ implies there exists } Q' \text{ s.t } Q \xrightarrow{\widehat{\mu}} Q', P' \mathcal{R} Q'\} \\ ws_\alpha(\mathcal{R}) &\stackrel{\text{def}}{=} \{(P, Q) \mid \text{for all } \alpha, P', P \xrightarrow{\alpha} P' \text{ implies there exists } Q' \text{ s.t } Q \xrightarrow{\widehat{\alpha}} Q', P' \mathcal{R} Q'\} \\ ws_\tau(\mathcal{R}) &\stackrel{\text{def}}{=} \{(P, Q) \mid \text{for all } P', P \xrightarrow{\tau} P' \text{ implies there exists } Q' \text{ s.t } Q \Rightarrow Q', P' \mathcal{R} Q'\} \end{aligned}$$

Weak bisimilarity is noted  $\approx_a$ .

Most results true in the strong case also hold in the weak case. We give details about those whose statement or proof need to be adapted.

First, it is known that up-to weak bisimilarity is not a sound technique. However, we can still use up-to strong bisimilarity but also use the expansion preorder  $\succsim$ .

Take  $s'(\mathcal{R}) \stackrel{\text{def}}{=} \{(P, Q) \mid \text{for all } \mu, P', P \xrightarrow{\mu} P' \text{ implies there exists } Q' \text{ s.t } Q \xrightarrow{\widehat{\mu}} Q', P' \mathcal{R} Q'\}$  where  $\widehat{\alpha} \stackrel{\text{def}}{=} \alpha$  and  $\widehat{\tau} \stackrel{\text{def}}{=} \tau$ . Then  $\succsim$  is the largest relation  $\mathcal{R}$  such that  $\mathcal{R} \subseteq ws(\mathcal{R}) \cap s'(\mathcal{R}^{-1})$ .

**Lemma 33.**  $\mathcal{F}_{\sim}, \mathcal{F}_{\succ}$  is  $wb_{\alpha}$ -compatible.

The aliased communication property needs also to be changed to use weak arrows so that the proof of substitution goes without trouble.

**Definition 34** (Weak Aliased Communication Property). We say that a set of processes  $\mathcal{P}$  satisfies the *weak aliased communication property* if for all process  $P$  in  $\mathcal{P}$ , we have the following properties:

- $P \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} P'$  implies  $P\sigma \Rightarrow P'\sigma$  for all  $\sigma$  s.t.  $\sigma(a) = \sigma(c)$ .
- $P \xrightarrow{(\nu b)\bar{a}(b)} \xrightarrow{c(b)} P'$  implies  $P\sigma \Rightarrow (\nu b)P'\sigma$  for all  $\sigma$  s.t.  $\sigma(a) = \sigma(c)$ .

Weak bisimilarity is not a congruence for sum. Indeed, we have  $\tau. a \approx_a a$  but  $\tau. a+b \not\approx_a a+b$ . Congruence is usually recovered by considering *non-degenerate* contexts, that is, contexts where the hole is not directly under a sum operator. Therefore, we want to prove the soundness of up-to non-degenerate contexts, and thus we use the up-to guarded sum technique instead of the previous up-to sum technique:

$$\text{sum}_{\mathbf{g}}(\mathcal{R}) = \{(\sum_i G_i, \sum_i G'_i) \mid \forall i, (G_i, G'_i) \in (\mathbf{tau} \cup \mathbf{out} \cup \mathbf{inp} \cup \mathbf{refl})(\mathcal{R})\}$$

**Lemma 35.**  $id \cup \text{sum}_{\mathbf{g}}$  is  $\bar{b}_{\alpha}$ -compatible up to  $\mathbf{sub} \cup \mathbf{refl}$  and  $\bar{b}_{\alpha}, \bar{b}_{\tau}$ -compatible up to  $id \cup \mathbf{refl}$ .

*Proof.* We prove  $\text{sum}_{\mathbf{g}} \subseteq b_{\tau} \circ (id \cup \mathbf{refl})$  and  $\text{sum}_{\mathbf{g}} \subseteq b_{\alpha} \circ (\mathbf{sub} \cup \mathbf{refl})$ .

Suppose  $\sum_i G_i \xrightarrow{\mu} G'$ , then  $G_{i_0} \xrightarrow{\mu} G'$ .

- If  $(G_{i_0}, G'_{i_0}) \in \mathbf{refl}(\mathcal{R})$ , then  $\sum_i G'_i \xrightarrow{\mu} G'$  and  $(G', G') \in \mathbf{refl}(\mathcal{R})$
- If  $(G_{i_0}, G'_{i_0}) \in \mathbf{tau}(\mathcal{R})$ , then  $\mu = \tau$ , so  $G'_{i_0} = \tau. G''$  with  $G' \mathcal{R} G''$  and  $\sum_i G'_i \xrightarrow{\tau} G''$  and  $(G', G'') \in id(\mathcal{R})$
- If  $(G_{i_0}, G'_{i_0}) \in \mathbf{out}(\mathcal{R})$ , then  $\mu = \bar{a}(b)$ ,  $G_{i_0} = \bar{a}(b). G'$ . So  $G'_{i_0} = \bar{a}(b). G''$  with  $G' \mathcal{R} G''$  and  $\sum_i G'_i \xrightarrow{\mu} G''$  and  $(G', G'') \in id(\mathcal{R})$ .
- If  $(G_{i_0}, G'_{i_0}) \in \mathbf{inp}(\mathcal{R})$ , then  $\mu = a(c)$ ,  $G_{i_0} = a(b). G''$  with  $G' = G''\{c/b\}$ . So  $G'_{i_0} = a(b). G'''$  with  $G'' \mathcal{R} G'''$  and  $\sum_i G'_i \xrightarrow{\mu} G'''\{c/b\}$  and  $(G', G'''\{c/b\}) \in \mathbf{sub}(\mathcal{R})$ .

□

The proof for replication needs also to be changed. We show instead that  $id \cup \mathbf{rep}$  is  $\bar{b}_{\alpha}, \bar{b}_{\tau}$ -compatible up to  $\mathcal{F}_{\sim} \circ (\mathbf{pcomp} \cup id)$ . Intuitively, the problem is similar to the case of the sum, but because we have the law  $!G \sim !G \mid G$ , it does not break soundness.

The other proofs can be carried out without any modification, and we can then conclude with the soundness of the whole up-to technique:

**Theorem 36.** If the weak aliased communication property holds, then  $(\mathcal{F}_{\succ} \cup id \cup \mathbf{refl} \cup \mathbf{sub} \cup \mathbf{res} \cup \mathbf{pcomp} \cup \text{sum}_{\mathbf{g}} \cup \mathbf{rep} \cup \mathbf{tau} \cup \mathbf{out} \cup \mathbf{inp})^{\omega}$  is  $\overline{wb}$ -sound.

We can now show that both subcalculi also satisfy the weak aliased communication property.

In asynchronous  $\pi$ -calculus, we know that outputs, being asynchronous, may always be postponed as expressed below.

**Lemma 37.** If  $P \xrightarrow{\bar{a}(b)} \xrightarrow{\mu} P'$  then  $P \xrightarrow{\mu} \xrightarrow{\bar{a}(b)} P'$ .  
If  $P \xrightarrow{\bar{a}(b)} \xrightarrow{\mu} P'$  and  $b \notin \text{fn}(\mu)$ , then  $P \xrightarrow{\mu} \xrightarrow{\bar{a}(b)} P'$ .

Thus, if  $P \xrightarrow{\bar{a}(b)} \xrightarrow{c(b)} P'$ , then  $P \Rightarrow \bar{a}(b) \xrightarrow{c(b)} \Rightarrow P'$  so  $\Lambda\pi$  satisfies the weak aliased communication property.

For immediately available names, the reasoning is reversed. As inputs are immediately available, we can show that they can be preponed:

**Lemma 38.** If  $P \xrightarrow{\mu} \xrightarrow{a(b)} P'$  and  $a \notin \text{bn}(\mu)$ , then  $P \xrightarrow{a(b)} \xrightarrow{\mu} P'$ .

Again, this lemma ensures that if  $P \xrightarrow{\bar{p}(b)} \xrightarrow{q(b)} P'$ , then  $P \Rightarrow \bar{p}(b) \xrightarrow{q(b)} \Rightarrow P'$  and so we can conclude.

We can notice a symmetry between Lemmas 37 and 38, the former delays outputs while the latter anticipates inputs.

This shows that both calculi are also a congruence for the weak bisimilarity and that the up-to context technique is sound.

*Remark 39.* Note that, in the (weak) aliased communication property, we quantify over all names  $a, c$ , being the subject of the output and input respectively. If we impose that the property holds for only some names, for instance if only a subset of names are asynchronous or immediately available, then we have the soundness of up-to substitution restricted to those asynchronous names (resp. immediately available names), and up-to input that only carry asynchronous names (resp. immediately available names).

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## A Language with lookahead

Theorem 15 is sufficient to derive result with operator enabling after more than transitions.

Consider the language

$$P ::= \text{op}(P) \mid a.P \mid \mathbf{0}$$

with the following semantic

$$\frac{}{a.P \xrightarrow{a} P} \qquad \frac{P \xrightarrow{a} P'}{\text{op}(P) \xrightarrow{a} P'}$$

It is known that up-to-bisimilarity-and-context is unsound.

However, one could tweak this language to make it sound.

### A.1 Two prefixes

A first way would be to consider two prefixes  $a$  and  $b$ , with  $a.P \xrightarrow{a} P$  and  $b.P \xrightarrow{b} P$  but only  $\frac{P \xrightarrow{b} P'}{\text{op}_1(P) \xrightarrow{a} P'}$ .

One could define simply  $b_a, b_b$  two bisimulations using actions  $a$  and  $b$  respectively.

Then with  $\text{op}_1(\mathcal{R}) = \{(\text{op}_1(P), \text{op}_1(Q)) \mid P \mathcal{R} Q\}$ , we have  $\text{op}_1 \circ b_b \subseteq b_b \circ \text{op}_1$  (there cannot be any  $b$  transition) and  $\text{op}_1 \circ (b_b^2 \cap b_a) \subseteq b_a \circ \text{op}_1$ .

### A.2 Second version

First, we need to show a variation of Theorem 15. Here, we add the condition that  $g \subseteq id$  along with a rather technical condition  $(\star)$  (stated to remain as general as possible), which is always verified in practice (for instance, if  $f$  is expansive). It allows us to show the soundness when we add an extra  $h$  in front of  $g^m$ , i.e in the case where  $f$  is  $h \circ g^m, h$ -compatible.

**Theorem 40.** If  $f, g, h$  are monotone,  $g \subseteq id$  and  $f$  is both  $g$ -compatible,  $h \circ g^m, h$ -compatible (with  $m \geq 1$ ) and verifies the following:

$$\forall n \in \mathbb{N}, f^n \circ \left( \bigcup_{i \leq mn+1} f^i \right) \subseteq \bigcup_{i \in \mathbb{N}} f^i \quad (\star)$$

then  $f$  is  $g \cap h$ -sound via  $f^\omega$ .

*Proof.* Note that if  $\mathcal{R} \subseteq ((g \cap h) \circ f)(\mathcal{R})$  then  $\mathcal{R} \subseteq (g \circ f)(\mathcal{R})$  and  $\mathcal{R} \subseteq (h \circ f)(\mathcal{R})$ .

We already know that  $f$  is  $g$ -sound via  $f^\omega$  so  $f^\omega(\mathcal{R}) \subseteq g(f^\omega(\mathcal{R}))$ .

As  $g \subseteq id$ , we have  $h \circ g^m \cap h = h \circ g^m$ . Thus  $f \circ h \circ g^m \subseteq h \circ f$ .

Now, let's show that the same inclusion holds for  $h$ . We prove by induction on  $n$  that

$$f^n \circ h \circ g^{mn} \subseteq h \circ f^n \quad (\Delta)$$

For  $n = 0$ , trivial. For  $n \geq 0$ ,

$$\begin{aligned} f^{n+1} \circ (h \circ g^{m(n+1)}) &= f^n \circ f \circ h \circ g^m \circ g^{mn} \\ &\subseteq f^n \circ h \circ f \circ g^{mn} && f \text{ is } g^m, h\text{-compatible} \\ &\subseteq f^n \circ h \circ g^{mn} \circ f && f \text{ is } g\text{-compatible} \\ &\subseteq h \circ f^n \circ f && \text{by induction} \end{aligned}$$

By monotonicity of  $g$  and  $f$ , we can prove by a simple induction that  $\mathcal{R} \subseteq (g \circ f)(\mathcal{R})$  implies  $\mathcal{R} \subseteq (g \circ f)^{mn}(\mathcal{R})$ . Then, we have  $\mathcal{R} \subseteq (h \circ f)(\mathcal{R}) \subseteq (h \circ f) \circ (g \circ f)^{mn}(\mathcal{R})$

So  $\mathcal{R} \subseteq (h \circ g^{mn} \circ f^{mn+1})(\mathcal{R})$  by compatibility and if we note  $f_{mn+1}^\omega = \bigcup_{i \leq mn+1} f^i$ , as  $g$  and  $h$  are monotone we have  $\mathcal{R} \subseteq (h \circ g^{mn} \circ f_{mn+1}^\omega)(\mathcal{R})$ .

Thus:

$$\begin{aligned}
\mathcal{R} &\subseteq (h \circ g^{mn} \circ f_{mn+1}^\omega)(\mathcal{R}) \\
f^n(\mathcal{R}) &\subseteq (f^n \circ h \circ g^{mn} \circ f_{mn+1}^\omega)(\mathcal{R}) && f \text{ monotone} \\
&\subseteq (h \circ f^n \circ f_{mn+1}^\omega)(\mathcal{R}) && \text{by } (\Delta) \\
&\subseteq (h \circ f^\omega)(\mathcal{R}) && \text{by } (\star)
\end{aligned}$$

Therefore,  $f^\omega(\mathcal{R}) \subseteq h(f^\omega(\mathcal{R}))$

In the end, we have  $f^\omega(\mathcal{R}) \subseteq (g \cap h)(f^\omega(\mathcal{R}))$ . □

This new theorem allows us to create a new operator  $\text{op}_2$  with  $\frac{P \xrightarrow{a,b} P'}{\text{op}_2(P) \xrightarrow{a} P'}$  meaning it can now perform action  $a$  as its first action. In that case,  $\text{op}_2 \circ (b_a \circ b_b \cap b_a) \subseteq b_a \circ \text{op}_2$ .

We may also take  $\frac{P \xrightarrow{\mu,b} P'}{\text{op}_2(P) \xrightarrow{a} P'}$  with  $\mu \in \{a, b\}$  and, noting  $b$  for the whole bisimulation (i.e  $b_a \cap b_b$ ), show  $\text{op}_2 \circ (b \circ b_b \cap b) \subseteq b \circ \text{op}_2$ .

### A.3 Chaining further

Compared to the previous examples where we split a bisimulation  $b$  by splitting the set of actions in two  $b_\tau, b_\alpha$ , one could also build incrementally smaller bisimulation, for instance proving  $b_\alpha^m, b$ -compatibility instead of  $b_\alpha^m, b_\tau$ -compatibility.

This approach may require a bit more redundancy to prove compatibility results, but it does make statements easier to read. Here, we aim to decompose  $b$  using more than 2 functions, so we will use this incremental approach.

**Lemma 41.** If  $f$  is  $g_1 \cap g_2$ -compatible and  $g_2^n, g_3$ -compatible, then  $f$  is  $(g_1 \cap g_2)^n \cap g_3$ -compatible.

*Proof.*

$$\begin{aligned}
f \circ ((g_1 \cap g_2)^n \cap g_3) &= f \circ ((g_1 \cap g_2)^n \cap g_2^n \cap g_3) \\
&\subseteq f \circ (g_1 \cap g_2)^n \cap f \circ (g_2^n \cap g_3) && \text{by Lemma 11} \\
&\subseteq (g_1 \cap g_2)^n \circ f \cap g_3 \circ f && \text{by compatibility} \\
&= ((g_1 \cap g_2)^n \cap g_3) \circ f
\end{aligned}$$

□

This Lemma allows us to chain Lemma 10 into one compatible function. For simplicity, we will assume  $f$  is expansive.

**Theorem 42.** If  $f, g_i$  are monotone,  $f$  is expansive,  $g_1 \supseteq \dots \supseteq g_n$ , and  $f$  is  $g_1$ -compatible and for some  $(m_i)_{i \leq n}$  with  $m_i \geq 1$ ,  $g_i^{m_i}, g_{i+1}$ -compatible, then  $f$  is  $g_n$ -sound via  $f^\omega$ .

*Proof.* First, we define  $h_i$  with  $h_1 = g_1$  and  $h_{i+1} = g_{i+1} \cap h_i^{m_i}$ . We show by induction on  $i$  that  $f$  is  $h_i$ -compatible.

When  $i = 1$ , this is true by assumption.

For  $i = 2$ , as  $f$  is  $h_1$ -compatible, it is also  $h_1^{m_1}$ -compatible. Thus, by Lemma 10,  $f$  is  $g_2 \cap h_1^{m_1}$ -compatible.

For  $i \geq 2$ , as  $f$  is  $g_i \cap h_{i-1}^{m_{i-1}}$ -compatible and  $g_i^{m_i}, g_{i+1}$ -compatible, by Lemma 41,  $f$  is  $g_{i+1} \cap (g_i \cap h_{i-1}^{m_{i-1}})^{m_i}$ -compatible.

Next, we will show by induction that  $\mathcal{R} \subseteq (h_i \circ f^{n_i})(\mathcal{R})$  for some  $n_i \geq 1$ . As  $\mathcal{R} \subseteq (g_n \circ f)(\mathcal{R})$ , then  $\mathcal{R} \subseteq (g_i \circ f)(\mathcal{R})$  for all  $i$ , so  $\mathcal{R} \subseteq (h_1 \circ f)(\mathcal{R})$ .

Then if  $\mathcal{R} \subseteq (h_i \circ f^{n_i})(\mathcal{R})$ ,  $\mathcal{R} \subseteq (h_i \circ f^{n_i})^{m_i}(\mathcal{R})$ , and as  $f$  is  $h_i$ -compatible,  $\mathcal{R} \subseteq (h_i^{m_i} \circ f^{n_i * m_i})(\mathcal{R})$ . Additionally,  $\mathcal{R} \subseteq (g_{i+1} \circ f)(\mathcal{R})$ . As  $f$  is expansive,  $f \subseteq f^{n_i * m_i}$ , so  $\mathcal{R} \subseteq (g_{i+1} \circ f^{n_i * m_i})(\mathcal{R})$ .

Thus,  $\mathcal{R} \subseteq ((g_{i+1} \circ h_i^{m_i}) \circ f^{n_i * m_i})(\mathcal{R})$ .

To sum up,  $f$  is  $h_n$ -compatible and  $\mathcal{R} \subseteq (h_n \circ f^N)(\mathcal{R})$  for some  $N \geq 1$ . As  $(f^N)^\omega = f^\omega$ , we obtain that  $f^\omega \subseteq g_n(f^\omega(\mathcal{R}))$ .  $\square$

#### A.4 Unverified approaches

Having integers  $n \geq 1$  as prefixes, i.e.  $n. P \xrightarrow{n} P$ , and

$$\frac{P \xrightarrow{n} \xrightarrow{n} P'}{\text{op}_3(P) \xrightarrow{n+1} P'} \qquad \frac{P \xrightarrow{n} \xrightarrow{m} P'}{\text{op}_4(P) \xrightarrow{n+m} P'}$$

With  $b_n$  the bisimulation obtained by looking at transitions  $\xrightarrow{m}$  with  $m \leq n$ .

We should have  $\text{op}_i \circ b_1 \subseteq b_1 \circ \text{op}_i$  for  $i = 3, 4$ , and for all  $n \geq 1$ ,  $\text{op}_i \circ (b_n^2 \cap b_{n+1}) \subseteq b_{n+1} \circ \text{op}_i$  for  $i = 3, 4$ .

Thus, for all  $n \geq 1$ ,  $\text{op}_3$  and  $\text{op}_4$  are valid up-to techniques for  $b_n$ .

$$\frac{P \xrightarrow{n} \xrightarrow{m} P'}{\text{op}_5(P) \xrightarrow{n} P'} n > m$$

We should have  $\text{op}_5 \circ b_1 \subseteq b_1 \circ \text{op}_5$ , and for all  $n$ ,  $\text{op}_5 \circ (b_{n+1} \circ b_n \cap b_{n+1}) \subseteq b_{n+1} \circ \text{op}_5$ .